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MODELLING AND ANALYSIS OF DYNAMIC SYSTEMS ON TIME-SPACE SCALES AND APPLICATION IN BURGERS EQUATION*

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Abstract As effective tools that can be used to solve both continuous and discrete dynamic systems, symmetry analysis, conserved quantities and Bäcklund transformations of dynamic systems on time-space scales are studied, which unify and generalize the continuous and discrete cases. Applying the method to heat equation and Burgers equation, we get symmetries, group invariant solutions and Bäcklund transformations of the system on time-space scales. The results are applied to approximately simulate motion process of traffic flow with given initial condition. The study of nonlinear systems on time-space scales provides a theoretical basis for revealing the internal physical mechanism of the systems. Applications of the method to other dynamic equations on time-space scales deserve to be further studied.

Keywords Burgers equation on time-space scales, symmetry analysis, conserved quantities, Bäcklund transformation.

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1. Introduction

The concept of time scale was first put forward by Hilger [11], which unifies and extends the continuous and discrete analysis. It not only realizes the study of continuous and discrete equations under a unified mathematical framework, but also provides an effective mathematical tool for the study of complex systems with both continuous and discrete factors. With the wide application and rapid development of the theory, the study of dynamic equations on time scale has raised more and more attention [2,3,8,22].

The first purpose of writing this paper is to give the symmetry analysis, conserved quantities and Bäcklund transformations for dynamic equations with partial delta derivatives on time-space scales. The basic properties, forms and application conditions of calculus theories for continuous or discrete case have changed

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greatly on time-space scales, which poses a great challenge to solve the corresponding problems. As a powerful method which can be applied to study both continuous and discrete systems, symmetry analysis and Bäcklund transformation method provide effective ways to study problems on time-space scales. There have been many study on the symmetry analysis or Bäcklund transformation of continuous or discrete models [1, 7, 9, 15, 20, 23]. However, to the best of our knowledge, research on symmetry analysis of dynamic equations with partial delta derivatives on time-space scales or Bäcklund transformations of dynamic equations on time scale is still new but meaningful in solving practical problems.

The second purpose of writing this paper is to give the precise mathematical description of Burgers equation on time-space scales. Burgers equation is applied widely in modelling traffic flow [18,21,25], sound waves in viscous medium, magnetic current wave with finite conductivity, turbulence problem [19] and multi-agents system [17] as differential or difference systems. However, many complex motions in these models can't be described in single continuous or discrete case. Such as, in models of traffic flow, the traffic density presents piecewise continuous on vehicle passable time scale $\bigcup_k [a_k, b_k]$. In models of multi-agents system, the change process of agent's spatial position with respect to time may not be a simple continuous or discrete process. For systems with variable value range including interval and isolated point set in engineering practice, the time-space scales dynamic system theory provides an effective way to simulate the corresponding models. Hence, unifying and generalizing the problems to the framework of time scales are meaningful. Finding different forms of exact solutions of Burgers equation on time-space scales is of great significance for understanding of physical phenomena described by the equation.

In this paper, we give symmetry analysis, conserved quantities and Bäcklund transformations of dynamic systems on time-space scales. The rest of the paper is organized as follows: In section 2, some related definitions and lemmas are given. In section 3, we investigate prolongation structures, single parameter invariant group and conserved quantities of general dynamic systems with partial Δ-derivative. The symmetry analysis on time-space scales using direct symmetry method and Lie symmetry method are presented in section 4, which are applied to heat equation and Burgers equation on time-space scales, respectively. In section 5, the Bäcklund transformation of heat equation and Burgers equation are derived on time-space scales. The application in approximately simulation of traffic flow model is presented in section 6.

2. Preliminaries

Definition 2.1 ([5], Time scale)**.** A time scale is an arbitrary nonempty closed subset of the real numbers.

Definition 2.2 ([5, 6], Jump operator)**.** Let $\mathbb{T}$ be a time scale.

(1) The forward jump operator on t is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

(2) The forward jump operator on x is defined by

$$\rho(x) := \inf\{s \in \mathbb{T} : s > x\}.$$

(3) The graininess function $\mu, \nu : \mathbb{T} \to [0, \infty)$ are defined by

$$\mu(t) := \sigma(t) - t,\ \ \nu(x) = \rho(x) - x.$$

(4) We set $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t, and $\rho(x) = x$ if $\mathbb{T}$ has a maximum x.

Definition 2.3 ([14], Partial Δ-derivative)**.** Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t = (t_1, t_2, ..., t_i, ..., t_n) \in (\mathbb{T}^\kappa)^n$. Then define $f^{\Delta_{t_i}}$ to be the number(provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t_i, with $U = (t_i - \delta, t_i + \delta) \bigcap \mathbb{T}_i$ for $\delta > 0$ such that

$$\left|(f^{\sigma_i}(t) - f_i^s(t)) - f^{\Delta_{t_i}}(t)(\sigma_i(t) - s)\right| \leq \varepsilon \left|\sigma_i(t) - s\right|,\ \text{for all}\ s \in U.$$

$f^{\Delta_{t_i}}$ is called partial Δ-derivative of f at t with respect to t_i.

Definition 2.4 ([5], Regressive and exponential functions on time scale)**.**

(1) A function $p : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(t)p(t) \neq 0$ holds for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

(2) If $p \in \mathcal{R}$, define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)\ \text{for}\ s, t \in \mathbb{T},$$

where $\xi_h(z) = \frac{1}{h} Log(1 + zh)$. *Log* is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

Lemma 2.1 ([5])**.** *If $p \in \mathcal{R}$, then*

$$(e_p(t, s))^\Delta = p(t)e_p(t, s). \tag{2.1}$$

Proof. Using Theorem 2.62 in [5], (2.1) can be easily derived. □

Lemma 2.2 ([5], Chain rule on time scale)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differential and $g : \mathbb{T} \to \mathbb{R}$ be Δ-differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable and*

$$(f \circ g)^\Delta(x) = \left[\int_0^1 f'(g(x) + h\mu(x)g^\Delta(x))dh\right] g^\Delta(x).$$

Referring to classical total derivative and chain rule on time scales, the total Δ-derivative on time-space scales can be expressed as

Definition 2.5 (Total Δ-derivative on time-space scales)**.** The total Δ-derivative operator of $F(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(n)})$ with respect to x_i is defined by

$$D_i F = F^{\Delta_{x_i}} + (F \circ u)^{\Delta_{x_i}} + (F \circ u^{(1)})^{\Delta_{x_i}} + \ldots + (F \circ u^{(n)})^{\Delta_{x_i}} = F^{\Delta_{x_i}} + \sum_{j=0}^n (F \circ u^{(j)})^{\Delta_{x_i}},$$

where $(F \circ u^{(j)})^{\Delta_{x_i}} = \left[\int_0^1 (\partial F / \partial(u^{(j)} + h\mu(x_i)(u^{(j)})^{\Delta_{x_i}}))dh\right] (u^{(j)})^{\Delta_{x_i}}$.

Lemma 2.3 ([5], Leibnitz formula on time scale)**.** *Let $S_j^{(k)}$ be the set consisting of all possible strings of length k, containing j times σ and $k-j$ times Δ. If f^{Λ} exists for all $\Lambda \in S_j^{(k)}$, then*

$$(fg)^{\Delta^k} = \sum_{j=0}^{k} \Big(\sum_{\Lambda \in S_j^{(k)}} f^{\Lambda} \Big) g^{\Delta^k}$$

holds for all $k \in \mathbb{N}$, where f^{Λ} denotes all possible permutations of j times σ and $k-j$ times Δ acting on f.

Define T as shift operator and denote

$$T_t u := u(\sigma(t), x),\ T_x u := u(t, \rho(x)),\ T_t^2 u := u(\sigma^2(t), x),\ T_x^2 u := u(t, \rho^2(x)).$$

Lemma 2.4. *If $v(t,x)$, $\mu(t)$, $\nu(x)$ are Δ-differential and $\Delta_x(\mu(t)) = \Delta_t(\nu(x)) = 0$, then*

$$\Delta_x(\Delta_t v) = \Delta_t(\Delta_x v),\ T_x(\Delta_t v) = \Delta_t(T_x v), \tag{2.2}$$

$$\Delta_x(T_x v) = (1 + \Delta_x(\nu(x)))T_x(\Delta_x v). \tag{2.3}$$

Proof. Similar to proof of Lemma 2.1 in [12] for ∇-differentiable case, (2.2) can be easily obtained. Since $T_x v = v + \nu(x)\Delta_x v$, we have $T_x(\Delta_x v) = \Delta_x v + \nu(x)\Delta_x^2 v$ and

$$\begin{aligned}\Delta_x(T_x v) = \Delta_x(v + \nu(x)\Delta_x v) &= \Delta_x v + \nu(x)\Delta_x^2 v + \Delta_x(\nu(x))T_x(\Delta_x v)\\ &= (1 + \Delta_x(\nu(x)))T_x(\Delta_x v),\end{aligned}$$

which means (2.3) holds. □

For more basic properties on time scale, we can refer to Theorem 1.16 and Theorem 1.20 in Chapter 1 by Bohner et al. in [5].

3. Symmetry analysis and conserved quantities of dynamic systems on time-space scales

3.1. Prolongation structures of dynamic systems with partial Δ-derivative

Referring to prolongation in symmetry analysis for partial differential and partial difference equations [4, 16], the prolongation structures of nonlinear systems with partial Δ-derivative on time-space scales can be expressed as follows.

