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FLIP BIFURCATION WITH RANDOM EXCITATION*

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Abstract In this paper, flip bifurcation with random excitation is studied by employing the methods of normal forms, Picard iterations and orthogonal polynomial approximation. For the codimension one case, a Neimark-Sacker bifurcation, a 1:2 resonance and a fold-flip bifurcation are detected. It is found that the system undergoes heteroclinic bifurcation and homoclinic bifurcation near 1:2 resonance point, a hopf bifurcation and a cusp bifurcation near fold-flip bifurcation point. For the codimension two case, the system undergoes only a flip bifurcation when random excitation is imposed on the nonlinear term. In addition, numerical simulations are given to show the disparity between the codimension one and two cases.

Keywords Flip bifurcation, random excitation, Neimark-Sacker bifurcation, 1:2 resonance, fold-flip bifurcation.

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1. Introduction

Flip bifurcation is also called period doubling bifurcation [11]. As the name suggests, period doubling means the cycle of period two which comes from the second iteration of its general map. Tracing the period-two cycle, it is found that there is an infinite sequence of bifurcation values $\mathcal{F}_{f_j}$, where $f_j = 2^j$ is the period of cycle. Observing the relationship among these values, the ratio

$$\frac{\mathcal{F}_{f_j} - \mathcal{F}_{f_{j-1}}}{\mathcal{F}_{f_{j+1}} - \mathcal{F}_{f_j}}$$

approaches to Feigenbaum constant [7]. At the limit of $\mathcal{F}_{f_j}$, the orbits turn to be irregular, nonperiodic and the system becomes chaotic. This phenomenon exists in many different systems where a cascade of flip bifurcations occurs. This universality, which has a deep reasoning, makes researchers to explore further in theoretical analysis [2,4,26] and applications in fields as biology, chemistry, physics and others [10,13,17,21].

Around the 1990s, Collet et al. [6] gave rigorous proof of the strong universality of analytic map for flip bifurcations. Combining with other bifurcation, Giraldo et al. [8] studied a specific bifurcation called a homoclinic flip bifurcation, which

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exhibited saddle periodic orbits, chaotic attractor. Afterwards, Ashish et al. [3] analyzed the chaotic behavior of the standard logistic map in superior orbit by using period doubling and time-series representations,

$$x_n = (1-\alpha)x_{n-1} + \alpha f_\lambda(x_{n-1}), \quad f_\lambda(x_{n-1}) = \lambda x_{n-1}(1-x_{n-1}),$$

where $\alpha \in (0,1)$ and λ is a positive constant parameter. In 2019, Ashish and Cao [1] proposed a more superior fixed point iterative strategy, which allows the freedom of two control parameters α, λ to establish the three dynamical phases for the above map. In 2020, Ma et al. [15] investigated an improved discrete Leslie-Gower predator-prey model with prey refuge and fear factor and explored when flip bifurcation and Neimark-Sacker bifurcation occur. In addition, these classical achievements in the deterministic system, systems with periodic excitation and random excitation draw more attention of people. For the first case, Li and Ren [14] considered a system which undergoes a cusp bifurcation and explored the effect of periodic perturbation by the Lyapunov-Schmidt method. Later on, He et al. [9] investigated an ecological system with state-dependent feedback control and periodic forcing, and revealed the complex dynamic behaviors of the system, including period-doubling bifurcations, period-halving bifurcations and others. For the second case, Ma et al. [16] introduced a known continuous random variable as bifurcation parameter in double-well Duffing system and their numerical results indicated the existence of flip bifurcation even in stochastic Duffing system.

The existing research about random excitation focus on specific systems which undergo flip bifurcation and uses numerical simulations to demonstrate the phenomenon. Due to the lack of research for the general flip bifurcation system with random excitation, we intend to make a theoretical analysis of its stochastic dynamics. It is self-evident that systems which undergo flip bifurcation can be covered by our system. However, it is hard to deal with the general flip bifurcation system with random excitation when the bifurcation parameter is replaced by a discrete random variable. Hence, we try to obtain its equivalent system based on the theory of functional analysis. Then we suppose a generalized orthogonal polynomial instead of some known orthogonal polynomial, such as Legendre orthogonal polynomials, Chebyshev orthogonal polynomials, Laguerre orthogonal polynomials, Hermite orthogonal polynomials [18, 19, 22], which is more general than specific polynomials. At the end, we give numerical simulations to visualize the theoretical results through using the Hermite orthogonal polynomials as an application.

Interestingly enough, codimension two bifurcations [20, 25] emerge after adding random excitation into codimension one flip bifurcation system, while there is only a flip bifurcation after adding random excitation into codimension two flip bifurcation system. More precisely, there is a wide gap when random parameter locates before the linear term and nonlinear term for codimension two flip bifurcation system. Its behaviors are similar to stochastic codimension one flip bifurcation when random excitation is affiliated with linear term, but it undergoes only a flip bifurcation when random excitation is imposed on nonlinear term. On the other hand, some obstacles arise during the process of exploring dynamics because the normal form is the most concise polynomial [24]. At fold-flip bifurcation point, we calculate its normal form without quadratic terms, which is essentially different from the equation (9) in Proposition 2.1.1 [12]. In the later reflection iteration of the map, we deal with a four-dimensional system when treating parameters as constants and obtain the dynamical behaviors of the approximate flow.

The paper is organized as follows. Section 2 includes some basic definitions and a lemma, which our study is based upon. In Section 3, we deduce the equivalent system for codimension one flip bifurcation with random excitation and investigate the dynamical behaviors of the system. At Neimark-Sacker point, we calculate the critical coefficient, which can ensure the stability of closed invariant curve. Then we acquire heteroclinic and homoclinic bifurcation curves as well as the uniqueness of limit cycle for the second iteration of 1:2 resonance point. At the fold-flip bifurcation point, we obtain the expressions of a hopf bifurcation curve and a cusp bifurcation curve. In Section 4, we study the stochastic codimension two flip bifurcation with random excitation and make simulations to illustrate the disparity between codimension one and two cases.

2. Preliminaries

In this section, we give some basic definitions and a lemma which our study is based upon.

Definition 2.1 (A weight standard orthogonal polynomial [5])**.** Suppose the polynomial sequence

$$F_i(u) = \sum_{k=0}^{i} a_{ik} u^k (a_{ii} \neq 0),\ i = 0, 1, 2, \cdots,$$

satisfies

$$\sum_{u=0}^{N} w(u) F_i(u) F_j(u) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

where $w(u)$ is a weight function and has the following properties

(1) $w(u) \geq 0, u \in \mathbb{N}$,

(2) $\sum_{u=0}^{N} w(u) > 0$,

(3) $\sum_{u=0}^{N} w(u) u^n$ exists, $n = 0, 1, 2, \cdots$.

Then $F_i(u)$ is called weight standard orthogonal polynomial in $\mathbb{N}$.

Definition 2.2 (A complete orthonormal sequence [23])**.** An orthonormal sequence $\{e_i\}$ in a Hilbert space H is complete if $\langle x, e_i \rangle = 0$ for all i imply $x = 0$. A complete orthonormal sequence is also called orthonormal basis in H.

Lemma 2.1 (Theorem 7.3 in [23])**.** *Let S be an orthonormal set in a Hilbert space H, and let M be the closed linear manifold generated by S. The following statements are equivalent:*

- *S is complete(i.e., maximal).*
- *$M = H$.*
- *If $x \perp S$, then $x = 0$.*
- *$x = \sum_{u \in S} \langle x, u \rangle u$ for all $x \in H$.*
- *$\|x\|^2 = \sum_{u \in S} \mid \langle x, u \rangle \mid^2$ for all $x \in H$.*

3. Stochastic codimension one flip bifurcation

In this section, we mainly consider codimension one flip bifurcation system which has the following normal form

$$x(n+1) = -(1+\alpha)x(n) + x^3(n), \tag{3.1}$$

where α is the bifurcation parameter. After adding random excitation, the bifurcation parameter can be expressed as $\alpha = \bar{\alpha} + \delta u$, where $\bar{\alpha}$ is the statistic parameter of α, δ is the intensity of α, and u is a discrete random variable in $\mathbb{N}$ with probability distribution function $p_u = P\{u = k\}, k = 0, 1, 2, \cdots$. Then system (3.1) becomes

$$x(n+1, u) = -(1 + \bar{\alpha} + \delta u)x(n, u) + x^3(n, u). \tag{3.2}$$

In fact, the probability distribution function p_u is a weight function according to the Definition 2.1. For $F_i(u)$ with weight function p_u, its recurrent formula is

$$uF_i(u) = \phi_i F_{i+1}(u) + \varphi_i F_i(u) + \psi_i F_{i-1}(u), \quad F_{-1}(u) = 0, \quad F_0(u) = 1, \tag{3.3}$$

where $\phi_i, \varphi_i, \psi_i$ are decided by the form of $F_i(u)$.

Combining with above two Definitions and the Lemma, we make an attempt to solve system (3.2) with $F_i(u)$, where $\{F_i(u)\}$ is an orthonormal sequence valued in $\mathbb{N}$. If $\{F_i(u)\}$ is complete, then for any discrete function $x(n)$, we have $x(n) = \sum_{i\in\mathbb{N}} x_i(n)F_i(u)$. Inspired by this, we conceive that whether the expression of finite terms can solve system (3.2).

Therefore, for a given $M \in \mathbb{N}$, we suppose

$$x(n, u) = \sum_{i=0}^{M} x_i(n)F_i(u), \tag{3.4}$$

is the solution of system (3.2), where $x_i(n) = \sum_{i=0}^{N} p_u x(n, u)F_i(u)$.

Substituting equation (3.4) into (3.2), we have

$$\sum_{i=0}^{M} x_i(n+1)F_i(u) = -(1 + \bar{\alpha} + \delta u)\sum_{i=0}^{M} x_i(n)F_i(u) + \left(\sum_{i=0}^{M} x_i(n)F_i(u)\right)^3. \tag{3.5}$$

By (3.3), the random term with u in (3.5) can be simplified to

$$\begin{aligned}
&u\sum_{i=0}^{M} x_i(n)F_i(u) = \sum_{i=0}^{M} x_i(n)uF_i(u)\\
&= \sum_{i=0}^{M} x_i(n)[\phi_i F_{i+1}(u) + \varphi_i F_i(u) + \psi_i F_{i-1}(u)]\\
&= \sum_{i=0}^{M} F_i(u)[\psi_{i+1}x_{i+1}(n) + \varphi_i x_i(n) + \phi_{i-1}x_{i-1}(n)] + \phi_M x_M(n)F_{M+1}(u)\\
&\quad - \phi_{-1}x_{-1}(n)F_0(u) + \psi_0 x_0(n)F_{-1}(u) - \psi_{M+1}x_{M+1}(n)F_M(u)\\
&= \sum_{i=0}^{M} F_i(u)[\psi_{i+1}x_{i+1}(n) + \varphi_i x_i(n) + \phi_{i-1}x_{i-1}(n)] + \phi_M x_M(n)F_{M+1}(u).
\end{aligned} \tag{3.6}$$

On the other hand, the nonlinear term can be expanded into

$$\left(\sum_{i=0}^{M} x_i(n)F_i(u)\right)^3 = \sum_{i=0}^{3M} X_i(n)F_i(u). \tag{3.7}$$

In fact, the expression of $F_i(u)$ is a polynomial with respect to u. For $\forall\ k \in \mathbb{N}$, $F_i^k(u)$ can be expressed as a function of $F_{ik}(u), F_{ik-1}(u), \ldots, F_0(u)$. Thus (3.7) holds. From (3.6) and (3.7), system (3.5) turns to

$$\begin{aligned}
&\sum_{i=0}^{M} x_i(n+1)F_i(u)\\
&= -(1+\bar{\alpha})\sum_{i=0}^{M} x_i(n)F_i(u) - \delta\sum_{i=0}^{M} F_i(u)[\psi_{i+1}x_{i+1}(n) + \varphi_i x_i(n) + \phi_{i-1}x_{i-1}(n)]\\
&\quad - \delta\phi_M x_M(n)F_{M+1}(u) + \sum_{i=0}^{3M} X_i(n)F_i(u).
\end{aligned} \tag{3.8}$$

Multiplying both sides of (3.8) by $F_i(u),\ i = 0, 1, 2, \ldots, M$ sequentially and taking expectation with respect to u, we can obtain the system which describes the relationship among the coefficients $x_i(n)$. By exploring dynamics of this system, it can reflect a profile of properties of the origin system $x(n,u)$.

