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SOME $L^Q(\mathbb{R})$-NORM DECAY ESTIMATES ($Q \in [1, +\infty]$) FOR TWO CAUCHY SYSTEMS OF TYPE RAO-NAKRA SANDWICH BEAM WITH A FRICTIONAL DAMPING OR AN INFINITE MEMORY

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Abstract In this paper, we consider two systems of type Rao-Nakra sandwich beam in the whole line $\mathbb{R}$ with a frictional damping or an infinite memory acting on the Euler-Bernoulli equation. When the speeds of propagation of the two wave equations are equal, we show that the solutions do not converge to zero when time goes to infinity. In the reverse situation, we prove some $L^2(\mathbb{R})$-norm and $L^1(\mathbb{R})$-norm decay estimates of solutions and theirs higher order derivatives with respect to the space variable. Thanks to interpolation inequalities and Carlson inequality, these $L^2(\mathbb{R})$-norm and $L^1(\mathbb{R})$-norm decay estimates lead to similar ones in the $L^q(\mathbb{R})$-norm, for any $q \in [1, +\infty]$. In our both $L^2(\mathbb{R})$-norm and $L^1(\mathbb{R})$-norm decay estimates, we specify the decay rates in terms of the regularity of the initial data and the nature of the control.

Keywords Rao-Nakra sandwich beam, frictional damping, infinite memory, unbounded domain, asymptotic behavior, $L^2(\mathbb{R})$-norm and $L^1(\mathbb{R})$-norm decay estimates, energy method, Fourier analysis.

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1. Introduction

Let ρ_1, ρ_2, ρ_3, k_0, k_1, k_2, k_3, γ and l be real positive constants and $g : \mathbb{R}_+ := [0, +\infty) \to \mathbb{R}_+$ satisfying $g \in C^1(\mathbb{R}_+)$,

$$0 < g_0 := \int_0^{+\infty} g(s)\, ds < k_3, \tag{1.1}$$

and, for some real positive constants β_1 and β_2,

$$-\beta_2 g \leq g' \leq -\beta_1 g. \tag{1.2}$$

Condition (1.2) implies that $g' \leq 0$, g is integrable over $\mathbb{R}_+$ and, by integrating,

$$g(0)e^{-\beta_2 s} \leq g(s) \leq g(0)e^{-\beta_1 s}, \quad s \in \mathbb{R}_+. \tag{1.3}$$

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So (1.1) is valid if $g(0) > 0$ and β_1 is big enough. For example, one can take $g(s) = d_1 e^{-d_2 s}$ with $d_1,\, d_2 > 0$ satisfying $\frac{d_1}{d_2} < k_3$, so (1.1) and (1.2) hold, for $\beta_1 = \beta_2 = d_2$.

This paper deals with the stability of two systems of type Rao-Nakra sandwich beam in the whole line $\mathbb{R}$ with a control acting only on the Euler-Bernoulli equation. These systems consist of two wave equations for the longitudinal displacements of the top and bottom layers, and one Euler-Bernoulli equation for the transversal displacement. The considered control is provided through a frictional damping of size γ:

$$\begin{cases} \rho_1 \varphi_{tt} - k_1 \varphi_{xx} + k_0(\varphi + \psi + l w_x) = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_0(\varphi + \psi + l w_x) = 0, \\ \rho_3 w_{tt} + k_3 w_{xxxx} - l k_0 (\varphi + \psi + l w_x)_x + \gamma w_t = 0, \\ \varphi(x,0) = \varphi_0(x),\, \psi(x,0) = \psi_0(x),\, w(x,0) = w_0(x), \\ \varphi_t(x,0) = \varphi_1(x),\, \psi_t(x,0) = \psi_1(x),\, w_t(x,0) = w_1(x) \end{cases} \tag{1.4}$$

or an infinite memory of kernel g:

$$\begin{cases} \rho_1 \varphi_{tt} - k_1 \varphi_{xx} + k_0(\varphi + \psi + l w_x) = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_0(\varphi + \psi + l w_x) = 0, \\ \rho_3 w_{tt} + k_3 w_{xxxx} - l k_0 (\varphi + \psi + l w_x)_x - \displaystyle\int_0^{+\infty} g(s)\, w_{xxxx}(x, t-s) ds = 0, \\ \varphi(x,0) = \varphi_0(x),\, \psi(x,0) = \psi_0(x),\, w(x,-t) = w_0(x,t), \\ \varphi_t(x,0) = \varphi_1(x),\, \psi_t(x,0) = \psi_1(x),\, w_t(x,0) = w_1(x), \end{cases} \tag{1.5}$$

where $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$ are, respectively, the spacial and time variables, φ_0 , φ_1, ψ_0, ψ_1, w_0 and w_1 are fixed initial data, and

$$(\varphi,\, \psi,\, w) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^3$$

is the unknown of (1.4) and (1.5). A subscript r denotes the derivative with respect to r. Also, we use ∂_r^m or $\frac{d^m}{dr^m}$ to denote the differential operator of order m with respect to r; i.e. $\frac{\partial^m}{\partial r^m}$. When a function has only one variable, its derivative is noted by $\prime$.

Since the works [23,31,36], several layer laminated beam and plate models have been introduced during the last sixty years. The known generalized Rao-Nakra beam (composed of a top and a bottom face plate), presented in [21], takes into account the shear effect of the bottom and top layers, and it is gien by

$$\begin{cases} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - \tau = 0, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + \tau = 0, \\ \rho h w_{tt} + EI w_{xxxx} - G_1 h_1 (w_x + \phi_1)_x - G_3 h_3 (w_x + \phi_3)_x - h_2 \tau_x = 0, \\ \rho_1 I_1 \phi_{1,tt} - E_1 I_1 \phi_{1,xx} - \frac{1}{2} h_1 \tau + G_1 h_1 (w_x + \phi_1) = 0, \\ \rho_3 I_3 \phi_{3,tt} - E_3 I_3 \phi_{3,xx} - \frac{1}{2} h_3 \tau + G_3 h_3 (w_x + \phi_3) = 0, \end{cases} \tag{1.6}$$

where ρ_j, h_j, E_j, G_j, I_j, EI and ρh are real positive constants representing some physical parameters and satisfying some relationships, u, ϕ_1, v and ϕ_3 are, respectively, the longitudinal displacement and shear angle of the top and bottom layers, w is the transverse displacement of the beam and τ is the shear stress in the core layer defined by

$$\tau = -u - \frac{1}{2}h_1\phi_1 + h_2 w_x + v - \frac{1}{2}h_3\phi_3.$$

By neglecting some components and/or parameters and/or considering some connections between them, several models were derived from (1.6) and studied in the literature like the Rao-Nakra sandwich beam [29]:

$$\begin{cases} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) = 0, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) = 0, \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = 0, \end{cases} \tag{1.7}$$

the laminated beam [17]:

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho (3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - u_x) = 0, \\ 3I_\rho s_{tt} - 3D s_{xx} + 3G(\psi - w_x) + 4\mu s + 4\delta s_t = 0, \end{cases} \tag{1.8}$$

the Bresse model [4]:

$$\begin{cases} \rho_1 u_{tt} - k_1(u_x + \psi + lw)_x - lk_3(w_x - lu) = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(u_x + \psi + lw) = 0, \\ \rho_3 w_{tt} - k_3(w_x - lu)_x + lk_1(u_x + \psi + lw) = 0, \end{cases} \tag{1.9}$$

the Mead-Markos model [23]:

$$\begin{cases} \rho h w_{tt} + EI w_{xxxx} - \alpha_1(-u + v + \alpha w_x) = 0, \\ (-u + v + \alpha w_x)_{xx} - \alpha_2(-u + v + \alpha w_x) - \alpha w_{xxxx} = 0 \end{cases} \tag{1.10}$$

and the Timoshenko beam [35]:

$$\begin{cases} \rho_1 u_{tt} - k_1(u_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(u_x + \psi) = 0, \end{cases} \tag{1.11}$$

where all the coefficients are real postive constants.

The study of long time behavior of (1.7)-(1.11) has been the subject of an active research and mathematical endeavour. Thereby, a huge number of research articles have been appeared. In order to highlight the main contribution and the novelty of the present paper, we will merely point out the articles whose content concerns the Rao-Nakra sandwich beam (1.7), and so their content is closely related to our problems (1.4) and (1.5). For (1.8)-(1.11) (in different contexts), we refer the reader, for example, to [1,2,5,8–11,14,21,24,32,33] and the references therein.

In [20], the authors considered the Rao-Nakra sandwich beam with $x \in (0,1)$ and an internal frictional damping acting either on the beam equation or on one of the wave equations, and proved that the polynomial stability occurs. Similar polynomial stability results were proved in [19] under internal damping or Kelvin-Voigt damping working on two of the three equations.

The authors of [30] proved the well-posedness and exponential stability of Rao-Nakra sandwich beam in $(0,L)$, $L>0$, under a heat conduction of secound sound acting on the second wave equation (see [6,22,27] for details on the second sound mechanism). The same well-posedness and exponential stability results of [30] were proved in [25] using boundary feedbacks at $x=L$.

The (global or local) boundary controllability problems of the Rao-Nakra beam were also the subject of several studies in the literature; see, for example, [12,13, 15,16,26,28] and the references therein.

The subject of this paper is to treat the stability of the Rao-Nakra sandwich beam (1.7) in the whole line $\mathbb{R}$ under a frictional damping (1.4) or an infinite memory (1.5) (where we denoted (u,v) by $(-\varphi,\psi)$ and simplified the notation of the coefficients). The frictinal damping and infinite memory terms generate the unique dissipation for (1.4) and (1.5) (see (2.15) and Remark 2.1 below). This dissipation is acting on the third equation of (1.4) and (1.5) (transversal displacement). To the best of our knowledge, these situations have never been considered in the literature.

The main objective of this paper is to get decay estimates of

$$t \mapsto \left\|\partial_x^j U(\cdot,t)\right\|_2 \quad \text{and} \quad t \mapsto \left\|\partial_x^j U(\cdot,t)\right\|_1, \tag{1.12}$$

where U is defined in (2.7) below, $j \in \mathbb{N}$ and $\|\cdot\|_q$ denotes the norm of $L^q(\mathbb{R})$, for $q \in [1,+\infty]$. We will prove the instability of (1.4) and (1.5) when

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}. \tag{1.13}$$

However, when

$$\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}, \tag{1.14}$$

we prove that the functions (1.12) satisfy some polynomial stability estimates with decay rates depending on the regularity of the initial data.

Our $L^2(\mathbb{R})$-norm and $L^1(\mathbb{R})$-norm decay estimates cover all the $L^q(\mathbb{R})$-norm decay estimates of $\partial_x^j U$, for any $q \in [1,+\infty]$. Indeed, using interpolation inequalities, the decay estimates of the L^∞-norm is obtained immediately from the inequality (δ_q denotes a real positive constant, which does not depend neither on U nor on j)

$$\|\partial_x^j U\|_\infty \leq \delta_\infty \|\partial_x^j U\|_2^{1/2} \|\partial_x^{j+1} U\|_2^{1/2}. \tag{1.15}$$

This eventually lead to the decay rate of the L^q-norm, $2<q<+\infty$, through the interpolation inequality

$$\|\partial_x^j U\|_q \leq \delta_q \|\partial_x^j U\|_\infty^{1-2/q} \|\partial_x^j U\|_2^{2/q}. \tag{1.16}$$

To fill the gap $1<q<2$, we use the interpolation inequality

$$\left\|\partial_x^j U\right\|_q \leq \delta_q \left\|\partial_x^j U\right\|_2^{2(q-1)/q} \left\|\partial_x^j U\right\|_1^{(2-q)/q}. \tag{1.17}$$

To get the $L^1(\mathbb{R})$-norm decay estimate, we treat the asymptotic behavior of $\|x\partial_x^j U\|_2$, and then we use the Carlson inequality (see [3] for instance)

$$\|\partial_x^j U\|_1 \leq \|\partial_x^j U\|_2^{1/2} \|x\partial_x^j U\|_2^{1/2}. \tag{1.18}$$

The proof is based on the energy method combined with the Fourier analysis (by using the transformation in the Fourier space).

The paper is organized as follows. In section 2, we formulate (1.4) and (1.5) in a first order Cauchy system. Section 3 will be devoted to the proof of the asymptotic behavior of $|\widehat{U}|$. In section 4, we prove our $L^2(\mathbb{R})$-norm decay estimate. The asymptotic behavior of $|\partial_\xi \widehat{U}|$ will be treated in section 5. Finally, in section 6, we prove our $L^1(\mathbb{R})$-norm decay estimate. The last section presents some concluding remarks.

2. Abstract formulation of (1.4) and (1.5)

To formulate (1.4) and (1.5) in an abstract first order system, we consider the following variables:

$$u = \varphi_t, \quad y = \psi_t, \quad \theta = w_t, \quad v = \varphi_x, \quad z = \psi_x, \quad \phi = w_{xx} \quad \text{and} \quad p = \varphi + \psi + l w_x. \tag{2.1}$$

We observe that $(1.4)_1$-$(1.4)_3$ can be written in the form

$$\begin{cases} v_t - u_x = 0, \\ \rho_1 u_t - k_1 v_x + k_0 p = 0, \\ z_t - y_x = 0, \\ \rho_2 y_t - k_2 z_x + k_0 p = 0, \\ \phi_t - \theta_{xx} = 0, \\ \rho_3 \theta_t + k_3 \phi_{xx} - l k_0 p_x + \gamma\theta = 0, \\ p_t - u - y - l\theta_x = 0. \end{cases} \tag{2.2}$$

In case (1.5), we consider the additional variable introduced in [7]

$$\eta(x,t,s) = w(x,t) - w(x,t-s) \tag{2.3}$$

with its initial data $\eta_0(x,s) = \eta(x,0,s)$. The variable η satisfies

$$\eta_t(x,t,s) + \eta_s(x,t,s) = w_t(x,t) \tag{2.4}$$

and

$$\int_0^{+\infty} g(s)\eta_{xxxx}(x,t,s)\, ds = g_0 w_{xxxx}(x,t) - \int_0^{+\infty} g(s) w_{xxxx}(x,t-s)\, ds. \tag{2.5}$$

Then $(1.5)_1$-$(1.5)_3$ can be formulated in the form

$$\begin{cases} v_t - u_x = 0, \\ \rho_1 u_t - k_1 v_x + k_0 p = 0, \\ z_t - y_x = 0, \\ \rho_2 y_t - k_2 z_x + k_0 p = 0, \\ \phi_t - \theta_{xx} = 0, \\ \rho_3 \theta_t + (k_3 - g_0)\phi_{xx} - l k_0 p_x + \int_0^{+\infty} g(s)\eta_{xxxx}\, ds = 0, \\ p_t - u - y - l\theta_x = 0, \\ \eta_t + \eta_s - \theta = 0. \end{cases} \tag{2.6}$$

We define the variable U and its initial data U_0 by

$$U(x,t) = \begin{cases} (v,\, u,\, z,\, y,\, \phi,\, \theta,\, p)^T(x,t) & \text{in case (1.4)}, \\ (v,\, u,\, z,\, y,\, \phi,\, \theta,\, p, \eta)^T(x,t) & \text{in case (1.5)} \end{cases} \quad \text{and} \quad U_0(x) = U(x,0). \tag{2.7}$$