Theorem 3.1. *Consider dynamic system with partial Δ-derivative on time-space scales*

$$\Delta_{x_1} u = F(x, u, ..., u^{(p)}, ..., u^{(n)}, T_x u, ..., T_x u^{(p)}, T_x u^{(n)}), \tag{3.1}$$

where $x = (x_1, x_2, ..., x_m) \in (\mathbb{T}^{\kappa})^m$, $x_1 = t$, $u^{(p)} = u_{i_1 i_2 \dots i_p} = \frac{\Delta^p u}{\Delta x_{i_1} \Delta x_{i_2} \dots \Delta x_{i_p}}$, $i_p \in \{1, 2, ..., m\}$ for $p = 1, 2, ..., n$ corresponding to all pth-order partial Δ-derivative of u with respect to x. $T_x u = (T_{x_1} u, T_{x_2} u, ..., T_{x_m} u)$, $T_x u^{(p)} = (T_{x_1} u^{(p)}, T_{x_2} u^{(p)}, ..., T_{x_m} u^{(p)})$. Let

$$\underline{V} = \sum_{i=1}^{m} \xi_i(x, u, T_x u) \frac{\Delta}{\Delta x_i} + \phi(x, u, T_x u) \frac{\partial}{\partial u} + \sum_{i=1}^{m} T_{x_i} \phi(x, u, T_x u) \frac{\partial}{\partial (T_{x_i} u)} \tag{3.2}$$

be a vector defined on $(\mathbb{T}^{\kappa})^m \times \mathbb{R}$*, then the corresponding kth-extended infinitesimal generator*

$$\begin{aligned}
&\mathrm{Pr}^{(n)}\underline{V}\\
=&\underline{V} + \sum_{i=1}^{m} \phi_i^{(1)}(x,u,u^{(1)},T_x u, T_x u^{(1)})\frac{\partial}{\partial u^{\Delta_{x_i}}} + ...\\
&+\sum_{k=2}^{n}\sum_{i_1,...,i_k=1}^{m} \phi_{i_1...i_k}^{(k)}(x,u,u^{(1)},...,u^{(k)},T_x u, T_x u^{(1)},...,T_x u^{(k)})\frac{\partial}{\partial(u_{i_1...i_k})}\\
&+\sum_{q=1}^{m}\sum_{i=1}^{m}(T_{x_q}\phi)_i^{(1)}(x,u,u^{(1)},T_x u, T_x u^{(1)})\frac{\partial}{\partial (T_{x_q}u)^{\Delta_{x_i}}} + ...\\
&+\sum_{q=1}^{m}\sum_{k=2}^{n}\sum_{i_1,...,i_k=1}^{m}(T_{x_q}\phi)_{i_1...i_k}^{(k)}(x,u,u^{(1)},...,u^{(k)},T_x u, T_x u^{(1)},...,T_x u^{(k)})\frac{\partial}{\partial((T_{x_q}u)_{i_1...i_k})},
\end{aligned}$$

where

$$\phi_i^{(1)} = D_i\phi - \sum_{j=1}^{m} u_j D_i \xi_j, \tag{3.3}$$

$$\phi_{i_1...i_k}^{(k)} = D_{i_k}\phi_{i_1...i_{k-1}}^{(k-1)} - \sum_{j=1}^{m} u_{i_1...i_{k-1}j} D_{i_k}\xi_j, \tag{3.4}$$

$$\left(T_{x_q}\phi\right)_i^{(1)} = D_i\left(T_{x_q}\phi\right) - \sum_{j=1}^{m}\left(T_{x_q}u\right)_j D_i\xi_j, \tag{3.5}$$

$$\left(T_{x_q}\phi\right)_{i_1...i_k}^{(k)} = D_{i_k}\left(T_{x_q}\phi\right)_{i_1...i_{k-1}}^{(k-1)} - \sum_{j=1}^{m}\left(T_{x_q}u\right)_{i_1...i_{k-1}j} D_{i_k}\xi_j, \tag{3.6}$$

$i,j,q,i_l = 1,...,m, l = 1,...,k$ *for* $k = 2,3,....$ *and* $\phi^{(0)} = \phi(x,u)$*. I.e.,*

$$\phi_i^{(1)} = D_i\phi - \sum_{j=1}^{m} u_j D_i\xi_j, \tag{3.7}$$

$$\phi_{i_1...i_k}^{(k)} = D_{i_1...i_k}\phi - \sum_{j=1}^{m}\sum_{l=1}^{k}(\sum_{\Lambda\in S_{l-1}^{(k-1)}} u_j^{\Lambda})(\xi_j)^{(l)}, \tag{3.8}$$

$$\left(T_{x_q}\phi\right)_i^{(1)} = D_i\left(T_{x_q}\phi\right) - \sum_{j=1}^{m}\left(T_{x_q}u\right)_j D_i\xi_j, \tag{3.9}$$

$$\left(T_{x_q}\phi\right)_{i_1...i_k}^{(k)} = D_{i_1...i_k}\left(T_{x_q}\phi\right) - \sum_{j=1}^{m}\sum_{l=1}^{k}(\sum_{\Lambda\in S_{l-1}^{(k-1)}}\left(T_{x_q}u\right)_j^{\Lambda})(\xi_j)^{(l)}, \tag{3.10}$$

where $(\xi_j)^{(l)}$ *denotes all* l *times total* Δ*-derivative acting on* ξ_j*,* u^{Λ} *denotes all possible permutations of* $l-1$ *times corresponding* σ *and the rest* $(k-l)$ *times* Δ*-derivative acting on* u*. For example, if the* l *times* Δ*-derivative acting on* ξ_j *is* $(\xi_j)_{i_1...i_l}$*, then* $l-1$ *times corresponding* σ *and the rest* $(k-l)$ *times* Δ*-derivative acting on* u *are* $\left(u(\sigma(x_{i_1}),...,\sigma(x_{i_{l-1}}),x_{i_l},x_{i_{l+1}},...,x_{i_m}\right)_{i_{l+1},...,i_k}$*.*

Proof. The one-parameter Lie group of infinitesimal transformations are

$$\begin{aligned}
&(x^*)_i = X_i(x,u,T_xu;\varepsilon) = x_i + \varepsilon\xi_i(x,u,T_xu) + O(\varepsilon^2),\\
&u^* = U(x,u,T_xu;\varepsilon) = u + \varepsilon\phi(x,u,T_xu) + O(\varepsilon^2),\\
&(T_{x_q}u)^* = T_{x_q}U(x,u,T_xu;\varepsilon) = T_{x_q}u + \varepsilon T_{x_q}\phi(x,u,T_xu) + O(\varepsilon^2),
\end{aligned} \tag{3.11}$$

with corresponding infinitesimal generator

$$\underset{\sim}{V} = \sum_{i=1}^{m}\xi_i(x,u,T_xu)\frac{\Delta}{\Delta x_i} + \phi(x,u,T_xu)\frac{\partial}{\partial u} + \sum_{i=1}^{m}T_{x_i}\phi(x,u)\frac{\partial}{\partial T_{x_i}u}. \tag{3.12}$$

The k-th extensions of one-parameter Lie group of point transformations are given by

$$\begin{aligned}
&(u^*)_i = U_1(x,u,u^{(1)},T_xu,T_xu^{(1)};\varepsilon) = u_i + \varepsilon\phi_i^{(1)} + O(\varepsilon^2),\\
&\vdots\\
&(u^*)_{i_1\ldots i_k} = U_k(x,u,u^{(1)},\ldots,u^{(k)},T_xu,T_xu^{(1)},\ldots,T_xu^{(k)};\varepsilon) = u_{i_1\ldots i_k} + \varepsilon\phi^{(k)}_{i_1\ldots i_k} + O(\varepsilon^2),\\
&\vdots\\
&((T_{x_q}u)^*)_i = \left(T_{x_q}U\right)_1(x,u,u^{(1)},T_xu,T_xu^{(1)};\varepsilon)\\
&\qquad = \left(T_{x_q}u\right)_i + \varepsilon\left(T_{x_q}\phi\right)_i^{(1)} + O(\varepsilon^2),\\
&((T_{x_q}u)^*)_{i_1\ldots i_k} = \left(T_{x_q}U\right)_k(x,u,u^{(1)},\ldots,u^{(k)},T_xu,T_xu^{(1)},\ldots,T_xu^{(k)};\varepsilon)\\
&\qquad = \left(T_{x_q}u\right)_{i_1\ldots i_k} + \varepsilon\left(T_{x_q}\phi\right)^{(k)}_{i_1\ldots i_k} + O(\varepsilon^2),
\end{aligned}$$

for $k = 2, 3\ldots, n$. Then

$$(U_1)^i = \frac{DU}{D_iX} = \frac{D[u+\varepsilon\phi+O(\varepsilon^2)]}{D_i[x+\varepsilon\xi+O(\varepsilon^2)]} = \frac{Du+\varepsilon D\phi}{D_ix+\varepsilon D_i\xi} + O(\varepsilon^2), \tag{3.13}$$

where

$$\begin{aligned}
&Du = (u_1,\ldots,u_m),\ D\phi = (D_1\phi,\ldots,D_m\phi),\\
&D_ix = (0,\ldots,\underbrace{1}_{i-th},0,\ldots0),\ D_i\xi = (D_i\xi_1,\ldots,D_i\xi_m).
\end{aligned}$$

It follows from (3.13) that

$$(U_1)^i = u_i + \varepsilon[D_i\phi - \sum_{j=1}^{m}u_jD_i\xi_j] + O(\varepsilon^2) = u_i + \varepsilon\phi_i^{(1)} + O(\varepsilon^2).$$

Using Taylor expansion at $\varepsilon = 0$, we have

$$\begin{aligned}
(U_k)^i &= \frac{DU_{k-1}}{D_{i_k}X} = \frac{D[u_{i_1\ldots i_{k-1}} + \varepsilon\phi^{(k-1)}_{i_1\ldots i_{k-1}} + O(\varepsilon^2)]}{D_{i_k}[x+\varepsilon\xi+O(\varepsilon^2)]}\\
&= u_{i_1\ldots i_k} + \varepsilon[D_{i_k}\phi^{(k-1)}_{i_1\ldots i_{k-1}} - \sum_{j=1}^{m}u_{i_1\ldots i_{k-1}j}D_{i_k}\xi_j] + O(\varepsilon^2)
\end{aligned}$$

$$= u_{i_1\ldots i_k} + \varepsilon\phi^{(k)}_{i_1\ldots i_k} + O(\varepsilon^2),$$

which means

$$\phi_i^{(1)} = D_i\phi - \sum_{j=1}^{m} u_j D_i\xi_j,$$

$$\phi^{(k)}_{i_1\ldots i_k} = D_{i_k}\phi^{(k-1)}_{i_1\ldots i_{k-1}} - \sum_{j=1}^{m} u_{i_1\ldots i_{k-1}j} D_{i_k}\xi_j.$$

