Journal of Applied Analysis and Computation
Volume 12, Number 6, December 2022, 2578–2592
Website:http://www.jaac-online.com
DOI:10.11948/20220112

ORDER TWO SUPERCONVERGENCE OF THE CDG METHOD FOR THE STOKES EQUATIONS ON TRIANGLE/TETRAHEDRON

Xiu Ye[1] and Shangyou Zhang[2,†]

Abstract A new conforming discontinuous Galerkin (CDG) finite element method is introduced for solving the Stokes equations. The CDG method gets its name by combining good features of both conforming finite element method and discontinuous finite element method. It has the flexibility of using discontinuous approximation and simplicity in formulation of the conforming finite element method. This new CDG method is not only stabilizer free but also has two order higher convergence rate than the optimal order. This CDG method uses discontinuous P_k element for velocity and continuous P_{k+1} element for pressure. Order two superconvergence is derived for velocity in an energy norm and the L^2 norm. The superconvergent P_k solution is lifted elementwise to a P_{k+2} velocity which converges at the optimal order. The numerical experiments confirm the theories.

Keywords Finite element, conforming discontinuous Galerkin method, stabilizer free, triangular mesh, tetrahedral mesh, Stokes equations.

MSC(2010) 65N15, 65N30.

1. Introduction

A new conforming discontinuous Galerkin method is introduced for the Stokes equations: seeks unknown functions $\mathbf{u}$ and p satisfying

$$-\Delta\mathbf{u}+\nabla p=\mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$
$$\nabla\cdot\mathbf{u}=0 \quad \text{in } \Omega, \tag{1.2}$$
$$\mathbf{u}=0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d=2,3)$.

The weak form in the primary velocity-pressure formulation for the Stokes problem (1.1)–(1.3) seeks $\mathbf{u}\in[H_0^1(\Omega)]^d$ and $p\in L_0^2(\Omega)$ satisfying

$$(\nabla\mathbf{u},\nabla\mathbf{v})-(\nabla\cdot\mathbf{v},p)=(\mathbf{f},\mathbf{v}), \tag{1.4}$$
$$(\nabla\cdot\mathbf{u},w)=0, \tag{1.5}$$

for all $\mathbf{v}\in[H_0^1(\Omega)]^d$ and $w\in L_0^2(\Omega)$. The conforming finite element method [8,10] for (1.1)–(1.3) developed over the last several decades is based on the weak formulation

†The corresponding author. Email: szhang@udel.edu (S. Zhang)
1Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA
2Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

(1.4)-(1.5) by constructing a pair of finite element spaces satisfying the so called inf-sup condition [3,4].

The CDG finite element method introduced in [26] uses discontinuous P_k element to approximate the solution of PDE and has simple formulation similar to conforming finite element method. Although discontinuous P_k element is used for velocity, our new CDG method has the following stabilizer free formulation comparable to (1.4)-(1.5): find $(\mathbf{u}_h, p_h) \in V_h \times W_h$ such that for any $(\mathbf{v}, w) \in V_h \times W_h$

$$(\nabla_w \mathbf{u}_h, \nabla_w \mathbf{v}) - (\nabla_w \cdot \mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}), \tag{1.6}$$

$$(\nabla_w \cdot \mathbf{u}_h, w) = 0, \tag{1.7}$$

where ∇_w and $\nabla_w \cdot$ are the weak gradient and weak divergence, which are the approximations for the gradient ∇ and the divergence $\nabla \cdot$.

The concept of weak derivative was first introduced in the weak Galerkin (WG) finite element method. In fact, the CDG method can be derived from the WG method by eliminating the unknowns defined on boundaries of elements. Since the weak Galerkin method was first developed in [24], it has been applied to solve the problems arising from science and engineering such as heat equation, porous media flow, interface problems, Helmholtz equation, Oseen equation, the Stokes equations, Maxwell equations, Cohn-Hilliard equations, etc, [1,5–7,9,11–16,18–20,22–25,28–31].