Therefore, the systems (2.2) and (2.6) lead to

$$\begin{cases} U_t(x,t) + AU(x,t) = 0, \\ U(x,0) = U_0(x), \end{cases} \tag{2.8}$$

where

$$AU = A_4 U_{xxxx} + A_2 U_{xx} + A_1 U_x + A_0 U$$

and the operators A_j are defined in case (2.2) by

$$\begin{cases} A_4 U_{xxxx} = (0,0,0,0,0,0,0)^T, \\ A_2 U_{xx} = \left(0,0,0,0,-\theta_{xx}, \frac{k_3}{\rho_3}\phi_{xx}, 0\right)^T, \\ A_1 U_x = \left(-u_x, -\frac{k_1}{\rho_1}v_x, -y_x, -\frac{k_2}{\rho_2}z_x, 0, -\frac{l k_0}{\rho_3}p_x, -l\theta_x\right)^T, \\ A_0 U = \left(0, \frac{k_0}{\rho_1}p, 0, \frac{k_0}{\rho_2}p, 0, \frac{\gamma}{\rho_3}\theta, -u-y\right)^T, \end{cases}$$

and A_j in case (2.6) are defined by

$$\begin{cases} A_4 U_{xxxx} = \left(0,0,0,0,0, \frac{1}{\rho_3}\int_0^{+\infty} g(s)\eta_{xxxx}\, ds, 0, 0\right)^T, \\ A_2 U_{xx} = \left(0,0,0,0,-\theta_{xx}, \frac{k_3-g_0}{\rho_3}\phi_{xx}, 0, 0\right)^T, \\ A_1 U_x = \left(-u_x, -\frac{k_1}{\rho_1}v_x, -y_x, -\frac{k_2}{\rho_2}z_x, 0, -\frac{l k_0}{\rho_3}p_x, -l\theta_x, 0\right)^T, \\ A_0 U = \left(0, \frac{k_0}{\rho_1}p, 0, \frac{k_0}{\rho_2}p, 0, 0, -u-y, \eta_s - \theta\right)^T. \end{cases}$$

For a given function $h : \mathbb{R} \to \mathbb{C}$, we use the notations $Re\, h$, $Im\, h$, $\bar{h}$ and $\widehat{h}$ to denote, respectively, the real part of h, the imaginary part of h, the conjugate of h and the Fourier transformation of h given by

$$\widehat{h}(\xi) = \int_{-\infty}^{+\infty} e^{-i\xi x} h(x)\, dx, \quad \xi \in \mathbb{R}. \tag{2.9}$$

Applying the Fourier transformation (with respect to the space variable x) to (2.8), we obtain the following first-order Cauchy system in the Fourier space:

$$\begin{cases} \widehat{U}_t(\xi,\, t) + \tilde{A}(\xi)\widehat{U}(\xi,\, t) = 0, \\ \widehat{U}(\xi,\, 0) = \widehat{U}_0(\xi), \end{cases} \tag{2.10}$$

where $\tilde{A}(\xi) = \xi^4\, A_4 - \xi^2\, A_2 + i\,\xi\, A_1 + A_0$. The solution of (2.10) is given by

$$\widehat{U}(\xi,\, t) = e^{-\tilde{A}(\xi)\, t}\; \widehat{U}_0(\xi). \tag{2.11}$$

Computing the term $e^{-\tilde{A}(\xi)t}$ is a challenging problem, and in many situations, this cannot be done. Consequently, in order to show the asymptotic behavior of $\widehat{U}$, it suffices to find a non negative function $f(\xi)$ and two positive constants $\tilde{c}$ and c such that, for each $(\xi, t) \in \mathbb{R}\times\mathbb{R}_+$,

$$\left| e^{-\tilde{A}(\xi)t} \right| \leq \tilde{c}e^{-cf(\xi)t}. \tag{2.12}$$

Let $\widehat{E}$ be the energy associated with (2.10) given by

$$\begin{aligned} \widehat{E}(\xi,t) =& \frac{1}{2}\left[k_1\,|\widehat{v}|^2 + \rho_1|\widehat{u}|^2 + k_2\,|\widehat{z}|^2 + \rho_2|\widehat{y}|^2 + (k_3 - \tau_0 g_0)|\widehat{\phi}|^2 + \rho_3|\widehat{\theta}|^2 + k_0|\widehat{p}|^2\right](\xi,\, t) \\ &+ \frac{\tau_0}{2}\xi^4\int_0^{+\infty} g(s)|\widehat{\eta}(\xi,\, t)|^2\, ds, \end{aligned} \tag{2.13}$$

where

$$\tau_0 = \begin{cases} 0 & \text{in case (2.2)}, \\ 1 & \text{in case (2.6)}. \end{cases} \tag{2.14}$$

Lemma 2.1. *The energy functional $\widehat{E}$ satisfies, for each $(\xi, t) \in \mathbb{R}\times\mathbb{R}_+$,*

$$\frac{d}{dt}\widehat{E}(\xi,\, t) = -(1-\tau_0)\gamma\,|\widehat{\theta}(\xi,\, t)|^2 + \frac{\tau_0}{2}\xi^4\int_0^{+\infty} g'(s)|\widehat{\eta}(\xi,\, t)|^2\, ds. \tag{2.15}$$

Proof. The equation $(2.10)_1$ in case (2.2) is equivalent to

$$\begin{cases} \widehat{v}_t - i\xi\widehat{u} = 0, \\ \rho_1\widehat{u}_t - ik_1\xi\,\widehat{v} + k_0\widehat{p} = 0, \\ \widehat{z}_t - i\xi\widehat{y} = 0, \\ \rho_2\widehat{y}_t - ik_2\xi\,\widehat{z} + k_0\widehat{p} = 0, \\ \widehat{\phi}_t + \xi^2\widehat{\theta} = 0, \\ \rho_3\widehat{\theta}_t - k_3\xi^2\widehat{\phi} - ilk_0\,\xi\widehat{p} + \gamma\widehat{\theta} = 0, \\ \widehat{p}_t - \widehat{u} - \widehat{y} - il\xi\widehat{\theta} = 0. \end{cases} \tag{2.16}$$

Multiplying the equations in (2.16) by $k_1\bar{\widehat{v}}$, $\bar{\widehat{u}}$, $k_2\bar{\widehat{z}}$, $\bar{\widehat{y}}$, $k_3\bar{\widehat{\phi}}$, $\bar{\widehat{\theta}}$ and $k_0\bar{\widehat{p}}$, respectively, adding the obtained equations, taking the real part of the resulting expression and using the identity, for two differentiable functions $h,\, d : \mathbb{R}\to\mathbb{C}$,

$$\frac{d}{dt}Re\,\left(h\bar{d}\right) = Re\,\left(h_t\bar{d} + d_t\bar{h}\right), \tag{2.17}$$

we obtain (2.15) with $\tau_0 = 0$. Similarly, $(2.10)_1$ in case (2.6) is reduced to

$$\begin{cases} \widehat{v}_t - i\xi\widehat{u} = 0, \\ \rho_1\widehat{u}_t - ik_1\xi\,\widehat{v} + k_0\widehat{p} = 0, \\ \widehat{z}_t - i\xi\widehat{y} = 0, \\ \rho_2\widehat{y}_t - ik_2\xi\,\widehat{z} + k_0\widehat{p} = 0, \\ \widehat{\phi}_t + \xi^2\widehat{\theta} = 0, \\ \rho_3\widehat{\theta}_t - (k_3 - g_0)\xi^2\widehat{\phi} - ilk_0\,\xi\widehat{p} + \xi^4\displaystyle\int_0^{+\infty} g(s)\widehat{\eta}\,ds = 0, \\ \widehat{p}_t - \widehat{u} - \widehat{y} - il\xi\widehat{\theta} = 0, \\ \widehat{\eta}_t + \widehat{\eta}_s - \widehat{\theta} = 0. \end{cases} \tag{2.18}$$

Multiplying $(2.18)_1$-$(2.18)_7$ by $k_1\bar{\widehat{v}}$, $\bar{\widehat{u}}$, $k_2\bar{\widehat{z}}$, $\bar{\widehat{y}}$, $(k_3 - g_0)\bar{\widehat{\phi}}$, $\bar{\widehat{\theta}}$ and $k_0\bar{\widehat{p}}$, respectively, multiplying $(2.18)_8$ by $\xi^4 g(s)\bar{\widehat{\eta}}$ and integrating on $\mathbb{R}_+$ with respect to s, adding all the obtained equations, taking the real part of the resulting expression and using (2.17), we get

$$\frac{d}{dt}\widehat{E}(\xi,\,t) = -\frac{1}{2}\xi^4\int_0^{+\infty} g(s)\frac{d}{ds}|\widehat{\eta}|^2\,ds,$$

therefore, by integrating with respect to s, we obtain

$$\frac{d}{dt}\widehat{E}(\xi,\,t) = -\frac{1}{2}\left[\xi^4\int_0^{+\infty} g(s)|\widehat{\eta}|^2\,ds\right]_{s=0}^{s=+\infty} + \frac{1}{2}\xi^4\int_0^{+\infty} g'(s)|\widehat{\eta}|^2\,ds.$$

Because (1.3) and (2.3) imply that

$$\lim_{s\to+\infty} g(s) = 0 \quad \text{and} \quad \widehat{\eta}(\xi, t, 0) = 0, \tag{2.19}$$

then we find (2.15) with $\tau_0 = 1$. □

Remark 2.1. Notice that (2.15) implies that

$$\frac{d}{dt}\widehat{E}(\xi,\,t) \le 0,$$

since $\gamma > 0$ and $g' \le 0$, so (2.10) is dissipative. If the frictional damping and infinite memory are not considered (i.e. $\gamma = g = 0$), then (2.10) is conservative; that is

$$\widehat{E}(\xi,\,t) = \widehat{E}(\xi,\,0).$$

On the other hand, we put

$$|\widehat{U}(\xi,\,t)|^2 = \left[|\widehat{v}|^2 + |\widehat{u}|^2 + |\widehat{z}|^2 + |\widehat{y}|^2 + |\widehat{\phi}|^2 + |\widehat{\theta}|^2 + |\widehat{p}|^2\right](\xi,\,t) \\ + \tau_0\xi^4\int_0^{+\infty} g(s)|\widehat{\eta}(\xi,\,t)|^2\,ds.$$

So we deduce that, thanks to the right inequality in (1.1),

$$|\widehat{U}|^2 \sim \widehat{E}. \tag{2.20}$$

3. Estimation of $|\widehat{U}|$

This section is dedicated to the study of the asymptotic behavior of $\widehat{U}(\xi, t)$, when time t goes to infinity. We prove the next theorem.

Theorem 3.1. *Let $\widehat{U}$ be the solution of* (2.10).

(i) In case (1.13) *and for every $\xi \in \mathbb{R}$, $|\widehat{U}(\xi, t)|$ doesn't converge to zero when t goes to infinity;*

(ii) In case (1.14)*, there exist $c, \widetilde{c} > 0$ (independent on ξ and t) such that*

$$|\widehat{U}(\xi, t)| \leq \widetilde{c}\, e^{-c\, f(\xi)\, t}\, |\widehat{U}_0(\xi)|, \quad \forall (\xi, t) \in \mathbb{R} \times \mathbb{R}_+, \tag{3.1}$$

where

$$f(\xi) = \frac{\xi^{4+2\tau_0}}{\xi^{10}+1}. \tag{3.2}$$

3.1. Case (1.13)

We prove here, under (1.13) and for every $\xi \in \mathbb{R}$, that $|\widehat{U}(\xi, t)|$ doesn't converge to zero when time t goes to infinity. It suffices to prove that, for any $\xi \in \mathbb{R}$, $-\tilde{A}(\xi)$ has at least a pure imaginary eigenvalue (see [34]); that is

$$\forall \xi \in \mathbb{R},\ \exists \lambda \in \mathbb{R}^*,\ \exists \widehat{U} \neq 0 : \quad i\lambda \widehat{U} + \tilde{A}(\xi)\widehat{U} = 0. \tag{3.3}$$

System (2.16). To get (3.3), it is enough to prove that

$$\forall \xi \in \mathbb{R},\ \exists \lambda \in \mathbb{R}^* : \quad det\Big(i\lambda I + \tilde{A}(\xi)\Big) = 0, \tag{3.4}$$

where I denotes the identity matrix. We see that (1.13) implies, in case (2.16), that

$$i\lambda I + \tilde{A}(\xi) = \begin{pmatrix} i\lambda & -i\xi & 0 & 0 & 0 & 0 & 0 \\ -i\frac{k_1}{\rho_1}\xi & i\lambda & 0 & 0 & 0 & 0 & \frac{k_0}{\rho_1} \\ 0 & 0 & i\lambda & -i\xi & 0 & 0 & 0 \\ 0 & 0 & -i\frac{k_1}{\rho_1}\xi & i\lambda & 0 & 0 & \frac{k_0}{\rho_2} \\ 0 & 0 & 0 & 0 & i\lambda & \xi^2 & 0 \\ 0 & 0 & 0 & 0 & -\frac{k_3}{\rho_3}\xi^2 & i\lambda + \frac{\gamma}{\rho_3} & -i\frac{lk_0}{\rho_3}\xi \\ 0 & -1 & 0 & -1 & 0 & -il\xi & i\lambda \end{pmatrix}.$$

A direct computation shows that

$$\begin{aligned} det\Big(i\lambda I+\tilde{A}(\xi)\Big) &= i\lambda\left(\lambda^2-\frac{k_1}{\rho_1}\xi^2\right)\left[\lambda^2-k_0\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-\frac{k_1}{\rho_1}\xi^2\right]\left[i\lambda\left(i\lambda+\frac{\gamma}{\rho_3}\right)+\frac{k_3}{\rho_3}\xi^4\right] \\ &\quad + i\frac{l^2k_0}{\rho_3}\lambda\xi^2\left(\lambda^2-\frac{k_1}{\rho_1}\xi^2\right)^2. \end{aligned} \tag{3.5}$$

It is clear that $det\left(i\lambda I+\tilde{A}(\xi)\right)=0$ for

$$\lambda=\begin{cases}\sqrt{\frac{k_1}{\rho_1}}\xi & \text{if } \xi\neq 0,\\ \sqrt{k_0\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)} & \text{if } \xi=0.\end{cases} \tag{3.6}$$

Consequently, (3.4) holds.

