Journal of Applied Analysis and Computation
Volume 12, Number 6, December 2022, 2541–2554
Website:http://www.jaac-online.com
DOI:10.11948/20220061

ON THE NUMBER OF NONTRIVIAL RATIONAL SOLUTIONS FOR ABEL EQUATIONS*

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Abstract In this paper, a systematic algorithm is provided to determine the sharp upper bound on the number of nontrivial rational solutions for the Abel differential equations $dy/dx = f_m(x)y^2 + g_n(x)y^3$, where $f_m(x)$ and $g_n(x)$ are real polynomials of degree m and n respectively. As an application, we present a thorough study for an important case, $(m, n) = (4, 9)$.

Keywords Abel differential equations, nontrivial rational solution, Liénard system, rational invariant curve.

MSC(2010) 34A05, 34C05, 37C10.

1. Introduction

The analysis of exact solutions such as polynomial and rational solutions of the differential equations

$$a(x)y' = b_0(x) + b_1(x)y + b_2(x)y^2 + \cdots + b_l(x)y^l, \tag{1.1}$$

where $a(x), b_i(x)$, with $0 \le i \le l$ are real polynomials, is one of the most classical topics in ordinary differential equations and has attracted much attention for almost one hundred years. Such a study is of particular importance to understand the whole set of solutions and the dynamical properties of the system. For example, it is well-known that once we know one solution of the classical Riccati equation, where $a(x) \equiv 1$ and $l = 2$ in (1.1), then we can know the whole set of its solutions in an explicit way. More and more study reveals that exploration of the exact solutions, esp. the polynomial and rational ones, already begins to exert deep influence in the related fields such as algebraic structure and geometric properties of the systems.

Speaking about the these special kinds of solutions, in 1936, Rainville in [19] proved the existence of one or two polynomial solutions for the Riccati equation (1.1). Campbell and Golomb [7] in 1954 provided an algorithm for finding all the polynomial solutions for the equation (1.1) with $a(x) \equiv 1$ and $l = 2$. Bhargava and Kaufman in 1960s in ([2,3]) give some sufficient conditions for the Riccati equation to have polynomial solutions. In 2011, Giné etc. in [13] considered the case $a(x) \equiv 1$

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*X.Qian is supported by Doctor of Entrepreneurship and Innovation Funds of Jiangsu (No. JSSCBS20210617), and PhD research startup foundation of Jinling Institute of Technology (Nos. jit-b-202049, jit-fhxm-202116). The authors are supported by National Science Foundation of China (No. 12071006).

in (1.1) and proved that the system has at most l polynomial solutions. Gasull etc. in 2016 in [12] investigated the case, $l = 2$ and $a(x)$ a polynomial, in (1.1) and obtained the upper bound of the number of polynomial solutions of the systems.

On the other hand, Behloul and Cheng [1] proved that all the polynomial solutions for equation (1.1) can be computed in a systematic manner. And recently, the maximum number of polynomial solutions of equation (1.1) for some values of l, $a(x)$ and $b_i(x)$ are estimated in [8, 13, 16]. For detailed progress in this direction, one can refer to [12] and the reference therein for more details.

Notice that another important system is the so-called Abel system, i.e., a special case of (1.1) where $a(x) \equiv 1$, $l = 3$, and $b_0(x) = b_1(x) = 0$. Namely,

$$\frac{dy}{dx} = f_m(x)y^2 + g_n(x)y^3, \tag{1.2}$$

where $f_m(x)$ and $g_n(x)$ are real polynomials of degree m and n, respectively, having the following explicit expressions

$$f_m(x) = \sum_{i=0}^{m} a_i x^i, \qquad g_n(x) = \sum_{i=0}^{n} b_i x^i, \qquad a_m b_n \neq 0. \tag{1.3}$$

For simplicity, we shall call such a system the Abel equation of type (m, n). The Abel equation appears in many theoretical and applied problems ([10, 14, 17]), and has been studied intensively with traditional emphasis on the number of limit cycles (see [9, 11]), or the center problem (see for instance [4–6]). Very recently, the highlight of the study begins to shift to polynomial limit cycles, and nontrivial rational limit cycles of the system, expecting to obtain deeper insight of of its mechanism. For example, the maximum number of polynomial solutions of equation (1.1) for some values of l, $a(x)$ and $b_i(x)$ are estimated in [8, 12, 13, 15, 16, 20]. For more information, we refer the reader to see these papers and the reference therein.

In this paper, we shall consider the nontrivial rational solutions of equation (1.2). By a *nontrivial* rational solution we means that it is a rational but not polynomial solution. The importance of study of nontrivial rational solutions for (1.2) relies on the intrinsic relation between (1.2) and other important systems, say the Liénard system. In fact, the number of nontrivial rational solutions for the equation (1.2) and the number of the rational invariant curves for Liénard systems coincide. It is already realized that knowing the number of invariant curves is one of the key ways to study the integrability of planar polynomial differential systems.

Concerning the nontrivial rational solutions of equation (1.2), Qian etc. in a recent paper, [18], gave an estimation of the upper bound of equation (1.2) in all the cases. More precisely, they proved the following:

Theorem 1.1 (see [18])**.** *Consider equation (1.2). The following statements stand.*

(i) If $n \leq m$ or $n = 2m + 2k$ $(k \geq 1)$, then (1.2) of type (m, n) has no nontrivial rational solutions.