The CDG methods use discontinuous P_k element, which introduces many more degrees of freedom. It is interested to know if there exists a finite element method that fully utilizes all the unknowns of discontinuous P_k element to achieve higher convergence rate than the optimal order. A novel CDG method has been developed for the Poisson equation on rectangular mesh in [27] with order two superconvergence.

In this paper, we develop a new CDG finite element method for the Stokes equations on triangle/tetrahedron. Discontinuous P_k element is used for velocity and continuous P_{k+1} element is used for pressure. Order two superconvergence is proved for velocity in an energy norm and the L^2 norm. Optimal order of convergence is obtained for pressure. It is proved that the superconvergent P_k solution is lifted elementwise to a P_{k+2} velocity which converges at the optimal order. The numerical tests verify the theorems.

2. Preliminary

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of triangles in 2D, or tetrahedra in 3D. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and by $h = \max_{T \in \mathcal{T}_h} h_T$ the mesh size of $\mathcal{T}_h$. Denote by $\mathcal{E}_h$ the set of all edges/faces in $\mathcal{T}_h$, and by $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ the set of all interior edges s or face-triangles.

For the purpose of error analysis, we define a WG (weak Galerkin) finite element space for $k \geq 1$ as follows,

$$\begin{aligned} \tilde{V}_h =&\{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0|_T \in [P_k(T)]^d,\ \mathbf{v}_b|_e \in [P_{k+1}(e)]^d, \\ & e \subset \partial T,\ T \in \mathcal{T}_h,\ \mathbf{v}_b|_{\partial\Omega} = 0\}. \end{aligned} \tag{2.1}$$

Please note that any function $\mathbf{v} \in \tilde{V}_h$ has a single value $\mathbf{v}_b$ on each edge $e \in \mathcal{E}_h$.

For $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \tilde{V}_h$, its weak gradient $\nabla_w \mathbf{v}$ is a piecewise matrix valued polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w \mathbf{v}|_T \in [P_{k+1}(T)]^{d\times d}$ satisfies

$$(\nabla_w \mathbf{v}, \boldsymbol{\tau})_T = -(\mathbf{v}_0, \nabla\cdot\boldsymbol{\tau})_T + \langle \mathbf{v}_b, \boldsymbol{\tau}\cdot\mathbf{n}\rangle_{\partial T} \qquad \forall \boldsymbol{\tau} \in [P_{k+1}(T)]^{d\times d}. \tag{2.2}$$

For a function $\mathbf{v} \in \tilde{V}_h$, its weak divergence $\nabla_w \cdot \mathbf{v}$ is a piecewise polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w \cdot \mathbf{v}|_T \in P_{k+1}(T)$ satisfies,

$$(\nabla_w \cdot \mathbf{v},\ w)_T = -(\mathbf{v}_0,\ \nabla w)_T + \langle \mathbf{v}_b\cdot\mathbf{n},\ w\rangle_{\partial T} \quad \forall w \in P_{k+1}(T). \tag{2.3}$$

We introduce two norms $|||\mathbf{v}|||$ and $\|\mathbf{v}\|_{1,h}$ for $\mathbf{v} \in \tilde{V}_h$ as follows:

$$|||\mathbf{v}|||^2 = \sum_{T\in\mathcal{T}_h} (\nabla_w \mathbf{v}, \nabla_w \mathbf{v})_T, \tag{2.4}$$

$$\|\mathbf{v}\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} \|\nabla \mathbf{v}_0\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2. \tag{2.5}$$

The following lemma is proved in [2].

Lemma 2.1. *For $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \tilde{V}_h$, we have*

$$C_1\|\mathbf{v}\|_{1,h} \le |||\mathbf{v}||| \le C_2\|\mathbf{v}\|_{1,h}. \tag{2.6}$$

3. The CDG Method and Its Well Posedness

In this section, a new CDG finite element method and its well posedness are investigated. First we introduce two finite element spaces for velocity and pressures as follows: for $k \ge 1$ and given $\mathcal{T}_h$,

$$V_h = \left\{\mathbf{v} \in [L^2(\Omega)]^d :\ \mathbf{v}|_T \in [P_k(T)]^d,\ T \in \mathcal{T}_h\right\}, \tag{3.1}$$

and

$$W_h = \left\{w \in H^1(\Omega) \cap L_0^2(\Omega) :\ w|_T \in P_{k+1}(T)\right\}. \tag{3.2}$$