System (2.18). For $\lambda\in\mathbb{R}^*$, we put

$$\widehat{\eta}(s)=\frac{1}{i\lambda}\left(1-e^{-i\lambda s}\right)\widehat{\theta} \quad \text{and} \quad \tilde{g}(\lambda)=\int_0^{+\infty}\left(1-e^{-i\lambda s}\right)g(s)ds. \tag{3.7}$$

We observe that $\tilde{g}(\lambda)$ is well defined (according to (1.1)) and $\widehat{\eta}$ is the unique function satisfying $(3.3)_8$ and $\widehat{\eta}(0)=0$. On the other hand, (1.13) implies that the first seven equations of (3.3) are equivalent to $\tilde{B}(\xi)\left(\widehat{v},\widehat{u},\widehat{z},\widehat{y},\widehat{\phi},\widehat{p}\right)^T=0$, where

$$\tilde{B}(\xi)=\begin{pmatrix} i\lambda & -i\xi & 0 & 0 & 0 & 0 & 0\\ -i\frac{k_1}{\rho_1}\xi & i\lambda & 0 & 0 & 0 & 0 & \frac{k_0}{\rho_1}\\ 0 & 0 & i\lambda & -i\xi & 0 & 0 & 0\\ 0 & 0 & -i\frac{k_1}{\rho_1}\xi & i\lambda & 0 & 0 & \frac{k_0}{\rho_2}\\ 0 & 0 & 0 & 0 & i\lambda & \xi^2 & 0\\ 0 & 0 & 0 & 0 & -\frac{k_3-g_0}{\rho_3}\xi^2 & i\lambda+\frac{\tilde{g}(\lambda)}{\rho_3}\xi^4 & -i\frac{lk_0}{\rho_3}\xi\\ 0 & -1 & 0 & -1 & 0 & -il\xi & i\lambda \end{pmatrix}.$$

Then the problem (3.3) is reduced to prove that

$$\forall\xi\in\mathbb{R},\ \exists\lambda\in\mathbb{R}^*: \quad det\,\tilde{B}(\xi)=0. \tag{3.8}$$

A direct computation shows that (as for (3.5) with k_3-g_0 and $\xi^4\tilde{g}(\lambda)$ instead of k_3 and γ, respectively)

$$\begin{aligned} det\tilde{B}(\xi)=&i\lambda\left(\lambda^2-\frac{k_1}{\rho_1}\xi^2\right)\left[\lambda^2-k_0\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-\frac{k_1}{\rho_1}\xi^2\right]\left[i\lambda\left(i\lambda+\frac{\tilde{g}(\lambda)}{\rho_3}\xi^4\right)+\frac{k_3-g_0}{\rho_3}\xi^4\right]\\ &+i\frac{l^2k_0}{\rho_3}\lambda\xi^2\left(\lambda^2-\frac{k_1}{\rho_1}\xi^2\right)^2. \end{aligned}$$

We remark that $det\,\tilde{B}(\xi)=0$ for λ given by (3.6). Thus (3.8) holds. Consequently, the proof of the first result in Theorem 3.1 is ended.

3.2. Some differential equations

We start the proof of (3.1) by proving some useful differential equations. To simplify the computations, we put

$$G=(1-\tau_0)\gamma\widehat{\theta}+\tau_0\xi^4\int_0^{+\infty}g(s)\widehat{\eta}ds, \tag{3.9}$$

where τ_0 is defined in (2.14). Let consider the system

$$\begin{cases} \widehat{v}_t - i\xi\widehat{u} = 0, \\ \rho_1\widehat{u}_t - ik_1\xi\,\widehat{v} + k_0\widehat{p} = 0, \\ \widehat{z}_t - i\xi\widehat{y} = 0, \\ \rho_2\widehat{y}_t - ik_2\xi\,\widehat{z} + k_0\widehat{p} = 0, \\ \widehat{\phi}_t + \xi^2\widehat{\theta} = 0, \\ \rho_3\widehat{\theta}_t - (k_3 - \tau_0 g_0)\xi^2\widehat{\phi} - ilk_0\,\xi\widehat{p} + G = 0, \\ \widehat{p}_t - \widehat{u} - \widehat{y} - il\xi\widehat{\theta} = 0, \\ \widehat{\eta}_t + \widehat{\eta}_s - \widehat{\theta} = 0. \end{cases} \tag{3.10}$$

It is evident that (2.16) is identical to $(3.10)_1$-$(3.10)_7$ if $\tau_0 = 0$, and (2.18) coincides with (3.10) if $\tau_0 = 1$.

Multiplying $(3.10)_4$ and $(3.10)_3$ by $i\,\xi\,\overline{\widehat{z}}$ and $-i\rho_2\,\xi\,\overline{\widehat{y}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we obtain

$$\frac{d}{dt}Re\left(i\rho_2\,\xi\,\widehat{y}\,\overline{\widehat{z}}\right) = \xi^2\left(\rho_2|\widehat{y}|^2 - k_2|\widehat{z}|^2\right) + k_0\,Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{p}}\right). \tag{3.11}$$

Similarly, multiplying $(3.10)_2$ and $(3.10)_1$ by $i\,\xi\,\overline{\widehat{v}}$ and $-i\rho_1\,\xi\,\overline{\widehat{u}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we find

$$\frac{d}{dt}Re\left(i\rho_1\,\xi\,\widehat{u}\,\overline{\widehat{v}}\right) = \xi^2\left(\rho_1|\widehat{u}|^2 - k_1|\widehat{v}|^2\right) + k_0 Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{p}}\right). \tag{3.12}$$

Also, multiplying $(3.10)_6$ and $(3.10)_5$ by $-\overline{\widehat{\phi}}$ and $-\rho_3\,\overline{\widehat{\theta}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we get

$$\frac{d}{dt}Re\left(-\rho_3\,\widehat{\theta}\,\overline{\widehat{\phi}}\right) = \xi^2\left(\rho_3|\widehat{\theta}|^2 - (k_3 - \tau_0 g_0)|\widehat{\phi}|^2\right) + lk_0\,Re\left(i\,\xi\,\widehat{\phi}\,\overline{\widehat{p}}\right) + Re\left(\overline{\widehat{\phi}}\,G\right). \tag{3.13}$$

Multiplying $(3.10)_4$ and $(3.10)_7$ by $\xi^2\,\overline{\widehat{p}}$ and $\rho_2\xi^2\,\overline{\widehat{y}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we obtain

$$\begin{aligned} \frac{d}{dt}Re\left(\rho_2\xi^2\,\widehat{p}\,\overline{\widehat{y}}\right) =& \xi^2\left(\rho_2\,|\widehat{y}|^2 - k_0|\widehat{p}|^2\right) + \rho_2\xi^2\,Re\left(\widehat{y}\,\overline{\widehat{u}}\right) + l\rho_2\xi^2\,Re\left(i\xi\,\widehat{\theta}\,\overline{\widehat{y}}\right) \\ &+ k_2\xi^2\,Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{p}}\right). \end{aligned} \tag{3.14}$$

After, multiplying $(3.10)_3$ and $(3.10)_6$ by $\rho_3\,\overline{\widehat{\theta}}$ and $\overline{\widehat{z}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we entail

$$\begin{aligned} \frac{d}{dt}Re\left(\rho_3\widehat{z}\,\overline{\widehat{\theta}}\right) =& -\rho_3\,Re\left(i\xi\widehat{\theta}\,\overline{\widehat{y}}\right) + (k_3 - \tau_0 g_0)\,\xi^2\,Re\left(\widehat{\phi}\,\overline{\widehat{z}}\right) \\ &- lk_0\,Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{p}}\right) - Re\left(\overline{\widehat{z}}\,G\right). \end{aligned} \tag{3.15}$$

Multiplying $(3.10)_5$ and $(3.10)_4$ by $i\rho_2\xi\,\overline{\widehat{y}}$ and $-i\xi\overline{\widehat{\phi}}$, respectively, adding the resulting equations, taking the real part and using (2.17), it follows that

$$\frac{d}{dt}Re\left(i\rho_2\xi\,\widehat{\phi}\,\overline{\widehat{y}}\right) = -\rho_2\xi^2\,Re\left(i\xi\widehat{\theta}\,\overline{\widehat{y}}\right) + k_2\xi^2\,Re\left(\widehat{\phi}\,\overline{\widehat{z}}\right) - k_0\,Re\left(i\xi\widehat{\phi}\,\overline{\widehat{p}}\right). \tag{3.16}$$

Next, multiplying $(3.10)_2$ and $(3.10)_3$ by $i\xi\overline{\widehat{z}}$ and $-i\rho_1\xi\,\overline{\widehat{u}}$, respectively, adding the resulting equations, taking the real part and using (2.17), it appears that

$$\frac{d}{dt}Re\left(i\rho_1\xi\,\widehat{u}\,\overline{\widehat{z}}\right)=-k_1\xi^2\,Re\left(\widehat{v}\,\overline{\widehat{z}}\right)+\rho_1\xi^2\,Re\left(\widehat{y}\,\overline{\widehat{u}}\right)+k_0\,Re\left(i\xi\widehat{z}\,\overline{\widehat{p}}\right). \tag{3.17}$$

Multiplying $(3.10)_2$ and $(3.10)_5$ by $i\xi\overline{\widehat{\phi}}$ and $-i\rho_1\xi\,\overline{\widehat{u}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we see that

$$\frac{d}{dt}Re\left(i\rho_1\xi\,\widehat{u}\,\overline{\widehat{\phi}}\right)=-k_1\xi^2\,Re\left(\widehat{\phi}\,\overline{\widehat{v}}\right)+\rho_1\,\xi^2\,Re\left(i\xi\widehat{\theta}\,\overline{\widehat{u}}\right)+k_0\,Re\left(i\xi\widehat{\phi}\,\overline{\widehat{p}}\right). \tag{3.18}$$

Also, multiplying $(3.10)_1$ and $(3.10)_4$ by $i\rho_2\xi\,\overline{\widehat{y}}$ and $-i\xi\overline{\widehat{v}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we find

$$\frac{d}{dt}Re\left(i\rho_2\xi\widehat{v}\,\overline{\widehat{y}}\right)=-\rho_2\xi^2\,Re\left(\widehat{y}\,\overline{\widehat{u}}\right)+k_2\xi^2\,Re\left(\widehat{v}\,\overline{\widehat{z}}\right)-k_0\,Re\left(i\xi\widehat{v}\,\overline{\widehat{p}}\right). \tag{3.19}$$

Multiplying $(3.10)_1$ and $(3.10)_6$ by $\rho_3\,\overline{\widehat{\theta}}$ and $\overline{\widehat{v}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we get

$$\begin{aligned}\frac{d}{dt}Re\left(\rho_3\,\widehat{v}\,\overline{\widehat{\theta}}\right)&=-\rho_3\,Re\left(i\xi\widehat{\theta}\,\overline{\widehat{u}}\right)+(k_3-\tau_0 g_0)\,\xi^2\,Re\left(\widehat{\phi}\,\overline{\widehat{v}}\right)\\&\quad-lk_0\,Re\left(i\xi\widehat{v}\,\overline{\widehat{p}}\right)-\,Re\left(\overline{\widehat{v}}\,G\right).\end{aligned} \tag{3.20}$$

Now, multiplying $(3.10)_6$ by $-\xi^2\displaystyle\int_0^{+\infty}g(s)\overline{\widehat{\eta}}\,ds$, multiplying $(3.10)_8$ by $-\rho_3\xi^2 g(s)\overline{\widehat{\theta}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.17), we infer that

$$\begin{aligned}&\frac{d}{dt}Re\left(-\rho_3\xi^2\widehat{\theta}\int_0^{+\infty}g(s)\overline{\widehat{\eta}}x\,ds\right)\\&=-\rho_3 g_0\xi^2|\widehat{\theta}|^2-(k_3-\tau_0 g_0)\xi^4 Re\left(\widehat{\phi}\int_0^{+\infty}g(s)\overline{\widehat{\eta}}ds\right)-lk_0\xi^2 Re\left(i\xi\widehat{p}\int_0^{+\infty}g(s)\overline{\widehat{\eta}}ds\right)\\&\quad-\rho_3\xi^2 Re\left(\overline{\widehat{\theta}}\int_0^{+\infty}g'(s)\widehat{\eta}\,ds\right)+\xi^2 Re\left(G\int_0^{+\infty}g(s)\overline{\widehat{\eta}}\,ds\right).\end{aligned} \tag{3.21}$$

Similarly, multiplying $(3.10)_4$ by $i\xi\displaystyle\int_0^{+\infty}g(s)\overline{\widehat{\eta}}\,ds$, multiplying $(3.10)_8$ by $-i\rho_2\xi g(s)\,\overline{\widehat{y}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.17), we get

$$\begin{aligned}&\frac{d}{dt}Re\left(i\rho_2\xi\widehat{y}\int_0^{+\infty}g(s)\overline{\widehat{\eta}}\,ds\right)\\&=-\rho_2 g_0 Re\left(i\xi\widehat{\theta}\overline{\widehat{y}}\right)-k_2\xi^2 Re\left(\widehat{z}\int_0^{+\infty}g(s)\overline{\widehat{\eta}}\,ds\right)\\&\quad-k_0 Re\left(i\xi\widehat{p}\int_0^{+\infty}g(s)\overline{\widehat{\eta}}\,ds\right)-\rho_2 Re\left(i\xi\overline{\widehat{y}}\int_0^{+\infty}g'(s)\widehat{\eta}\,ds\right).\end{aligned} \tag{3.22}$$

Finally, multiplying $(3.10)_2$ by $i\xi \int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds$, multiplying $(3.10)_8$ by $-\rho_1\xi g(s)\overline{\widehat{u}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.17), we see that

$$\begin{aligned}
&\frac{d}{dt}Re\left(i\rho_1\xi\widehat{u}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds\right)\\
=&-k_1\xi^2 Re\left(\widehat{v}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds\right)-k_0 Re\left(i\xi\widehat{p}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds\right)\\
&-\rho_1 Re\left(i\xi\overline{\widehat{u}}\int_0^{+\infty} g'(s)\widehat{\eta}\,ds\right)-\rho_1 g_0 Re\left(i\xi\widehat{\theta}\overline{\widehat{u}}\right).
\end{aligned}\tag{3.23}$$

Let $\lambda_1,\cdots,\lambda_{13}$ be real numbers to be chosen later (which do not depend on time t but may depend on ξ and the parameters of (1.4) and (1.5)). We define the functionals F_0, F_1, F_2 and F_3 as follows:

$$\begin{aligned}
F_0(\xi,\,t)=&Re\left(i\rho_2\lambda_1\xi\widehat{y}\overline{\widehat{z}}+i\rho_1\lambda_2\xi\widehat{u}\overline{\widehat{v}}-\rho_3\lambda_3\widehat{\theta}\overline{\widehat{\phi}}+\rho_2\lambda_4\xi^2\widehat{p}\overline{\widehat{y}}+\rho_3\lambda_5\widehat{z}\overline{\widehat{\theta}}\right)\\
&+Re\left(i\rho_2\lambda_6\xi\widehat{\phi}\overline{\widehat{y}}+i\rho_1\lambda_7\xi\widehat{u}\overline{\widehat{z}}+i\rho_1\lambda_8\xi\widehat{u}\overline{\widehat{\phi}}+i\rho_2\lambda_9\xi\widehat{v}\overline{\widehat{y}}+\rho_3\lambda_{10}\widehat{v}\overline{\widehat{\theta}}\right)\\
&+\tau_0 Re\left[\left(-\rho_3\lambda_{11}\xi^2\widehat{\theta}+i\rho_2\lambda_{12}\xi\widehat{y}+i\rho_1\lambda_{13}\xi\widehat{u}\right)\int_0^{+\infty} g(s)\overline{\widehat{\eta}}ds\right],
\end{aligned}\tag{3.24}$$

$$F_1(\xi,\,t)=-\xi^2\left(B_1|\widehat{z}|^2+B_2|\widehat{v}|^2+B_3|\widehat{\phi}|^2+B_4|\widehat{u}|^2+B_5|\widehat{p}|^2+B_6|\widehat{y}|^2+B_7|\widehat{\theta}|^2\right),\tag{3.25}$$