Next we focus on the analysis of system (3.2) when $M = 1$. It becomes a two-dimension system

$$\begin{cases} x_0(n+1) = -(1+\bar{\alpha})x_0(n) - \delta(\psi_1 x_1(n) + \varphi_0 x_0(n)) + X_0(n),\\ x_1(n+1) = -(1+\bar{\alpha})x_1(n) - \delta(\varphi_1 x_1(n) + \phi_0 x_0(n)) + X_1(n), \end{cases} \tag{3.9}$$

where

$$\begin{aligned}
&\phi_0 = \frac{1}{a_{11}}, \quad \varphi_0 = -\frac{a_{10}}{a_{11}},\\
&\phi_1 = \frac{a_{11}}{a_{22}}, \quad \varphi_1 = \frac{a_{10}}{a_{11}} - \frac{a_{21}}{a_{22}}, \quad \psi_1 = \frac{a_{10}a_{21} - a_{11}a_{20}}{a_{22}} - \frac{a_{10}^2}{a_{11}},
\end{aligned}$$

and

$$\begin{aligned}
X_0(n) &= \beta_{03}x_1^3(n) + \beta_{12}x_0(n)x_1^2(n) + \beta_{30}x_0^3(n),\\
X_1(n) &= \gamma_{21}x_0^2(n)x_1(n) + \gamma_{12}x_0(n)x_1^2(n) + \gamma_{03}x_1^3(n),\\
\beta_{03} &= \frac{a_{11}^3a_{20}a_{32}}{a_{22}a_{33}} + \frac{3a_{10}^2a_{11}a_{21}}{a_{22}} + \frac{a_{10}a_{11}^2a_{31}}{a_{33}} - \frac{a_{10}a_{11}^2a_{21}a_{32}}{a_{22}a_{33}} - \frac{3a_{10}a_{11}^2a_{20}}{a_{22}}\\
&\quad - \frac{a_{11}^3a_{30}}{a_{33}} - 2a_{10}^3,\\
\beta_{12} &= \frac{3a_{10}a_{11}a_{21}}{a_{22}} - \frac{3a_{11}^2a_{20}}{a_{22}} - 3a_{10}^2, \quad \beta_{30} = 1, \quad \gamma_{21} = 3,\\
\gamma_{03} &= 3a_{10}^2 - \frac{3a_{10}a_{11}a_{21}}{a_{22}} - \frac{a_{11}^2a_{31}}{a_{33}} + \frac{a_{11}^2a_{21}a_{32}}{a_{22}a_{33}}, \quad \gamma_{12} = 6a_{10} - \frac{3a_{11}a_{21}}{a_{22}}.
\end{aligned}$$

Let $x = x_0(n)$, $y = x_1(n)$, system (3.9) becomes

$$\begin{pmatrix} x\\ y \end{pmatrix} \mapsto \begin{pmatrix} -(1+\bar{\alpha}+\delta\varphi_0)x - \delta\psi_1 y + \beta_{03}y^3 + \beta_{12}xy^2 + \beta_{30}x^3\\ -(1+\bar{\alpha}+\delta\varphi_1)y - \delta\phi_0 x + \gamma_{21}x^2y + \gamma_{12}xy^2 + \gamma_{03}y^3 \end{pmatrix}. \tag{3.10}$$

In the next section, we mainly study the dynamical behaviors at fixed point E_0 for map (3.10), where E_0 is the origin. It is found that map (3.10) undergoes Neimark-Sacker bifurcation, 1:2 resonance and fold-flip bifurcation.

The characteristic equation at fixed points E_0 is given by

$$\mu^2 + (2 + P(\delta, \bar{\alpha}))\mu + Q(\delta, \bar{\alpha}) + P(\delta, \bar{\alpha}) + 1 = 0,$$

where

$$\begin{aligned} &P(\delta, \bar{\alpha}) = P_1(\delta, \bar{\alpha}) + P_2(\delta, \bar{\alpha}), \quad P_1(\delta, \bar{\alpha}) = \bar{\alpha} + \delta\varphi_0, \\ &P_2(\delta, \bar{\alpha}) = \bar{\alpha} + \delta\varphi_1, \quad Q(\delta, \bar{\alpha}) = P_1(\delta, \bar{\alpha})P_2(\delta, \bar{\alpha}) - \delta^2\phi_0\psi_1. \end{aligned}$$

The map (3.10) undergoes Neimark-Sacker bifurcation at E_0 if $Q(\delta, \bar{\alpha}) = -P(\delta, \bar{\alpha}) = 3$. Here we choose δ as the bifurcation parameter and have the following theorem.

Theorem 3.1. *If $d(\delta) < 0$, there is a unique stable closed invariant curve bifurcated from E_0; If $d(\delta) > 0$, there is a unique unstable closed invariant curve bifurcated from E_0, where*

$$\begin{aligned} d(\delta) = &\frac{\gamma_{21}(3P_1^3 + 16P_1^2 + 27P_1 + 15)}{2\delta\phi_0^2} + \frac{\gamma_{12}(3P_1^2 + 11P_1 + 9)}{2\phi_0} \\ &+ \frac{3\delta\gamma_{03}(P_1 + 2)}{2} - \frac{\delta\beta_{12}(3P_1 + 5)}{2} - \frac{3\beta_{03}\delta^2\phi_0}{2} - \frac{3\beta_{30}(P_1 + 2)}{2\phi_0}. \end{aligned} \tag{3.11}$$

Proof. If $Q = -P = 3$, we have $\mu_1 = \frac{1-\sqrt{3}i}{2}$, $\mu_2 = \frac{1+\sqrt{3}i}{2}$ and $\mu_{1,2}^k \neq 1$, $k = 1, 2, 3, 4$. The eigenvectors satisfying $A_0 q = \mu_2 q$, $A_0^T p = \mu_1 p$, $\langle p, q \rangle = 1$ are

$$p = \left(-\frac{\sqrt{3}\delta\phi_0 i}{3}, \frac{1}{2} + \frac{\sqrt{3}\,(2P_1 + 3)\,i}{6} \right)^T, \quad q = \left(\frac{2P_1 + 3 - \sqrt{3}i}{2\delta\phi_0}, 1 \right)^T.$$

Denoted $zq_1 + \bar{z}\bar{q}_1 = x$, $zq_2 + \bar{z}\bar{q}_2 = y$ and calculating the Taylor expansion of the inner product between p and map (3.10) at E_0, we can obtain the critical coefficient $d(\delta)$ of the only resonance term $|z|^2\bar{z}$, that is (3.11). Therefore, the proof is complete. $\square$

Furthermore, we analyze 1:2 resonance at E_0 when choosing δ and $\bar{\alpha}$ as bifurcation parameters. If $Q(\delta, \bar{\alpha}) = P(\delta, \bar{\alpha}) = 0$, then $\mu_{1,2} = -1$. The critical conditions of bifurcation parameters $\bar{\alpha}, \delta$ are

$$\begin{cases} \delta_0^2\left(4\phi_0\psi_1 + (\varphi_0 - \varphi_1)^2\right) = 0, \\ 2\bar{\alpha}_0 + \delta_0(\varphi_0 + \varphi_1) = 0. \end{cases}$$

Denote $P_0 = P_{10} + P_{20}$, $P_{10} = P_1(\delta_0, \bar{\alpha}_0)$, $P_{20} = P_2(\delta_0, \bar{\alpha}_0)$. Let

$$P_{10}^2 y_1 = \delta_0\phi_0 x, \quad y_2 = P_{10}y - \delta_0\phi_0 x. \tag{3.12}$$

Then

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} -1 + \varepsilon_{10}(\delta, \bar{\alpha}) & 1 + \varepsilon_{01}(\delta, \bar{\alpha}) \\ \epsilon_{10}(\delta, \bar{\alpha}) & -1 + \epsilon_{01}(\delta, \bar{\alpha}) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} g(y_1, y_2, \delta, \bar{\alpha}) \\ h(y_1, y_2, \delta, \bar{\alpha}) \end{pmatrix},$$

where

$$\varepsilon_{10}(\delta,\bar{\alpha}) = \frac{\delta}{\delta_0}P_{10} - P_1(\delta,\bar{\alpha}), \quad \varepsilon_{01}(\delta,\bar{\alpha}) = \frac{\delta}{\delta_0 P_{10}} - \frac{1}{P_{10}},$$
$$\epsilon_{10}(\delta,\bar{\alpha}) = (P_1(\delta,\bar{\alpha}) - P_2(\delta,\bar{\alpha}))P_{10}^2 - \frac{2\delta P_{10}^3}{\delta_0}, \quad \epsilon_{01}(\delta,\bar{\alpha}) = 2P_{10} - P_2(\delta,\bar{\alpha}) - \frac{\delta P_{10}}{\delta_0}.$$

Denoted

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1+\varepsilon_{01} & 0 \\ -\varepsilon_{10} & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

we have

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 1 \\ \nu_1(\delta,\bar{\alpha}) & -1+\nu_2(\delta,\bar{\alpha}) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} g(u_1,u_2,\delta,\bar{\alpha}) \\ h(u_1,u_2,\delta,\bar{\alpha}) \end{pmatrix}. \tag{3.13}$$

Here

$$\begin{aligned}
&\nu_1(\delta,\bar{\alpha}) = \epsilon_{01}(\delta,\bar{\alpha}) + \varepsilon_{01}(\delta,\bar{\alpha})\epsilon_{10}(\delta,\bar{\alpha}) - \varepsilon_{10}(\delta,\bar{\alpha})\epsilon_{01}(\delta,\bar{\alpha}),\\
&\nu_2(\delta,\bar{\alpha}) = \varepsilon_{01}(\delta,\bar{\alpha}) + \epsilon_{01}(\delta,\bar{\alpha}),\\
&g(u_1,u_2,\delta,\bar{\alpha}) = \sum_{j+k=3} g_{jk}(\delta,\bar{\alpha})u_1^j u_2^k, \quad h(u_1,u_2,\delta,\bar{\alpha}) = \sum_{j+k=3} g_{jk}(\delta,\bar{\alpha})u_1^j u_2^k.
\end{aligned}$$

The coefficients $g_{jk}(\delta,\bar{\alpha})$ and $h_{jk}(\delta,\bar{\alpha})$ are as follows.