Let Π_k denote a generic local L^2 projection onto $[P_k(D)]^i$ where the set D could be an element $T \in \mathcal{T}_h$ or an edge/face $e \in \mathcal{E}_h$ and i can be 1, 2, or 3. Define

$$Q_h\mathbf{u} = \{\Pi_k \mathbf{u}, \Pi_{k+1}\mathbf{u}\} \in \tilde{V}_h. \tag{3.3}$$

We define an embedding operator $E_h : V_h \to \tilde{V}_h$ such that for $\mathbf{v} \in V_h$

$$E_h\mathbf{v} = \{\mathbf{v}, \mathbf{v}_b\} \in \tilde{V}_h, \tag{3.4}$$

where $\mathbf{v}_b$ is an edge-function of $\mathbf{v} \in V_h$. For our new CDG method, we choose $\mathbf{v}_b$ such that

$$\mathbf{v}_b|_e = \begin{cases} \Pi_{k+1}(E_{k+2}\mathbf{v}) & \text{if } e \in \mathcal{E}_h^0 \\ 0 & \text{if } e \subset \partial\Omega, \end{cases} \tag{3.5}$$

where a pseudo-projection $E_{k+2} : \prod_{T\subset U_e} P_k(T) \to P_{k+2}(U_e)$ is to be defined next in (3.7). For an $e \in \mathcal{E}_h^0$, we let U_e be the union of enough triangles near e such that four aligned squares of size Ch, shown as in Figure 1, are contained inside some

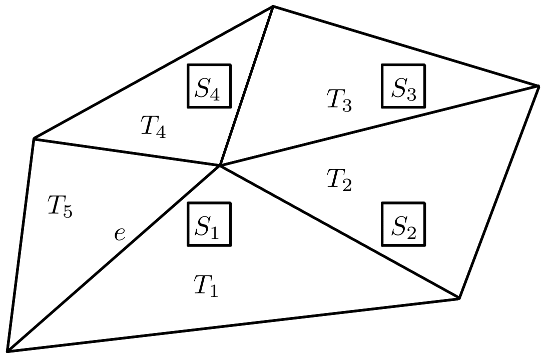

Figure 1. A closed polygon $U_e = \cup_{i=1}^5 \overline{T_i}$ contains 4 aligned squares $\{S_i, i = 1, \ldots, 4\}$, for an edge e, where $\overline{T_i}$ is the closure of T_i.

triangles. In 3D, we need the neighbor tetrahedra contain eight aligned cubes of size Ch.

It is proved in [26] that the local L_2 projection $\Pi_k(\cup_{i=1}^4 S_i) : \ P_{k+2}(U_e) \to \prod_{i=1}^4 P_k(S_i)$ is an injection mapping, i.e., for $u \in P_{k+2}(U_e)$, if $\Pi_k u = 0$ on $\cup_{i=1}^4 S_i$, then $u = 0$, where

$$(\Pi_k u, v)_{S_i} = (u, v)_{S_i} \quad \forall v \in P_k(S_i), \ i = 1, ..., 4. \tag{3.6}$$

The lifting operator E_{k+2} is the pseudo-inverse of $\Pi_k|_{P_{k+2}(U_e)}$. That is, for an $e \in \mathcal{E}^0$ and a $v \in \prod_{i=1}^4 P_k(S_i)$, $E_{k+2} v \in P_{k+2}(U_e)$ such that

$$(E_{k+2} v, q)_{\cup_{i=1}^4 S_i} = (v, q)_{\cup_{i=1}^4 S_i} \quad \forall q \in R_e \subset \prod_{i=1}^4 P_k(S_i), \tag{3.7}$$

where R_e is the image space of $P_{k+2}(U_e)$ under the local L^2 projection, i.e., $R_e = \{v = \Pi_k(\cup_{i=1}^4 S_i) u \ : \ \text{ for some } \ u \in P_{k+2}(U_e)\}$.