(ii) If $m + 1 \leq n \leq 2m$, or $n = 2m + 2k + 1$ $(k \geq 1)$, then equation (1.2) of type (m, n) has at most 2 nontrivial rational solutions, and this upper bound is sharp.

(iii) If $n = 2m + 1$ (i.e., $k = 0$ in (ii)), then equation (1.2) of type $(m, 2m + 1)$ has at most $m + 3$ nontrivial rational solutions. In particular, for $m = 1, 2$ and 3, the upper bounds are $4, 5$ and 5, respectively, and these bounds are sharp.

Roughly speaking, the above theorem exhausts all the cases, presenting a systematic estimation about the number of nontrivial rational solutions of (1.2). More precisely, except the degenerate case, $n = 2m+1$, the upper bounds together with their sharpness of the nontrivial rational solutions are completely solved.

The case $n = 2m+1$ is exceptionally difficult and complicated, all the methods and algorithm known turn out to either fail or to be inefficient. On the other hand, it is exact those degenerate cases that admit extra values in understanding the whole setting of the related problems as many researcher agrees with this, at least to certain degree. Therefore, it is worthy paying more attention to the degeneracy, and in this note we shall fully focus on taking one more step to feel the delicacy of the situation. In other words, we give a careful consideration the type $(m, 2m+1)$. With some heavy calculation and sophisticated analysis, we prove the following result.

Theorem 1.2. *The equations (1.2) of type* $(4, 9)$ *have at most* 5 *nontrivial rational solutions, and this bound is sharp.*

For clarity and a better comparison, we collect all the known cases and put them into the following table:

(m, 2m+1) type	(1,3)	(2,5)	(3,7)	(4,9)
No. Rational Solutions	4	5	5	5

The paper is organized as follows: In Section 2, we develop an algorithm for finding the rational solutions of the systems, while in Section 3, we present a detailed proof of Theorem 1.2, where we also give concrete forms of the solutions of type $(4, 9)$. Our methods yield an efficient way in estimating the sharp upper bound on the number of such solutions for the Abel equations.

2. The Algorithm

2.1. Preliminary

In this section, we shall present an algorithm for finding the sharp upper bound on the number of nontrivial rational solutions for Abel equations (1.2). To this end, we need the following useful lemma.

Firstly, from the proof of Proposition 2.5 in [18], we know that if the equation (1.2) of type $(m, 2m+1)$ has more than 2 nontrivial rational solutions, then the following relations must stand: $b_{2m+1} = \frac{ja_m^2}{(m+1+j)^2}$, where a_m and b_{2m+1} are the leading coefficient of $f_m(x)$ and $g_{2m+1}(x)$, $1 \leq j \leq m$. Notice that for different j in the above relations, one can derive different number of nontrivial rational solutions.

Lemma 2.1. *If* $b_{2m+1} = \frac{ja_m^2}{(m+1+j)^2}$*, where* $1 \leq j \leq m$*, then the equations (1.2) of type* $(m, 2m+1)$ *have at most* $[\frac{m+1}{m+1-j}] + 2$ *nontrivial rational solutions. In particular, for* $j = m$*, we have the upper bound* $m+3$*.*

Proof. From [15], we know that to study nontrivial rational solution of equations (1.2), we need only to consider solution of the form $y = \frac{1}{R(x)}$. Now assume that $y = \frac{1}{R(x)}$ is a nontrivial rational solution of equations (1.2), where

$$R(x) = u_{m+1}x^{m+1} + \cdots + u_1 x + u_0, \tag{2.1}$$

and $u_{m+1} \neq 0$, then it follows that

$$R(x)R'(x) + R(x)f_m(x) = -g_n(x). \tag{2.2}$$

By substituting (2.1) and (1.3) into (2.2) and then comparing the coefficients of the equation, we obtain the following relations:

$$\begin{aligned}
-b_{2m+1} &= (a_m + (m+1)u_{m+1})u_{m+1}\\
-b_{2m} &= (a_m + (2m+1)u_{m+1})u_m + a_{m-1}u_{m+1}\\
-b_{2m-1} &= (a_m + 2mu_{m+1})u_{m-1} + (mu_m + a_{m-1})u_m + a_{m-2}u_{m+1}\\
\cdots &= \cdots\\
-b_{m+j} &=(a_m + (m+1+j)u_{m+1})u_j \cdots + ((j+1)u_{j+1} + a_j)u_m + a_{j-1}u_{m+1}\\
\cdots &= \cdots\\
-b_0 &= (a_0 + u_1)u_0.
\end{aligned}$$

Since $b_{2m+1} = \frac{ja_m^2}{(m+1+j)^2}$, we know that the first equation of the above system has two solutions:

$$(i). \quad u'_{m+1} = \frac{-ja_m}{(m+1)(m+1+j)}. \qquad (ii). \quad u''_{m+1} = \frac{-a_m}{m+1+j}.$$

(i). If $u'_{m+1} = \frac{-ja_m}{(m+1)(m+1+j)}$, for $0 \leq k \leq m$, then we have

$$j(m+1+k) \neq (m+1)(m+1+j),$$

i.e.,

$$a_m + (m+1+k)u'_{m+1} \neq 0.$$

Hence the system can have at most one nontrivial rational solution $y = \frac{1}{R(x)}$, where the leading coefficient of $R(x)$ is u'_{m+1}.