Similarly,

$$\left(T_{x_q}\phi\right)_i^{(1)} = D_i\left(T_{x_q}\phi\right) - \sum_{j=1}^{m}\left(T_{x_q}u\right)_j D_i\xi_j,$$

$$\left(T_{x_q}\phi\right)^{(k)}_{i_1\ldots i_k} = D_{i_k}\left(T_{x_q}\phi\right)^{(k-1)}_{i_1\ldots i_{k-1}} - \sum_{j=1}^{m}\left(T_{x_q}u\right)_{i_1\ldots i_{k-1}j} D_{i_k}\xi_j.$$

Using Lemma 2.3 and mathematical induction, we can obtain

$$\phi^{(k)}_{i_1\ldots i_k} = D_{i_1\ldots i_k}\phi - \sum_{j=1}^{m}\sum_{l=1}^{k}\Big(\sum_{\Lambda\in S^{(k-1)}_{l-1}} u_j^{\Lambda}\Big)(\xi_j)^{(l)}. \tag{3.14}$$

Actually, for $k=1$, $i_1 = i$,

$$\phi_i^{(1)} = D_i\phi - \sum_{j=1}^{m} u_j D_i\xi_j \tag{3.15}$$

holds. If

$$\phi^{(k)}_{i_1\ldots i_k} = D_{i_1\ldots i_k}\phi - \sum_{j=1}^{m}\sum_{l=1}^{k}\Big(\sum_{\Lambda\in S^{(k-1)}_{l-1}} u_j^{\Lambda}\Big)(\xi_j)^{(l)} \tag{3.16}$$

holds for k, we prove it also holds for $k+1$.

$$\begin{aligned}
\phi^{(k+1)}_{i_1\ldots i_{k+1}} &= D_{i_{k+1}}\phi^{(k)}_{i_1\ldots i_k} - \sum_{j=1}^{m} u_{i_1\ldots i_k j} D_{i_{k+1}}\xi_j \\
&= D_{i_{k+1}}\Big(D_{i_1\ldots i_k}\phi - \sum_{j=1}^{m}\sum_{l=1}^{k}\Big(\sum_{\Lambda\in S^{(k-1)}_{l-1}} u_j^{\Lambda}\Big)(\xi_j)^{(l)}\Big) - \sum_{j=1}^{m} u_{i_1\ldots i_k j} D_{i_{k+1}}\xi_j \\
&= D_{i_1\ldots i_{k+1}}\phi - \sum_{j=1}^{m} u_{i_1\ldots i_k j} D_{i_{k+1}}\xi_j \\
&\quad - \sum_{j=1}^{m}\sum_{l=1}^{k}\Big[\Big(D_{i_{k+1}}\Big(\sum_{\Lambda\in S^{(k-1)}_{l-1}} u_j^{\Lambda}\Big)\Big)(\xi_j)^{(l)} + \Big(\sum_{\Lambda\in S^{(k-1)}_{l-1}} u_j^{\Lambda}\Big)^{\sigma(x^{i_{k+1}})} D_{i_{k+1}}(\xi_j)^{(l)}\Big] \\
&= D_{i_1\ldots i_{k+1}}\phi - \sum_{j=1}^{m}\Big(\sum_{\Lambda\in S^{(k)}_0} u_j^{\Lambda}\Big)(\xi_j)^{(1)} - \sum_{j=1}^{m}\sum_{l=2}^{k+1}\Big(\sum_{\Lambda\in S^{(k)}_{l-1}} u_j^{\Lambda}\Big)(\xi_j)^{(l)}
\end{aligned}$$

$$= D_{i_1\ldots i_{k+1}}\phi - \sum_{j=1}^{m}\sum_{l=1}^{k+1}\Big(\sum_{\Lambda\in S_{l-1}^{(k)}} u_j^{\Lambda}\Big)(\xi_j)^{(l)}. \tag{3.17}$$

Then (3.8) holds. Similarly,

$$\left(T_{x_q}\phi\right)_{i_1\ldots i_k}^{(k)} = D_{i_1\ldots i_k}\left(T_{x_q}\phi\right) - \sum_{j=1}^{m}\sum_{l=1}^{k}\Big(\sum_{\Lambda\in S_{l-1}^{(k-1)}} \left(T_{x_q}u\right)_j^{\Lambda}\Big)(\xi_j)^{(l)}. \tag{3.18}$$

□

3.2. Single parameter transformation groups and invariant solutions

Let M be set of all Δ-differentiable functions $u(t,x)$, and $G = \{g_\varepsilon | \varepsilon \in \mathbb{R}\}$ be single parameter transformation group acting on M, where

$$g_\varepsilon : u \to \bar{u} = \bar{u}(u,\varepsilon) = u + \varepsilon V(u) + \ldots$$

and $V(u) = \frac{d\bar{u}}{d\varepsilon}\big|_{\varepsilon=0}$ is a vector field on M corresponding to G. Inspired by definitions and properties related to symmetries for differential and difference equations [4, 16], we introduce some concepts and properties related to symmetries on time-space scales.

Lemma 3.1. *The vector field $V(u)$ corresponding to single parameter invariant group $G = \{g_\varepsilon | g_\varepsilon : u \to \bar{u}\}$ satisfies*

$$\Delta_t V = F'(u)V, \tag{3.19}$$

where $F'(u) = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial(\Delta_t u)}\Delta_t + \frac{\partial F}{\partial(\Delta_x u)}\Delta_x + \frac{\partial F}{\partial(\Delta_x^2 u)}\Delta_x^2 + \ldots$

Proof. Considering G is invariant, for $\Delta_t u = F(u)$, we have $\Delta_t \bar{u} = F(\bar{u})$. Since $\frac{d\bar{u}}{d\varepsilon}\big|_{\varepsilon=0} = V(u)$, then

$$\Delta_t V = \frac{d(\Delta_t \bar{u})}{d\varepsilon}\bigg|_{\varepsilon=0} = \frac{d(F(\bar{u}))}{d\varepsilon}\bigg|_{\varepsilon=0}. \tag{3.20}$$

Since $\bar{u}|_{\varepsilon=0} = u$, we have

$$\frac{dF(\bar{u})}{d\varepsilon}\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}F(u+\varepsilon V)\bigg|_{\varepsilon=0} = F'(u)\cdot\frac{d\bar{u}}{d\varepsilon}\bigg|_{\varepsilon=0} = F'(u)V. \tag{3.21}$$

It follows from (3.20) and (3.21) that (3.19) holds. □

Definition 3.1. $V(t,x,u,Tu,u^{(1)},Tu^{(1)},\ldots)$ is called a symmetry of (3.1) if $\Delta_t V = F'(u)V$.

Proposition 3.1. *Let $V(u)$ be a symmetry of (3.1). If $\bar{u}(u,\varepsilon)$ satisfies*

$$\begin{cases} \dfrac{d\bar{u}}{d\varepsilon} = V(\bar{u}), \\ \bar{u}|_{\varepsilon=0} = u, \end{cases}$$

then $G = \{g_\varepsilon | g_\varepsilon : u \to \bar{u}\}$ is a single parameter invariant group.

Proof. Since $\bar{u}|_{\varepsilon=0} = u$, then g_0 is identity mapping.

$$(g_{\varepsilon_1} \circ g_{\varepsilon_2})u = g_{\varepsilon_1}(g_{\varepsilon_2}u) = g_{\varepsilon_1}(\bar{u}(u, \varepsilon_2)) = \bar{\bar{u}}(u, \varepsilon_1, \varepsilon_2),$$
$$g_{\varepsilon_1+\varepsilon_2}u = \bar{u}(u, \varepsilon_1 + \varepsilon_2) = \tilde{u}(u, \varepsilon_1, \varepsilon_2),$$

and $\bar{\bar{u}}(u, \varepsilon_1, \varepsilon_2)$, $\tilde{u}(u, \varepsilon_1, \varepsilon_2)$ satisfy

$$\frac{d\bar{\bar{u}}}{d\varepsilon} = V(\bar{\bar{u}}),\ \frac{d\tilde{u}}{d\varepsilon} = V(\tilde{u}),$$

respectively. And for initial data, $\bar{\bar{u}}(u, \varepsilon_1, \varepsilon_2)|_{\varepsilon_1=0} = \bar{u}(u, \varepsilon_2) = \tilde{u}(u, \varepsilon_1, \varepsilon_2)|_{\varepsilon_1=0}$, then $\bar{\bar{u}}(u, \varepsilon_1, \varepsilon_2) = \tilde{u}(u, \varepsilon_1, \varepsilon_2)$, which means

$$g_{\varepsilon_1} \circ g_{\varepsilon_2} = g_{\varepsilon_1+\varepsilon_2}.$$

Then G is a group. Next we prove G is invariant. Since

$$\frac{d(\Delta_t \bar{u})}{d\varepsilon} = \Delta_t(\frac{d\bar{u}}{d\varepsilon}) = \Delta_t(V(\bar{u})), \tag{3.22}$$

$$\frac{dF(\bar{u})}{d\varepsilon} = F'(\bar{u})\frac{d\bar{u}}{d\varepsilon} = F'(\bar{u})V(\bar{u}), \tag{3.23}$$

substituting $\Delta_t V = F'V$ into (3.23), we have

$$\frac{dF(\bar{u})}{d\varepsilon} = \Delta_t(V(\bar{u})). \tag{3.24}$$

It follows from (3.22)-(3.24) that as functions of ε, $\Delta_t \bar{u}$ and $F(\bar{u})$ satisfy the same ODE. Considering the initial conditions

$$\Delta_t \bar{u}|_{\varepsilon=0} = \Delta_t u,\ \ F(\bar{u})|_{\varepsilon=0} = F(u),$$

we have $\Delta_t \bar{u} = F(\bar{u})$, which means $\bar{u}$ is also a solution of (3.1). Then G is a single parameter invariant group. □

3.3. Conserved quantities of dynamic systems on time-space scales

For $t \in \mathbb{T}$, $\bar{x} = (x_2, ..., x_m) \in \mathbb{X} \subset \mathbb{T}^{m-1}$, $f(t, \bar{x}) : \mathbb{T} \times \mathbb{X} \to \mathbb{R}$, set space

$$L^2_{\Delta,\bar{x}}(\mathbb{X}) = \left\{ f(t, \bar{x})| \int_{\mathbb{X}} |f(t, \bar{x})|^2 \Delta\bar{x} < +\infty \right\}.$$

Using Theorem 2.1 in [24], it can be easily derived that the space $L^2_{\Delta,\bar{x}}(\mathbb{X})$ is a Banach space with norm

$$\|f\|_{L^2_{\Delta,\bar{x}}} = \left(\int_{\mathbb{X}} |f(t, \bar{x})|^2 \Delta\bar{x} \right)^{\frac{1}{2}},$$

and inner product $\langle f, g \rangle = \int_{\mathbb{X}} f(t, \bar{x})g(t, \bar{x})\Delta\bar{x}$. The solutions of dynamic systems on time-space scales can be connected with a conserved quantity as follows.