$$\begin{aligned}
F_2(\xi,\,t)=&Re\left[i\xi\left(A_1\widehat{z}+A_2\widehat{v}+A_3\widehat{\phi}\right)\overline{\widehat{p}}+A_4\widehat{\phi}\overline{\widehat{z}}+A_5\widehat{v}\overline{\widehat{z}}\right]\\
&+Re\left[A_6\widehat{\phi}\overline{\widehat{v}}+A_7\xi^2\widehat{y}\overline{\widehat{u}}+iA_8\xi\widehat{\theta}\overline{\widehat{u}}+iA_9\xi\widehat{\theta}\overline{\widehat{y}}\right]
\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}
F_3(\xi,\,t)=&Re\left[\left(\lambda_3\overline{\widehat{\phi}}-\lambda_5\overline{\widehat{z}}-\lambda_{10}\overline{\widehat{v}}\right)G\right]\\
&-\tau_0 Re\left[\left(\rho_3\lambda_{11}\xi^2\overline{\widehat{\theta}}+i\rho_2\lambda_{12}\xi\overline{\widehat{y}}+i\rho_1\lambda_{13}\xi\overline{\widehat{u}}\right)\int_0^{+\infty} g'(s)\widehat{\eta}ds\right]\\
&+\tau_0 Re\left[\lambda_{11}\left[-(k_3-\tau_0 g_0)\xi^4\widehat{\phi}-ilk_0\xi^3\widehat{p}+\xi^2 G\right]\int_0^{+\infty} g(s)\overline{\widehat{\eta}}ds\right]\\
&-\tau_0 Re\left[\left[\lambda_{12}\left(k_2\xi^2\widehat{z}+ik_0\xi\widehat{p}\right)+\lambda_{13}\left(k_1\xi^2\widehat{v}+ik_0\xi\widehat{p}\right)\right]\int_0^{+\infty} g(s)\overline{\widehat{\eta}}ds\right],
\end{aligned}\tag{3.27}$$

where

$B_1=k_2\lambda_1,\quad B_2=k_1\lambda_2,\quad B_3=(k_3-\tau_0 g_0)\lambda_3,\quad B_4=-\rho_1\lambda_2,$
$B_5=k_0\lambda_4,\quad B_6=-\rho_2(\lambda_1+\lambda_4),\quad B_7=\rho_3(\tau_0 g_0\lambda_{11}-\lambda_3),$
$A_1=k_0\lambda_1+k_2\lambda_4\xi^2-lk_0\lambda_5+k_0\lambda_7,\ A_2=k_0(\lambda_2-\lambda_9-l\lambda_{10}),\ A_3=k_0(l\lambda_3-\lambda_6+\lambda_8),$
$A_4=(k_3-\tau_0 g_0)\lambda_5\xi^2+k_2\lambda_6\xi^2, A_5=-k_1\lambda_7\xi^2+k_2\lambda_9\xi^2, A_6=-k_1\lambda_8\xi^2+(k_3-\tau_0 g_0)\lambda_{10}\xi^2,$
$A_7=\rho_1\lambda_7+\rho_2(\lambda_4-\lambda_9),\quad A_8=\rho_1(\lambda_8\xi^2-\tau_0 g_0\lambda_{13})-\rho_3\lambda_{10}$

and

$$A_9 = \rho_2(l\lambda_4 - \lambda_6)\xi^2 - \rho_3\lambda_5 - \tau_0\rho_2 g_0\lambda_{12}.$$

Multiplying (3.11)-(3.23) by $\lambda_1, \cdots, \lambda_{10}$, $\tau_0\lambda_{11}$, $\tau_0\lambda_{12}$ and $\tau_0\lambda_{13}$, respectively, and adding the obtained equations, we deduce that

$$\frac{d}{dt}F_0(\xi, t) = F_1(\xi, t) + F_2(\xi, t) + F_3(\xi, t). \tag{3.28}$$

According to the notations (2.1), we have $p_x = v + z + l\phi$. Because (2.9) implies that $\widehat{p_x} = i\xi\widehat{p}$, then

$$i\xi\overline{\widehat{p}} = -\overline{\widehat{v}} - \overline{\widehat{z}} - l\overline{\widehat{\phi}},$$

this identity allows to formulate the first term in F_2 as

$$\begin{aligned} &Re\left[i\xi\left(A_1\widehat{z} + A_2\widehat{v} + A_3\widehat{\phi}\right)\overline{\widehat{p}}\right] \\ &= -\left(A_1|\widehat{z}|^2 + A_2|\widehat{v}|^2 + lA_3|\widehat{\phi}|^2\right) - (A_1 + A_2)\, Re\left(\widehat{v}\overline{\widehat{z}}\right) \\ &\quad - (A_3 + lA_1)\, Re\left(\widehat{\phi}\overline{\widehat{z}}\right) - (lA_2 + A_3)\, Re\left(\widehat{\phi}\overline{\widehat{v}}\right). \end{aligned} \tag{3.29}$$

By combining (3.28) and (3.29), we get

$$\frac{d}{dt}F_0(\xi, t) = F_5(\xi, t) + F_4(\xi, t) + F_3(\xi, t), \tag{3.30}$$

where

$$\begin{aligned} F_5(\xi, t) = &-\left[\left(B_1\xi^2 + A_1\right)|\widehat{z}|^2 + \left(B_2\xi^2 + A_2\right)|\widehat{v}|^2 + \left(B_3\xi^2 + lA_3\right)|\widehat{\phi}|^2\right] \\ &-\xi^2\left[B_4|\widehat{u}|^2 + B_5|\widehat{p}|^2 + B_6|\widehat{y}|^2 + B_7|\widehat{\theta}|^2\right] \end{aligned}$$

and

$$\begin{aligned} F_4(\xi, t) = Re&\left[(A_4 - A_3 - lA_1)\,\widehat{\phi}\overline{\widehat{z}} + (A_5 - A_1 - A_2)\,\widehat{v}\overline{\widehat{z}} + (A_6 - lA_2 - A_3)\,\widehat{\phi}\overline{\widehat{v}}\right] \\ &+Re\left[A_7\xi^2\widehat{y}\overline{\widehat{u}} + iA_8\xi\widehat{\theta}\overline{\widehat{u}} + iA_9\xi\widehat{\theta}\overline{\widehat{y}}\right]. \end{aligned}$$

Now, we choose the different real numbers $\lambda_1, \cdots, \lambda_{15}$ in order to have

$$A_4 - A_3 - lA_1 = A_5 - A_1 - A_2 = A_6 - lA_2 - A_3 = A_7 = 0, \tag{3.31}$$

$$B_1\xi^2 + A_1 = \tilde{B}_1\xi^2, \quad B_2\xi^2 + A_2 = \tilde{B}_2\xi^2, \quad B_3\xi^2 + lA_3 = \tilde{B}_3\xi^2, \tag{3.32}$$

$$\tilde{B}_1 > 0, \quad \tilde{B}_2 > 0, \quad \tilde{B}_3 > 0, \quad B_4 > 0, \quad B_5 > 0, \quad B_6 > 0 \tag{3.33}$$

and

$$A_8 = A_9 = 0 \text{ and } B_7 > 0 \quad \text{if } \tau_0 = 1. \tag{3.34}$$

We start by choosing λ_7, λ_8 and λ_9 as follows:

$$\lambda_7 = k_2\xi^2 + l\lambda_5 - \lambda_1,\ \lambda_8 = -\frac{2(k_3 - \tau_0 g_0)}{l}\xi^2 + \lambda_6 - l\lambda_3 \text{ and } \lambda_9 = -k_1\xi^2 + \lambda_2 - l\lambda_{10}. \tag{3.35}$$

The choices (3.35) guarantee (3.32) with

$$\tilde{B}_1 = B_1 + k_2\lambda_4 + k_0k_2, \quad \tilde{B}_2 = B_2 + k_0k_1 \quad \text{and} \quad \tilde{B}_3 = B_3 - 2k_0(k_3 - \tau_0 g_0). \tag{3.36}$$

Also, we select λ_1, λ_5, λ_6 and λ_{10} by

$$\lambda_6 = \frac{1}{k_2}\left[lk_2\lambda_4 - (k_3 - \tau_0 g_0)\lambda_5 + k_0\left(lk_2 - \frac{2(k_3 - \tau_0 g_0)}{l}\right)\right], \tag{3.37}$$

$$\lambda_{10} = \frac{1}{lk_2}\left[-2k_1k_2\xi^2 + k_1\lambda_1 + k_2\lambda_2 - k_2\lambda_4 - lk_1\lambda_5 - k_0(k_1 + k_2)\right], \tag{3.38}$$

$$\lambda_1 = -\frac{k_2}{k_1}\lambda_2 - \frac{l^2k_2}{k_3 - \tau_0 g_0}\lambda_3 + \left(\frac{l^2k_2}{k_3 - \tau_0 g_0} + \frac{k_2}{k_1}\right)\lambda_4 + k_0\left(\frac{2l^2k_2}{k_3 - \tau_0 g_0} - 1 - \frac{k_2}{k_1}\right) \tag{3.39}$$

and

$$\lambda_5 = -\frac{k_2}{l}\xi^2 + \frac{1}{l}\lambda_1 - \frac{k_0\rho_2(k_1 + k_2)}{l(k_1\rho_2 - k_2\rho_1)} \tag{3.40}$$

(λ_5 and λ_1 are well defined thanks to (1.14) and the right inequality in (1.1), respectively). According to (3.35), the selections (3.37) and (3.38) imply that

$$A_4 - A_3 - lA_1 = 0 \quad \text{and} \quad A_5 - A_1 - A_2 = 0,$$

respectively, (3.37)-(3.39) lead to

$$A_6 - lA_2 - A_3 = 0,$$

and (3.38) and (3.40) guarantee $A_7 = 0$, so (3.31) is satisfied. We put

$$B_0 = \min\left\{\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, B_4, B_5, B_6\right\}. \tag{3.41}$$

Then, multiplying (3.30) by ξ^2 and exploiting (3.31) and (3.32), we obtain

$$\begin{aligned}\frac{d}{dt}\left[\xi^2 F_0(\xi,\, t)\right] \leq &- B_0\xi^4\left[|\widehat{z}|^2 + |\widehat{v}|^2 + |\widehat{\phi}|^2 + |\widehat{u}|^2 + |\widehat{p}|^2 + |\widehat{y}|^2\right] - B_7\xi^4|\widehat{\theta}|^2 \\ &+ \xi^2 Re\left[iA_8\xi\widehat{\theta}\overline{\widehat{u}} + iA_9\xi\widehat{\theta}\overline{\widehat{y}}\right] + \xi^2 F_3(\xi, t).\end{aligned} \tag{3.42}$$

In order to estimate $\xi^2 F_3(\xi, t)$, first, we notice the following evident inequality:

$$|\xi|^{m_2} \leq |\xi|^{m_1} + |\xi|^{m_3}, \quad \forall \xi \in \mathbb{R},\ \forall 0 \leq m_1 \leq m_2 \leq m_3. \tag{3.43}$$

Second, using Hölder's inequality and the right inequality in (1.2), we see that, for $\tilde{\eta} \in \left\{\widehat{\eta}, \overline{\widehat{\eta}}\right\}$,

$$\begin{aligned}\left|\int_0^{+\infty} g(s)\tilde{\eta}(\xi, s)\, ds\right|^2 &= \left|\int_0^{+\infty} \sqrt{g(s)}\sqrt{g(s)}\tilde{\eta}(\xi, s)\, ds\right|^2 \\ &\leq \left(\int_0^{+\infty} g(s)\, ds\right)\int_0^{+\infty} g(s)|\tilde{\eta}(\xi, s)|^2\, ds \\ &\leq -\frac{g_0}{\beta_1}\int_0^{+\infty} g'(s)|\tilde{\eta}(\xi, s)|^2\, ds.\end{aligned}$$

Similarly, using the limit in (2.19), we have

$$\begin{aligned}\left|\int_0^{+\infty} g'(s)\tilde{\eta}(\xi,s)\,ds\right|^2 &= \left|\int_0^{+\infty} \sqrt{-g'(s)}\sqrt{-g'(s)}\tilde{\eta}(\xi,s)\,ds\right|^2\\ &\le \left(-\int_0^{+\infty} g'(s)\,ds\right)\int_0^{+\infty}(-g'(s))|\tilde{\eta}(\xi,s)|^2\,ds\\ &\le -g(0)\int_0^{+\infty} g'(s)|\tilde{\eta}(\xi,s)|^2\,ds.\end{aligned}$$

Then, using these two inequalities and Young's inequality, we get, for any $h:\mathbb{R}\to\mathbb{C}$ and $\varepsilon>0$,

$$Re\left(h(\xi)\int_0^{+\infty} g(s)\tilde{\eta}(\xi,s)\,ds\right)\le \varepsilon|h(\xi)|^2-\frac{g_0}{4\varepsilon\beta_1}\int_0^{+\infty} g'(s)|\tilde{\eta}(\xi,s)|^2\,ds \tag{3.44}$$

and

$$Re\left(h(\xi)\int_0^{+\infty} g'(s)\tilde{\eta}(\xi,s)\,ds\right)\le \varepsilon|h(\xi)|^2-\frac{g(0)}{4\varepsilon}\int_0^{+\infty} g'(s)|\tilde{\eta}(\xi,s)|^2\,ds. \tag{3.45}$$

Third, in the sequel, we use C (sometimes $C_1,\, C_2,\,\cdots$) to denote a generic real positive constant, and C_ε to denote a generic real positive constant depending on some real positive constant ε, where C and C_ε may be different from step to step. The constants C ($C_1,\, C_2,\,\cdots$), ε and C_ε are independent on x, ξ and t.

Applying Young's inequality for all the terms in $\xi^2 F_3(\xi,\,t)$ and using (3.43), (3.44), (3.45) and the fact that $\tau_0^2=\tau_0$ and $\tau_0(1-\tau_0)=0$, we get, for any $\varepsilon>0$,

$$\begin{aligned}\xi^2F_3(\xi,\,t)\le{}& \varepsilon\xi^4\left[|\widehat{z}|^2+|\widehat{v}|^2+|\widehat{\phi}|^2+\tau_0\left(|\widehat{u}|^2+|\widehat{y}|^2+|\widehat{\theta}|^2+|\widehat{p}|^2\right)\right]\\ &+C_\varepsilon\left(\lambda_3^2+\lambda_5^2+\lambda_{10}^2\right)|G|^2+\tau_0|\lambda_{11}|\xi^4|G|\left|\int_0^{+\infty} g(s)\overline{\widehat{\eta}}(\xi,s)\,ds\right| \\ &-\tau_0C_\varepsilon\left[\lambda_{11}^2\xi^4\left(\xi^4+1\right)+\left(\lambda_{12}^2+\lambda_{13}^2\right)\xi^2\left(\xi^2+1\right)\right]\int_0^{+\infty} g'(s)|\widehat{\eta}(\xi,s)|^2\,ds.\end{aligned}\tag{3.46}$$

Now, we select λ_2, λ_3, λ_4, λ_{11}, λ_{12} and λ_{13} in order to get (3.33) and (3.34) (and then, in particular, $B_0>0$ since (3.41)). To do so, we distinguish the two cases related to (2.14).