$$\begin{aligned}
g_{30}(\delta,\bar{\alpha}) =& \frac{\beta_{12}(1+\varepsilon_{01})\varepsilon_{10}^2}{\delta_0\phi_0} - \frac{2\beta_{12}P_{10}^2(1+\varepsilon_{01})^2\varepsilon_{10}}{\delta_0\phi_0} + \frac{\beta_{12}P_{10}^4(1+\varepsilon_{01})^3}{\delta_0\phi_0}\\
&+ \frac{\beta_{30}P_{10}^6(1+\varepsilon_{01})^3}{\delta_0^3\phi_0^3} + \frac{3\beta_{03}\varepsilon_{10}^2(1+\varepsilon_{01})}{P_{10}} + P_{10}^3\beta_{03}(1+\varepsilon_{01})^3\\
&- \frac{\beta_{03}\varepsilon_{10}^3}{P_{10}^3} - 3P_{10}\beta_{03}(1+\varepsilon_{01})^2\varepsilon_{10},\\
g_{21}(\delta,\bar{\alpha}) =& \frac{2\beta_{12}P_{10}^2\varepsilon_{01}^2}{\delta_0\phi_0} + \frac{4\beta_{12}P_{10}^2\varepsilon_{01}}{\delta_0\phi_0} + \frac{2\beta_{12}P_{10}^2}{\delta_0\phi_0} - \frac{2\beta_{12}\varepsilon_{10}}{\delta_0\phi_0} - \frac{2\beta_{12}\varepsilon_{01}\varepsilon_{10}}{\delta_0\phi_0}\\
&+ 6\beta_{03}P_{10}\varepsilon_{01} + 3\beta_{03}P_{10} + \frac{3\beta_{03}\varepsilon_{10}^2}{P_{10}^3} - \frac{6\beta_{03}\varepsilon_{10}}{P_{10}} - \frac{6\beta_{03}\varepsilon_{01}\varepsilon_{10}}{P_{10}}\\
&+ 3\beta_{03}P_{10}\varepsilon_{01}^2,\\
g_{12}(\delta,\bar{\alpha}) =& \frac{\beta_{12}\varepsilon_{01}}{\delta_0\phi_0} + \frac{\beta_{12}}{\delta_0\phi_0} + \frac{3\beta_{03}\varepsilon_{01}}{P_{10}} - \frac{3\beta_{03}\varepsilon_{10}}{P_{10}^3} + \frac{3\beta_{03}}{P_{10}}, \quad g_{03}(\delta,\bar{\alpha}) = \frac{\beta_{03}}{P_{10}^3},\\
h_{30}(\delta,\bar{\alpha}) =& \frac{P_{10}^3\gamma_{21}(1+\varepsilon_{01})^2\varepsilon_{10}}{\delta_0^2\phi_0^2} + \frac{P_{10}^2\gamma_{12}(1+\varepsilon_{01})^2\varepsilon_{10}}{\delta_0\phi_0} - \frac{\gamma_{12}(1+\varepsilon_{01})\varepsilon_{10}^2}{\delta_0\phi_0}\\
&+ \frac{P_{10}^5\gamma_{21}(1+\varepsilon_{01})^3}{\delta_0^2\phi_0^2} + \frac{P_{10}^4\gamma_{12}(1+\varepsilon_{01})^3}{\delta_0\phi_0} - \frac{P_{10}\gamma_{12}(1+\varepsilon_{01})^2\varepsilon_{10}}{\delta_0\phi_0}\\
&+ \frac{\gamma_{03}(1+\varepsilon_{01})\varepsilon_{10}^2}{P_{10}} - \frac{2\gamma_{03}(1+\varepsilon_{01})\varepsilon_{10}^2}{P_{10}} + \frac{\gamma_{03}\varepsilon_{10}^3}{P_{10}^3}\\
&- 2P_{10}\gamma_{03}(1+\varepsilon_{01})^2\varepsilon_{10},
\end{aligned}$$

$$
\begin{aligned}
h_{21}(\delta,\bar{\alpha}) =& \frac{P_{10}^3\gamma_{21}(1+\varepsilon_{01})^2}{\delta_0^2\phi_0^2} + \frac{2P_{10}^2\gamma_{12}(1+\varepsilon_{01})^2}{\delta_0\phi_0} - \frac{2\gamma_{12}(1+\varepsilon_{01})\varepsilon_{10}}{\delta_0\phi_0} \\
&+ \frac{3\gamma_{03}\varepsilon_{10}^2}{P_{10}^3} - \frac{6\gamma_{03}(1+\varepsilon_{01})\varepsilon_{10}}{P_{10}} + 3P_{10}\gamma_{03}(1+\varepsilon_{01})^2, \\
h_{12}(\delta,\bar{\alpha}) =& \frac{\gamma_{12}\varepsilon_{01}}{\delta_0\phi_0} + \frac{\gamma_{12}}{\delta_0\phi_0} + \frac{3\gamma_{03}\varepsilon_{01}}{P_{10}} - \frac{3\gamma_{03}\varepsilon_{10}}{P_{10}^3} + \frac{3\gamma_{03}}{P_{10}}, \quad h_{03}(\delta,\bar{\alpha}) = \frac{\gamma_{03}}{P_{10}^3}.
\end{aligned}
$$

To remove all cubic terms except resonant terms ξ_1^3 and $\xi_1^2\xi_2$, we make a transformation

$$
\begin{cases}
u_1 = \xi_1 + \theta_{30}(\delta,\bar{\alpha})\xi_1^3 + \theta_{21}(\delta,\bar{\alpha})\xi_1^2\xi_2 + \theta_{12}(\delta,\bar{\alpha})\xi_1\xi_2^2, \\
u_2 = \xi_2 + \vartheta_{30}(\delta,\bar{\alpha}))\xi_1^3 + \vartheta_{21}(\delta,\bar{\alpha})\xi_1^2\xi_2 + \vartheta_{12}(\delta,\bar{\alpha})\xi_1\xi_2^2,
\end{cases}
$$

where

$$
\begin{aligned}
&\theta_{30}(\delta,\bar{\alpha}) = \frac{g_{30}(\delta,\bar{\alpha})}{2} + \frac{g_{21}(\delta,\bar{\alpha})}{3} + \frac{h_{21}(\delta,\bar{\alpha})}{6}, \\
&\theta_{21}(\delta,\bar{\alpha}) = g_{30}(\delta,\bar{\alpha}) + \frac{g_{12}(\delta,\bar{\alpha}) + g_{21}(\delta,\bar{\alpha}) + h_{12}(\delta,\bar{\alpha}) + h_{03}(\delta,\bar{\alpha})}{2}, \\
&\theta_{12}(\delta,\bar{\alpha}) = g_{03}(\delta,\bar{\alpha}) + \frac{g_{30}(\delta,\bar{\alpha}) + g_{12}(\delta,\bar{\alpha}) + h_{03}(\delta,\bar{\alpha})}{2} + \frac{g_{21}(\delta,\bar{\alpha}) - h_{21}(\delta,\bar{\alpha})}{6}, \\
&\vartheta_{30}(\delta,\bar{\alpha}) = g_{30}(\delta,\bar{\alpha}), \quad \vartheta_{21}(\delta,\bar{\alpha}) = \frac{3g_{30}(\delta,\bar{\alpha}) + h_{12}(\delta,\bar{\alpha})}{2}, \\
&\vartheta_{12}(\delta,\bar{\alpha}) = h_{03}(\delta,\bar{\alpha}) + \frac{g_{30}(\delta,\bar{\alpha}) + h_{12}(\delta,\bar{\alpha})}{2}.
\end{aligned}
$$

The normal form of map (3.13) is

$$
\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 1 \\ \nu_1(\delta,\bar{\alpha}) & -1+\nu_2(\delta,\bar{\alpha}) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ C(\delta,\bar{\alpha})\xi_1^3 + D(\delta,\bar{\alpha})\xi_1^2\xi_2 \end{pmatrix}. \tag{3.14}
$$

Here $C(\delta,\bar{\alpha}) = h_{30}(\delta,\bar{\alpha})$, $D(\delta,\bar{\alpha}) = h_{21}(\delta,\bar{\alpha}) + 3g_{30}(\delta,\bar{\alpha})$.

The second iteration of map (3.14) is

$$
\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 4\nu_1(\delta,\bar{\alpha}) & (-2\nu_1(\delta,\bar{\alpha}) - 2\nu_2(\delta,\bar{\alpha})) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4C\xi_1^3 - (2D+6C))\xi_1^2\xi_2 \end{pmatrix}. \tag{3.15}
$$

Theorem 3.2. *Assume $D(\delta_0,\bar{\alpha}_0) + 3C(\delta_0,\bar{\alpha}_0) < 0$, map (3.15) has the following dynamical behaviors at 1:2 resonance point*

(*i*) *If $C(\delta_0,\bar{\alpha}_0) > 0$, there is a heteroclinic bifurcation curve $\mathcal{C}$*

$$
\mathcal{C} = \{(\kappa_1,\kappa_2) : \kappa_2(\delta,\bar{\alpha}) = -\frac{1}{5}\kappa_1(\delta,\bar{\alpha}) + o \mid \kappa_1(\delta,\bar{\alpha}) \mid\};
$$

(*ii*) *If $C(\delta_0,\bar{\alpha}_0) < 0$, there is a homoclinic bifurcation curve $\bar{\mathcal{C}}$*

$$
\bar{\mathcal{C}} = \{(\kappa_1,\kappa_2) : \kappa_2(\delta,\bar{\alpha}) = \frac{4}{5}\kappa_1(\delta,\bar{\alpha}) + o \mid \kappa_1(\delta,\bar{\alpha}) \mid\}.
$$

Proof. (*i*) If $D(\delta_0, \bar{\alpha}_0) + 3C(\delta_0, \bar{\alpha}_0) < 0$, $C(\delta_0, \bar{\alpha}_0) > 0$, let

$$\zeta_1 = \frac{D(\delta, \bar{\alpha}) + 3C(\delta, \bar{\alpha})}{\sqrt{C(\delta, \bar{\alpha})}}\xi_1, \quad \zeta_2 = \frac{(D(\delta, \bar{\alpha}) + 3C(\delta, \bar{\alpha}))^2}{2\sqrt{C(\delta, \bar{\alpha})}C(\delta, \bar{\alpha})}\xi_2,$$

and

$$\tau = \frac{2C(\delta, \bar{\alpha})}{D(\delta, \bar{\alpha}) + 3C(\delta, \bar{\alpha})}t.$$

(3.15) becomes

$$\begin{cases} \dot{\zeta}_1 = \zeta_2, \\ \dot{\zeta}_2 = \kappa_1(\delta, \bar{\alpha})\zeta_1 + \kappa_2(\delta, \bar{\alpha})\zeta_2 + \zeta_1^3 - \zeta_1^2\zeta_2, \end{cases} \tag{3.16}$$

where

$$\begin{aligned} \kappa_1(\delta, \bar{\alpha}) &= \frac{(D(\delta, \bar{\alpha}) + 3C(\delta, \bar{\alpha}))\nu_1(\delta, \bar{\alpha})}{2C^2(\delta, \bar{\alpha})}, \\ \kappa_2(\delta, \bar{\alpha}) &= \frac{(D(\delta, \bar{\alpha}) + 3C(\delta, \bar{\alpha}))(\nu_1(\delta, \bar{\alpha}) + \nu_2(\delta, \bar{\alpha}))}{C(\delta, \bar{\alpha})}. \end{aligned}$$

Making a singular rescaling of the variables, times and parameters,

$$\varsigma_1 = \frac{1}{\sqrt{-\kappa_1(\delta, \bar{\alpha})}}\zeta_1, \quad \varsigma_2 = \frac{1}{-\kappa_1(\delta, \bar{\alpha})}\zeta_2, \quad \tilde{\tau} = \sqrt{-\kappa_1(\delta, \bar{\alpha})}\tau,$$