Since $E_h \mathbf{v} \in \tilde{V}_h$, $\nabla_w E_h \mathbf{v}$ and $\nabla_w \cdot E_h \mathbf{v}$ can be calculated by (2.2) and (2.3) respectively. For $\mathbf{v} \in V_h$, its weak gradient $\nabla_w \mathbf{v}$ and weak divergence $\nabla_w \cdot \mathbf{v}$ are defined as

$$\nabla_w \mathbf{v} = \nabla_w E_h \mathbf{v}, \tag{3.8}$$
$$\nabla_w \cdot \mathbf{v} = \nabla_w \cdot E_h \mathbf{v}. \tag{3.9}$$

The CDG Algorithm 1. A numerical approximation for (1.1)–(1.3) is seeking $(\mathbf{u}_h, p_h) \in V_h \times W_h$ such that for all $(\mathbf{v}, w) \in V_h \times W_h$,

$$(\nabla_w \mathbf{u}_h, \ \nabla_w \mathbf{v}) - (\nabla_w \cdot \mathbf{v}, \ p_h) = (\mathbf{f}, \ \mathbf{v}) \qquad \forall \mathbf{v} \in V_h, \tag{3.10}$$
$$(\nabla_w \cdot \mathbf{u}_h, \ w) = 0 \qquad \forall w \in W_h. \tag{3.11}$$

For any function $\varphi \in H^1(T)$, the following trace inequality holds true :

$$\|\varphi\|_e^2 \le C \left(h_T^{-1} \|\varphi\|_T^2 + h_T \|\nabla \varphi\|_T^2 \right). \tag{3.12}$$

To derive the inf-sup condition for the finite element formulation (3.10)-(3.11), we need the following lemma, which is proved in [27].

Lemma 3.1. *For $k \geq 1$, the following estimate holds true,*

$$\|E_{k+2}\Pi_k \mathbf{u} - \mathbf{u}\|_{0,U_e} \leq Ch^{k+3}|\mathbf{u}|_{k+3,U_e}. \tag{3.13}$$

Lemma 3.2. *There exists a positive constant β independent of h such that for all $\rho \in W_h$,*

$$\sup_{\mathbf{v}\in V_h} \frac{(\nabla_w \cdot \mathbf{v}, \rho)}{|\!|\!|\mathbf{v}|\!|\!|} \geq \beta\|\rho\|. \tag{3.14}$$

Proof. For any given $\rho \in W_h \subset L_0^2(\Omega)$, it is known [8] that there exists a function $\tilde{\mathbf{v}} \in [H_0^1(\Omega)]^d$ such that

$$\frac{(\nabla \cdot \tilde{\mathbf{v}}, \rho)}{\|\tilde{\mathbf{v}}\|_1} \geq C\|\rho\|, \tag{3.15}$$

where $C > 0$ is a constant independent of h. Let $\mathbf{v} = \Pi_k \tilde{\mathbf{v}} \in V_h$. It follows from (3.8), (2.6), (3.13) and (3.12),

$$\begin{aligned}
|\!|\!|\mathbf{v}|\!|\!|^2 &= |\!|\!|E_h\mathbf{v}|\!|\!|^2 \leq C\|E_h\mathbf{v}\|_{1,h}^2 \\
&= C\Big(\sum_{T\in\mathcal{T}_h} \|\nabla \mathbf{v}\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{v} - \Pi_{k+1}(E_{k+2}\mathbf{v})\|_{\partial T}^2\Big) \\
&= C\Big(\sum_{T\in\mathcal{T}_h} \|\nabla(\Pi_k\tilde{\mathbf{v}})\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1}\|\Pi_k\tilde{\mathbf{v}} - \Pi_{k+1}(E_{k+2}\Pi_k\tilde{\mathbf{v}})\|_{\partial T}^2\Big) \\
&\leq C\|\tilde{\mathbf{v}}\|_1^2,
\end{aligned}$$