(ii). If $u''_{m+1} = \frac{-a_m}{m+1+j}$, then u_j is an independent variable, and we can express $u_0, \cdots, u_{j-1}$ in terms of u_j. In fact, we can derive the expression of $u_{j-(m+1-j)k_1}$ in terms of u_j from the $(1 + (m+1-j)(k_1+1))$-th equation. Namely,

$$u_{j-(m+1-j)k_1} = \frac{c_{k_1,0}}{u'^{k_1}_{m+1}} u_j^{k_1+1} + \cdots + c_{k_1,k_1},$$

where $1 \leq k_1 \leq [\frac{j}{m+1-j}]$ and $c_{k_1,0} > 0$. In general, from the $(1-l+(m+1-j)(k_1+1))$-th equation, we have

$$u_{j-(m+1-j)k_1+l} = R_{j-(m+1-j)k_1+l}(u_j),$$

where $R_{j-(m+1-j)k_1+l}(u_j)$ is a polynomial of degree at most k_1, and $1 \leq l \leq m-j$.

By substituting $u_0, \cdots, u_{j-1}$ into the $\Big(1 + ([\frac{m+1}{m+1-j}]+1)(m+1-j)\Big)$-th equation, we obtain an equation of $[\frac{m+1}{m+1-j}]+1$ degree, where the coefficient of the highest term is $d_m \cdot \big(u'_{m+1}\big)^{-[\frac{m+1}{m+1-j}]}$ where $d_m > 0$. Thus the equation (1.2) has at most $[\frac{m+1}{m+1-j}]+1$ nontrivial rational solutions in this case.

Based on the above discussion of two possibilities, we know that the equation (1.2) has at most $[\frac{m+1}{m+1-j}]+2$ nontrivial rational solutions, and the lemma follows. $\square$

Recall that the authors of [18] present equations (1.2) of type $(m, 2m+1)$ which can possess exactly 4 nontrivial rational solutions, where $m \geq 1$. Thus the sharp upper bound on the number of nontrivial rational solutions for equation (1.2) of type $(m, 2m+1)$ must be ≥ 4. By Lemma 2.1, we only have to consider the sharp upper bound in the case $b_{2m+1} = \frac{ja_m^2}{(m+1+j)^2}$, where $\frac{2}{3}(m+1) \leq j \leq m$.

Now we fix some notation and definitions. First, we set

$$g_{2m+1}(x) = b_{2m+1}(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{2m+1}),$$

where $b_{2m+1} = \frac{ja_m^2}{(m+1+j)^2}$, $\frac{2}{3}(m+1) \leq j \leq m$. Denote by $y = \frac{1}{P_i(x)}$, $i \geq 1$, solutions of equation (1.2). Denote the greatest common divisor of two polynomials $P_{i_1}(x)$ and $P_{i_2}(x)$ by $G_{i_1,i_2}(x)$. For example, $G_{1,2}(x)$ means the common divisor of the first polynomial $P_1(x)$ and the second one $P_2(x)$. In this way, the meaning of the collection of $G_{1,2}(x)$, $G_{1,3}(x)$ and $G_{2,3}(x)$, denoted by A, is clear. That is, $A = \{G_{1,2}(x), G_{1,3}(x), G_{2,3}(x)\}$. Notice that in the notation $G_{i_1,i_2}(x)$, we have $1 \leq i_1 < i_2$.

To describe the algorithm, we also have to introduce the following two Lemmas in [18].

Lemma 2.2 ([18])**.** *Let $g_{2m+1}(x) = b_{2m+1}(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{2m+1})$, where $b_{2m+1} = \frac{ja_m^2}{(m+1+j)^2}$, and $\frac{2}{3}(m+1) \leq j \leq m$. If $y = \frac{1}{P_\mu(x)}$, with $1 \leq \mu \leq 2$, are two nontrivial rational solutions of equation (1.2), where $P_\mu(x) = G_{1,2}(x)\widetilde{P_\mu}$,*

$$\begin{aligned} &G_{1,2}(x) = (x-\alpha_1)\cdots(x-\alpha_i),\\ &\widetilde{P_1} = (x-\alpha_{i+1})\cdots(x-\alpha_{m+1}),\ \ \widetilde{P_2} = (x-\alpha_{m+2})\cdots(x-\alpha_{2m+2-i}), \end{aligned}$$

with $1 \leq i \leq j$ and $\gcd(\widetilde{P_1}(x), \widetilde{P_2}(x)) = 1$. Then if $i = 1$, we set $H(x) = 1$. If $2 \leq i \leq j$, let $H(x) = (x-\alpha_{2m+3-i})\cdots(x-\alpha_{2m+1})$. Finally the following equations hold:

$$\widetilde{P_1} - \widetilde{P_2} = C(x-\alpha_{i_1})^{k_1}\cdots(x-\alpha_{i_l})^{k_l},\ \ jH(x) - G'_{1,2}(x) = G_{1,2}(x)\sum_{v=1}^{l}\frac{k_v}{x-\alpha_{i_v}},$$

where C is an constant, $1 \leq i_1 < i_2 < \cdots < i_l \leq i$, $k_v \geq 0$, $1 \leq v \leq l$ and $k_1 + \cdots + k_l = j - i$.