(3.16) turns to

$$\begin{cases} \dot{\varsigma}_1 = \varsigma_2, \\ \dot{\varsigma}_2 = \varsigma_1(\varsigma_1^2 - 1) + \eta_2(\delta, \bar{\alpha})\varsigma_2 - \eta_1(\delta, \bar{\alpha})\varsigma_1^2\varsigma_2. \end{cases} \tag{3.17}$$

Here

$$\eta_1(\delta, \bar{\alpha}) = \sqrt{-\kappa_1(\delta, \bar{\alpha})}, \quad \eta_2(\delta, \bar{\alpha}) = \frac{\kappa_2(\delta, \bar{\alpha})}{\sqrt{-\kappa_1(\delta, \bar{\alpha})}}.$$

After substituting $\eta_i(\delta, \bar{\alpha}) = 0, i = 1, 2$, the Hamiltonian function of (3.17) is

$$S(\varsigma_1, \varsigma_2) = \frac{\varsigma_1^2}{2} + \frac{\varsigma_2^2}{2} - \frac{\varsigma_1^4}{4}.$$

There is a heteroclinic orbit $4S(\varsigma_1, \varsigma_2) = 1$ connecting the saddles $(1, 0)$ and $(-1, 0)$, where

$$\varsigma_1(t) = \frac{e^{\sqrt{2}t} - 1}{e^{\sqrt{2}t} + 1}, \quad \varsigma_2(t) = \frac{2\sqrt{2}e^{\sqrt{2}t}}{(e^{\sqrt{2}t} + 1)^2}.$$

Define an orbit split function $\aleph(s, \eta_1, \eta_2)$ by the difference between the Hamiltonian values at points $(\varsigma_{1-}, \varsigma_{2-})$ and $(\varsigma_{1+}, \varsigma_{2+})$, where $(\varsigma_{1-}, \varsigma_{2-})$ and $(\varsigma_{1+}, \varsigma_{2+})$ are the intersection points of the horizontal axis and orbit Γ, *i.e.* $S(\varsigma_1, \varsigma_2) = s$.

$$\aleph(s, \eta_1, \eta_2) = S(\varsigma_{1-}, \varsigma_{2-}) - S(\varsigma_{1+}, \varsigma_{2+}).$$

For $(\eta_1, \eta_2) \neq 0$, $S(\varsigma_1, \varsigma_2)$ varies along orbits of (3.17) and satisfies

$$\dot{S}(\varsigma_1, \varsigma_2) = \frac{\partial S(\varsigma_1, \varsigma_2)}{\partial \varsigma_1}\dot{\varsigma}_1 + \frac{\partial S(\varsigma_1, \varsigma_2)}{\partial \varsigma_2}\dot{\varsigma}_2 = \eta_2\varsigma_2^2 - \eta_1\varsigma_1^2\varsigma_2^2.$$

Therefore,

$$\begin{aligned}\aleph(s,\eta_1,\eta_2) &= \int_{t(\varsigma_{1+},\varsigma_{2+})}^{t(\varsigma_{1-},\varsigma_{2-})} \dot{S}(\varsigma_1,\varsigma_2)dt\\ &= -\,\eta_2\int_{S(\varsigma_1,\varsigma_2)=s}\varsigma_2 d\varsigma_1 + \eta_1\int_{S(\varsigma_1,\varsigma_2)=o}\varsigma_1^2\varsigma_2 d\varsigma_1 + o(||\eta_1,\eta_2||),\end{aligned}$$

where $t(\varsigma_{1+},\varsigma_{2+})$ and $t(\varsigma_{1-},\varsigma_{2-})$ mean the time when $(\varsigma_1,\varsigma_2)=(\varsigma_{1+},\varsigma_{2+})$ and $(\varsigma_1,\varsigma_2)=(\varsigma_{1-},\varsigma_{2-})$, respectively.

Next denote the integrals

$$Q_1(s)=\int_{S(\varsigma_1,\varsigma_2)=s}\varsigma_2 d\varsigma_1, \quad Q_2(s)=\int_{S(\varsigma_1,\varsigma_2)=s}\varsigma_1^2\varsigma_2 d\varsigma_1.$$

The equation

$$\aleph(\frac{1}{4},0,0)=0$$

with the constraint $\eta_2>0$ defines a curve $\mathcal{C}$ on the $(\eta_1,\eta_2)-$plane starting at $(1,0)$(or $(-1,0)$) along which (3.17) has a heteroclinic orbit. For $s\in(0,\frac{1}{4})$, define curve $\mathcal{T}_s$ in the upper parameter half-plane $\eta_2>0$ at which (3.17) has a cycle between $(-1,0)$ and $(1,0)$ corresponding to s.

By the implicit function theorem, the curves $\mathcal{T}_h$ and $\mathcal{C}$ exist and have the representation

$$\eta_2=\frac{Q_2(s)}{Q_1(s)}\eta_1+o(|\ \eta_1\ |).$$

Define function

$$\mathcal{Q}(s)=\frac{Q_2(s)}{Q_1(s)}. \tag{3.18}$$

To get $\mathcal{Q}(\frac{1}{4})$, we calculate

$$\begin{aligned}Q_1(\frac{1}{4}) &= \int_{S(\varsigma_1,\varsigma_2)=\frac{1}{4}}\varsigma_2 d\varsigma_1 = \int_{S(\varsigma_1,\varsigma_2)\le\frac{1}{4}} d\varsigma_2 d\varsigma_1\\ &= \int_{-1}^{1}\int_0^{\frac{\varsigma_1^2}{2}+\frac{\varsigma_2^2}{2}-\frac{\varsigma_1^4}{4}=\frac{1}{4}} d\varsigma_2 d\varsigma_1 = \frac{2\sqrt{2}}{3},\\ Q_2(\frac{1}{4}) &= \int_{S(\varsigma_1,\varsigma_2)=\frac{1}{4}}\varsigma_1^2\varsigma_2 d\varsigma_1 = \int_{S(\varsigma_1,\varsigma_2)\le\frac{1}{4}}\varsigma_1^2 d\varsigma_2 d\varsigma_1\\ &= \int_{-1}^{1}\int_0^{\frac{\varsigma_1^2}{2}+\frac{\varsigma_2^2}{2}-\frac{\varsigma_1^4}{4}=\frac{1}{4}} \varsigma_1^2\varsigma_2 d\varsigma_1 = \frac{2\sqrt{2}}{15}.\end{aligned}$$

Then we obtain $\mathcal{Q}(\frac{1}{4})=\frac{1}{5}$. For $s\in[0,\frac{1}{4}]$, (3.18) is sufficient to prove the uniqueness of cycles. Considering ς_2 as a function of ς_1 and s, define the function

$$\frac{\varsigma_1^2}{2}+\frac{\varsigma_2^2}{2}-\frac{\varsigma_1^4}{4}=s.$$

Differentiating it with respect to s, we get

$$\varsigma_2\frac{\partial\varsigma_2}{\partial s}=1.$$

On the other hand, differentiating it with respect to ς_1 yields

$$\varsigma_1 + \varsigma_2 \frac{\partial \varsigma_2}{\partial \varsigma_1} - \varsigma_1^3 = 0.$$

Multiplying by $\varsigma_1^m \varsigma_2^{-1}$ and integrating by parts, we acquire

$$\int_{S(\varsigma_1,\varsigma_2)=s} \frac{\varsigma_1^{m+3}}{\varsigma_2} d\varsigma_1 = \int_{S(\varsigma_1,\varsigma_2)=s} \frac{\varsigma_1^{m+1}}{\varsigma_2} d\varsigma_1 - m \int_{S(\varsigma_1,\varsigma_2)=s} \varsigma_1^{m-1} \varsigma_2 d\varsigma_1.$$

Utilizing $m = 0, 1, 3$, we have

$$\begin{aligned}
s\frac{dQ_1(s)}{ds} =& s\int_{S(\varsigma_1,\varsigma_2)=s} \frac{d\varsigma_1}{\varsigma_2}, \\
=& \frac{1}{2}\int_{S(\varsigma_1,\varsigma_2)=s} \varsigma_1^2 \frac{d\varsigma_1}{\varsigma_2} + \frac{1}{2}\int_{S(\varsigma_1,\varsigma_2)=s} \varsigma_2 d\varsigma_1 - \frac{1}{4}\int_{S(\varsigma_1,\varsigma_2)=s} \varsigma_1^4 \frac{d\varsigma_1}{\varsigma_2}, \\
=& \frac{3}{4}Q_1(s) + \frac{1}{4}\frac{dQ_2(s)}{ds}, \\
s\frac{dQ_2(s)}{ds} =& s\int_{S(\varsigma_1,\varsigma_2)=s} \varsigma_1^2 \frac{d\varsigma_1}{\varsigma_2}, \\
=& \frac{1}{2}\int_{S(\varsigma_1,\varsigma_2)=s} \varsigma_1^4 \frac{d\varsigma_1}{\varsigma_2} + \frac{1}{2}\int_{S(\varsigma_1,\varsigma_2)=s} \varsigma_1^2 \varsigma_2 d\varsigma_1 - \frac{1}{4}\int_{S(\varsigma_1,\varsigma_2)=s} \varsigma_1^6 \frac{d\varsigma_1}{\varsigma_2}, \\
=& \frac{5}{4}Q_2(s) + \frac{1}{4}\frac{dQ_2(s)}{ds} - \frac{1}{4}Q_1(s).
\end{aligned}$$

Therefore, $Q_1(s)$ and $Q_2(s)$ satisfy the following system of differential equations:

$$\begin{cases} s(s-\frac{1}{4})\dot{Q}_1(s) = (\frac{3}{4}s - \frac{1}{4})Q_1(s) + \frac{5}{16}Q_2(s), \\ s(s-\frac{1}{4})\dot{Q}_2(s) = \frac{5}{4}sQ_2(s) - \frac{1}{4}sQ_1(s). \end{cases}$$

The function $\mathcal{Q}(s)$ satisfies

$$s(s-\frac{1}{4})\dot{\mathcal{Q}}(s) = -\frac{5}{16}\mathcal{Q}^2 + (\frac{s}{2}+\frac{1}{4})\mathcal{Q} - \frac{s}{4}. \tag{3.19}$$

Substituting $\mathcal{Q}(s) = \iota s + O(s^2)$ into (3.19) yields $\iota = \frac{1}{2}$.

For all $s \in (0, \frac{1}{4})$, we have $0 \leq \mathcal{Q}(s) \leq \frac{1}{5}$. Suppose $\bar{s} \in (0, \frac{1}{4})$ is the first intersection point of $\mathcal{Q}(s)$ with $s-$axis. Then

$$\bar{s}(\bar{s}-\frac{1}{4})\dot{\mathcal{Q}}(\bar{s}) = -\frac{\bar{s}}{4} < 0.$$

This is a contradiction with $\dot{\mathcal{Q}}(\bar{s}) > 0$. If $\bar{s} \in (0, \frac{1}{4})$ is the first point of $\mathcal{Q}(s) = \frac{1}{5}$, then

$$\bar{s}(\bar{s}-\frac{1}{4})\dot{\mathcal{Q}}(\bar{s}) = \frac{3}{20}(\frac{1}{4}-\bar{s}) > 0.$$

This is a contradiction with $\dot{\mathcal{Q}}(\bar{s}) < 0$.