3.3. Case 1: $\tau_0=0$

Notice that, because $\tau_0=0$, the numbers λ_{11}, λ_{12} and λ_{13} are not used. On the other hand, we select λ_2, λ_3 and λ_4 as follows:

$$-k_0<\lambda_2<0, \tag{3.47}$$

$$0<\lambda_4<\frac{lk_2}{k_1k_3+k_2\left(l^2k_1+k_3\right)}\left[\frac{k_3}{l}\lambda_2+lk_1\lambda_3+k_0\left(\frac{k_1k_3}{lk_2}+\frac{k_3}{l}-2lk_1\right)\right], \tag{3.48}$$

$$\lambda_4>\frac{k_1k_3}{k_1k_3+k_2\left(l^2k_1+k_3\right)}\left[\frac{k_2}{k_1}\lambda_2+\frac{l^2k_2}{k_3}\lambda_3+k_0\left(\frac{k_2}{k_1}-\frac{2l^2k_2}{k_3}\right)\right] \tag{3.49}$$

and

$$\lambda_3 > \max\left\{2k_0, \frac{1}{l^2 k_1 k_2}\left[-k_2 k_3 \lambda_2 + k_0\left(2l^2 k_1 k_2 - k_1 k_3 - k_2 k_3\right)\right]\right\}. \tag{3.50}$$

We go back in the reverse order to select the numbers in the following manner:

1. Choose λ_2 by (3.47).
2. After, select λ_3 large enough in such a way that (3.50) holds.
3. Take λ_4 so that (3.48) and (3.49) are true. According to (3.50), λ_4 exists.
4. Now, it is possible to find λ_1 through (3.39).
5. After, take λ_5 as in (3.40).
6. Next, it is time to pick λ_6 and λ_{10} verifying (3.37) and (3.38), respectively.
7. Finally, we can select λ_7, λ_8 and λ_9 by (3.35).

We observe that the left inequality in (3.47), (3.50), the right inequality in (3.47) and the left one in (3.48) imply that $\tilde{B}_2 > 0$, $\tilde{B}_3 > 0$, $B_4 > 0$ and $B_5 > 0$, respectively. Moreover, (3.49) and the right inequality in (3.48) combined with (3.39) imply $\tilde{B}_1 > 0$ and $B_6 > 0$, respectively. Therefore (3.33) holds. Thus (3.31)-(3.33) are satisfied.

On the other hand, because λ_1, λ_2, λ_3 and λ_4 do not depend on ξ, we have

$$|\lambda_j| \le C\left(\xi^2 + 1\right),\ j = 5, \cdots, 10, \quad \text{and} \quad |A_j| \le C\left(\xi^4 + 1\right),\ j = 8, 9 \tag{3.51}$$

(since (3.43)), then, applying Young's inequality and using (3.43), we get, for any $\varepsilon > 0$,

$$\xi^2 Re\left[iA_8 \xi \widehat{\theta}\widehat{\overline{u}} + iA_9 \xi \widehat{\theta}\widehat{\overline{y}}\right] \le \varepsilon \xi^4\left(|\widehat{u}|^2 + |\widehat{y}|^2\right) + C_\varepsilon \xi^2\left(\xi^8 + 1\right)|\widehat{\theta}|^2. \tag{3.52}$$

Moreover, we conclude from (3.9) and (3.46) (with $\tau_0 = 0$) that

$$\xi^2 F_3(\xi, t) \le \varepsilon \xi^4\left[|\widehat{z}|^2 + |\widehat{v}|^2 + |\widehat{\phi}|^2\right] + C_\varepsilon\left(\xi^4 + 1\right)|\widehat{\theta}|^2. \tag{3.53}$$

By combining (3.42), (3.52) and (3.53), choosing $0 < \varepsilon < B_0$ (ε exists thanks to (3.33) and (3.41)), and using (2.13) (with $\tau_0 = 0$) and (3.43), we deduce that there exist real positive constants c_1 and c_2 such that (notice that B_7 does not depend on ξ)

$$\frac{d}{dt}\left[\xi^2 F_0(\xi, t)\right] \le -c_1 \xi^4 \widehat{E}(\xi, t) + c_2\left(\xi^{10} + 1\right)|\widehat{\theta}|^2. \tag{3.54}$$

Now, let λ be a real positive constant and

$$F(\xi, t) = \lambda\, \widehat{E}(\xi, t) + \frac{\xi^2}{\xi^{10} + 1} F_0(\xi, t). \tag{3.55}$$

From (2.15) (with $\tau_0 = 0$), (3.54) and (3.55), we find

$$\frac{d}{dt} F(\xi, t) \le -c_1 f(\xi) \widehat{E}(\xi, t) - (\gamma\lambda - c_2)|\widehat{\theta}|^2, \tag{3.56}$$

where f is defined by (3.2) with $\tau_0 = 0$. Moreover, using the definitions of $\widehat{E}$, F_0 and F, and exploiting (3.43) and (3.51), we observe that there exists $c_3 > 0$ (not depend on λ) such that

$$|F(\xi, t) - \lambda\, \widehat{E}(\xi, t)| \le \frac{c_3\, \xi^2(|\xi|^3 + 1)}{\xi^{10} + 1}\, \widehat{E}(\xi, t) \le 2c_3\, \widehat{E}(\xi, t). \tag{3.57}$$

Therefore, for λ satisfying $\lambda > \max\left\{\frac{c_2}{\gamma},\, 2c_3\right\}$, we deduce from (3.56) and (3.57) that

$$\frac{d}{dt}F(\xi,\, t) \leq -c_1\, f(\xi)\, \widehat{E}(\xi,\, t) \tag{3.58}$$

and $F \sim \widehat{E}$, since

$$(\lambda - 2c_3)\, \widehat{E}(\xi,\, t) \leq F(\xi,\, t) \leq (\lambda + 2c_3)\, \widehat{E}(\xi,\, t). \tag{3.59}$$

Consequently, a combination of (3.58) and the second inequality in (3.59) leads to, for $c = \frac{c_1}{2(\lambda+2c_3)}$,

$$\frac{d}{dt}F(\xi,\, t) \leq -2c\, f(\xi)\, F(\xi,\, t). \tag{3.60}$$

Multiplying (3.60) by $e^{2cf(\xi)t}$, we find

$$\frac{d}{dt}\left(e^{2cf(\xi)t}F(\xi,\, t)\right) \leq 0, \tag{3.61}$$

then, by integration (3.61) with respect to t,

$$F(\xi,\, t) \leq e^{-2cf(\xi)t}F(\xi,\, 0). \tag{3.62}$$

Finally, using (2.20) and (3.59), (3.1) in case $\tau_0 = 0$ follows from (3.62).

3.4. Case 2: $\tau_0 = 1$

We choose $\lambda_1, \cdots, \lambda_{10}$ as in case 1 but with $k_3 - g_0$ instead of k_3 ($k_3 - g_0 > 0$ thanks to the right inequality in (1.1)), and we get (3.31)-(3.33). Next, we select

$$\lambda_{13} = \frac{1}{g_0}\lambda_8\xi^2 - \frac{\rho_3}{g_0\rho_1}\lambda_{10}, \tag{3.63}$$

$$\lambda_{12} = \frac{1}{g_0}(l\lambda_4 - \lambda_6)\xi^2 - \frac{\rho_3}{g_0\rho_2}\lambda_5 \tag{3.64}$$

and

$$\lambda_{11} > \frac{1}{g_0}\lambda_3 \tag{3.65}$$

($g_0 > 0$ according to the left inequality in (1.1)). The choices (3.63)-(3.65) imply that $A_8 = A_9 = 0$ and $B_7 > 0$, respectively, thus (3.34) is satisfied. Therefore, (3.42) implies

$$\begin{aligned}\frac{d}{dt}\left[\xi^2 F_0(\xi,\, t)\right] \leq & -\min\{B_0, B_7\}\xi^4\left[|\widehat{z}|^2 + |\widehat{v}|^2 + |\widehat{\phi}|^2 + |\widehat{u}|^2 + |\widehat{p}|^2 + |\widehat{y}|^2 + |\widehat{\theta}|^2\right] \\ & + \xi^2 F_3(\xi, t).\end{aligned} \tag{3.66}$$

On the other hand, λ_1, λ_2, λ_3, λ_4 and λ_{11} do not depend on ξ, then

$$|\lambda_j| \leq C\left(\xi^2 + 1\right),\ j = 5, \cdots, 10, \quad \text{and} \quad |\lambda_j| \leq C\left(\xi^4 + 1\right),\ j = 12, 13, \tag{3.67}$$

therefore, applying Young's inequality and using (3.9) (with $\tau_0 = 1$) and (3.43), we conclude from (3.46), for any $\varepsilon > 0$, that

$$\xi^2 F_3(\xi,\, t) \leq \varepsilon\xi^4\left[|\widehat{z}|^2 + |\widehat{v}|^2 + |\widehat{\phi}|^2 + |\widehat{p}|^2 + |\widehat{u}|^2 + |\widehat{y}|^2 + |\widehat{\theta}|^2\right]$$

$$-C_\varepsilon \xi^2 \left(\xi^{10}+1\right) \int_0^{+\infty} g'(s)|\widehat{\eta}(\xi,s)|^2\, ds. \tag{3.68}$$

By combining (3.66) and (3.68), choosing $0<\varepsilon<\min\{B_0,B_7\}$, and using (2.13) and (3.43), we deduce that there exist real positive constants c_1 and c_2 such that

$$\frac{d}{dt}\left[\xi^2 F_0(\xi,\,t)\right] \le -c_1\xi^4 \widehat{E}(\xi,\,t) - c_2\xi^2\left(\xi^{10}+1\right)\int_0^{+\infty} g'(s)|\widehat{\eta}(\xi,s)|^2\, ds. \tag{3.69}$$

Now, let λ be a real positive constant and

$$F(\xi,\,t) = \lambda\, \widehat{E}(\xi,\,t) + \frac{\xi^4}{\xi^{10}+1}\, F_0(\xi,\,t). \tag{3.70}$$

From (2.15) (with $\tau_0=1$), (3.69) and (3.70), we find

$$\frac{d}{dt}F(\xi,\,t) \le -c_1 f(\xi)\widehat{E}(\xi,\,t) + \left(\frac{\lambda}{2}-c_2\right)\xi^4\int_0^{+\infty} g'(s)|\widehat{\eta}(\xi,s)|^2\, ds, \tag{3.71}$$

where f is defined in (3.2) with $\tau_0=1$. Also, using the definitions of $\widehat{E}$, F_0 and F, and exploiting (3.43) and (3.67), we see that there exists $c_3>0$ (not depend on λ) such that

$$|F(\xi,\,t)-\lambda\,\widehat{E}(\xi,\,t)| \le \frac{c_3\,\xi^4\left(|\xi|^5+1\right)}{\xi^{10}+1}\,\widehat{E}(\xi,\,t) \le 2c_3\,\widehat{E}(\xi,\,t). \tag{3.72}$$

Therefore, choosing $\lambda>\max\{2c_2,\,2c_3\}$ and using (3.71) and (3.72) and the fact that $g'\le 0$, we find (3.58) and (3.59). Consequently, (3.58) and the second inequality in (3.59) lead to (3.60). Finally, by integration (3.60) with respect to t and using (2.20) and (3.59), we obtain (3.1) in case $\tau_0=1$. The proof of the second result in Theorem 3.1 is finished.

4. Estimation of $\left\|\partial_x^j U\right\|_2$

In this section, we use the second result in Theorem 3.1 to get some decay estimates on $\|\partial_x^k U\|_2$ when (1.14) is valid.

Theorem 4.1. *Assume that* (1.14) *is satisfied. Let* $N\in\mathbb{N}^*$,

$$U_0\in H^N(\mathbb{R})\cap L^1(\mathbb{R}) \tag{4.1}$$

and U *be the solution of* (2.8)*. Then, for any* $\ell\in\{1,2,\ldots,N\}$ *and* $j\in\{0,1,\ldots,N-\ell\}$*, there exists* $c_0>0$ *such that*

$$\|\partial_x^j U\|_2 \le c_0\,(1+t)^{-1/8-j/4}\,\|U_0\|_1 + c_0\,(1+t)^{-\ell/6}\,\|\partial_x^{j+\ell}U_0\|_2, \quad t\in\mathbb{R}_+ \tag{4.2}$$

if $\tau_0=0$*, and*

$$\|\partial_x^j U\|_2 \le c_0\,(1+t)^{-1/12-j/6}\,\|U_0\|_1 + c_0\,(1+t)^{-\ell/4}\,\|\partial_x^{j+\ell}U_0\|_2, \quad t\in\mathbb{R}_+ \tag{4.3}$$

if $\tau_0=1$.

Proof. For the proof of (4.2), we see that (3.2) with $\tau_0 = 0$ implies that (low and high frequences)

$$f(\xi) \geq \begin{cases} \frac{1}{2}\,\xi^4 & \text{if} \quad |\xi| \leq 1, \\ \frac{1}{2}\,\xi^{-6} & \text{if} \quad |\xi| > 1. \end{cases} \tag{4.4}$$

Applying Plancherel's theorem and (3.1), we entail

$$\begin{aligned} \|\partial_x^j U\|_2^2 &= \left\| \widehat{\partial_x^j U}(x,\,t) \right\|_2^2 = \int_{\mathbb{R}} \xi^{2\,j}\, |\widehat{U}(\xi,\,t)|^2 d\xi \\ &\leq \widetilde{c} \int_{\mathbb{R}} \xi^{2\,j}\, e^{-\,c\,f(\xi)\,t}\, |\widehat{U}_0(\xi)|^2 d\xi \\ &\leq \widetilde{c} \int_{|\xi|\leq 1} \xi^{2\,j}\, e^{-\,c\,f(\xi)\,t}\, |\widehat{U}_0(\xi)|^2 d\xi + \widetilde{c} \int_{|\xi|>1} \xi^{2\,j}\, e^{-\,c\,f(\xi)\,t}\, |\widehat{U}_0(\xi)|^2\, d\xi \\ &:= J_1 + J_2. \end{aligned} \tag{4.5}$$

Applying [9, Lemma 2.3] and $(4.4)_1$, it follows, for the low frequency region,

$$J_1 \leq C\, \|\widehat{U}_0\|_\infty^2 \int_{|\xi|\leq 1} \xi^{2\,j}\, e^{-\,\frac{c}{2}\,t\,\xi^4}\, d\xi \leq C\,(1+t)^{-\,\frac{1}{4}(1+2\,j)}\, \|U_0\|_1^2. \tag{4.6}$$

For the high frequency region, using $(4.4)_2$, we observe that

$$\begin{aligned} J_2 &\leq C \int_{|\xi|>1} |\xi|^{2\,j}\, e^{-\frac{c}{2}\,t\,\xi^{-6}}\, |\widehat{U}(\xi,\,0)|^2\, d\xi \\ &\leq C \sup_{|\xi|>1} \left\{ |\xi|^{-2\,\ell}\, e^{-\frac{c}{2}\,t\,|\xi|^{-6}} \right\} \int_{\mathbb{R}} |\xi|^{2\,(j+\ell)}\, |\widehat{U}(\xi,\,0)|^2\, d\xi, \end{aligned}$$

then, using [9, Lemma 2.4],

$$J_2 \leq C\,(1+t)^{-\frac{1}{3}\,\ell}\, \|\,\partial_x^{j+\ell} U_0\,\|_2^2, \tag{4.7}$$

and so, by combining (4.5)-(4.7), we get

$$\|\partial_x^j U\|_2^2 \leq C\left[(1+t)^{-\,\frac{1}{4}(1+2\,j)}\, \|U_0\|_1^2 + (1+t)^{-\frac{1}{3}\,\ell}\, \|\,\partial_x^{j+\ell} U_0\,\|_2^2 \right]. \tag{4.8}$$

Finally, by combining (4.8) and the inequality

$$\sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2}, \quad \forall a_1, a_2 \in \mathbb{R}_+, \tag{4.9}$$

we find (4.2).