Lemma 2.3 ([18])**.** *Assume the equation (1.2) with at least three nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with $P_i(0) = 0$ and $1 \leq i \leq 3$. Set*

$$P_1(x) = u_{1,m+1}x^{m+1} + u_{1,m}x^m + \cdots + u_{1,n_1}x^{n_1},$$

with $1 \leq n_1 \leq m$, and $u_{1,n_1} \neq 0$. Then the term $P_k(x)$ of all the other nontrivial rational solutions $y = \frac{1}{P_k(x)}$ with $P_k(0) = 0$ can be express as

$$P_k(x) = u_{k,m+1}x^{m+1} + u_{k,m}x^m + \cdots + u_{k,n_1}x^{n_1},$$

where $k \geq 2$ and $u_{k,n_1} \neq 0$. Furthermore, u_{k,n_1} with $k \geq 2$ can take only two possible values:

(i) *all the values are the same;*

(ii) *there exists a natural number* k_1 *such that* $u_{k_1,n_1} \neq u_{1,n_1}$ *and* $u_{k,n_1} \equiv u_{1,n_1}$ *or* $u_{k,n_1} \equiv u_{k_1,n_1}$*, with* $k, k_1 \geq 2$ *and* $k \neq k_1$*.*

Furthermore, we say that $P_1(x), \cdots, P_i(x)$ is equivalent to $R_1(x), \cdots, R_i(x)$ if there exists $\sigma \in S_{2m+1}$, such that $\alpha_k \to \alpha_{\sigma(k)}$, and $P_1(x), \cdots, P_i(x) \to R_1(x), \cdots, R_i(x)$, where $1 \leq k \leq 2m+1$ and S_{2m+1} is the symmetric group on $2m+1$ letters. In the following discussion, we only list a representative of each equivalent class.

2.2. The procedures

We are ready now to describe the algorithm.

(1) Assume there exist a system (1.2) of type $(m, 2m+1)$ having at least five nontrivial rational solutions $y = \frac{1}{P_i(x)}$, where the leading coefficient of $P_i(x)$ is 1, $1 \leq i \leq 5$.

(2) We use the Pigeonhole principle to obtain the form of $P_i(x)$, $1 \leq i \leq 5$.

(3) When $b_{2m+1} = \frac{ja_m^2}{(m+1+j)^2}$, with $\frac{3}{4}(m+1) \leq j \leq m$, then $\deg G_{i_1,i_2}(x) \leq j$, where $1 \leq i_1 < i_2$. We distinguish three cases according to the degrees of $G_{1,2}(x)$, $G_{1,3}(x)$ and $G_{2,3}(x)$ as follows.

(I): At least two elements of A are of degree j;

(II): One element of A has degree j;

(III): None element of A admits degree j.

(4) Assume $P_1(x) = \prod_{i=1}^{m+1}(x - \alpha_i)$. For each case, we first give all the possible equivalent classes of $P_2(x)$ according to the degree of $G_{1,2}(x)$.

(5) For each equivalent class of $P_2(x)$, we give all the possible equivalent classes of $P_3(x)$ according to the degree of $G_{1,3}(x)$ and $G_{2,3}(x)$. Then we delete the equivalent classes which violate Lemma 2.3. For the remaining equivalent classes, by Lemma 2.2, we can obtain some algebraic equations. By solving these equations, we shall delete the equivalent classes which violate the requirement that $P_1(x), P_2(x), P_3(x)$ are different polynomials.

(6) For each equivalent class of $P_i(x)$, we give the equivalent classes of $P_{i+1}(x)$ according to the degree of $G_{1,i+1}(x), G_{2,i+1}(x), \cdots G_{i,i+1}(x)$, where $3 \leq i \leq m+2$. Secondly, we delete the equivalent classes of P_{i+1} that $P_{j_1}, \cdots, P_{j_{i-1}}, P_{i+1}$ doesn't exist from the above discussion, where $1 \leq j_1 < j_2 < \cdots < j_{i-1} \leq i$. Then we delete the equivalent classes of $P_{i+1}(x)$ which violate Lemma 2.2 and Lemma 2.3.

(7) If the equivalent classes of $P_5(x)$ does not exist, then the upper bound on the number of nontrivial rational solutions for Abel equations (1.2) of type $(m, 2m+1)$ is 5. Based on the concrete forms of $P_1(x), \cdots, P_4(x)$ which exist, determine whether the solution $y = \frac{1}{R(x)}$ exist or not, where the leading coefficient of $R(x)$ is $\frac{j}{m+1}$. If $R(x)$ exists, then the sharp upper bound is 5. Otherwise, it is 4.

(8) If the equivalent classes of $P_5(x)$ exists, then the upper bound on the number of nontrivial rational solutions for Abel equations (1.2) of type $(m, 2m+1)$ is $k+1$, with k is the maximum index such that the equivalent classes of $P_k(x)$ exist. Based on the concrete forms of $P_1(x), \cdots, P_k(x)$ which exist, determine whether the solution $y = \frac{1}{R(x)}$ exists or not, where the leading coefficient of $R(x)$ is $\frac{j}{m+1}$. If $R(x)$ exists, then the sharp upper bound is $k+1$. Otherwise, it is k.