From

$$\bar{s}(\bar{s}-\frac{1}{4})\ddot{\mathcal{Q}}(\bar{s}) \mid_{\dot{\mathcal{Q}}(\bar{s})=0} = \frac{1}{2}(\mathcal{Q}(\bar{s}) - \frac{1}{2}) < 0,$$

we have $\ddot{\mathcal{Q}}(\bar{s}) > 0$ at any point where $\dot{\mathcal{Q}}(\bar{s}) = 0$ which means all extrema are maximum points. Therefore, $\mathcal{Q}(0) = 0, \mathcal{Q}(\frac{1}{4}) = \frac{1}{5} = \max_{0\le s\le \frac{1}{4}} \mathcal{Q}(s)$.

(ii) If $D(\delta_0, \bar{\alpha}_0) + 3C(\delta_0, \bar{\alpha}_0) < 0$, $C(\delta_0, \bar{\alpha}_0) < 0$, let

$$\bar{\zeta}_1 = \frac{D(\delta,\bar{\alpha}) + 3C(\delta,\bar{\alpha})}{\sqrt{-C(\delta,\bar{\alpha})}}\xi_1, \quad \bar{\zeta}_2 = -\frac{(D(\delta,\bar{\alpha}) + 3C(\delta,\bar{\alpha}))^2}{2\sqrt{-C(\delta,\bar{\alpha})}C(\delta,\bar{\alpha})}\xi_2,$$

and

$$\bar{\tau} = -\frac{2C(\delta,\bar{\alpha})}{D(\delta,\bar{\alpha}) + 3C(\delta,\bar{\alpha})}t.$$

(3.15) becomes

$$\begin{cases} \dot{\bar{\zeta}}_1 = \bar{\zeta}_2, \\ \dot{\bar{\zeta}}_2 = \kappa_1(\delta,\bar{\alpha})\bar{\zeta}_1 - \kappa_2(\delta,\bar{\alpha})\bar{\zeta}_2 - \bar{\zeta}_1^3 - \bar{\zeta}_1^2\bar{\zeta}_2. \end{cases} \tag{3.20}$$

Making a singular rescaling of the variables, times and parameters,

$$\bar{\varsigma}_1 = \frac{1}{\sqrt{\kappa_1(\delta,\bar{\alpha})}}\bar{\zeta}_1, \quad \bar{\varsigma}_2 = \frac{1}{\kappa_1(\delta,\bar{\alpha})}\bar{\zeta}_2, \quad \hat{\tau} = \sqrt{\kappa_1(\delta,\bar{\alpha})}\bar{\tau},$$

(3.20) turns to

$$\begin{cases} \dot{\bar{\varsigma}}_1 = \bar{\varsigma}_2, \\ \dot{\bar{\varsigma}}_2 = -\bar{\varsigma}_1(\bar{\varsigma}_1^2 - 1) + \eta_2(\delta,\bar{\alpha})\bar{\varsigma}_2 - \eta_1(\delta,\bar{\alpha})\bar{\varsigma}_1^2\bar{\varsigma}_2. \end{cases} \tag{3.21}$$

After substituting $\eta_i(\delta,\bar{\alpha}) = 0, i = 1, 2$, the Hamiltonian function of (3.21) is

$$H(\bar{\varsigma}_1, \bar{\varsigma}_2) = -\frac{\bar{\varsigma}_1^2}{2} + \frac{\bar{\varsigma}_2^2}{2} + \frac{\bar{\varsigma}_1^4}{4}.$$

There is a homoclinic orbit $H(\bar{\varsigma}_1, \bar{\varsigma}_2) = 0$ to the saddle $(0, 0)$.

$$\bar{\varsigma}_1(t) = \frac{2\sqrt{2}e^t}{e^{2t}+1}, \quad \bar{\varsigma}_2(t) = \frac{2\sqrt{2}e^t(1-e^{2t})}{(e^{2t}+1)^2}.$$

Define an orbit split function $\bar{\aleph}(h, \eta_1, \eta_2)$ by the difference between the Hamiltonian values at points $(\bar{\varsigma}_{1-}, \bar{\varsigma}_{2-})$ and $(\bar{\varsigma}_{1+}, \bar{\varsigma}_{2+})$, where $(\bar{\varsigma}_{1-}, \bar{\varsigma}_{2-})$ and $(\bar{\varsigma}_{1+}, \bar{\varsigma}_{2+})$ are the intersection points of the horizontal axis and orbit $\bar{\Gamma}$, *i.e.* $H(\bar{\varsigma}_1, \bar{\varsigma}_2) = h$.

$$\bar{\aleph}(h, \eta_1, \eta_2) = H(\bar{\varsigma}_{1-}, \bar{\varsigma}_{2-}) - H(\bar{\varsigma}_{1+}, \bar{\varsigma}_{2+}).$$

For $(\eta_1, \eta_2) \neq 0$, $H(\bar{\varsigma}_1, \bar{\varsigma}_2)$ varies along orbits of (3.17),

$$\dot{H}(\bar{\varsigma}_1, \bar{\varsigma}_2) = \frac{\partial H(\bar{\varsigma}_1, \bar{\varsigma}_2)}{\partial \bar{\varsigma}_1}\dot{\bar{\varsigma}}_1 + \frac{\partial H(\bar{\varsigma}_1, \bar{\varsigma}_2)}{\partial \bar{\varsigma}_2}\dot{\bar{\varsigma}}_2 = \eta_2\bar{\varsigma}_2^2 - \eta_1\bar{\varsigma}_1^2\bar{\varsigma}_2^2.$$

Therefore, similarly define $\bar{\aleph}(h, \eta_1, \eta_2), \bar{Q}_1(h), \bar{Q}_2(h)$. The equation $\bar{\aleph}(0, 0, 0) = 0$ with the constraint $\eta_2 > 0$ defines a curve $\bar{\mathcal{C}}$ on the $(\eta_1, \eta_2)-$plane starting at the origin along which (3.17) has a homoclinic orbit. For $h \in (-\frac{1}{4}, 0)$, define curve $\mathcal{T}_h$ in the upper parameter half-plane $\eta_2 > 0$ at which (3.17) has two cycles passing through a point between $(0, 0)$ and $(1, 0)$(or $(-1, 0)$) corresponding to h.

By the implicit function theorem, curves $\bar{\mathcal{T}}_h$ and $\bar{\mathcal{C}}$ exist and the representation is

$$\eta_2 = \frac{\bar{Q}_2(h)}{\bar{Q}_1(h)}\eta_1 + o(|\ \eta_1\ |).$$

Define function

$$\bar{\mathcal{Q}}(h) = \frac{\bar{Q}_2(h)}{\bar{Q}_1(h)}.$$

To get $\bar{\mathcal{Q}}(0), \bar{\mathcal{Q}}(-\frac{1}{4})$, we calculate

$$\bar{Q}_1(0) = \frac{4}{3}, \quad \bar{Q}_2(0) = \frac{16}{15}.$$

Then we obtain $\bar{\mathcal{Q}}(0) = \frac{4}{5}$.

For $h \in [-\frac{1}{4}, 0]$, the function $\bar{\mathcal{Q}}(h)$ satisfies

$$h(h+\frac{1}{4})\dot{\bar{\mathcal{Q}}}(h) = -\frac{1}{16}\bar{\mathcal{Q}}^2 - \frac{h}{2}\bar{\mathcal{Q}} + \frac{h}{4}. \tag{3.22}$$

Substituting $\bar{\mathcal{Q}}(h) = \bar{\iota}h + O(h^2)$ into (3.22) yields $\bar{\iota} = -4 < 0$ at $h = -\frac{1}{4}$ and $\bar{\iota} = 1 > 0$ at $h = 0$. For all $h \in (-\frac{1}{4}, 0)$, $\bar{\mathcal{Q}}(h)$ firstly decrease then increase. It is not monotonous, which implies that the limit cycle is not unique. □

Next, we analyze the fold-flip bifurcation at E_0. If $P(\delta, \bar{\alpha}) = -2$, $Q(\delta, \bar{\alpha}) = 0$, we have $\mu_1 = -1$, $\mu_2 = 1$. The critical conditions of bifurcation parameters $\bar{\alpha}, \delta$ satisfy

$$\begin{cases} \delta_0^2(4\phi_0\psi_1 - (\varphi_0 - \varphi_1)^2) + 4\delta_0(\varphi_0 + \varphi_1) - 1 = 0, \\ 2\bar{\alpha}_0 + 2 + \delta_0(\varphi_0 + \varphi_1) = 0. \end{cases}$$

For all sufficiently near to the neighbourhood of the critical values, the eigenvalues at E_0 are

$$\mu_1(\delta, \bar{\alpha}) = \frac{-2 - P(\delta, \bar{\alpha}) - \sqrt{P(\delta, \bar{\alpha})^2 - 4Q(\delta, \bar{\alpha})}}{2},$$
$$\mu_2(\delta, \bar{\alpha}) = \frac{-2 - P(\delta, \bar{\alpha}) - \sqrt{P(\delta, \bar{\alpha})^2 + 4Q(\delta, \bar{\alpha}))}}{2}.$$

The eigenvectors satisfying $A_{\bar{\alpha},\delta}q_1 = \mu_1 q_1$, $A_{\bar{\alpha},\delta}q_2 = \mu_2 q_2$, $A^T_{\bar{\alpha},\delta}p_1 = \mu_1 p_1$, $A^T_{\bar{\alpha},\delta}p_2 = \mu_2 p_2$, $\langle p_i, q_i\rangle = 1$, $i = 1, 2$, $\langle p_1, q_2\rangle = 0$, $\langle p_2, q_1\rangle = 0$ are

$$q_1 = \left(\frac{P_1 - P_2 + \sqrt{P^2 - 4Q}}{2\delta\phi_0}, 1\right)^T, \quad q_2 = \left(\frac{P_1 - P_2 - \sqrt{P^2 - 4Q}}{2\delta\phi_0}, 1\right)^T,$$
$$\tilde{p_1} = \left(\frac{P_1 - P_2 + \sqrt{P^2 - 4Q}}{2\delta\psi_1}, 1\right)^T, \quad \tilde{p_2} = \left(\frac{P_1 - P_2 - \sqrt{P^2 - 4Q}}{2\delta\psi_1}, 1\right)^T,$$
$$p_1 = \frac{\tilde{p_1}}{\langle\tilde{p_1}, q_1\rangle}, \quad p_2 = \frac{\tilde{p_2}}{\langle\tilde{p_2}, q_2\rangle}.$$

Let

$$x = \frac{\delta\psi_1}{\sqrt{P^2 - 4Q}}\left(\langle\tilde{p_1}, q_1\rangle z_1 - \langle\tilde{p_2}, q_2 z_2\rangle\right),$$
$$y = \frac{P_1 - P_2}{2\sqrt{P^2 - 4Q}}\left(\langle\tilde{p_2}, q_2\rangle z_2 - \langle\tilde{p_1}, q_1 z_1\rangle\right) + \frac{\langle\tilde{p_2}, q_2\rangle z_2 + \langle\tilde{p_1}, q_1 z_1\rangle}{2}.$$

Denoted $\langle \tilde{p_1}, q_1\rangle = I_1(\delta,\bar{\alpha})$, $\langle \tilde{p_2}, q_2\rangle = I_2(\delta,\bar{\alpha})$, $C_1(\delta,\bar{\alpha}) = \frac{\delta\psi_1}{\sqrt{P^2-4Q}}$, $C_2(\delta,\bar{\alpha}) = \frac{P_1-P_2}{2\sqrt{P^2-4Q}}$, $C_3 = \frac{1}{2}$, system (3.10) becomes