The proof of (4.3) is identical to the one of (4.2) where, instead of (4.4), we have (according to (3.2) with $\tau_0 = 1$)

$$f(\xi) \geq \begin{cases} \frac{1}{2}\,\xi^6 & \text{if} \quad |\xi| \leq 1, \\ \frac{1}{2}\,\xi^{-4} & \text{if} \quad |\xi| > 1. \end{cases} \tag{4.10}$$

□

5. Estimation of $|\partial_\xi \widehat{U}|$

In this section, we study the asymptotic behavior (with respect to t) of $\partial_\xi \widehat{U}$. In order to simplify the computations, let us denoting $\partial_\xi \widehat{U} = \widehat{\mathbf{U}}$, $\partial_\xi \widehat{U}_0 = \widehat{\mathbf{U}}_0$ and

$$\left(\partial_\xi \widehat{v}, \partial_\xi \widehat{u}, \partial_\xi \widehat{z}, \partial_\xi \widehat{y}, \partial_\xi \widehat{\phi}, \partial_\xi \widehat{\theta}, \partial_\xi \widehat{p}, \partial_\xi \widehat{\eta}\right) = \left(\widehat{\mathbf{v}}, \widehat{\mathbf{u}}, \widehat{\mathbf{z}}, \widehat{\mathbf{y}}, \widehat{\Phi}, \widehat{\Theta}, \widehat{\mathbf{p}}, \widehat{\Lambda}\right).$$

As (2.13), the energy associated to $\widehat{\mathbf{U}}$ is define by

$$\begin{aligned}\widehat{\mathbf{E}}(\xi, t) =& \frac{1}{2}\left[k_1|\widehat{\mathbf{v}}|^2 + \rho_1|\widehat{\mathbf{u}}|^2 + k_2|\widehat{\mathbf{z}}|^2 + \rho_2|\widehat{\mathbf{y}}|^2 + (k_3 - \tau_0 g_0)|\widehat{\Phi}|^2 + \rho_3|\widehat{\Theta}|^2 + k_0|\widehat{\mathbf{p}}|^2\right](\xi,\, t)\\ &+ \frac{\tau_0}{2}\xi^4 \int_0^{+\infty} g(s)|\widehat{\Lambda}(\xi,\, t)|^2\, ds, \end{aligned} \tag{5.1}$$

where τ_0 is defined by (2.14). Applying the operator ∂_ξ to (3.10), we obtain the system

$$\begin{cases}\widehat{\mathbf{v}}_t - i\xi\widehat{\mathbf{u}} = i\widehat{u},\\ \rho_1\widehat{\mathbf{u}}_t - ik_1\xi\,\widehat{\mathbf{v}} + k_0\widehat{\mathbf{p}} = ik_1\widehat{v},\\ \widehat{\mathbf{z}}_t - i\xi\widehat{\mathbf{y}} = i\widehat{y},\\ \rho_2\widehat{\mathbf{y}}_t - ik_2\xi\,\widehat{\mathbf{z}} + k_0\widehat{\mathbf{p}} = ik_2\widehat{z},\\ \widehat{\Phi}_t + \xi^2\widehat{\Theta} = -2\xi\widehat{\theta},\\ \rho_3\widehat{\Theta}_t - (k_3 - \tau_0 g_0)\xi^2\widehat{\Phi} - ilk_0\,\xi\widehat{\mathbf{p}} + \mathbf{G} = G_0,\\ \widehat{\mathbf{p}}_t - \widehat{\mathbf{u}} - \widehat{\mathbf{y}} - il\xi\widehat{\Theta} = il\widehat{\theta},\\ \widehat{\Lambda}_t + \widehat{\Lambda}_s - \widehat{\Theta} = 0,\end{cases} \tag{5.2}$$

where

$$\begin{cases}\mathbf{G} = (1-\tau_0)\gamma\widehat{\Theta} + \tau_0\xi^4\int_0^{+\infty} g(s)\widehat{\Lambda}ds,\\ G_0 = 2(k_3 - \tau_0 g_0)\xi\widehat{\phi} + ilk_0\widehat{p} - 4\tau_0\xi^3 \displaystyle\int_0^{+\infty} g(s)\widehat{\eta}(\xi, s)\, ds.\end{cases} \tag{5.3}$$

As for the proof of (2.15), multiplying $(5.2)_1$-$(5.2)_7$ by $k_1\bar{\widehat{\mathbf{v}}}$, $\bar{\widehat{\mathbf{u}}}$, $k_2\bar{\widehat{\mathbf{z}}}$, $\bar{\widehat{\mathbf{y}}}$, $(k_3 - \tau_0 g_0)\bar{\widehat{\Phi}}$, $\bar{\widehat{\Theta}}$ and $k_0\bar{\widehat{\mathbf{p}}}$, respectively, multiplying $(5.2)_8$ by $\tau_0\xi^4 g(s)\bar{\widehat{\Lambda}}$ and integrating on $\mathbb{R}_+$ with respect to s, adding all the obtained equations, taking the real part of the resulting expression and using (2.17), we have

$$\begin{aligned}\frac{d}{dt}\widehat{\mathbf{E}}(\xi,\, t) =& -(1-\tau_0)\gamma\,|\widehat{\Theta}|^2 + \frac{\tau_0}{2}\xi^4\int_0^{+\infty} g'(s)|\widehat{\Lambda}|^2 ds\\ &+ Re\Big[ik_1\widehat{u}\bar{\widehat{\mathbf{v}}} + ik_1\widehat{v}\bar{\widehat{\mathbf{u}}} + ik_2\widehat{y}\bar{\widehat{\mathbf{z}}} + ik_2\widehat{z}\bar{\widehat{\mathbf{y}}} - 2(k_3 - \tau_0 g_0)\xi\widehat{\theta}\bar{\widehat{\Phi}} + G_0\bar{\widehat{\Theta}} + ilk_0\widehat{\theta}\bar{\widehat{\mathbf{p}}}\Big].\end{aligned} \tag{5.4}$$

This identity shows that $\widehat{\mathbf{E}}$ is not necessarily nonincreasing with respect to t. On the other hand, because

$$|\widehat{\mathbf{U}}(\xi,t)|^2 = \Big[|\widehat{\mathbf{v}}|^2 + |\widehat{\mathbf{u}}|^2 + |\widehat{\mathbf{z}}|^2 + |\widehat{\mathbf{y}}|^2 + |\widehat{\Phi}|^2 + |\widehat{\Theta}|^2 + |\widehat{\mathbf{p}}|^2\Big](\xi, t) + \tau_0\xi^4\int_0^{+\infty} g(s)|\widehat{\Lambda}(\xi,t)|^2 ds,$$

then we see that, according to the right inequality in (1.1),

$$|\widehat{\mathbf{U}}|^2 \sim \widehat{\mathbf{E}}. \tag{5.5}$$

Theorem 5.1. *Assume that* (1.14) *is satisfied. Let* $\widehat{U}$ *be the solution of* (2.10). *Then there exist* $\mathbf{c}$, $\widetilde{\mathbf{c}} > 0$ *such that, for any* $t \in \mathbb{R}_+$ *and for any* $\xi \in \mathbb{R}^*$,

$$|\widehat{\mathbf{U}}(\xi, t)| \leq \widetilde{\mathbf{c}} e^{-\mathbf{c}\, f(\xi)\, t} \left[|\widehat{\mathbf{U}}_0(\xi)| + \left(\xi^{7-2\tau_0} + \xi^{-(4+2\tau_0)} \right) |\widehat{U}_0(\xi)| \right], \tag{5.6}$$

where f *is defined in* (3.2).

Proof. We observe that the left hand sides of (5.2) are identical to the ones of (3.10) if we replace $\widehat{v}$, $\widehat{u}$, $\widehat{z}$, $\widehat{y}$, $\widehat{\phi}$, $\widehat{\theta}$, $\widehat{p}$, $\widehat{\eta}$ and G by $\widehat{\mathbf{v}}$, $\widehat{\mathbf{u}}$, $\widehat{\mathbf{z}}$, $\widehat{\mathbf{y}}$, $\widehat{\Phi}$, $\widehat{\Theta}$, $\widehat{\mathbf{p}}$, $\widehat{\Lambda}$ and $\mathbf{G}$, respectively. So, first, we use the same arguments to get similar differential identities to (3.11)-(3.23), and second, we treat the additional terms generated by the ones in the right hand sindes of (5.2).

Multiplying $(5.2)_4$ and $(5.2)_3$ by $i\,\xi\,\overline{\widehat{\mathbf{z}}}$ and $-i\rho_2\,\xi\,\overline{\widehat{\mathbf{y}}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we obtain

$$\frac{d}{dt} Re\Big(i\rho_2\,\xi\,\widehat{\mathbf{y}}\overline{\widehat{\mathbf{z}}}\Big) = \xi^2\left(\rho_2|\widehat{\mathbf{y}}|^2 - k_2|\widehat{\mathbf{z}}|^2\right) + k_0 Re\left(i\,\xi\,\widehat{\mathbf{z}}\,\overline{\widehat{\mathbf{p}}}\right) + Re\left(\rho_2\,\xi\widehat{y}\,\overline{\widehat{\mathbf{y}}} - k_2\xi\widehat{z}\overline{\widehat{\mathbf{z}}}\right). \tag{5.7}$$

Similarly, multiplying $(5.2)_2$ and $(5.2)_1$ by $i\,\xi\,\overline{\widehat{\mathbf{v}}}$ and $-i\rho_1\,\xi\,\overline{\widehat{\mathbf{u}}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we find

$$\frac{d}{dt} Re\left(i\rho_1\,\xi\,\widehat{\mathbf{u}}\,\overline{\widehat{\mathbf{v}}}\right) = \xi^2\left(\rho_1|\widehat{\mathbf{u}}|^2 - k_1|\widehat{\mathbf{v}}|^2\right) + k_0 Re\left(i\,\xi\,\widehat{\mathbf{v}}\,\overline{\widehat{\mathbf{p}}}\right) + Re\left(\rho_1\,\xi\widehat{u}\,\overline{\widehat{\mathbf{u}}} - k_1\,\xi\widehat{v}\,\overline{\widehat{\mathbf{v}}}\right). \tag{5.8}$$

Also, multiplying $(5.2)_6$ and $(5.2)_5$ by $-\overline{\widehat{\Phi}}$ and $-\rho_3\,\overline{\widehat{\Theta}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we get

$$\begin{aligned} \frac{d}{dt} Re\left(-\rho_3\,\widehat{\Theta}\,\overline{\widehat{\Phi}}\right) =& \xi^2\left(\rho_3|\widehat{\Theta}|^2 - (k_3 - \tau_0 g_0)|\widehat{\Phi}|^2\right) + lk_0\, Re\left(i\,\xi\,\widehat{\Phi}\,\overline{\widehat{\mathbf{p}}}\right) \\ &+ Re\left(\overline{\widehat{\Phi}}\,\mathbf{G}\right) + Re\left(2\rho_3\xi\widehat{\theta}\,\overline{\widehat{\Theta}} - G_0\,\overline{\widehat{\Phi}}\right). \end{aligned} \tag{5.9}$$

Multiplying $(5.2)_4$ and $(5.2)_7$ by $\xi^2\,\overline{\widehat{\mathbf{p}}}$ and $\rho_2\xi^2\,\overline{\widehat{\mathbf{y}}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we obtain

$$\begin{aligned} \frac{d}{dt} Re\left(\rho_2\xi^2\,\widehat{\mathbf{p}}\,\overline{\widehat{\mathbf{y}}}\right) =& \xi^2\left(\rho_2\,|\widehat{\mathbf{y}}|^2 - k_0|\widehat{\mathbf{p}}|^2\right) + \rho_2\xi^2\, Re\left(\widehat{\mathbf{y}}\,\overline{\widehat{\mathbf{u}}}\right) + l\rho_2\xi^2\, Re\left(i\xi\,\widehat{\Theta}\,\overline{\widehat{\mathbf{y}}}\right) \\ &+ k_2\xi^2\, Re\left(i\,\xi\,\widehat{\mathbf{z}}\,\overline{\widehat{\mathbf{p}}}\right) + Re\left(ik_2\xi^2\widehat{z}\,\overline{\widehat{\mathbf{p}}} + il\rho_2\,\xi^2\widehat{\theta}\,\overline{\widehat{\mathbf{y}}}\right). \end{aligned} \tag{5.10}$$

After, multiplying $(5.2)_3$ and $(5.2)_6$ by $\rho_3\,\overline{\widehat{\Theta}}$ and $\overline{\widehat{\mathbf{z}}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we entail

$$\begin{aligned} \frac{d}{dt} Re\left(\rho_3\widehat{\mathbf{z}}\,\overline{\widehat{\Theta}}\right) =& -\rho_3\, Re\left(i\xi\widehat{\Theta}\,\overline{\widehat{\mathbf{y}}}\right) + (k_3 - \tau_0 g_0)\,\xi^2\, Re\left(\widehat{\Phi}\,\overline{\widehat{\mathbf{z}}}\right) - lk_0\, Re\left(i\,\xi\,\widehat{\mathbf{z}}\,\overline{\widehat{\mathbf{p}}}\right) \\ &- Re\left(\overline{\widehat{\mathbf{z}}}\,\mathbf{G}\right) + Re\left(i\rho_3\widehat{y}\,\overline{\widehat{\Theta}} + G_0\,\overline{\widehat{\mathbf{z}}}\right). \end{aligned} \tag{5.11}$$

Multiplying $(5.2)_5$ and $(5.2)_4$ by $i\rho_2\xi\,\overline{\widehat{\mathbf{y}}}$ and $-i\xi\overline{\widehat{\Phi}}$, respectively, adding the resulting equations, taking the real part and using (2.17), it follows that

$$\frac{d}{dt} Re\left(i\rho_2\xi\,\widehat{\Phi}\,\overline{\widehat{\mathbf{y}}}\right) = -\rho_2\xi^2\, Re\left(i\xi\widehat{\Theta}\,\overline{\widehat{\mathbf{y}}}\right) + k_2\xi^2\, Re\left(\widehat{\Phi}\,\overline{\widehat{\mathbf{z}}}\right) - k_0\, Re\left(i\xi\widehat{\Phi}\,\overline{\widehat{\mathbf{p}}}\right)$$

$$+Re\left(k_2\,\xi\widehat{z}\,\overline{\widehat{\boldsymbol{\Phi}}}-2i\rho_2\,\xi^2\widehat{\theta}\,\overline{\widehat{\mathbf{y}}}\right). \tag{5.12}$$

Next, multiplying $(5.2)_2$ and $(5.2)_3$ by $i\xi\overline{\widehat{\mathbf{z}}}$ and $-i\rho_1\xi\,\overline{\widehat{\mathbf{u}}}$, respectively, adding the resulting equations, taking the real part and using (2.17), it appears that

$$\begin{aligned}\frac{d}{dt}Re\left(i\rho_1\xi\,\widehat{\mathbf{u}}\,\overline{\widehat{\mathbf{z}}}\right)&=-k_1\xi^2\,Re\left(\widehat{\mathbf{v}}\,\overline{\widehat{\mathbf{z}}}\right)+\rho_1\xi^2\,Re\left(\widehat{\mathbf{y}}\,\overline{\widehat{\mathbf{u}}}\right)+k_0\,Re\left(i\xi\widehat{\mathbf{z}}\,\overline{\widehat{\mathbf{p}}}\right)\\&\quad+Re\left(\rho_1\,\xi\widehat{y}\,\overline{\widehat{\mathbf{u}}}-k_1\,\xi\widehat{v}\,\overline{\widehat{\mathbf{z}}}\right).\end{aligned} \tag{5.13}$$