Remark 2.1. For $m \geq 4$, theoretically speaking, the sharp upper bound on the number of nontrivial rational solutions for equations (1.2) of type $(m, 2m+1)$ can be presented by this algorithm, although in practice, the computation can be very heavy and complicated.

3. Application

In this section, we use the above algorithm to prove Theorem 1.2, namely we give the sharp upper bound on the number of nontrivial rational solutions for equations (1.2) of type $(4, 9)$.

Proof of Theorem 1.2. By Lemma 2.1, we know that only in the case $b_{2m+1} = \frac{4}{81}a_m^2$, the equations (1.2) of type $(4, 9)$ can have more than 4 nontrivial rational solutions. Thus we only have to study this case.

Assume there exist a system (1.2) of type $(4, 9)$ having at least five nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \leq i \leq 5$. We use the Pigeonhole principle to give the form of $P_i(x)$. Since there are five nontrivial rational solutions, we have 25 objectives. Moreover, there are 9 boxes $\alpha_1, \cdots, \alpha_9$. By putting objectives into these boxes, we obtain that there is a box with at least three objectives, namely $P_1(x)$, $P_2(x)$ and $P_3(x)$ must have a common factor $(x - \alpha_1)$. Then use the Pigeonhole principle again, we deduce without loss of generality that $P_1(x)$ and $P_2(x)$ have a comma factor $(x - \alpha_2)$.

Set $P_1(x) = \prod\limits_{i=1}^{5}(x - \alpha_i)$, which will be abbreviated as $\{12345\}$. Such kind of abbreviation will be used throughout this paper. We will distinguish three cases according to the degrees of $G_{1,2}(x)$, $G_{1,3}(x)$ and $G_{2,3}(x)$.

Case (I) [*At least two elements of A are of degree 4*]: We can assume without loss of generality that the degree of $G_{1,2}(x)$ and $G_{1,3}(x)$ is 4. Hence we can choose $P_2(x) = \{12346\}$ as a representative. Then $P_3(x)$ have three equivalent classes, namely $P_3(x) = \{12347\}$, or $P_3(x) = \{12356\}$, or $P_3(x) = \{12357\}$. By Lemma 2.3, we obtain that the coefficients of the non-zero term of the smallest degree of $P_i(x)$ have at most two values, where $1 \leq i \leq 3$. Hence $P_3(x) = \{12357\}$.

When $P_3(x) = \{12357\}$, by Lemma 2.3, we deduce that $\alpha_2\alpha_3\alpha_4\alpha_6 = \alpha_2\alpha_3\alpha_5\alpha_7$. Then by submitting $P_1(x)$ and $P_2(x)$, $P_2(x)$ and $P_3(x)$ into the Lemma 2.2 respectively, we obtain the following equations,

$$
\begin{aligned}
&3(\alpha_2 + \alpha_3 + \alpha_4) = 4(\alpha_7 + \alpha_8 + \alpha_9), \quad \prod_{2 \leq i < j \leq 4} \alpha_i\alpha_j = 2 \prod_{7 \leq i < j \leq 9} \alpha_i\alpha_j, \\
&\alpha_2\alpha_3\alpha_4 = 4\alpha_7\alpha_8\alpha_9, \quad 3(\alpha_2 + \alpha_3) = 4(\alpha_8 + \alpha_9), \\
&2\alpha_2\alpha_3 = 4\alpha_8\alpha_9, \quad 0 = \alpha_4\alpha_6 - \alpha_5\alpha_7.
\end{aligned}
$$

Hence we have

$$
\alpha_2 = \alpha_3 = \alpha_8 = \alpha_9 = 0, \qquad \alpha_4 = \frac{4}{3}\alpha_7, \qquad \alpha_5 = \frac{4}{3}\alpha_6.
$$

Thus the nontrivial rational solution $y = \frac{1}{P_3(x)}$ exists.

Now we go on with the study of the existence of the nontrivial rational solution $y = \frac{1}{P_4(x)}$ in this case. Firstly, by Lemma 2.3, we deduce that the constant term of $P_4(x)$ is not zero. Thus the nontrivial rational solution $y = \frac{1}{P_4(x)}$ doesn't exist in this case. As a consequence, there are at most three nontrivial rational solutions $y = \frac{1}{P_i(x)}$, where the leading coefficient of $P_i(x)$ is 1, $1 \le i \le 3$.

There are two other cases **(II)** and **(III)** described below. Since the structure and techniques involved in the remaining two cases are similar to the discussion of the case **(I)**, we only give a list of all equivalent classes of $P_3(x)$ and $P_4(x)$ and point out the existing cases in the following proof.

Case (II) [*One element of A is of degree 4*]: We can assume without loss of generality that the degree of $G_{1,2}(x)$ is 4. We choose $P_2(x) = \{12346\}$ as a representative.