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} m(z_1,z_2,\delta,\bar{\alpha}) \\ n(z_1,z_2,\delta,\bar{\alpha}) \end{pmatrix},$$

where

$$m(z_1,z_2,\delta,\bar{\alpha}) = \sum_{j+k=3} m_{jk}(\delta,\bar{\alpha}) z_1^j z_2^k, \quad n(z_1,z_2,\delta,\bar{\alpha}) = \sum_{j+k=3} n_{jk}(\delta,\bar{\alpha}) z_1^j z_2^k,$$

and

$$\begin{aligned}
m_{30} &= \beta_{03} I_1^3 (C_3 - C_2)^3 + \beta_{12} I_1^3 C_1 (C_3 - C_2)^2 + \beta_{30} I_1^3 C_1^3, \\
m_{03} &= \beta_{03} I_2^3 (C_3 + C_2)^3 - \beta_{12} I_2^3 C_1 (C_3 + C_2)^2 - \beta_{30} I_2^3 C_1^3, \\
m_{21} &= 3\beta_{03} I_1^2 I_2 (C_3 - C_2)^2 (C_2 + C_3) + \beta_{12} I_1^2 I_2 C_1 (C_3^2 - 3C_2^2 + 2C_2C_3) \\
&\quad - 3\beta_{30} I_1^2 I_2 C_1^3, \\
m_{12} &= 3\beta_{03} I_1 I_2^2 (C_3 - C_2)(C_2 + C_3)^2 + \beta_{12} I_1 I_2^2 C_1 (3C_2^2 + 2C_2C_3 - C_3^2) \\
&\quad + 3\beta_{30} I_1 I_2^2 C_1^3, \\
n_{30} &= \gamma_{21} I_1^3 C_1^2 (C_3 - C_2) + \gamma_{12} I_1^2 I_2 C_1 (C_3 - C_2)^2 + \gamma_{03} I_1^3 (C_3 - C_2)^3, \\
n_{03} &= \gamma_{21} I_2^3 C_1^2 (C_3 + C_2) - \gamma_{12} I_2^3 C_1 (C_3 + C_2)^2 + \gamma_{03} I_2^3 (C_2 + C_3)^3, \\
n_{21} &= \gamma_{21} I_1^2 I_2 C_1^2 (3C_2 - C_3) + \gamma_{12} I_1^2 I_2 C_1 (C_3^2 - 3C_2^2 + 2C_2C_3) \\
&\quad + 3\gamma_{03} I_1^2 I_2 (C_2 - C_3)^2 (C_2 + C_3), \\
n_{12} &= -\gamma_{21} I_1 I_2^2 C_1^2 (3C_2 + C_3) + \beta_{12} I_1 I_2^2 C_1 (3C_2^2 + 2C_2C_3 - C_3^2) \\
&\quad + 3\gamma_{03} I_1 I_2^2 (C_3 - C_2)(C_2 + C_3)^2.
\end{aligned}$$

Making the change of variables,

$$w_1 = z_1 - \frac{1}{2} m_{21} z_1^2 z_2 - \frac{1}{2} m_{03} z_2^3, \quad w_2 = z_2 + \frac{1}{2} m_{12} z_1 z_2^2 + \frac{1}{2} m_{30} z_1^3,$$

then we obtain

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} \mu_1 w_1 + \frac{1}{6}\tilde{c}_1 w_1^3 + \frac{1}{2}\tilde{c}_2 w_1 w_2^2 \\ \mu_2 w_2 + \frac{1}{2}\tilde{c}_3 w_1^2 w_2 + \frac{1}{6}\tilde{c}_4 w_2^3 \end{pmatrix} + O(||z||^4), \tag{3.23}$$

where

$$\begin{aligned}
\tilde{c}_1 = {} & 12(p_{11}(3m_{30}q_{11}^3 + m_{21}q_{11}^2 + m_{12}q_{11} + 3m_{03}) + p_{12}(3n_{30}q_{11}^3 + n_{21}q_{11}^2 \\
& + n_{12}q_{11} + 3n_{03})), \\
\tilde{c}_2 = {} & 4p_{11}(9m_{30}q_{11}q_{21}^2 + m_{21}q_{21}(2q_{11} + q_{21}) + m_{12}(q_{11} + 2q_{21}) + 9m_{03}) \\
& + 4p_{12}(9n_{30}q_{11}q_{21}^2 + n_{21}q_{21}(2q_{11} + q_{21}) + n_{12}(q_{11} + 2q_{21}) + 9n_{03}), \\
\tilde{c}_3 = {} & 4p_{21}(9m_{30}q_{11}^2q_{21} + m_{21}q_{11}(q_{11} + 2q_{21}) + m_{12}(2q_{11} + q_{21}) + 9m_{03}) \\
& + 4p_{22}(9n_{30}q_{11}^2q_{21} + n_{21}q_{11}(q_{11} + 2q_{21}) + n_{12}(2q_{11} + q_{21}) + 9n_{03}), \\
\tilde{c}_4 = {} & 12(p_{21}(3m_{30}q_{21}^3 + m_{21}q_{21}^2 + m_{12}q_{21} + 3m_{03}) + p_{22}(3n_{30}q_{21}^3 + n_{21}q_{21}^2 \\
& + n_{12}q_{21} + 3n_{03})).
\end{aligned}$$

To study the dynamics at fold-flip bifurcation point, we need to make time rescaling $\tau = -\mu_2 t$ and recombine matrix R to map (3.23), where $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The new system is

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} \upsilon_1 + (1+\upsilon_2)w_1 + c_1(\upsilon_1,\upsilon_2)w_1^3 + c_2(\upsilon_1,\upsilon_2)w_1w_2^2 \\ w_2 + c_3(\upsilon_1,\upsilon_2)w_1^2w_2 + c_4(\upsilon_1,\upsilon_2)w_2^3 \end{pmatrix} + O(||z||^4), \tag{3.24}$$

where

$$\upsilon_1(\delta,\bar\alpha) \equiv 0, \quad \upsilon_2(\delta,\bar\alpha) = \frac{\mu_1}{\mu_2} - 1, \quad c_1(\upsilon_1,\upsilon_2) = -\frac{\tilde c_1}{\mu_2},$$
$$c_2(\upsilon_1,\upsilon_2) = -\frac{\tilde c_2}{\mu_2}, \quad c_3(\upsilon_1,\upsilon_2) = \frac{\tilde c_3}{\mu_2}, \quad c_4(\upsilon_1,\upsilon_2) = \frac{\tilde c_4}{\mu_2}.$$

Theorem 3.3. *The map (3.24) can be represented as*

$$B_\upsilon(z) = \varphi_\upsilon^1(z) + O(||z||^4),$$

where $\varphi_\upsilon^1(z)$ is the flow of a planar system

$$\begin{cases} \dot z_1 = \upsilon_1 + \upsilon_2 z_1 - \frac{1}{2}\upsilon_1\upsilon_2 + \Phi_{30}(z_1^3 - \frac{3}{2}z_1^2\upsilon_1 + \frac{1}{2}z_1\upsilon_1^2 + \frac{1}{4}\upsilon_1^3) \\ \qquad +\Phi_{12}z_1z_2^2 + \frac{1}{4}(\upsilon_1^2\upsilon_2 + \upsilon_1\upsilon_2^2) - \frac{1}{2}\upsilon_1\upsilon_2 z_1, \\ \dot z_2 = \Psi_{21}(z_1^2z_2 - z_1z_2\upsilon_1 + \frac{1}{6}z_2\upsilon_1^2) + \Psi_{03}z_2^3, \end{cases}$$

and $z = (z_1,z_2)^T$, $\upsilon = (\upsilon_1,\upsilon_2)^T$, $\Phi_{30} = c_1(\bar\alpha,\delta)$, $\Phi_{12} = c_2(\bar\alpha,\delta)$, $\Psi_{21} = -c_3(\bar\alpha,\delta)$, $\Psi_{03} = -c_4(\bar\alpha,\delta)$.

Proof. Consider a four-dimensional system

$$Z \mapsto \Theta^t(Z) = \begin{pmatrix} \varphi_\upsilon^1(z) \\ \upsilon \end{pmatrix}, Z = \begin{pmatrix} z \\ \upsilon \end{pmatrix} \in \mathbb{R}^4.$$

This flow is

$$\dot Z = W(Z) = MZ + M_2(Z) + M_3(Z) + \cdots, Z \in \mathbb{R}^4, \tag{3.25}$$

where $M = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$, $M_k(Z) = \begin{pmatrix} V_k(z) \\ 0 \end{pmatrix}$, and $V_k(z) : \mathbb{R}^4 \to \mathbb{R}^2$ is a homogeneous polynomial function with unknown coefficients. Let N_υ be the map (3.24), then $J(Z) = \begin{pmatrix} N_\upsilon \\ \upsilon \end{pmatrix}$. Next we need to make three times Picard iterations for (3.25).

First we have $Z^{(1)}(t) = (z_1 + t\upsilon_1, z_2, 0, 0)^T$, $M_2(Z) = (A_{10}\upsilon_2 z_1 + A_{11}\upsilon_1\upsilon_2, 0, 0, 0)^T$. Then

$$
\begin{aligned}
&Z^{(2)}(t)\\
&= e^{Mt}Z + \int_0^t e^{M(t-s)} M_2(Z^{(1)}(s))ds\\
&= \begin{pmatrix} z_1 + t\upsilon_1 \\ z_2 \\ \upsilon_1 \\ \upsilon_2 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & 0 & t-s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{10}\upsilon_2(z_1 + s\upsilon_1) + A_{11}\upsilon_1\upsilon_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} ds\\
&= \begin{pmatrix} z_1 + t\upsilon_1 + z_1\upsilon_1 t - \frac{1}{2}\upsilon_1\upsilon_2 t \\ z_2 \\ \upsilon_1 \\ \upsilon_2 \end{pmatrix}.
\end{aligned}
$$

Comparing quadratic terms of N_υ and $Z^{(2)}(1)$, we obtain $A_{10} = 1$ and $A_{11} = -\frac{1}{2}$.