Multiplying $(5.2)_2$ and $(5.2)_5$ by $i\xi\overline{\widehat{\boldsymbol{\Phi}}}$ and $-i\rho_1\xi\,\overline{\widehat{\mathbf{u}}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we see that

$$\begin{aligned}\frac{d}{dt}Re\left(i\rho_1\xi\,\widehat{\mathbf{u}}\,\overline{\widehat{\boldsymbol{\Phi}}}\right)&=-k_1\xi^2\,Re\left(\widehat{\boldsymbol{\Phi}}\,\overline{\widehat{\mathbf{v}}}\right)+\rho_1\,\xi^2\,Re\left(i\xi\widehat{\Theta}\,\overline{\widehat{\mathbf{u}}}\right)+k_0\,Re\left(i\xi\widehat{\boldsymbol{\Phi}}\,\overline{\widehat{\mathbf{p}}}\right)\\&\quad+Re\left(2i\rho_1\,\xi^2\widehat{\theta}\,\overline{\widehat{\mathbf{u}}}-k_1\,\xi\widehat{v}\,\overline{\widehat{\boldsymbol{\Phi}}}\right).\end{aligned} \tag{5.14}$$

Also, multiplying $(5.2)_1$ and $(5.2)_4$ by $i\rho_2\xi\,\overline{\widehat{\mathbf{y}}}$ and $-i\xi\overline{\widehat{\mathbf{v}}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we find

$$\begin{aligned}\frac{d}{dt}Re\left(i\rho_2\xi\widehat{\mathbf{v}}\,\overline{\widehat{\mathbf{y}}}\right)&=-\rho_2\xi^2\,Re\left(\widehat{\mathbf{y}}\,\overline{\widehat{\mathbf{u}}}\right)+k_2\xi^2\,Re\left(\widehat{\mathbf{v}}\,\overline{\widehat{\mathbf{z}}}\right)-k_0\,Re\left(i\xi\widehat{\mathbf{v}}\,\overline{\widehat{\mathbf{p}}}\right)\\&\quad+Re\left(k_2\xi\widehat{z}\,\overline{\widehat{\mathbf{v}}}-\rho_2\,\xi\widehat{u}\,\overline{\widehat{\mathbf{y}}}\right).\end{aligned} \tag{5.15}$$

Multiplying $(5.2)_1$ and $(5.2)_6$ by $\rho_3\,\overline{\widehat{\Theta}}$ and $\overline{\widehat{\mathbf{v}}}$, respectively, adding the resulting equations, taking the real part and using (2.17), we get

$$\begin{aligned}\frac{d}{dt}Re\left(\rho_3\,\widehat{\mathbf{v}}\,\overline{\widehat{\Theta}}\right)&=-\rho_3\,Re\left(i\xi\widehat{\Theta}\,\overline{\widehat{\mathbf{u}}}\right)+(k_3-\tau_0g_0)\,\xi^2\,Re\left(\widehat{\boldsymbol{\Phi}}\,\overline{\widehat{\mathbf{v}}}\right)-lk_0\,Re\left(i\xi\widehat{\mathbf{v}}\,\overline{\widehat{\mathbf{p}}}\right)\\&\quad-Re\left(\overline{\widehat{\mathbf{v}}}\,\mathbf{G}\right)+Re\left(i\rho_3\widehat{u}\,\overline{\widehat{\Theta}}+G_0\,\overline{\widehat{\mathbf{v}}}\right).\end{aligned} \tag{5.16}$$

Now, multiplying $(5.2)_6$ by $-\xi^2\displaystyle\int_0^{+\infty}g(s)\overline{\widehat{\Lambda}}\,ds$, multiplying $(5.2)_8$ by $-\rho_3\xi^2g(s)\overline{\widehat{\Theta}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.17), we infer that

$$\begin{aligned}&\frac{d}{dt}Re\left(-\rho_3\xi^2\widehat{\Theta}\int_0^{+\infty}g(s)\overline{\widehat{\Lambda}}ds\right)\\&=-\rho_3g_0\xi^2|\widehat{\Theta}|^2-(k_3-\tau_0g_0)\xi^4Re\left(\widehat{\boldsymbol{\Phi}}\int_0^{+\infty}g(s)\overline{\widehat{\Lambda}}\,ds\right)-lk_0\xi^2Re\left(i\xi\widehat{\mathbf{p}}\int_0^{+\infty}g(s)\overline{\widehat{\Lambda}}ds\right)\\&\quad-\rho_3\xi^2Re\left(\overline{\widehat{\Theta}}\int_0^{+\infty}g'(s)\widehat{\Lambda}ds\right)+\xi^2Re\left(\mathbf{G}\int_0^{+\infty}g(s)\overline{\widehat{\Lambda}}ds\right)-\xi^2Re\left(G_0\int_0^{+\infty}g(s)\overline{\widehat{\Lambda}}ds\right).\end{aligned} \tag{5.17}$$

Similarly, multiplying $(5.2)_4$ by $i\xi\displaystyle\int_0^{+\infty}g(s)\overline{\widehat{\Lambda}}\,ds$, multiplying $(5.2)_8$ by $-i\rho_2\xi g(s)\,\overline{\widehat{\mathbf{y}}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.17), we arrive at

$$\frac{d}{dt}Re\left(i\rho_2\xi\widehat{\mathbf{y}}\int_0^{+\infty}g(s)\overline{\widehat{\Lambda}}\,ds\right)$$

$$
\begin{aligned}
&= -\rho_2 g_0 Re\left(i\xi\widehat{\Theta}\overline{\widehat{\mathbf{y}}}\right) - k_2\xi^2 Re\left(\widehat{\mathbf{z}}\int_0^{+\infty} g(s)\overline{\widehat{\Lambda}}\,ds\right) - k_0 Re\left(i\xi\widehat{\mathbf{p}}\int_0^{+\infty} g(s)\overline{\widehat{\Lambda}}\,ds\right)\\
&\quad -\rho_2 Re\left(i\xi\overline{\widehat{\mathbf{y}}}\int_0^{+\infty} g'(s)\widehat{\Lambda}\,ds\right) - k_2\, Re\left(\xi\,\widehat{z}\int_0^{+\infty} g(s)\overline{\widehat{\Lambda}}\,ds\right). \qquad (5.18)
\end{aligned}
$$

Finally, multiplying $(5.2)_2$ by $i\xi\displaystyle\int_0^{+\infty} g(s)\overline{\widehat{\Lambda}}\,ds$, multiplying $(5.2)_8$ by $-\rho_1\xi g(s)\overline{\widehat{\mathbf{u}}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.17), we see that

$$
\begin{aligned}
&\frac{d}{dt}Re\left(i\rho_1\xi\widehat{\mathbf{u}}\int_0^{+\infty} g(s)\overline{\widehat{\Lambda}}\,ds\right)\\
&= -k_1\xi^2 Re\left(\widehat{\mathbf{v}}\int_0^{+\infty} g(s)\overline{\widehat{\Lambda}}\,ds\right) - k_0 Re\left(i\xi\widehat{\mathbf{p}}\int_0^{+\infty} g(s)\overline{\widehat{\Lambda}}\,ds\right)\\
&\quad -\rho_1 Re\left(i\xi\overline{\widehat{\mathbf{u}}}\int_0^{+\infty} g'(s)\widehat{\Lambda}\,ds\right) - \rho_1 g_0 Re\left(i\xi\widehat{\Theta}\overline{\widehat{\mathbf{u}}}\right) - k_1\, Re\left(\xi\,\widehat{v}\int_0^{+\infty} g(s)\overline{\widehat{\Lambda}}\,ds\right). \qquad (5.19)
\end{aligned}
$$

Let $\mathbf{F}_0,\cdots,\mathbf{F}_5$ as $F_0,\cdots,F_5$, respectively, with $\widehat{\mathbf{v}}$, $\widehat{\mathbf{u}}$, $\widehat{\mathbf{z}}$, $\widehat{\mathbf{y}}$, $\widehat{\Phi}$, $\widehat{\Theta}$, $\widehat{\mathbf{p}}$, $\widehat{\Lambda}$ and $\mathbf{G}$ instead of $\widehat{v}$, $\widehat{u}$, $\widehat{z}$, $\widehat{y}$, $\widehat{\phi}$, $\widehat{\theta}$, $\widehat{p}$, $\widehat{\eta}$ and G, respectively. Exploiting (5.7)-(5.19), we find (instead of (3.30))

$$
\frac{d}{dt}\mathbf{F}_0(\xi,\,t) = \mathbf{F}_5(\xi,\,t) + \mathbf{F}_4(\xi,\,t) + \mathbf{F}_3(\xi,\,t) + R(\xi,t), \qquad (5.20)
$$

where R gathers the mixed products of the components of U and $\mathbf{U}$, and it is given by

$$
\begin{aligned}
R(\xi,\,t) = &Re\left[-k_2\lambda_1\xi\widehat{z}\overline{\widehat{\mathbf{z}}} + \rho_2\lambda_1\xi\widehat{y}\overline{\widehat{\mathbf{y}}} - k_1\lambda_2\xi\widehat{v}\overline{\widehat{\mathbf{v}}} + \rho_1\lambda_2\xi\widehat{u}\overline{\widehat{\mathbf{u}}} - \lambda_3 G_0\overline{\widehat{\Phi}} + 2\rho_3\lambda_3\xi\widehat{\theta}\overline{\widehat{\Theta}}\right]\\
&+Re\left[ik_2\lambda_4\xi^2\widehat{z}\overline{\widehat{\mathbf{p}}} + i\rho_2 l\lambda_4\xi^2\widehat{\theta}\overline{\widehat{\mathbf{y}}} + i\rho_3\lambda_5\widehat{y}\overline{\widehat{\Theta}} + \lambda_5 G_0\overline{\widehat{\mathbf{z}}} - 2i\rho_2\lambda_6\xi^2\widehat{\theta}\overline{\widehat{\mathbf{y}}}\right]\\
&+Re\left[k_2\lambda_6\xi\widehat{z}\overline{\widehat{\Phi}} - k_1\lambda_7\xi\widehat{v}\overline{\widehat{\mathbf{z}}} + \rho_1\lambda_7\xi\widehat{y}\overline{\widehat{\mathbf{u}}} - k_1\lambda_8\xi\widehat{v}\overline{\widehat{\Phi}} + 2i\rho_1\lambda_8\xi^2\widehat{\theta}\overline{\widehat{\mathbf{u}}}\right]\\
&+Re\left[-\rho_2\lambda_9\xi\widehat{u}\overline{\widehat{\mathbf{y}}} + k_2\lambda_9\xi\widehat{z}\overline{\widehat{\mathbf{v}}} + i\rho_3\lambda_{10}\widehat{u}\overline{\widehat{\Theta}} + \lambda_{10}G_0\overline{\widehat{\mathbf{v}}}\right].\\
&+Re\left[\tau_0\left(-\lambda_{11}\xi^2 G_0 - k_2\lambda_{12}\xi\widehat{z} - k_1\lambda_{13}\xi\widehat{v}\right)\int_0^{+\infty} g(s)\overline{\widehat{\Lambda}}ds\right]. \qquad (5.21)
\end{aligned}
$$

Considering the same choices of $\lambda_1\,\cdots,\,\lambda_{13}$ and using (3.44) and (3.45) for $\tilde{\eta}\in\left\{\widehat{\Lambda},\overline{\widehat{\Lambda}}\right\}$, we get (instead of (3.54) and (3.69))

$$
\begin{aligned}
&\frac{d}{dt}\left[\xi^{2+2\tau_0}\mathbf{F}_0(\xi,\,t)\right]\\
&\le -c_1\xi^{4+2\tau_0}\widehat{\mathbf{E}}(\xi,\,t) + \xi^{2+2\tau_0}R(\xi,t)\\
&\quad + c_2\xi^{2\tau_0}\left(\xi^{10}+1\right)\left[(1-\tau_0)|\widehat{\Theta}|^2 - \tau_0\xi^2\int_0^{+\infty} g'(s)|\widehat{\Lambda}(\xi,s)|^2\,ds\right]. \qquad (5.22)
\end{aligned}
$$

Using Young's inequality and exploiting (3.43), (3.51) and (3.67), we obtain, for any $\epsilon>0$,

$$
\xi^{2+2\tau_0}|R(\xi,\,t)| \le \epsilon\xi^{4+2\tau_0}\left(|\widehat{\mathbf{z}}|^2 + |\widehat{\mathbf{y}}|^2 + |\widehat{\mathbf{v}}|^2 + |\widehat{\mathbf{u}}|^2 + |\widehat{\Phi}|^2 + |\widehat{\Theta}|^2 + |\widehat{\mathbf{p}}|^2\right)
$$

$$+\tau_0\epsilon\xi^{6+4\tau_0}\int_0^{+\infty} g(s)|\widehat{\Lambda}(\xi,s)|^2\,ds + C_\epsilon\xi^{2\tau_0}\left(\xi^8+1\right)|\widehat{U}|^2,$$

thus, using (5.5), we obtain in both cases $\tau_0 = 0$ and $\tau_0 = 1$

$$\xi^{2+2\tau_0}|R(\xi,\,t)| \leq \epsilon C_1\xi^{4+2\tau_0}\widehat{\mathbf{E}}(\xi,\,t) + C_\epsilon\xi^{2\tau_0}\left(\xi^8+1\right)|\widehat{U}(\xi,\,t)|^2, \tag{5.23}$$

therefore, chosing $\epsilon = \frac{c_1}{2C_1}$, we deduce from (5.22) and (5.23) that

$$\begin{aligned}
&\frac{d}{dt}\left[\xi^{2+2\tau_0}\mathbf{F}_0(\xi,\,t)\right]\\
&\leq -\frac{c_1}{2}\,\xi^{4+2\tau_0}\,\widehat{\mathbf{E}}(\xi,\,t) + C\xi^{2\tau_0}\left(\xi^8+1\right)|\widehat{U}(\xi,\,t)|^2\\
&\quad + c_2\xi^{2\tau_0}\left(\xi^{10}+1\right)\left[(1-\tau_0)|\widehat{\Theta}|^2 - \tau_0\xi^2\int_0^{+\infty} g'(s)|\widehat{\Lambda}(\xi,s)|^2\,ds\right].
\end{aligned} \tag{5.24}$$

Now, let $\lambda > 0$ and $\mathbf{F}$ defined as F in (3.55) and (3.70); that is

$$\mathbf{F}(\xi,\,t) = \lambda\widehat{\mathbf{E}}(\xi,t) + \frac{\xi^{2+2\tau_0}}{\xi^{10}+1}\mathbf{F}_0(\xi,t).$$

By combining (5.4) and (5.24), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\mathbf{F}(\xi,\,t)\\
&\leq -\frac{c_1\xi^{4+2\tau_0}}{2\left(\xi^{10}+1\right)}\,\widehat{\mathbf{E}}(\xi,\,t) + C\frac{\xi^{2\tau_0}\left(\xi^8+1\right)}{\xi^{10}+1}|\widehat{U}(\xi,\,t)|^2 + R_1(\xi,\,t)\\
&\quad + (1-\tau_0)\left(c_2\xi^{2\tau_0}-\gamma\lambda\right)|\widehat{\Theta}|^2 + \tau_0\left(\frac{1}{2}\lambda\xi^4 - c_2\xi^{2+2\tau_0}\right)\int_0^{+\infty} g'(s)|\widehat{\Lambda}(\xi,s)|^2\,ds,
\end{aligned} \tag{5.25}$$

where

$$R_1(\xi,t) = \lambda Re\left[ik_1\widehat{u}\bar{\widehat{\mathbf{v}}} + ik_1\widehat{v}\bar{\widehat{\mathbf{u}}} + ik_2\widehat{y}\bar{\widehat{\mathbf{z}}} + ik_2\widehat{z}\bar{\widehat{\mathbf{y}}} - 2(k_3-\tau_0 g_0)\xi\widehat{\theta}\bar{\widehat{\Phi}} + G_0\bar{\widehat{\Theta}} + ilk_0\widehat{\theta}\bar{\widehat{\mathbf{p}}}\right].$$