If the degree of $G_{1,3}(x)$ is 3, then $P_3(x)$ can have at most three equivalent classes. Namely $P_3(x) \in B_1$, where

$$B_1 = \{\{12378\}, \{12567\}, \{12578\}\}.$$

If the degree of $G_{1,3}(x)$ is 2, then $P_3(x)$ can also have at most three equivalent classes. Namely $P_3(x) \in B_2$, where

$$B_2 = \{\{12789\}, \{15678\}, \{15789\}\}.$$

Finally, if the degree of $G_{1,3}(x)$ is 1, then we can choose $P_3(x) = \{56789\}$ as a representative.

Below we shall consider the above 7 cases one by one. In other words, the proof is done with a case by case study on the form of $P_3(x)$.

(**II-i**) $P_3(x) = \{12378\}$. Firstly if the degree of $G_{1,4}(x)$ is 3, then $P_4(x)$ have the following equivalent classes, $P_4(x) \in D_1$, where

$$D_1 = \left\{ \begin{array}{l} \{12379\}, \{12478\}, \{12479\}, \{12567\}, \{12569\}, \{12578\}, \\ \{12579\}, \{14567\}, \{14569\}, \{14578\}, \{14579\}. \end{array} \right\}$$

Secondly, if the degree of $G_{1,4}(x)$ is 2, then $P_4(x)$ have the following equivalent classes, $P_4(x) \in D_2$, where

$$D_2 = \left\{ \begin{array}{l} \{12789\}, \{14789\}, \{15678\}, \{15789\}, \\ \{15679\}, \{45678\}, \{45679\}, \{45789\}. \end{array} \right\}$$

Finally if the degree of $G_{1,4}(x)$ is 1, then we choose $P_4(x) = \{56789\}$ as a representative.

Examining all these cases, we get that $P_4(x)$ can only be $\{12567\}$ and $\{15679\}$. Moreover, when $P_4(x) = \{12567\}$, By Lemma 2.2, we get that $\alpha_2 = \alpha_9 = 0$, namely $\{12567\} = \{15679\}$. Hence there are at most four nontrivial rational solutions $y = \frac{1}{P_i(x)}$, where the leading coefficient of $P_i(x)$ is 1, $1 \le i \le 4$.

(**II-ii**) $P_3(x) = \{12567\}$. Firstly, if the degree of $G_{1,4}(x)$ is 3, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_3$, where

$$D_3 = \left\{ \begin{array}{l} \{12568\}, \{12578\}, \{12589\}, \{13567\}, \{13568\}, \{13578\}, \\ \{13589\}, \{34567\}, \{34568\}, \{34578\}, \{34589\}. \end{array} \right\}$$

If the degree of $G_{1,4}(x)$ is 2, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_4$, where

$$D_4 = \left\{ \begin{array}{l} \{12789\}, \{13789\}, \{15678\}, \{15689\}, \{15789\}, \\ \{34789\}, \{35678\}, \{35689\}, \{35789\}. \end{array} \right\}$$

Finally if the degree of $G_{1,4}(x)$ is 1, then we choose $P_4(x) = \{56789\}$ as a representative.

Examining all these cases, we get that $P_4(x)$ can only be $\{13789\}$. Hence there are at most four nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \leq i \leq 3$.

(**II-iii**) $P_3(x) = \{12578\}$. If the degree of $G_{1,4}(x)$ is 3, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_5$, where

$$D_5 = \left\{ \begin{array}{l} \{12579\}, \{13578\}, \{13579\}, \{34567\}, \\ \{34569\}, \{34578\}, \{34579\}. \end{array} \right\}$$

If the degree of $G_{1,4}(x)$ is 2, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_6$, where

$$D_6 = \left\{ \begin{array}{l} \{12789\}, \{13789\}, \{15678\}, \{15679\}, \{15789\}, \\ \{34789\}, \{35678\}, \{35679\}, \{35789\}. \end{array} \right\}$$

Finally if the degree of $G_{1,4}(x)$ is 1, then we choose $P_4(x) = \{56789\}$ as a representative.

Examining all these cases, we get that the nontrivial rational solution $y = \frac{1}{P_4(x)}$ doesn't exist. Hence there are at most three nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \leq i \leq 3$.

(**II-iv**) $P_3(x) = \{12789\}$. In this case, the degree of $G_{1,4}(x)$ can be 2, or 1. Firstly, if the degree of $G_{1,4}(x)$ is 2, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_7$, where

$$D_7 = \left\{ \begin{array}{l} \{13789\}, \{15678\}, \{15789\}, \\ \{34789\}, \{35678\}, \{35789\}. \end{array} \right\}$$

If the degree of $G_{1,4}(x)$ is 1, then we choose $P_4(x) = \{56789\}$ as a representative.

Examining all these cases, we get that $P_4(x)$ can only be $\{15678\}$ and $\{35678\}$. Moreover, if $P_4(x) = \{15678\}$, or $P_4(x) = \{35678\}$, by Lemma 2.2, we get two necessary systems of equations. From the solutions of these two systems of equations, we know that $\{15678\} = \{35678\}$. Hence there are at most four nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \leq i \leq 4$.