Defined $M_3(Z) = \left(\sum_{i+j+k+l=3} A_{ijkl} z_1^i z_2^j \upsilon_1^k \upsilon_2^l, \sum_{i+j+k+l=3} B_{ijkl} z_1^i z_2^j \upsilon_1^k \upsilon_2^l, 0, 0 \right)^T$, then

$$
\begin{aligned}
&Z^{(3)}(t)\\
&= e^{Mt}Z + \int_0^t e^{M(t-s)} (M_2(Z^{(2)}(s)) + M_3(Z^{(2)}(s)))ds\\
&= \begin{pmatrix} z_1 + t\upsilon_1 \\ z_2 \\ 0 \\ 0 \end{pmatrix} + \int_0^t e^{M(t-s)} \begin{pmatrix} A_{10}\upsilon_2(z_1 + s\upsilon_1 + z_1\upsilon_1 s - \frac{1}{2}\upsilon_1\upsilon_2 s) + A_{11}\upsilon_1\upsilon_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} ds\\
&\quad + \int_0^t e^{M(t-s)} \begin{pmatrix} \sum_{i+j+k+l=3} A_{ijkl}(z_1 + s\upsilon_1 + z_1\upsilon_1 s - \frac{1}{2}\upsilon_1\upsilon_2 s)^i z_2^j \upsilon_1^k \upsilon_2^l \\ \sum_{i+j+k+l=3} B_{ijkl}(z_1 + s\upsilon_1 + z_1\upsilon_1 s - \frac{1}{2}\upsilon_1\upsilon_2 s)^i z_2^j \upsilon_1^k \upsilon_2^l \\ 0 \\ 0 \end{pmatrix} ds,\\
&= \begin{pmatrix} z_1 + t\upsilon_1 + \upsilon_2 z_1 t + \frac{1}{2}\upsilon_1\upsilon_2 t^2 - \frac{1}{2}\upsilon_1\upsilon_2 t + \frac{1}{2}\upsilon_1\upsilon_2 z_1 t^2 - \frac{1}{4}\upsilon_1\upsilon_2^2 t^2 \\ z_2 \\ \upsilon_1 \\ \upsilon_2 \end{pmatrix}
\end{aligned}
$$

$$+\begin{pmatrix} \int_0^t \sum_{i+j+k+l=3} A_{ijkl}(z_1+s\upsilon_1+z_1\upsilon_1 s-\frac{1}{2}\upsilon_1\upsilon_2 s)^i z_2^j \upsilon_1^k \upsilon_2^l)ds \\ \int_0^t \sum_{i+j+k+l=3} B_{ijkl}(z_1+s\upsilon_1+z_1\upsilon_1 s-\frac{1}{2}\upsilon_1\upsilon_2 s)^i z_2^j \upsilon_1^k \upsilon_2^l)ds \\ 0 \\ 0 \end{pmatrix}.$$

Comparing cubic terms of N_υ and $Z^{(3)}(1)$, we obtain

$$\begin{aligned} &A_{3000}=\Phi_{30}, \quad A_{2010}=-\frac{3}{2}\Phi_{30}, \quad A_{0210}=-\frac{1}{2}\Phi_{12}, \quad A_{1020}=\frac{1}{2}\Phi_{30}, \\ &A_{0021}=A_{0012}=\frac{1}{4}, \quad A_{1011}=-\frac{1}{2}, \quad A_{0030}=\frac{1}{4}\Phi_{30}, \\ &B_{2100}=\Psi_{21}, \quad B_{0300}=\Psi_{03}, \quad B_{1110}=-\Psi_{21}, \quad B_{0120}=\frac{1}{6}\Psi_{21}. \end{aligned}$$

Others A_{ijkl}, B_{ijkl} are zeros. The proof is completed. □

Theorem 3.4. *Consider system* $\varphi_\upsilon^1(z)+O(||\upsilon||^2)+O(||z||^2||\upsilon||)+O(||z||^4)$, *that is*

$$\begin{cases} \dot{z}_1=\upsilon_1+\upsilon_2 z_1+\Phi_{30}z_1^3+\Phi_{12}z_1 z_2^2, \\ \dot{z}_2=\Psi_{21}z_1^2 z_2+\Psi_{03}z_2^3. \end{cases}$$

It has dynamical behaviors near fold-flip bifurcation point as follows.

(i) *If the first Lyapunov coefficient is not zero, there is a hopf bifurcation curve*

$$\mathfrak{H}=\{(\upsilon_1,\upsilon_2):\upsilon_2=\frac{8(\Psi_{03}\Psi_{21}^2-\Phi_{12}\Psi_{21}^2)}{\Phi_{12}\Psi_{21}}\upsilon_1^2+O(|\upsilon_1|^3)\}.$$

(ii) *If* $\Phi_{30}\neq 0$, *there is a cusp bifurcation curve*

$$\mathfrak{C}=\{(\upsilon_1,\upsilon_2):27\Phi_{30}\upsilon_1^2+4\upsilon_2^3=0\}.$$

Proof. (i) A hopf bifurcation occurs at a bifurcation curve $\mathfrak{H}$ on $(\upsilon_1,\upsilon_2)-$plane. It is given by the projection of the curve

$$\begin{cases} \upsilon_1+\upsilon_2 z_1+\Phi_{30}z_1^3+\Phi_{12}z_1 z_2^2=0, \\ \Psi_{21}z_1^2 z_2+\Psi_{03}z_2^3=0, \\ \upsilon_2+3\Phi_{30}z_1^2+\Phi_{12}z_2^2+\Psi_{21}z_1^2+3\Psi_{03}z_2^2=0, \\ (\upsilon_2+3\Phi_{30}z_1^2+\Phi_{12}z_2^2)(\Psi_{21}z_1^2+3\Psi_{03}z_2^2)-4\Phi_{12}\Psi_{21}z_1^2 z_2^2=1, \end{cases}$$

onto the parameter plane. Elimination of z_1 and z_2 gives the curve $\mathfrak{H}$.

(ii) A cusp bifurcation occurs at a bifurcation curve $\mathfrak{C}$ on $(\upsilon_1,\upsilon_2)-$plane. It is given by the projection of the curve

$$\begin{cases} \upsilon_1+\upsilon_2 z_1+\Phi_{30}z_1^3=0, \\ \upsilon_2+3\Phi_{30}z_1^2=0, \end{cases}$$

onto the parameter plane. Elimination of z_1 gives the curve $\mathfrak{C}$. □

4. Stochastic codimension two flip bifurcation

The normal form of codimension two flip bifurcation is

$$x(n+1) = -(1+\alpha_1)x(n) + \alpha_2 x^3(n) + x^5(n), \tag{4.1}$$

where α_1 and α_2 are bifurcation parameters.

If we choose α_1 as the random parameter, it can be expressed as $\alpha_1 = \bar{\alpha}_1 + \delta u$, where $\bar{\alpha}_1$ is the statistic parameter, δ is the intensity. Then system (4.1) becomes

$$\begin{cases} x_0(n+1) = -(1+\bar{\alpha}_1)x_0(n) - \delta(\psi_1 x_1(n) + \varphi_0 x_0(n)) + \alpha_2 X_0 + \bar{X}_0, \\ x_1(n+1) = -(1+\bar{\alpha}_1)x_1(n) - \delta(\varphi_1 x_1(n) + \phi_0 x_0(n)) + \alpha_2 X_1 + \bar{X}_1. \end{cases} \tag{4.2}$$

The dynamical behaviors of system (4.2) at fixed point E_0 are analogous to system (3.10). It can undergo flip bifurcation, 1:2 resonance and fold-flip bifurcation through numerical simulations.

We use the following Hermite orthogonal polynomials to make simulations for systems (3.10) and (4.2). The first six polynomials are

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12,$$
$$H_4(x) = 16x^4 - 48x^2 + 12, \quad H_5(x) = 32x^5 - 160x^3 + 120x.$$

Its coefficients in recurrent formula are

$$\phi_i = \frac{1}{2}, \quad \varphi_i = 0, \quad \psi_i = i.$$

Substituting these values into the representations of $\beta_{ij}, \gamma_{ij}, \bar{\beta}_{ij}, \bar{\gamma}_{ij}$, we can get the explicit systems of (3.10) and (4.2). The results of numerical simulations are shown in Fig. 1 and Fig. 2. There are some special notations in these figures. The circle, square and triangle represent flip bifurcation, 1:2 resonance and fold-flip bifurcation, respectively.

In Fig. 1(a), black, red and green curves are flip bifurcation curves when δ is the bifurcation parameter and $\bar{\alpha} = 0.1$ in (3.10). Two flip bifurcation points on the black curves are both the origin when $\delta = -0.14, \delta = -0.65$. Flip bifurcation point on red curve is $(-0.90, -0.36)$ and flip bifurcation point on green curve is $(0.90, 0.36)$ when $\delta = 2.83$. In Fig. 1(b), 1:2 resonance point appears at the intersection of the red and black lines with $\delta = \bar{\alpha} = 0$. Fold-flip bifurcation turns up on green curve when $\delta = 2.43, \bar{\alpha} = -0.19$.

In Fig. 2(a), there are two flip bifurcations on the black curve when $\delta = -0.14$ and $\delta = -0.65$, and other parameters value $\bar{\alpha}_1 = 0.1,\ \alpha_2 = 0.1$ in (4.2). In Fig. 2(b), 1:2 resonance point appears at $\delta = 0,\ \bar{\alpha}_1 = 0$ and fold-flip bifurcation turns up when $\delta = 2.43,\ \bar{\alpha}_1 = -1.86$. Its dynamical behaviors are similar to (3.10).

If we choose α_2 as the random parameter, it can be expressed as $\alpha_2 = \bar{\alpha}_2 + \delta u$, where $\bar{\alpha}_2$ is the statistic parameter, δ is the intensity. System (4.1) becomes

$$\begin{cases} x_0(n+1) = -(1+\alpha_1)x_0(n) + \bar{\alpha}_2 X_0 + \delta(\psi_1 X_1 + \varphi_0 X_0) + \bar{X}_0, \\ x_1(n+1) = -(1+\alpha_1)x_1(n) + \bar{\alpha}_2 X_1 + \delta(\psi_2 X_2 + \varphi_1 X_1 + \phi_0 X_0) + \bar{X}_1. \end{cases} \tag{4.3}$$

System (4.3) at fixed point E_0 does not undergo any bifurcations with the change of δ or $\bar{\alpha}_2$. But there is only one flip bifurcation at the origin when α_1 is the bifurcation parameter, seeing Fig. 3.

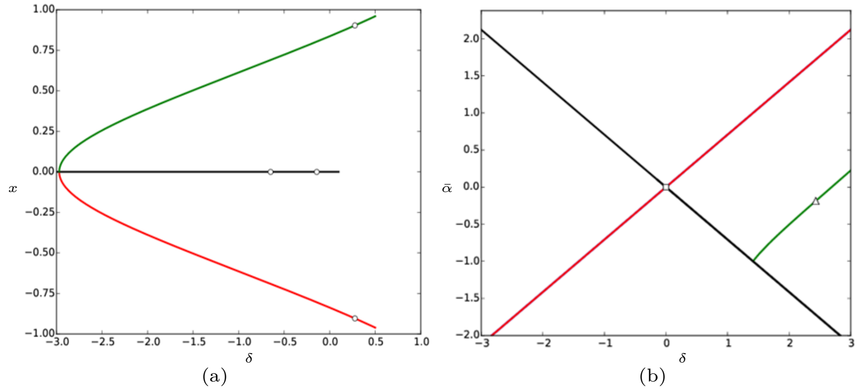

Figure 1. The bifurcation diagrams for system (3.10).

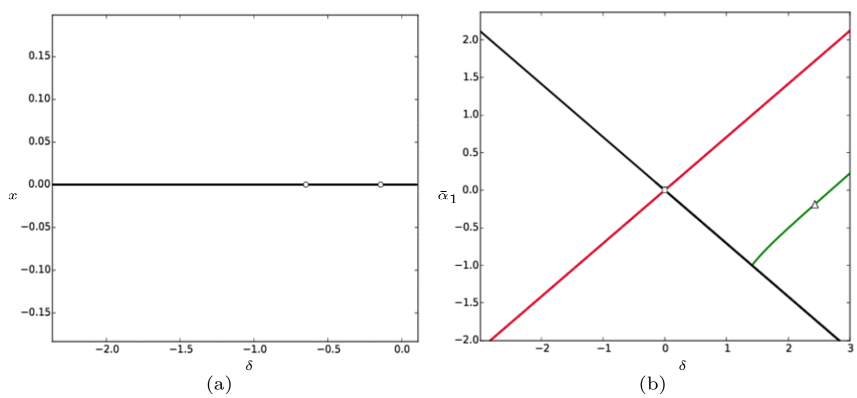

Figure 2. The bifurcation diagrams for system (4.2) with $x = x_0(n)$.