On the other hand, as (3.57) and (3.72), we have

$$|\mathbf{F}(\xi,\,t) - \lambda\,\widehat{\mathbf{E}}(\xi,\,t)| \leq 2c_3\,\widehat{\mathbf{E}}(\xi,\,t). \tag{5.26}$$

Hence, for $\lambda > \left\{\frac{c_2}{\gamma}, 2c_3\right\}$ in case $\tau_0 = 0$, and $\lambda > \{2c_2, 2c_3\}$ in case $\tau_0 = 1$, we deduce from (5.25) and (5.26) that

$$\frac{d}{dt}\mathbf{F}(\xi,\,t) \leq -\frac{c_1\xi^{4+2\tau_0}}{2\left(\xi^{10}+1\right)}\,\widehat{\mathbf{E}}(\xi,\,t) + C\frac{\xi^{2\tau_0}\left(\xi^8+1\right)}{\xi^{10}+1}|\widehat{U}(\xi,\,t)|^2 + R_1(\xi,\,t) \tag{5.27}$$

and

$$\mathbf{F} \sim \widehat{\mathbf{E}}. \tag{5.28}$$

Applying Young's inequality and using the definitin of $\widehat{U}$, $\widehat{\mathbf{E}}$ and G_0, we see that, for any $\varepsilon > 0$,

$$\xi^{4+2\tau_0}R_1(\xi,\,t) \leq \frac{\varepsilon\,\xi^{2(4+2\tau_0)}}{\xi^{10}+1}\widehat{\mathbf{E}}(\xi,\,t) + C_\varepsilon\left(\xi^2+1\right)\left(\xi^{10}+1\right)|\widehat{U}(\xi,\,t)|^2, \tag{5.29}$$

then, by multiplying (5.27) by $\xi^{4+2\tau_0}$, combining with (5.29), choosing $\varepsilon = \frac{c_1}{4}$ and using (3.43) and (5.28), we obtain, for some $c_0 = \frac{c_1}{8}$,

$$\frac{d}{dt}\left[\xi^{4+2\tau_0}\mathbf{F}(\xi,\, t)\right] \leq -\frac{2c_0\xi^{2(4+2\tau_0)}}{\xi^{10}+1}\,\mathbf{F}(\xi,\, t) + C\frac{\xi^{22}+1}{\xi^{10}+1}|\widehat{U}(\xi,\, t)|^2, \tag{5.30}$$

therefore, by multiplying (5.30) by $e^{2c_0 f(\xi)t}$ (f is defined in (3.2)) and using (3.1), we find

$$\frac{d}{dt}\left[\xi^{4+2\tau_0}e^{2c_0 f(\xi)t}\mathbf{F}(\xi,\, t)\right] \leq C\frac{\xi^{22}+1}{\xi^{10}+1}e^{2(c_0-c)f(\xi)t}|\widehat{U}_0(\xi)|^2,$$

then, by integrating the above inequality with respect to t, we find

$$\mathbf{F}(\xi,\, t) \leq e^{-2c_0 f(\xi)t}\mathbf{F}(\xi,\, 0) + C\frac{\xi^{22}+1}{\xi^{2(4+2\tau_0)}}e^{-2cf(\xi)t}|\widehat{U}_0(\xi)|^2, \tag{5.31}$$

thus, according to (5.5) and (5.28) and using the inequality (4.9), the above inequality (5.31) implies (5.6). The proof of Theorem 5.1 is now achieved. $\square$

6. Estimation of $\left\|\partial_x^j U\right\|_1$

In this section, we show our decay estimate for $\|\partial_x^j U\|_1$ by exploiting (4.2), (4.3) and (5.6).

Theorem 6.1. *Assume that* (1.14) *is satisfied. Let* $N \in \{5+2\tau_0, 6+2\tau_0, \cdots\}$ *and* U *be a solution of* (2.8) *corresponding to an initial data* U_0 *satisfying*

$$U_0 \in H^{N+7-2\tau_0}(\mathbb{R}) \cap L^1(\mathbb{R}) \quad \text{and} \quad \tilde{U}_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R}), \tag{6.1}$$

where $\tilde{U}_0(x) = xU_0(x)$. *Then, for any* $\ell \in \{1,\, 2,\, \cdots,\, N-4-2\tau_0\}$ *and* $j \in \{4+2\tau_0,\, 5+2\tau_0,\, \cdots,\, N-\ell\}$, *there exist* $\mathbf{c}_0,\, \tilde{\mathbf{c}}_0 > 0$ *such that, for any* $t \in \mathbb{R}_+$,

$$\begin{aligned}\|\partial_x^j U\|_1 \leq & \tilde{\mathbf{c}}_0 \left[(1+t)^{-1/8-j/4}\,\|\tilde{U}_0\|_1 + (1+t)^{7/8-j/4}\,\|U_0\|_1\right] \\ & + \tilde{\mathbf{c}}_0(1+t)^{-\ell/6}\left[\|\partial_x^{j+\ell}\tilde{U}_0\|_2 + \|\partial_x^{j+\ell+7}U_0\|_2 + \|\partial_x^{j+\ell}U_0\|_2 + \|\partial_x^{j+\ell-1}U_0\|_2\right].\end{aligned} \tag{6.2}$$

if $\tau_0 = 0$, *and*

$$\begin{aligned}\|\partial_x^j U\|_1 \leq & \tilde{\mathbf{c}}_0 \left[(1+t)^{-1/12-j/6}\,\|\tilde{U}_0\|_1 + (1+t)^{11/12-j/6}\,\|U_0\|_1\right] \\ & + \tilde{\mathbf{c}}_0(1+t)^{-\ell/4}\left[\|\partial_x^{j+\ell}\tilde{U}_0\|_2 + \|\partial_x^{j+\ell+5}U_0\|_2 + \|\partial_x^{j+\ell}U_0\|_2 + \|\partial_x^{j+\ell-1}U_0\|_2\right]\end{aligned} \tag{6.3}$$

if $\tau_0 = 1$.

Proof. First, using (1.18) and applying Young's inequality, it follows that

$$\|\partial_x^j U\|_1 \leq \frac{1}{2}\|\partial_x^j U\|_2 + \frac{1}{2}\|x\partial_x^j U\|_2. \tag{6.4}$$

Because the term $\|\partial_x^j U\|_2$ has yet been estimated in (4.2) and (4.3), we have only to estimate the term $\|x\partial_x^j U\|_2$. Using the method in [18] and Plancherel's theorem, we may write

$$\begin{aligned}\int_{\mathbb{R}} x^2|\partial_x^j U(x,t)|^2dx &\leq C\int_{\mathbb{R}}\left|\partial_\xi\left(|\xi|^j|\widehat{U}(\xi,t)|\right)\right|^2 d\xi\\ &\leq C\int_{\mathbb{R}}\left(|\xi|^{j-1}|\widehat{U}(\xi,t)|+|\xi|^j|\partial_\xi\widehat{U}(\xi,t)|\right)^2 d\xi\\ &\leq C\|\partial_x^{j-1}U\|_2^2+C\int_{\mathbb{R}}\xi^{2j}|\widehat{\mathbf{U}}(\xi,t)|^2d\xi. \end{aligned}\tag{6.5}$$

It is clear that $\|\partial_x^{j-1}U\|_2$ can be easily estimated by using (4.2) and (4.3) (with $j-1$ instead of j). To estimate the last integral in (6.5), we use (5.6) and apply Plancherel's theorem, it appears that

$$\begin{aligned}&\int_{\mathbb{R}}\xi^{2\,j}\,|\widehat{U}(\xi,\,t)|^2d\xi\\ \leq& C\int_{\mathbb{R}^*}\xi^{2\,j}\,e^{-2\,\mathbf{c}\,f(\xi)\,t}\,\left[|\widehat{\mathbf{U}}_0(\xi)|^2+\left(\xi^{14-4\tau_0}+\xi^{-(8+4\tau_0)}\right)|\widehat{U}_0(\xi)|^2\right]\,d\xi\\ \leq& C\int_{0<|\xi|\leq 1}e^{-2\,\mathbf{c}\,f(\xi)\,t}\,\left[\xi^{2\,j}|\widehat{\mathbf{U}}_0(\xi)|^2+\xi^{2(j-4-2\tau_0)}|\widehat{U}_0(\xi)|^2\right]d\xi\\ &+C\int_{|\xi|>1}e^{-2\,\mathbf{c}\,f(\xi)\,t}\,\left[\xi^{2j}|\widehat{\mathbf{U}}_0(\xi)|^2+\xi^{2\,(j+7-2\tau_0)}|\widehat{U}_0(\xi)|^2\right]\,d\xi\\ :=&J_1+J_2. \end{aligned}\tag{6.6}$$

Case $\tau_0=0$: using $(4.4)_1$ and [9, Lemma 2.3], we observe that

$$\begin{aligned}J_1&\leq C\,\|\widehat{\mathbf{U}}_0\|_\infty^2\int_{|\xi|\leq 1}\xi^{2\,j}\,e^{-\,\mathbf{c}\,t\,\xi^4}\,d\xi+C\,\|\widehat{U}_0\|_\infty^2\int_{|\xi|\leq 1}\xi^{2\,(j-4)}\,e^{-\,\mathbf{c}\,t\,\xi^4}\,d\xi\\ &\leq C\,(1+t)^{-\,\frac14(1+2\,j)}\,\|\tilde{U}_0\|_1^2+C\,(1+t)^{-\,\frac14[1+2\,(j-4)]}\,\|U_0\|_1^2. \end{aligned}\tag{6.7}$$

In the high frequency region, using $(4.4)_2$, we entail

$$\begin{aligned}J_2&\leq C\int_{|\xi|>1}\xi^{2\,j}\,e^{-\mathbf{c}\,t\,\xi^{-6}}\,|\widehat{\mathbf{U}}_0(\xi)|^2\,d\xi+C\int_{|\xi|>1}\xi^{2\,(j+7)}\,e^{-\mathbf{c}\,t\,\xi^{-6}}\,|\widehat{U}_0(\xi)|^2\,d\xi\\ &\leq C\,\sup_{|\xi|>1}\left\{\xi^{-2\,\ell}\,e^{-\mathbf{c}\,t\,\xi^{-6}}\right\}\left[\int_{\mathbb{R}}\xi^{2\,(j+\ell)}\,|\widehat{\mathbf{U}}_0(\xi)|^2\,d\xi+\int_{\mathbb{R}}\xi^{2\,(j+\ell+7)}\,|\widehat{U}_0(\xi)|^2\,d\xi\right]\\ &\leq C\,\sup_{|\xi|>1}\left\{\xi^{-2\,\ell}\,e^{-\mathbf{c}\,t\,\xi^{-6}}\right\}\left(\|\,\partial_x^{j+\ell}\tilde{U}_0\,\|_2^2+\|\,\partial_x^{j+\ell+7}U_0\,\|_2^2\right), \end{aligned}$$

thus, simple computations (see, for example, [9, Lemma 2.4]) imply that

$$J_2\leq C\,(1+t)^{-\frac{\ell}{3}}\,\left(\|\,\partial_x^{j+\ell}\tilde{U}_0\,\|_2^2+\|\,\partial_x^{j+\ell+7}U_0\,\|_2^2\right),\tag{6.8}$$

and so, by combining (6.6)-(6.8), we get

$$\int_{\mathbb{R}}\xi^{2\,j}\,|\widehat{U}(\xi,\,t)|^2d\xi\leq C\,(1+t)^{-\,\frac14(1+2\,j)}\,\|\tilde{U}_0\|_1^2+(1+t)^{-\,\frac14[1+2\,(j-4)]}\,\|U_0\|_1^2$$

$$+C\,(1+t)^{-\frac{\ell}{3}}\ \left(\|\,\partial_x^{j+\ell}\tilde{U}_0\,\|_2^2+\|\,\partial_x^{j+\ell+7}U_0\,\|_2^2\right) \tag{6.9}$$

thus, by combining (4.2), (6.4) and (6.9), and using the inequality (4.9), we get (6.2).

Case $\tau_0 = 1$: using (4.3), (4.10) and (5.6), and following the same arguments as for the proof of (6.2), we obtain (6.3). □

7. Concluding discussion

1. This article is concerned with the stability of two systems of type Rao-Nakra sandwich beam in the whole line $\mathbb{R}$ under the presence of a frictional damping or an infinite memory acting on the Euler-Bernoulli equation. We prove the instability of both systems if the speeds of propagation of the two wave equations are equal. In the reverse situation, we are able, despite the presence of only one controle, to obtain the desired $L^2(\mathbb{R})$-norm and $L^1(\mathbb{R})$-norm decay estimates (4.2), (4.3), (6.2) and (6.3). The main ingredient of the proof is the energy method and the Fourier analysis.

2. The decay estimates (4.2), (4.3), (6.2) and (6.3) are still satisfied when (1.13) holds if we add $\tilde{\gamma}\varphi_t$ to $(1.4)_1$ and $(1.5)_1$, or $\tilde{\gamma}\psi_t$ to $(1.4)_2$ and $(1.5)_2$, where $\tilde{\gamma} > 0$. Indeed, in these cases, we do not need $A_7 = 0$ because either $|\widehat{u}|^2$ or $|\widehat{y}|^2$ will be directly controlled via the derivative of the energy functional $\widehat{E}$. Similarly, (4.2), (4.3), (6.2) and (6.3) hold even in case (1.13) if the added control is of memory type:

$$\int_0^{+\infty}\tilde{g}(s)\,\varphi_{xx}(x,t-s)ds \quad \text{or} \quad \int_0^{+\infty}\tilde{g}(s)\,\psi_{xx}(x,t-s)ds$$

instead of $\tilde{\gamma}\varphi_t$ and $\tilde{\gamma}\psi_t$, respectively, where $\tilde{g}$ is as g.

3. Using the interpolation inequalities (1.15), (1.16) and (1.17), we see that our $L^2(\mathbb{R})$-norm and $L^1(\mathbb{R})$-norm decay estimates lead to similar $L^q(\mathbb{R})$-norm ones, for any $q \in [1,+\infty]$. The L^q-norm decay estimates, for $1 \le q < 2$, are based on the L^1-norm decay estimate, and so they require initial data U_0 having the regularity (6.1) with $N \in \{5,6,\cdots\}$ in case (1.4), and $N \in \{7,8,\cdots\}$ in case (1.5). However, the L^q-norm decay estimates, for $q \in [2,+\infty]$ require the weaker regularity (4.1), where $N \in \mathbb{N}^*$.

4. In a future work, we aspire to treat the case where the control occurs on a one wave equation of systems. On the other hand, we think that similar results can be obtained with a control subject to a thermal effect like Fourier law, Cattaneo law and Gurtin-Pipkin law. These kinds of controls deserve to be treated and we aspire to do it in a future work.

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