(**II-v**) $P_3(x) = \{15678\}$. In this case, the degree of $G_{1,4}(x)$ can be 2, or 1. Firstly, if the degree of $G_{1,4}(x)$ is 2, then $P_4(x)$ have the following equivalent classes, namely

$P_4(x) \in D_8$, where

$$D_8 = \left\{ \begin{array}{l} \{15679\}, \{15789\}, \{25678\}, \\ \{25679\}, \{25789\}. \end{array} \right\}$$

If the degree of $G_{1,4}(x)$ is 1, then we choose $P_4(x) = \{56789\}$ as a representative.

Examining all these cases, we get that the nontrivial rational solution $y = \frac{1}{P_4(x)}$ doesn't exist. Hence there are at most three nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \leq i \leq 3$.

(**II-vi**) $P_3(x) = \{15789\}$. In this case, the degree of $G_{1,4}(x)$ can be 2, or 1. Then if the degree of $G_{1,4}(x)$ is 2, we choose $P_4(x) = \{25789\}$ as a representative.

If the degree of $G_{1,4}(x)$ is 1, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_9$, where

$$D_9 = \{\{16789\}, \{26789\}, \{56789\}\}.$$

Examining all these cases, we get that the nontrivial rational solution $y = \frac{1}{P_4(x)}$ doesn't exist. Hence there are at most three nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \leq i \leq 3$.

(**II-vii**) $P_3(x) = \{56789\}$. In this case, all possible forms of $P_4(x)$ have been discussed. Hence there are at most four nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \leq i \leq 4$.

Case (III) [*None of A is of the degree 4*]: We can assume without loss of generality that the degree of $G_{1,2}(x)$ is 3. Hence we choose $P_2(x) = \{12367\}$ as a representative.

If the degree of $G_{1,3}(x)$ is 3, then $P_3(x)$ have the following equivalent classes, namely $P_3(x) \in B_3$, where

$$B_3 = \left\{ \begin{array}{l} \{12389\}, \{12468\}, \{12489\}, \\ \{14567\}, \{14568\}, \{14589\}. \end{array} \right\}$$

If the degree of $G_{1,3}(x)$ is 2, then $P_3(x)$ have the following equivalent classes, namely $P_3(x) \in B_4$, where

$$B_4 = \{\{14689\}, \{45678\}, \{45689\}\}.$$

Finally if the degree of $G_{1,3}(x)$ is 1, then all possible forms of $P_4(x)$ have been discussed in the above cases.

On the other hand, if the degree of $G_{1,2}(x)$ is 2, then we choose $P_2(x) = \{12678\}$ as a representative, and $P_3(x)$ can only have one equivalent class, namely $P_3(x) = \{34679\}$. In this case, all possible forms of $P_4(x)$ have been discussed in the above cases. Furthermore, if the degree of $G_{1,2}(x)$ is 1, then all possible forms of $P_3(x)$ have been discussed in the above cases. Hence we only have to consider the case which the degree of $G_{1,2}(x)$ is 3. The proof is done with a case by case study on the form of $P_3(x)$.

(**Ⅲ-i**) $P_3(x) = \{12389\}$. Firstly, if the degree of $G_{1,4}(x)$ is 3, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_{10}$, where

$$D_{10} = \{\{12468\}, \{14567\}, \{14568\}\}.$$

Secondly, if the degree of $G_{1,4}(x)$ is 2, then we choose $P_4(x) = \{45678\}$ as a representative. Finally, if the degree of $G_{1,4}(x)$ is 1, then all possible forms of $P_4(x)$ have been discussed in the above cases.

Examining all these cases, we obtain that the nontrivial rational solution $y = \frac{1}{P_4(x)}$ doesn't exist. Hence there are at most three nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 3$.

(**Ⅲ-ii**) $P_3(x) = \{12468\}$. Firstly if the degree of $G_{1,4}(x)$ is 3, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_{11}$, where

$$D_{11} = \left\{ \begin{array}{l} \{12578\},\ \{12589\}, \{13469\}, \{13569\}, \{13578\}, \{13579\}, \\ \{13589\}, \{14567\}, \{34567\}, \{34569\}, \{34578\}, \{34579\}. \end{array} \right\}$$

Secondly, if the degree of $G_{1,4}(x)$ is 2, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_{12}$, where

$$D_{12} = \{\{15789\}, \{35689\},\{35789\}\}.$$

Finally if the degree of $G_{1,4}(x)$ is 1, then all possible forms of $P_4(x)$ have been discussed in the above cases.

Examining all these cases, we get that $P_4(x)$ can be $\{12589\}$, or $\{13589\}$. But these two solutions cannot exist at the same time, because $\forall \sigma \in S_9$, changing $\alpha_k \to \alpha_{\sigma(k)}$ with $1 \le k \le 9$ we cannot change the solutions of the necessary system of equations which derive from $P_4(x) = \{12589\}$ into the solution of the necessary system of equations which derive from $P_4(x) = \{13589\}$. Hence there are at most four nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 4$.