Theorem 3.2. *u is a solution of dynamic system (3.1) on time-space scales if and only if*

$$E = \|u(t,\bar{x})\|^2_{L^2_{\Delta,\bar{x}}} - \|u(t_0,\bar{x})\|^2_{L^2_{\Delta,\bar{x}}} - 2\int_{t_0}^{t} \langle F, u(s,\bar{x})\rangle \,\Delta s - \int_{t_0}^{t} \mu(s)\,\|F\|^2_{L^2_{\Delta,\bar{x}}}\,\Delta s \tag{3.25}$$

is a conserved quantity.

Proof. If (3.25) is a conserved quantity, then $\frac{\Delta E}{\Delta t} = 0$. Since

$$\begin{aligned} &\|u(t,\bar{x})\|^2_{L^2_{\Delta,\bar{x}}} - \|u(t_0,\bar{x})\|^2_{L^2_{\Delta,\bar{x}}} \\ =&\int_{t_0}^{t} \Delta_s \langle u(s,\bar{x}), u(s,\bar{x})\rangle \,\Delta s \\ =&\int_{t_0}^{t} \langle \Delta_s u(s,\bar{x}), (u(s,\bar{x}) + u(\sigma(s),\bar{x}))\rangle \,\Delta s, \end{aligned} \tag{3.26}$$

it follows from (3.25)-(3.26) that

$$\begin{aligned} &\frac{\Delta}{\Delta t}\int_{t_0}^{t} \langle \Delta_s u(s,\bar{x}), (u(s,\bar{x}) + u(\sigma(s),\bar{x}))\rangle \,\Delta s = \frac{\Delta}{\Delta t}\int_{t_0}^{t} \langle F, 2u(s,\bar{x}) + \mu(s)F\rangle \,\Delta s \\ =&\frac{\Delta}{\Delta t}\int_{t_0}^{t} \langle F, 2u(s,\bar{x}) + \mu(s)\Delta_s u\rangle \,\Delta s = \frac{\Delta}{\Delta t}\int_{t_0}^{t} \langle F, (u(s,\bar{x}) + u(\sigma(s),\bar{x}))\rangle \,\Delta s, \end{aligned}$$

which means u satisfies (3.1).

Conversely, if u satisfies (3.1), multiplying both sides of (3.1) by $(u(s,\bar{x}) + u(\sigma(s),\bar{x}))$, and integrating on $[t_0,t]_{\mathbb{T}}$, we have

$$\int_{t_0}^{t} \langle \Delta_s u(s,\bar{x}), (u(s,\bar{x}) + u(\sigma(s),\bar{x}))\rangle \,\Delta s = \int_{t_0}^{t} \langle F, (u(s,\bar{x}) + u(\sigma(s),\bar{x}))\rangle \,\Delta s + C.$$

Then

$$\begin{aligned} \|u(t,\bar{x})\|^2_{L^2_{\Delta,\bar{x}}} - \|u(t_0,\bar{x})\|^2_{L^2_{\Delta,\bar{x}}} &= \int_{t_0}^{t} \langle F, (u(s,\bar{x}) + u(\sigma(s),\bar{x}))\rangle \,\Delta s + C \\ &= 2\int_{t_0}^{t} \langle F, u(s,\bar{x})\rangle \,\Delta s + \int_{t_0}^{t} \mu(s)\,\|F\|^2_{L^2_{\Delta,\bar{x}}}\,\Delta s + C, \end{aligned}$$

which means (3.25) is a conserved quantity. □

4. Symmetry analysis of heat equation and Burgers equation on time-space scales

4.1. The relation of symmetries between heat equation and Burgers equation on time-space scales

Since the Burgers equation and heat equation can be transformed into each other with the Cole-Hopf transformation, we start with the heat equation on time-space scales

$$\Delta_t v = q\Delta_{xx} v, \tag{4.1}$$

where $t,\, x \in \mathbb{T}\times\mathbb{X}=\mathbb{T}\times\mathbb{T}$. If $v(t,x)$, $\mu(t)$, $\nu(x)$ are Δ-differential and $\Delta_x(\mu(t)) = \Delta_t(\nu(x)) = 0$, using Lemma 2.4, we get

$$\Delta_x(\Delta_t v) = \Delta_t(\Delta_x v). \tag{4.2}$$

Using Cole-Hopf transformation

$$\Delta_x v = \frac{1}{p}uv, \tag{4.3}$$

we can obtain

$$\Delta_t v = q\Delta_{xx} v = \frac{q}{p}\Delta_x(uv) = \frac{q}{p}v\Delta_x u + \frac{q}{p}T_x u\cdot\Delta_x v = v(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u). \tag{4.4}$$

Substituting (4.3) and (4.4) into (4.2), we have

$$\begin{aligned}\Delta_t(\Delta_x v) &= \frac{1}{p}\Delta_t(uv) = \frac{1}{p}v\Delta_t u + \frac{1}{p}T_t u\cdot\Delta_t v = \frac{1}{p}v\Delta_t u + \frac{1}{p}T_t u(\frac{q}{p}v\Delta_x u + \frac{q}{p^2}uvT_x u)\\ &= \frac{1}{p}v\Delta_t u + \frac{1}{p}(\mu(t)\Delta_t u + u)(\frac{q}{p}v\Delta_x u + \frac{q}{p^2}uvT_x u)\\ &= \frac{1}{p}(1+\mu(t)(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u))v\Delta_t u + \frac{1}{p}uv(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u),\end{aligned}$$

$$\begin{aligned}&\Delta_x(\Delta_t v)\\ =&\Delta_x((\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u)v) = \Delta_x(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u)v + T_x(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u)\Delta_x v\\ =&\Delta_x(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u)v + \frac{1}{p}\nu(x)uv\Delta_x(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u) + \frac{1}{p}uv(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u)\\ =&(1+\frac{1}{p}\nu(x)u)v\Delta_x(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u) + \frac{1}{p}uv(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u).\end{aligned}$$

Then Burgers equation on time-space scales can be obtained as

$$\Delta_t u = \frac{p+\nu(x)u}{(p^2+\mu(t)(pq\Delta_x u + quT_x u))}\Delta_x(pq\Delta_x u + quT_x u). \tag{4.5}$$

When $\mathbb{X}=\mathbb{R}$ ($\nu(x)=0$), (4.5) becomes

$$\Delta_t u = \frac{pq(pu_{xx}+2uu_x)}{p^2+\mu(t)(pqu_x+qu^2)}. \tag{4.6}$$

Furthermore, when $\mathbb{T}=\mathbb{R}$ and $\mathbb{X}=\mathbb{R}$ ($\mu(t)=0, \nu(x)=0$), $p_0=\frac{2q}{p}$, (4.5) becomes classical constant-coefficient Burgers equation

$$u_t = qu_{xx} + p_0 uu_x. \tag{4.7}$$

Now we study the symmetry of heat equation and Burgers equation on time-space scales. The symmetries of heat equation on time-space scales can be written as $v_\lambda = Sv$, where $S = S(t,x,v,\Delta_t,\Delta_x)$. Inspired by relations of symmetries between the discrete heat equation and Burgers equation [10], we establish relations of symmetries between the heat equation (4.1) and Burgers equation (4.5) on time-space scales.

Proposition 4.1. *Let $v_\lambda = Sv$ represents the symmetry of heat equation (4.1) on time-space scales, then the symmetry of Burgers equation (4.5) on time-space scales can be represented as*

$$u_{\lambda_1} = (\nu(x)u + p)\Delta_x(\frac{Sv}{v}), \tag{4.8}$$

$$u_{\lambda_2} = (\frac{q}{p}\mu(t)(\Delta_x u + \frac{1}{p}uT_x u) + 1)\Delta_t(\frac{Sv}{v}). \tag{4.9}$$

Proof. Considering

$$(\Delta_x v)_{\lambda_1} = \Delta_x(v_{\lambda_1}) \tag{4.10}$$

and Cole-Hopf transformation (4.3), we have

$$\frac{(\Delta_x v)_{\lambda_1}}{v} = (\frac{\Delta_x v}{v})_{\lambda_1} + \frac{\Delta_x v \cdot v_{\lambda_1}}{v^2} = \frac{1}{p}u_{\lambda_1} + \frac{1}{p}\frac{uv_{\lambda_1}}{v} = \frac{1}{p}u_{\lambda_1} + \frac{1}{p}\frac{uSv}{v}. \tag{4.11}$$

It follows from (4.11) that

$$u_{\lambda_1} = p\frac{(\Delta_x v)_{\lambda_1}}{v} - \frac{uSv}{v} = p\frac{\Delta_x(v_{\lambda_1})}{v} - \frac{uSv}{v} = \frac{p\Delta_x(Sv) - uSv}{v}. \tag{4.12}$$