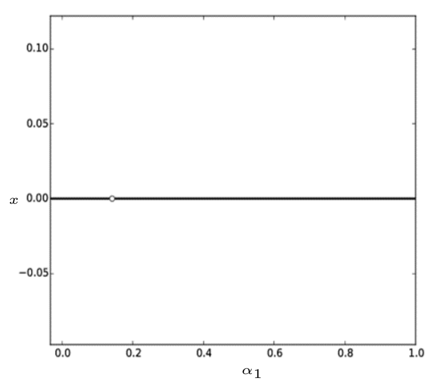

Figure 3. The bifurcation diagram for system (4.3) with $x = x_0(n)$. There is only one flip bifurcation at the origin when $\delta = -0.2$, $\alpha_1 = 0.14$ and $\bar{\alpha}_2 = 0.1$.

Here

$$
\begin{aligned}
&X_2(n) = \zeta_{12}x_0(n)x_1^2(n) + \zeta_{03}x_1^3(n),\\
&\bar{X}_0(n) = \bar{\beta}_{05}x_1^5(n) + \bar{\beta}_{14}x_0(n)x_1^4(n) + \bar{\beta}_{23}x_0^2(n)x_1^3(n) + \bar{\beta}_{32}x_0^3(n)x_1^2(n) + \bar{\beta}_{50}x_0^5(n),\\
&\bar{X}_1(n) = \bar{\gamma}_{05}x_1^5(n) + \bar{\gamma}_{14}x_0(n)x_1^4(n) + \bar{\gamma}_{23}x_0^2(n)x_1^3(n) + \bar{\gamma}_{32}x_0^3(n)x_1^2(n) + \bar{\gamma}_{41}x_0^4(n)x_1(n),\\
&\zeta_{03} = \frac{3a_{10}a_{11}^2}{a_{22}} - \frac{a_{11}^3a_{32}}{a_{22}a_{33}}, \quad \zeta_{12} = \frac{3a_{11}^2}{a_{22}},\\
&\bar{\beta}_{05} = \frac{10a_{10}^4a_{11}a_{21}}{a_{22}} + \frac{10a_{10}^3a_{11}^2a_{31}}{a_{33}} + \frac{10a_{10}^2a_{11}^3a_{20}a_{32}}{a_{22}a_{33}} + \frac{5a_{10}^2a_{11}^3a_{41}}{a_{44}} + \frac{5a_{10}a_{11}^4a_{20}a_{42}}{a_{22}a_{44}}\\
&\qquad + \frac{5a_{10}a_{11}^4a_{30}a_{43}}{a_{33}a_{44}} + \frac{5a_{10}^2a_{11}^3a_{21}a_{32}a_{43}}{a_{22}a_{33}a_{44}} + \frac{a_{11}^5a_{20}a_{52}}{a_{22}a_{55}} + \frac{a_{10}a_{11}^4a_{21}a_{32}a_{53}}{a_{22}a_{33}a_{55}}\\
&\qquad + \frac{a_{11}^5a_{40}a_{54}}{a_{44}a_{55}} + \frac{a_{10}a_{11}^4a_{21}a_{42}a_{54}}{a_{22}a_{44}a_{55}} + \frac{a_{10}a_{11}^4a_{31}a_{43}a_{54}}{a_{33}a_{44}a_{55}} + \frac{a_{11}^5a_{20}a_{32}a_{43}a_{54}}{a_{22}a_{33}a_{44}a_{55}}\\
&\qquad - \frac{10a_{10}^2a_{11}^3a_{30}}{a_{33}} - \frac{10a_{10}^3a_{11}^2a_{21}a_{32}}{a_{22}a_{33}} - \frac{5a_{10}a_{11}^4a_{40}}{a_{44}} - \frac{5a_{10}^2a_{11}^3a_{21}a_{42}}{a_{22}a_{44}}\\
&\qquad - \frac{5a_{10}a_{11}^4a_{20}a_{32}a_{43}}{a_{22}a_{33}a_{44}} - \frac{a_{10}a_{11}^4a_{21}a_{52}}{a_{22}a_{55}} - \frac{a_{10}a_{11}^4a_{31}a_{53}}{a_{33}a_{55}} - \frac{a_{11}^5a_{20}a_{32}a_{53}}{a_{22}a_{33}a_{55}}\\
&\qquad - \frac{a_{10}a_{11}^4a_{41}a_{54}}{a_{44}a_{55}} - \frac{a_{11}^5a_{20}a_{42}a_{54}}{a_{22}a_{44}a_{55}} - \frac{a_{11}^5a_{30}a_{43}a_{54}}{a_{33}a_{44}a_{55}} - \frac{a_{10}a_{11}^4a_{21}a_{32}a_{43}a_{54}}{a_{22}a_{33}a_{44}a_{55}}\\
&\qquad + \frac{a_{11}^5a_{30}a_{53}}{a_{33}a_{55}} + \frac{a_{10}a_{11}^4a_{51}}{a_{55}} - \frac{10a_{10}^3a_{11}^2a_{20}}{a_{22}} - \frac{5a_{10}^2a_{11}^3a_{31}a_{43}}{a_{33}a_{44}} - \frac{a_{11}^5a_{50}}{a_{55}} - 4a_{10}^5,\\
&\bar{\beta}_{14} = \frac{30a_{10}^3a_{11}a_{21}}{a_{22}} + \frac{20a_{10}^2a_{11}^2a_{31}}{a_{33}} + \frac{20a_{10}a_{11}^3a_{20}a_{32}}{a_{22}a_{33}} + \frac{5a_{10}a_{11}^3a_{41}}{a_{44}} + \frac{5a_{11}^4a_{20}a_{42}}{a_{22}a_{44}}\\
&\qquad + \frac{5a_{11}^4a_{30}a_{43}}{a_{33}a_{44}} + \frac{5a_{10}a_{11}^3a_{21}a_{32}a_{43}}{a_{22}a_{33}a_{44}} - \frac{30a_{10}^2a_{11}^2a_{20}}{a_{22}} - \frac{20a_{10}^2a_{11}^2a_{21}a_{32}}{a_{22}a_{33}}\\
&\qquad - \frac{5a_{11}^4a_{40}}{a_{44}} - \frac{5a_{10}a_{11}^3a_{21}a_{42}}{a_{22}a_{44}} - \frac{5a_{10}a_{11}^3a_{31}a_{43}}{a_{33}a_{44}} - \frac{5a_{11}^4a_{20}a_{32}a_{43}}{a_{22}a_{33}a_{44}}\\
&\qquad - \frac{20a_{10}a_{11}^3a_{30}}{a_{33}} - 15a_{10}^4,\\
&\bar{\beta}_{23} = \frac{30a_{10}^2a_{11}a_{21}}{a_{22}} + \frac{10a_{10}a_{11}^2a_{31}}{a_{33}} + \frac{10a_{11}^3a_{20}a_{32}}{a_{22}a_{33}} - \frac{30a_{10}a_{11}^2a_{20}}{a_{22}} - \frac{10a_{11}^3a_{30}}{a_{33}} -\\
&\qquad \frac{10a_{10}a_{11}^2a_{21}a_{32}}{a_{22}a_{33}} - 20a_{10}^3,\\
&\bar{\beta}_{32} = \frac{10a_{10}a_{11}a_{21}}{a_{22}} - \frac{10a_{11}^2a_{20}}{a_{22}} - 10a_{10}^2, \ \bar{\beta}_{50} = 1,\\
&\bar{\gamma}_{05} = 5a_{10}^4 - \frac{10a_{10}^3a_{11}a_{21}}{a_{22}} - \frac{10a_{10}^2a_{11}^2a_{31}}{a_{33}} + \frac{10a_{10}^2a_{11}^2a_{21}a_{32}}{a_{22}a_{33}} - \frac{5a_{10}a_{11}^3a_{41}}{a_{44}}\\
&\qquad + \frac{5a_{10}a_{11}^3a_{31}a_{43}}{a_{33}a_{44}} - \frac{5a_{10}a_{11}^3a_{21}a_{32}a_{43}}{a_{22}a_{33}a_{44}} - \frac{a_{11}^4a_{51}}{a_{55}} + \frac{a_{11}^4a_{21}a_{52}}{a_{22}a_{55}} + \frac{a_{11}^4a_{31}a_{53}}{a_{33}a_{55}}\\
&\qquad + \frac{a_{11}^4a_{41}a_{54}}{a_{44}a_{55}} - \frac{a_{11}^4a_{21}a_{42}a_{54}}{a_{22}a_{44}a_{55}} - \frac{a_{11}^4a_{31}a_{43}a_{54}}{a_{33}a_{44}a_{55}} + \frac{a_{11}^4a_{21}a_{32}a_{43}a_{54}}{a_{22}a_{33}a_{44}a_{55}}\\
&\qquad + \frac{5a_{10}a_{11}^3a_{21}a_{42}}{a_{22}a_{44}} - \frac{a_{11}^4a_{21}a_{32}a_{53}}{a_{22}a_{33}a_{55}},
\end{aligned}
$$

$$\bar{\gamma}_{14} = 20a_{10}^3 - \frac{30a_{11}a_{21}a_{10}^2}{a_{22}} - \frac{20a_{11}^2a_{31}a_{10}}{a_{33}} + \frac{20a_{11}^2a_{21}a_{32}a_{10}}{a_{22}a_{33}} - \frac{5a_{11}^3a_{41}}{a_{44}}$$
$$+ \frac{5a_{11}^3a_{21}a_{42}}{a_{22}a_{44}} + \frac{5a_{11}^3a_{31}a_{43}}{a_{33}a_{44}} - \frac{5a_{11}^3a_{21}a_{32}a_{43}}{a_{22}a_{33}a_{44}}, \ \bar{\gamma}_{41} = 5,$$
$$\bar{\gamma}_{23} = 30a_{10}^2 - \frac{30a_{11}a_{21}a_{10}}{a_{22}} - \frac{10a_{11}^2a_{31}}{a_{33}} + \frac{10a_{11}^2a_{21}a_{32}}{a_{22}a_{33}}, \ \bar{\gamma}_{32} = 20a_{10} - \frac{10a_{11}a_{21}}{a_{22}},$$
$$\phi_2 = \frac{a_{22}}{a_{33}}, \quad \varphi_2 = \frac{a_{21}}{a_{22}} - \frac{a_{32}}{a_{33}}, \quad \psi_2 = \frac{a_{20}}{a_{11}} - \frac{a_{22}a_{31}}{a_{11}a_{33}} - \frac{a_{21}^2}{a_{11}a_{22}} + \frac{a_{21}a_{32}}{a_{11}a_{33}}.$$

5. Conclusion

In this paper, we investigate the dynamics for codimension one and two flip bifurcation systems in normal form, when bifurcation parameters in different positions are randomly disturbed. In either case, the random excitation imposed on linear term can generate more complex dynamical behaviors: flip bifurcation, Neimark-Sacker bifurcation, 1:2 resonance and fold-flip bifurcation. We first give the critical coefficient which determines the stability of closed invariant curve caused by Neimark-Sacker bifurcation. Then we prove that both systems undergo heteroclinic bifurcation and homoclinic bifurcation near 1:2 resonance point. Moreover, we calculate hopf bifurcation curve and cusp bifurcation curve near fold-flip bifurcation point. However, when the random excitation imposed on nonlinear term, the situation departures from that case for the random excitation imposed on linear term. It is found that only flip bifurcation occurs in this situation. This interesting phenomenon, which is determined by the positions where random parameter locates, may have a significance impact on applications in engineering control problems.

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