(**Ⅲ-iii**) $P_3(x) = \{12489\}$. Firstly, if the degree of $G_{1,4}(x)$ is 3, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_{13}$, where

$$D_{13} = \{\{14567\}, \{34567\}, \{34568\}\}.$$

Secondly, if the degree of $G_{1,4}(x)$ is 2, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_{14}$, where

$$D_{14} = \left\{ \begin{array}{l} \{14678\},\ \{15678\},\ \{34678\}, \\ \{35678\},\ \{35689\}. \end{array} \right\}$$

Finally if the degree of $G_{1,4}(x)$ is 1, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_{15}$, where

$$D_{15} = \{\{16789\}, \{36789\}, \{56789\}\}.$$

Examining all these cases, we get that $P_4(x)$ exists when $P_4(x) \in D_{16}$, where

$$D_{16} = \{\{14567\}, \{14678\}, \{15678\}, \{35689\}\}.$$

Since $\forall\sigma \in S_9$, changing $\alpha_k \to \alpha_{\sigma(k)}$ with $1 \le k \le 9$ we can't change the solutions of the necessary system of equations which derive from $P_4(x) = a$, with $a \in D_{16}$ into the solution of the necessary system of equations which derive from $P_4(x) = b$, with $b \in D_{16} - \{a\}$, we obtain that there are at most four nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 4$.

(**Ⅲ-iv**) $P_3(x) = \{14567\}$. In this case, the degree of $G_{1,4}(x)$ can only be 2, then we choose $P_4(x) = \{24689\}$ as a representative. Examining this case, we get that the nontrivial rational solution $y = \frac{1}{P_4(x)}$ doesn't exist. Hence there are at most three nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 3$.

(**Ⅲ-v**) $P_3(x) = \{14568\}$. In this case, the degree of $G_{1,4}(x)$ can be 2, or 1. If the degree of $G_{1,4}(x)$ is 2, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_{17}$, where

$$D_{17} = \{\{24678\}, \{24679\}, \{24789\}\}.$$

If the degree of $G_{1,4}(x)$ is 1, then $P_4(x)$ can have at most two equivalent classes, namely $P_4(x) = \{16789\}$, or $P_4(x) = \{26789\}$.

Examining all these cases, we get that $P_4(x)$ exists when $P_4(x) \in D_{18}$, where

$$D_{18} = \{\{24678\}, \{24679\}, \{24789\}, \{26789\}\}.$$

Since $\forall\sigma \in S_9$, changing $\alpha_k \to \alpha_{\sigma(k)}$ with $1 \le k \le 9$ we cannot change the solutions of the necessary system of equations which derive from $P_4(x) = c$, with $c \in D_{18}$ into the solution of the necessary system of equations which derive from $P_4(x) = d$, with $d \in D_{18} - \{c\}$, we obtain that there are at most four nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 4$.

(**Ⅲ-vi**) $P_3(x) = \{14589\}$. In this case, the degree of $G_{1,4}(x)$ can only be 1, then $P_4(x)$ can have at most two equivalent classes, namely $P_4(x) = \{16789\}$, or $P_4(x) = \{26789\}$.

Examining all these cases, we get that the nontrivial rational solution $y = \frac{1}{P_4(x)}$ doesn't exist. Hence there are at most three nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 3$.

(**Ⅲ-vii**) $P_3(x) = \{14689\}$. In this case, the degree of $G_{1,4}(x)$ can only be 2, then $P_4(x)$ have the following equivalent classes, namely $P_4(x) \in D_{19}$, where

$$D_{19} = \left\{ \begin{matrix} \{15789\}, \{24789\}, \{25789\}, \\ \{45678\}, \{45789\}. \end{matrix} \right\}$$

Examining all these cases, we get that $P_4(x)$ exists when $P_4(x) \in D_{20}$, where

$$D_{20} = \{\{24789\}, \{45678\}, \{45789\}\}.$$

Since $\forall\sigma \in S_9$, changing $\alpha_k \to \alpha_{\sigma(k)}$ with $1 \le k \le 9$ we cannot change the solutions of the necessary system of equations which derive from $P_4(x) = e$, with $e \in D_{20}$ into the solution of the necessary system of equations which derive from $P_4(x) = f$, with $f \in D_{20} - \{e\}$, we obtain that there are at most four nontrivial rational solutions $y = \frac{1}{P_i(x)}$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 4$.

(**Ⅲ-viii**) $P_3(x) = \{45678\}$ or $P_3(x) = \{45689\}$. In these cases, all possible forms of $P_4(x)$ have been discussed in the above cases.

From the above discussion, we obtain that the equivalent classes of $P_5(x)$ does not exist. Hence the upper bound on the number of nontrivial rational solutions for Abel equations (1.2) of type $(4, 9)$ is 5.

On the other hand, basing on the concrete forms of $P_1(x), P_2(x), P_3(x), P_4(x)$, we obtain that only in the case (Ⅲ-v), with the representative of $P_4(x)$ is $\{24789\}$, there exists an equation (1.2) of type $(4, 9)$ having five nontrivial rational solutions, namely $y = \frac{1}{R(x)}$ is also a nontrivial rational solution of this equation, where the leading coefficient of $R(x)$ is $\frac{4}{5}$. (The explicit example for equation (1.2) of type $(4, 9)$ having exactly 5 nontrivial rational solutions can be see in Theorem 1.5 in [18].) Thus the theorem follows.

Acknowledgments

X. Qian is supported by Doctor of Entrepreneurship and Innovation Funds of Jiangsu (No. JSSCBS20210617), and PhD research startup foundation of Jinling Institute of Technology (Nos. jit-b-202049, jit-fhxm-202116). The authors are supported by National Science Foundation of China (No. NSFC 12071006).

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