Since

$$\begin{aligned} \nu(x)\Delta_x(\frac{Sv}{v}) &= \frac{T_x(Sv)}{T_x v} - \frac{Sv}{v} \\ &= \frac{1}{vT_x v}(v \cdot (T_x(Sv) - Sv) - Sv(T_x v - v)) \\ &= \frac{\nu(x)}{T_x v}(\Delta_x(Sv) - \frac{1}{p}u(Sv)), \end{aligned} \tag{4.13}$$

multiplying both sides of (4.13) by $\frac{T_x v - v}{v}$ and using (4.12), we get

$$\begin{aligned} &\frac{T_x v - v}{v} \cdot \nu(x)\Delta_x(\frac{Sv}{v}) \\ =&\frac{T_x v - v}{T_x v} \cdot \frac{\nu(x)(\Delta_x(Sv) - \frac{1}{p}u(Sv))}{v} \\ =&\frac{\nu(x)(\Delta_x(Sv) - \frac{1}{p}u(Sv))}{v} - \frac{\nu(x)}{T_x v}(\Delta_x(Sv) - \frac{1}{p}u(Sv)) \\ =&\frac{\nu(x)(\Delta_x(Sv) - \frac{1}{p}u(Sv))}{v} - \nu(x)\Delta_x(\frac{Sv}{v}) \\ =&\frac{\nu(x)}{p}u_{\lambda_1} - \nu(x)\Delta_x(\frac{Sv}{v}). \end{aligned} \tag{4.14}$$

It follows from (4.14) that

$$\begin{aligned} \nu(x)u_{\lambda_1} &= p(\frac{T_x v - v}{v} + 1)\nu(x)\Delta_x(\frac{Sv}{v}) \\ &= p(\nu(x)\frac{\Delta_x v}{v} + 1)\nu(x)\Delta_x(\frac{Sv}{v}) \\ &= (\nu(x)u + p)\nu(x)\Delta_x(\frac{Sv}{v}), \end{aligned}$$

which means (4.8) holds. Considering

$$(\Delta_t v)_\lambda = \Delta_t(v_\lambda), \tag{4.15}$$

and (4.4), we have

$$\begin{aligned}\frac{(\Delta_t v)_\lambda}{v} &= (\frac{\Delta_t v}{v})_\lambda + \frac{\Delta_t v \cdot v_\lambda}{v^2} \\ &= (\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u)_\lambda + \frac{(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u)v_\lambda}{v} \\ &= (\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u)_\lambda + \frac{(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u)Sv}{v}.\end{aligned} \tag{4.16}$$

From (4.16),

$$\begin{aligned}(\frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u)_\lambda &= \frac{(\Delta_t v)_\lambda}{v} - \frac{q}{p}\frac{(\Delta_x u + \frac{1}{p}uT_x u)Sv}{v} \\ &= \frac{\Delta_t(Sv) - \frac{q}{p}(\Delta_x u + \frac{1}{p}uT_x u)Sv}{v}.\end{aligned} \tag{4.17}$$

Since

$$\begin{aligned}\mu(t)\Delta_t(\frac{Sv}{v}) &= \frac{T_t(Sv)}{T_t v} - \frac{Sv}{v} \\ &= \frac{1}{vT_t v}(v \cdot (T_t(Sv) - Sv) - Sv(T_t v - v)) \\ &= \frac{\mu(t)}{T_t v}(\Delta_t(Sv) - \frac{q}{p}(\Delta_x u + \frac{1}{p}uT_x u)(Sv)),\end{aligned} \tag{4.18}$$

multiplying both sides of (4.18) by $\frac{T_t v - v}{v}$ and using (4.17), we have

$$\begin{aligned}\frac{T_t v - v}{v} \cdot \mu(t)\Delta_t(\frac{Sv}{v}) &= \frac{T_t v - v}{T_t v} \cdot \frac{\mu(t)[\Delta_t(Sv) - \frac{q}{p}(\Delta_x u + \frac{1}{p}uT_x u)(Sv)]}{v} \\ &= \frac{\mu(t)[\Delta_t(Sv) - \frac{q}{p}(\Delta_x u + \frac{1}{p}uT_x u)(Sv)]}{v} \\ &\quad - \frac{\mu(t)}{T_t v}(\Delta_t(Sv) - \frac{q}{p}(\Delta_x u + \frac{1}{p}uT_x u)(Sv)) \\ &= \frac{\mu(t)[\Delta_t(Sv) - \frac{q}{p}(\Delta_x u + \frac{1}{p}uT_x u)(Sv)]}{v} - \mu(t)\Delta_t(\frac{Sv}{v}) \\ &= \frac{q}{p}\mu(t)(\Delta_x u + \frac{1}{p}uT_x u)_\lambda - \mu(t)\Delta_t(\frac{Sv}{v}).\end{aligned} \tag{4.19}$$

It follows from (4.19) that

$$\begin{aligned}\mu(t)u_{\lambda_2} &= \frac{q}{p}\mu(t)(\Delta_x u + \frac{1}{p}uT_x u)_\lambda = (\frac{T_t v - v}{v} + 1)\mu(t)\Delta_t(\frac{Sv}{v}) \\ &= (\mu(t)\frac{\Delta_t v}{v} + 1)\mu(t)\Delta_t(\frac{Sv}{v}) = (\frac{q}{p}\mu(t)(\Delta_x u + \frac{1}{p}uT_x u) + 1)\mu(t)\Delta_t(\frac{Sv}{v}),\end{aligned} \tag{4.20}$$

which means (4.9) holds. □

4.2. Symmetry analysis of Burgers equation using direct symmetry method on time-space scales

Theorem 4.1. *If $\Delta_x^2(\rho(x)) = 0$, then*

$$\begin{aligned} &v_{\lambda_1} = \Delta_t v,\ v_{\lambda_2} = q(2+\Delta_x(\nu(x)))t\Delta_t v + xT_tT_x^{-1}\Delta_x v,\ v_{\lambda_3} = \Delta_x v,\\ &v_{\lambda_4} = q(2+\Delta_x(\nu(x)))t\Delta_x v + xT_tT_x^{-1}v,\ v_{\lambda_5} = v \end{aligned} \tag{4.21}$$

are basis of symmetry algebra for heat equation (4.1) on time-space scales.

$$\begin{aligned} &u_{\lambda_1} = (1+\mu(t)w)\Delta_t u,\ u_{\lambda_3} = (p+\nu(x)u)\Delta_x u,\\ &u_{\lambda_2} = q(1+\mu(t)w)(2+\Delta_x(\nu(x)))\Delta_t(tw) + \frac{1}{p}(p+\nu(x)u)\Delta_x[x(1+\mu(t)w)T_tT_x^{-1}u],\\ &u_{\lambda_4} = \frac{qt}{p}(p+\nu(x)u)(2+\Delta_x(\nu(x)))\Delta_x u + (p+\nu(x)u)\Delta_x[xT_tT_x^{-1}\frac{1}{1+\frac{1}{p}\nu(x)u}] \end{aligned} \tag{4.22}$$

are basis of symmetry algebra for Burgers equation (4.5) on time-space scales, where $w = \frac{q}{p}\Delta_x u + \frac{q}{p^2}uT_x u$.

Proof. The symmetry V of heat equation (4.1) on time-space scales satisfies

$$\Delta_t V = q\Delta_{xx}V. \tag{4.23}$$

Let $V = a(t,x)\Delta_t v + b(t,x)\Delta_x v + c(t,x)v + d(t,x)$. We get

$$\begin{aligned} \Delta_t V &= \Delta_t(a(t,x)\Delta_t v + b(t,x)\Delta_x v + c(t,x)v + d(t,x))\\ &= \Delta_t a(t,x)\cdot T_t(\Delta_t v) + a(t,x)\cdot\Delta_t^2 v + \Delta_t b(t,x)\cdot T_t(\Delta_x v) + b(t,x)\cdot\Delta_t\Delta_x v\\ &\quad + \Delta_t c(t,x)\cdot T_t v + c(t,x)\cdot\Delta_t v + \Delta_t d(t,x),\\ \Delta_x V &= \Delta_x(a(t,x)\Delta_t v + b(t,x)\Delta_x v + c(t,x)v + d(t,x))\\ &= \Delta_x a(t,x)\cdot T_x(\Delta_t v) + a(t,x)\cdot\Delta_x\Delta_t v + \Delta_x b(t,x)\cdot T_x(\Delta_x v) + b(t,x)\cdot\Delta_x^2 v\\ &\quad + \Delta_x c(t,x)\cdot T_x v + c(t,x)\cdot\Delta_x v + \Delta_x d(t,x),\\ \Delta_{xx}V &= \Delta_x^2 a\cdot T_x^2(\Delta_t v) + \Delta_x a\cdot\Delta_x(T_x(\Delta_t v)) + \Delta_x a\cdot T_x(\Delta_x\Delta_t v) + a\cdot\Delta_x^2\Delta_t v\\ &\quad + \Delta_x^2 b\cdot T_x^2(\Delta_x v) + \Delta_x b\cdot\Delta_x(T_x(\Delta_x v)) + \Delta_x b\cdot T_x(\Delta_x^2 v) + b\cdot\Delta_x^3 v\\ &\quad + \Delta_x^2 c\cdot T_x^2 v + \Delta_x c\cdot\Delta_x(T_x v) + \Delta_x c\cdot T_x(\Delta_x v) + c\Delta_x^2 v + \Delta_x^2 d. \end{aligned}$$

Using Lemma 2.4, we have

$$\begin{aligned} &\Delta_x(T_x(\Delta_t v)) : \Delta_x a = 0,\\ &T_x(\Delta_x\Delta_t v) : \Delta_x a = 0,\\ &\Delta_t : \Delta_t a\cdot T_t - q\Delta_x^2 a\cdot T_x^2 - 2q\Delta_x b\cdot T_x - q\Delta_x(\nu(x))\cdot\Delta_x b\cdot T_x = 0,\\ &\Delta_x v : \Delta_t b\cdot T_t - q\Delta_x^2 b\cdot T_x^2 - 2q\Delta_x c\cdot T_x - q\Delta_x(\nu(x))\cdot\Delta_x c\cdot T_x = 0,\\ &v : \Delta_t c\cdot T_t - q\Delta_x^2 c\cdot T_x^2 = 0,\\ &1 : \Delta_t d = q\Delta_x^2 d. \end{aligned} \tag{4.24}$$

It follows from (4.24) and $\Delta_x^2(\rho(x)) = 0$ that

$$\begin{aligned} &a = q(2+\Delta_x(\nu(x)))\hat{c}_2 t + \hat{c}_1,\ b = q(2+\Delta_x(\nu(x)))\hat{c}_4 t + \hat{c}_2 xT_tT_x^{-1} + \hat{c}_3,\\ &c = \hat{c}_4 xT_tT_x^{-1} + \hat{c}_5,\ \Delta_t d = q\Delta_x^2 d. \end{aligned} \tag{4.25}$$

Then symmetry of heat equation (4.1) on time-space scales can be expressed as

$$\begin{aligned} V =& a(t,x)\Delta_t v + b(t,x)\Delta_x v + c(t,x)v + d(t,x) = (q(2+\Delta_x(\nu(x)))\hat{c}_2 t + \hat{c}_1)\Delta_t v \\ &+ (q(2+\Delta_x(\nu(x)))\hat{c}_4 t + \hat{c}_2 x T_t T_x^{-1} + \hat{c}_3)\Delta_x v + (\hat{c}_4 x T_t T_x^{-1} + \hat{c}_5)v + d(t,x). \end{aligned} \tag{4.26}$$

$$\begin{aligned} &V_1 = \frac{\Delta}{\Delta t},\ V_2 = q(2+\Delta_x(\nu(x)))t\frac{\Delta}{\Delta t} + xT_tT_x^{-1}\frac{\Delta}{\Delta x},\ V_3 = \frac{\Delta}{\Delta x}, \\ &V_4 = q(2+\Delta_x(\nu(x)))t\frac{\Delta}{\Delta x} + xT_tT_x^{-1}v\frac{\partial}{\partial v},\ V_5 = v\frac{\partial}{\partial v} \end{aligned} \tag{4.27}$$

are generators of invariant groups for (4.1). Basis of symmetry algebra of heat equation (4.1) on time-space scales can be given as (4.21). Using Proposition 4.1, (4.4) and

$$\begin{aligned} &\Delta_x v = \frac{1}{p}vu,\ T_x v = v(1+\frac{1}{p}\nu(x)u), \\ &T_x^{-1}v = vT_x^{-1}\frac{1}{1+\frac{1}{p}\nu(x)u},\ T_t^{-1}v = vT_t^{-1}\frac{1}{1+\mu(t)w}, \end{aligned} \tag{4.28}$$

the symmetry of Burgers equation on time-space scales can be obtained. Basis of symmetry algebra for Burgers equation (4.5) on time-space scales can be given by (4.22). □

Time scale case. When $\mathbb{T}\times\mathbb{X} = \mathbb{T}\times\mathbb{R}$, dynamic systems on time-space scales reduced to time scale cases. Specially, heat equation (4.1) and Burgers equation (4.5) on time-space scales reduced to

$$\Delta_t u = q u_{xx} \tag{4.29}$$

and (4.6), respectively. The symmetry analysis of heat equation (4.29) and Burgers equation (4.6) on time scale can be obtained as follows.

Corollary 4.1. *Let $\hat{c}_i (i = 1,2,...,5)$ be arbitrary constants. If $\Delta_t^2(\sigma(t)) = 0$, then*

$$\begin{aligned} &v_{\lambda_1} = \Delta_t v,\ v_{\lambda_2} = 2qt\Delta_t v + xT_t\partial_x v,\ v_{\lambda_3} = \partial_x v, v_{\lambda_4} = 2qtT_t^{-1}\partial_x v + xT_t v,\ v_{\lambda_5} = v, \\ &v_{\lambda_6} = 4q^2t^2T_t^{-1}\frac{\Delta}{\Delta t} + 2qx(2t+\mu(t))\frac{\partial}{\partial x} + x^2(1+\frac{1}{2}\Delta_t(\mu(t)))T_t v\frac{\partial}{\partial v} \\ &\qquad + q(2t+\mu(t))T_t^{-1}v\frac{\partial}{\partial v} \end{aligned} \tag{4.30}$$

are basis of symmetry algebra of heat equation (4.29) on time scale.

$$\begin{aligned} &u_{\lambda_1} = (1+\mu(t)w)\Delta_t u,\ u_{\lambda_2} = 2q(1+\mu(t)w)\Delta_t(tw) + x\partial_x u + u, \\ &u_{\lambda_3} = p\partial_x u,\ u_{\lambda_4} = 2qtT_t^{-1}\partial_x u + p, \\ &u_{\lambda_6} = 4q^2(1+\mu(t)w)\Delta_t(t^2T_t^{-1}\frac{w}{1+\mu(t)w}) + 2q(2t+\mu(t))(u + x\partial_x u) \\ &\qquad + 2p(1+\frac{1}{2}\Delta_t(\mu(t)))x + pq(2t+\mu(t))\partial_x(T_t^{-1}\frac{1}{1+\mu(t)w}) \end{aligned} \tag{4.31}$$

are basis of symmetry algebra for Burgers equation (4.6) on time scale.

Proof. (4.24) reduces to

$$\begin{aligned}
&\Delta_x(v_t) : a_x = 0,\\
&\Delta_t v : \Delta_t a \cdot T_t - q a_{xx} - 2q b_x = 0,\\
&v_x : \Delta_t b \cdot T_t - q b_{xx} - 2q c_x = 0,\\
&v : \Delta_t c \cdot T_t - q c_{xx} = 0,\\
&1 : \Delta_t d = q d_{xx}.
\end{aligned} \tag{4.32}$$

It follows from (4.32) and $\Delta_t^2(\sigma(t)) = 0$ that

$$\begin{aligned}
&a = 4c_6 q^2 t^2 T_t^{-1} + 2c_2 qt + c_1,\ b = (2c_6 qx(2t+\mu(t)) + 2c_4 qt) + c_2 x T_t + c_3,\\
&c = (c_4 x + c_6 x^2 (1 + \frac{1}{2}\Delta_t(\mu(t))))T_t + qc_6(2t+\mu(t)) + c_5.
\end{aligned}$$

Then

$$\begin{aligned}
V &= [4c_6 q^2 t^2 T_t^{-1} + 2c_2 qt + c_1]\Delta_t v + [(2c_6 qx(2t+\mu(t)) + 2c_4 qt) + c_2 x T_t + c_3]v_x\\
&\quad + [(c_4 x + c_6 x^2(1 + \frac{1}{2}\Delta_t(\mu(t))))T_t + qc_6(2t+\mu(t)) + c_5]v + d(t,x)
\end{aligned} \tag{4.33}$$

is symmetry of heat equation on time scale.

$$\begin{aligned}
&V_1 = \frac{\Delta}{\Delta t},\ V_2 = 2qt\frac{\Delta}{\Delta t} + xT_t\frac{\partial}{\partial x},\ V_3 = \frac{\partial}{\partial x}, V_4 = 2qtT_t^{-1}\frac{\partial}{\partial x} + xT_t v\frac{\partial}{\partial v},\ V_5 = v\frac{\partial}{\partial v},\\
&V_6 = 4q^2t^2T_t^{-1}\frac{\Delta}{\Delta t} + 2qx(2t+\mu(t))\frac{\partial}{\partial x} + x^2(1+\frac{1}{2}\Delta_t(\mu(t)))T_t v\frac{\partial}{\partial v}\\
&\qquad + q(2t+\mu(t))v\frac{\partial}{\partial v}.
\end{aligned}$$

are generators of invariant groups for (4.1). Basis of symmetry algebra for heat equation on time scale can be given as (4.30). Using Proposition 4.1, (4.4) and (4.28), basis of symmetry algebra for Burgers equation on time scale can be given as (4.31). □

Remark 4.1. (1) When $\mathbb{T}\times\mathbb{X} = \mathbb{R}\times\mathbb{R}$, the symmetry (4.33) for (4.1) reduces to familiar continuous forms

$$\begin{aligned}
&V_1 = \frac{\partial}{\partial t},\ V_2 = 2qt\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},\ V_3 = \frac{\partial}{\partial x},\ V_4 = 2qt\frac{\partial}{\partial x} + xv\frac{\partial}{\partial v},\\
&V_5 = v\frac{\partial}{\partial v},\ V_6 = 4q^2t^2\frac{\partial}{\partial t} + 4qtx\frac{\partial}{\partial x} + (x^2 v + 2qtv)\frac{\partial}{\partial v}.
\end{aligned} \tag{4.34}$$

(2) When $\mathbb{T}\times\mathbb{X} = \mathbb{R}\times\mathbb{R}$, the symmetry (4.31) for (4.5) reduces to familiar continuous forms

$$\begin{aligned}
&\bar{V}_1 = \frac{\partial}{\partial t},\ \bar{V}_2 = 2qt\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},\ \bar{V}_3 = p\frac{\partial}{\partial x},\\
&\bar{V}_4 = 2qt\frac{\partial}{\partial x} + p\frac{\partial}{\partial u},\ \bar{V}_6 = 4q^2t^2\frac{\partial}{\partial t} + 4qtx\frac{\partial}{\partial x} + 2(2qtu + px)\frac{\partial}{\partial u}.
\end{aligned} \tag{4.35}$$

(3) Using Theorem 3.2, the conserved quantities of heat equation and Burgers equation on time-space scales can be easily obtained.

(4) Since $\nu(x) = \rho(x) - x$, $\mu(t) = \sigma(t) - t$, then $\Delta_t^2(\mu(t)) = 0$, $\Delta_x^2(\nu(x)) = 0$ is equivalent to $\Delta_t^2(\sigma(t)) = 0$, $\Delta_x^2(\rho(x)) = 0$, respectively. Actually, $\Delta_t^2(\sigma(t)) = 0$ or $\Delta_x^2(\rho(x)) = 0$ is valid for common time or space scale including continuous $\mathbb{R}$, discrete $h\mathbb{Z}$ and quantum time or space scale $q^{\mathbb{Z}}$ et al, which means Theorem 4.1 unifies and generalizes both continuous and discrete cases and can be applied to solve more general practical problems.

Single parameter invariant groups on time-space scales. From (4.27), the single parameter invariant groups can be derived. To get more concise forms, we assume $\Delta_x(\rho(x)) = 1$, which holds naturally for both continuous and discrete cases. Thus

$$\begin{aligned}
&g_1 : (t, x, v) \to (t + \varepsilon, x, v),\\
&g_2 : (t, x, v) \to (qe^{2\varepsilon} t, e^{\varepsilon} x, v),\\
&g_3 : (t, x, v) \to (t, x + \varepsilon, v),\\
&g_4 : (t, x, v) \to (t, x + 2\varepsilon qt, e_{-\varepsilon T_t}(x, x_0) \cdot e_{\varepsilon^2 qT_x}(t, t_0)v)
\end{aligned}$$

are single parameter transformation groups for heat equation (4.1) on time-space scales. If $f(t, x)$ is a solution of (4.1), then

$$f(t - \varepsilon, x),\ f(qe^{2\varepsilon} t, e^{\varepsilon} x),\ f(t, x - \varepsilon),\ e_{-\varepsilon T_t}(x, x_0) \cdot e_{\varepsilon^2 qT_x}(t, t_0) f(t, x - 2\varepsilon t)$$

are also solutions of (4.1). Therefore

$$\begin{aligned}
&\frac{p\Delta_x f(t, x)}{f(t, x)},\ \frac{p\Delta_x f(t - \varepsilon, x)}{f(t - \varepsilon, x)},\ \frac{p\Delta_x f(qe^{2\varepsilon} t, e^{\varepsilon} x)}{f(qe^{2\varepsilon} t, e^{\varepsilon} x)},\\
&\frac{p\Delta_x f(t, x - \varepsilon)}{f(t, x - \varepsilon)},\ \frac{p\Delta_x (e_{-\varepsilon T_t}(x, x_0) f(t, x - 2\varepsilon t))}{e_{-\varepsilon T_t}(x, x_0) f(t, x - 2\varepsilon t)}
\end{aligned}$$

are also solutions of Burgers equation (4.5) on time-space scales.

Using Lemma 2.1, it is easy to get that

$$v(t, x) = e_{-q\lambda^2}(t, t_0) e_{i\lambda}(x, x_0) + C$$

is solution for heat equation on time-space scales. Using Cole-Hopf transformation (4.3),

$$u(t, x) = \frac{ip\lambda e_{-q\lambda^2}(t, t_0) e_{i\lambda}(x, x_0)}{e_{-q\lambda^2}(t, t_0) e_{i\lambda}(x, x_0) + C} \tag{4.36}$$

is solution of Burgers equation on time-space scales. Then

$$u(t, x) = \frac{ip\lambda e_{-q\lambda^2}(t - \varepsilon, t_0) e_{i\lambda}(x, x_0)}{e_{-q\lambda^2}(t - \varepsilon, t_0) e_{i\lambda}(x, x_0) + C}, \tag{4.37}$$

$$u(t, x) = \frac{ip\lambda e^{\varepsilon} e_{-q\lambda^2}(e^{2\varepsilon} t, t_0) e_{i\lambda}(e^{\varepsilon} x, x_0)}{e_{-q\lambda^2}(e^{2\varepsilon} t, t_0) e_{i\lambda}(e^{\varepsilon} x, x_0) + C}, \tag{4.38}$$

$$u(t, x) = \frac{ip\lambda e_{-q\lambda^2}(t, t_0) e_{i\lambda}(x - \varepsilon, x_0)}{e_{-q\lambda^2}(t, t_0) e_{i\lambda}(x - \varepsilon, x_0) + C}, \tag{4.39}$$

$$u(t, x) = -\varepsilon p + \frac{ip\lambda e_{-q\lambda^2}(t, t_0) e_{i\lambda}(x - 2\varepsilon t, x_0)}{e_{-q\lambda^2}(t, t_0) e_{i\lambda}(x - 2\varepsilon t, x_0) + C} \tag{4.40}$$

are also solutions of Burgers equation (4.5) on time-space scales.

4.3. Symmetry analysis of Burgers equation using Lie symmetry method on time-space scales

For heat equation (4.1) on time-space scales, let infinitesimal generator be

$$\underset{\sim}{V}=\sum_{i=1}^{m}\xi_i(x,v)\frac{\Delta}{\Delta x_i}+\phi(x,v)\frac{\partial}{\partial v}.$$

Then the corresponding 2th-extended infinitesimal generator

$$\mathrm{Pr}^{(2)}\underset{\sim}{V}=\underset{\sim}{V}+\sum_{i=1}^{m}\phi_i^{(1)}(x,v,v^{(1)})\frac{\partial}{\partial v_i}+\sum_{i_1,i_2=1}^{m}\phi_{i_1i_2}^{(2)}(x,v,v^{(1)},v^{(2)}).$$

$\mathrm{Pr}^{(2)}\underset{\sim}{V}(\Delta)\Big|_{\Delta=0}=0$ if and only if $(\phi^t-q\phi^{xx})|_{\Delta=0}=0$, where

$$\begin{aligned}\phi^t=&D_t\phi-\Delta_t v\cdot D_t\xi_1-\Delta_x v\cdot D_t\xi_2\\=&(\Delta_t\phi+\Delta_t(\phi\circ v))-(\Delta_t(\xi_1)\cdot\Delta_t v+\Delta_t(\xi_1\circ v)\cdot\Delta_t v)\\&-(\Delta_t(\xi_2)\cdot\Delta_x v+\Delta_t(\xi_2\circ v)\cdot\Delta_x v),\\ \phi^{xx}=&D_{xx}\phi-2(\Delta_t(\Delta_x v))D_x(\xi_1)-T_x(\Delta_t v)D_{xx}(\xi_1)\\&-2(\Delta_x(\Delta_x v))D_x(\xi_2)-T_x(\Delta_x v)D_{xx}(\xi_2)\\=&(\Delta_x(\Delta_x\phi)+\Delta_x(\Delta_x\phi\circ v)+\Delta_x(\Delta_x(\phi\circ v))+\Delta_x(\Delta_x(\phi\circ v)\circ v))\\&-2(\Delta_t(\Delta_x v))(\Delta_x(\xi_1)+\Delta_x(\xi_1\circ v))-2\Delta_x(\Delta_x v)(\Delta_x(\xi_2)+\Delta_x(\xi_2\circ v))\\&-T_x(\Delta_t v)(\Delta_x(\Delta_x(\xi_1))+\Delta_x(\Delta_x(\xi_1)\circ v)+\Delta_x(\Delta_x(\xi_1\circ v))\\&+\Delta_x(\Delta_x(\xi_1\circ v)\circ v))-T_x(\Delta_x v)(\Delta_x(\Delta_x(\xi_2))+\Delta_x(\Delta_x(\xi_2)\circ v)\\&+\Delta_x(\Delta_x(\xi_2\circ v))+\Delta_x(\Delta_x(\xi_2\circ v)\circ v)).\end{aligned}$$

For $\phi=\hat{\phi}v$, using Lemma 2.4 and $\Delta_x^2(\rho(x))=0$, we can obtain

$$\begin{aligned}\xi_1&=q(2+\Delta_x(\nu(x)))c_2t+c_1,\\ \xi_2&=c_2x+q(2+\Delta_x(\nu(x)))c_4tT_t+c_3,\\ \phi&=(-c_4xT_tT_x^{-1}+c_5)v.\end{aligned}$$

Then

$$\begin{aligned}&\underline{V}_1=\frac{\Delta}{\Delta t},\ \underline{V}_2=q(2+\Delta_x(\nu(x)))t\frac{\Delta}{\Delta t}+x\frac{\Delta}{\Delta x},\ \underline{V}_3=\frac{\Delta}{\Delta x},\\&\underline{V}_4=q(2+\Delta_x(\nu(x)))tT_t\frac{\Delta}{\Delta x}-xT_tT_x^{-1}v\frac{\Delta}{\Delta v},\ \underline{V}_5=v\frac{\Delta}{\Delta v},\end{aligned}\tag{4.41}$$

which means

$$\begin{aligned}&v_{\lambda_1}=\Delta_t v,\ v_{\lambda_2}=q(2+\Delta_x(\nu(x)))t\Delta_t v+x\Delta_x v,\ v_{\lambda_3}=\Delta_x v,\\&v_{\lambda_4}=q(2+\Delta_x(\nu(x)))tT_t\Delta_x v-xT_tT_x^{-1}v,\ v_{\lambda_5}=v.\end{aligned}\tag{4.42}$$

Using Proposition 4.1, (4.4) and (4.28), we get the symmetry for Burgers equation (4.5) on time-space scales

$$\begin{aligned}&u_{\lambda_1}=(1+\mu(t)w)\Delta_t u,\ u_{\lambda_3}=(p+\nu(x)u)\Delta_x u,\\&u_{\lambda_2}=q(1+\mu(t)w)(2+\Delta_x(\nu(x)))\Delta_t(tw)+\frac{1}{p}(p+\nu(x)u)\Delta_x[x(1+\mu(t)w)u],\\&u_{\lambda_4}=\frac{qt}{p}(p+\nu(x)u)(2+\Delta_x(\nu(x)))T_t\Delta_x u+(p+\nu(x)u)\Delta_x[xT_tT_x^{-1}\frac{1}{1+\frac{1}{p}\nu(x)u}]\end{aligned}\tag{4.43}$$

are basis of symmetry algebra for (4.5).

Similar to the method in Section 4.2, the reduced continuous form, single parameter transformation groups and invariant solutions can be obtained similarly.

5. Bäcklund transformation of heat and Burgers equations on time-space scales

The truncated Painlevé expansion for heat equation (4.1) on time-space scales can be expressed as

$$v = v_0 + \frac{v_1}{\phi},$$

where v_0, v_1, ϕ are functions of t and x.

Theorem 5.1. *(1) If v_0 is a solution of heat equation (4.1) on time-space scales, then*

$$v = v_0 + \frac{Ce_{\Delta_t\phi/\phi}(t,t_0)e_{\Delta_x\phi/\phi}(x,x_0)}{\phi} \tag{5.1}$$

is a Bäcklund transformation between solutions v_0 and v for (4.1).

(2) If u_0 is a solution of Burgers equation (4.5) on time-space scales, then

$$u = \frac{u_0 e_{\frac{1}{p}u_0}(x,x_0)\phi}{e_{\frac{1}{p}u_0}(x,x_0)\phi + Ce_{\Delta_t\phi/\phi}(t,t_0)e_{\Delta_x\phi/\phi}(x,x_0)} \tag{5.2}$$

is a Bäcklund transformation between solutions u_0 and u for (4.5).

Proof. (1) Substituting $v = v_0 + \frac{v_1}{\phi}$ into $\Delta_t v = q\Delta_{xx}v$, we have

$$\begin{aligned}
&\Delta_t(v_0) + \frac{\Delta_t(v_1)}{T_t\phi} - \frac{v_1\Delta_t\phi}{\phi T_t\phi} \\
&= q[\Delta_{xx}(v_0) + \frac{\Delta_{xx}(v_1)}{T_x^2\phi} - \frac{\Delta_x(v_1)\cdot\Delta_x(T_x\phi)}{T_x\phi\cdot T_x^2\phi} - \frac{\Delta_x(v_1)\cdot T_x(\Delta_x\phi) + v_1\Delta_{xx}\phi}{T_x\phi\cdot T_x^2\phi}] \\
&\quad + q[\frac{v_1\cdot(\Delta_x\phi)^2}{\phi\cdot T_x\phi\cdot T_x^2\phi} + \frac{v_1\cdot\Delta_x\phi\cdot\Delta_x(T_x\phi)}{\phi\cdot T_x\phi\cdot T_x^2\phi}] \\
&= q[\Delta_{xx}(v_0) + \frac{\Delta_{xx}(v_1)}{T_x^2\phi} + \frac{\Delta_x v_1\cdot\Delta_x\phi}{T_x\phi\cdot T_x^2\phi} - \frac{\Delta_x(v_1)\cdot T_x(\Delta_x\phi) + v_1\Delta_{xx}\phi}{T_x\phi\cdot T_x^2\phi}] \\
&\quad + q[-\frac{\Delta_x v_1\cdot\Delta_x\phi}{T_x\phi\cdot T_x^2\phi} - \frac{\Delta_x(v_1)\cdot\Delta_x(T_x\phi)}{T_x\phi\cdot T_x^2\phi}] \\
&\quad + q[\frac{v_1\cdot(\Delta_x\phi)^2}{\phi\cdot T_x\phi\cdot T_x^2\phi} + \frac{v_1\cdot\Delta_x\phi\cdot\Delta_x(T_x\phi)}{\phi\cdot T_x\phi\cdot T_x^2\phi}].
\end{aligned} \tag{5.3}$$

It follows from (5.3) that

$$\begin{aligned}
&1 : \Delta_t(v_0) = q\Delta_{xx}(v_0),\\
&\frac{1}{\phi T_t\phi} : \phi\Delta_t(v_1) - v_1\Delta_t\phi = 0,\\
&\frac{1}{T_x\phi T_x^2\phi} : \Delta_{xx}(v_1)T_x\phi + \Delta_x(v_1)\Delta_x\phi - \Delta_x(v_1)T_x(\Delta_x\phi) - v_1\Delta_{xx}\phi\\
&\qquad\qquad = \Delta_x(\phi\Delta_x(v_1) - v_1\Delta_x\phi) = 0,\\
&\frac{1}{\phi T_x\phi \cdot T_x^2\phi} : -(\phi\Delta_x(v_1) - v_1\Delta_x\phi)\cdot\Delta_x\phi - (\phi\Delta_x(v_1) - v_1\Delta_x\phi)\cdot\Delta_x(T_x\phi) = 0.
\end{aligned} \tag{5.4}$$

It can be obtained that

$$\begin{cases} \Delta_t(v_1) = \dfrac{\Delta_t\phi}{\phi}v_1, \\ \Delta_x(v_1) = \dfrac{\Delta_x\phi}{\phi}v_1 \end{cases} \tag{5.5}$$

is one solution of (5.4).

Let

$$v_1 = Ce_{\Delta_t\phi/\phi}\,(t,t_0)e_{\Delta_x\phi/\phi}\,(x,x_0),$$

the B*ä*cklund transformation (5.1) of heat equation (4.1) on time-space scales can be obtained.

(2) Using Cole-Hopf transformation (4.3), if u_0 is a solution of Burgers equation (4.5) on time-space scales, then $v_0 = e_{\frac{1}{p}u_0}(x,x_0)$ is a solution for (4.1). Therefore, from (5.1),

$$v = e_{\frac{1}{p}u_0}(x,x_0) + \frac{Ce_{\Delta_t\phi/\phi}\,(t,t_0)e_{\Delta_x\phi/\phi}\,(x,x_0)}{\phi}$$

is also a solution for (4.1). Reusing Cole-Hopf transformation (4.3),

$$u = \frac{u_0 e_{\frac{1}{p}u_0}(x,x_0)\phi}{e_{\frac{1}{p}u_0}(x,x_0)\phi + Ce_{\Delta_t\phi/\phi}\,(t,t_0)e_{\Delta_x\phi/\phi}\,(x,x_0)}$$

is also a solution of Burgers equation (4.5) on time-space scales, which gives relation between the seed solution u_0 and general solution u. □

Remark 5.1. If we choose $u_0 = ip\lambda$, $\phi = e_{\frac{q}{p^2}u_0^2}(t,t_0)$ in (5.2), then solution (5.2) reduces to solution (4.36).

6. Application in traffic flow model

Burgers equation can be used to describe traffic flow approximately [13]. On vehicle passable time-space scales $\bigcup_k [a_k,b_k]$, let b_k represents the moment of entering waiting when encountering waiting signal. $\Delta t_w^{(k)} = a_{k+1} - b_k$ represents the waiting time. $\Delta t_p^{(k)} = b_k - a_k$ represent the passable duration. Then the graininess functions $\nu(x) = 0$ and

$$\mu(t) = \begin{cases} 0, & t \in [a_k,b_k), \\ \Delta t_w^{(k)}, & t = b_k. \end{cases}$$

(1) Sparse and compressible waves. If the initial data

$$u|_{t=0} = p\frac{e^{(x-22.5)}}{e^{(x-22.5)}+1} + (-\frac{p}{2}+1),$$

the parameters in (4.40) can be obtained as $\lambda = -i$, $C = 1$, $\varepsilon = \frac{1}{2} - \frac{1}{p}$, $x_0 = 22.5$, $t_0 = 0$, $\Delta t_p = 30$. The anti-kink solution corresponding to the compressible wave ($p = 2$, $q = -1$) and the kink solution corresponding to the sparse wave ($p = -2$, $q = 1$) are obtained (c.f. Figure 1), which approximately simulate the traffic flow encounters waiting signal and passable signal, respectively. In the compressional wave, the moving traffic queue is gradually compressed. The density of traffic flow increases gradually, and the speed of traffic flow decreases gradually. In the sparse wave, the static traffic queue is gradually disbanded. The density of traffic flow decreases gradually, and the speed of traffic flow increases gradually.

(2) Oscillatory wave. If

$$u|_{t=10} = 2\frac{0.6ie^{0.6ix}}{e^{0.6ix}+1},$$

the parameters in (4.36) can be obtained as $\lambda = 0.6$, $C = 1$, $x_0 = 0$, $t_0 = 10$, $p = 2$, $q = -1$. The spatial oscillation solution corresponding to the oscillatory wave is obtained (c.f. Figure 2), which approximately simulates the traffic flow that goes and stops and the oscillation blockage in traffic jam.

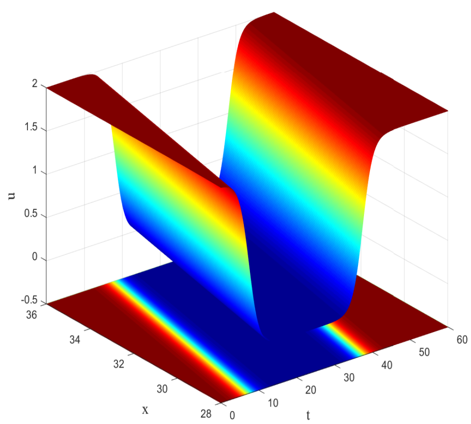

Figure 1. Space-time evolution of u

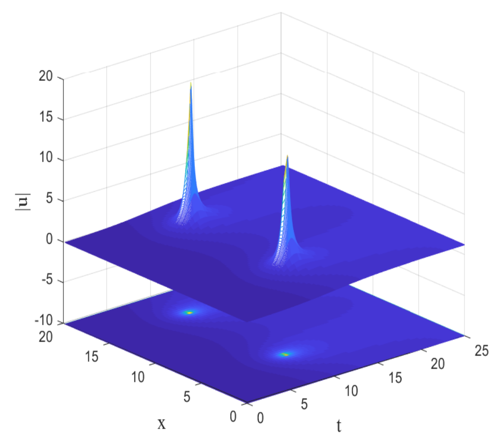

Figure 2. Space-time evolution of $|u|$

7. Conclusion and discussion

As an effective tool that can be used to study both continuous and discrete systems, the symmetry analysis method to general dynamic system with partial derivatives is generalized to time-space scales using direct symmetry method and Lie symmetry method, respectively. Forms of single parameter transformation groups, invariant solutions and conserved quantities on time-space scales are obtained, which are applied to heat equation and Burgers equation on time-space scales, respectively. The Bäcklund transformation of heat equation and Burgers equation on time-space scales are derived by Painlevé analysis. Based on the analysis of Burgers equation on time scale, the approximate simulation of the traffic flow model is obtained. The methods given in this paper provide a way to study the dynamic systems on time-space scales with partial delta derivatives. The study of nonlinear systems on time-space scales provides a theoretical basis for revealing the internal physical mechanism of the systems. The nonclassical symmetries and conservation laws of dynamic systems on time-space scales deserve to be further studied.

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