which implies

$$|\!|\!|\mathbf{v}|\!|\!| \leq C\|\tilde{\mathbf{v}}\|_1. \tag{3.16}$$

It follows from (3.8) that for $\mathbf{v}_b$ defined in (3.5),

$$\begin{aligned}
(\nabla_w \cdot \mathbf{v},\ \rho) &= -(\mathbf{v},\ \nabla\rho) + \langle \mathbf{v}_b \cdot \mathbf{n}, \rho\rangle_{\partial\mathcal{T}_h} \\
&= -(\mathbf{v},\ \nabla\rho) = -(\Pi_k\tilde{\mathbf{v}},\ \nabla\rho) \\
&= -(\tilde{\mathbf{v}},\ \nabla\rho) = (\nabla\cdot\tilde{\mathbf{v}},\ \rho).
\end{aligned} \tag{3.17}$$

Using (3.17), (3.16) and (3.15), we have

$$\frac{(\nabla_w \cdot \mathbf{v}, \rho)}{|\!|\!|\mathbf{v}|\!|\!|} = \frac{(\nabla \cdot \tilde{\mathbf{v}}, \rho)}{|\!|\!|\mathbf{v}|\!|\!|} \geq \frac{(\nabla \cdot \tilde{\mathbf{v}}, \rho)}{C\|\tilde{\mathbf{v}}\|_1} \geq \beta\|\rho\|,$$

for a positive constant β. This completes the proof of the lemma. □

Lemma 3.3. *The weak Galerkin method (3.10)-(3.11) has a unique solution.*

Proof. It suffices to show that zero is the only solution of (3.10)-(3.11) if $\mathbf{f} = 0$. To this end, let $\mathbf{f} = 0$ and take $\mathbf{v} = \mathbf{u}_h$ in (3.10) and $w = p_h$ in (3.11). By adding the two resulting equations, we obtain

$$(\nabla_w \mathbf{u}_h,\ \nabla_w \mathbf{u}_h) = 0,$$

which implies that $\nabla_w \mathbf{u}_h = 0$ on each element T. By (2.6), we have $\|E_h\mathbf{u}_h\|_{1,h} = 0$ which implies that $\mathbf{u}_h = 0$.

Since $\mathbf{u}_h = 0$ and $\mathbf{f} = 0$, the equation (3.10) becomes $(\nabla\cdot\mathbf{v},\ p_h) = 0$ for any $\mathbf{v} \in V_h$. Then the inf-sup condition (3.14) implies $p_h = 0$. We have proved the lemma. □

4. Superconvergence in energy norm

In this section, order two superconvergence is derived for the CDG finite element solution defined in (3.10)-(3.11). We will use the superconvergence results of the corresponding WG method [21] in our error analysis. The WG method in [21] is to find $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \tilde{V}_h \times W_h$ such that for any $(\mathbf{v}, w) \in \tilde{V}_h \times W_h$,

$$(\nabla_w \tilde{\mathbf{u}}_h,\ \nabla_w \mathbf{v}) - (\nabla_w \cdot \mathbf{v},\ \tilde{p}_h) = (\mathbf{f},\ \mathbf{v}_0) \qquad \forall \mathbf{v} \in \tilde{V}_h, \tag{4.1}$$

$$(\nabla_w \cdot \tilde{\mathbf{u}}_h,\ w) = 0 \qquad \forall w \in W_h. \tag{4.2}$$

Lemma 4.1. *Let $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \tilde{V}_h \times W_h$ be the solution of (4.1)-(4.2). Then, the following error estimates hold true*

$$\|\!|Q_h \mathbf{u} - \tilde{\mathbf{u}}_h|\!\| + \|\Pi_{k+1} p - \tilde{p}_h\|_0 \le Ch^{k+2}(|\mathbf{u}|_{k+3} + |p|_{k+2}), \tag{4.3}$$

$$\|\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h\|_0 \le Ch^{k+3}(|\mathbf{u}|_{k+3} + |p|_{k+2}). \tag{4.4}$$

The proof of the above lemma can be found in [21].

Lemma 4.2. *Let $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \tilde{V}_h \times W_h$ and $(\mathbf{u}_h, p_h) \in V_h \times W_h$ be the solution of (4.1)-(4.2) and (3.10)-(3.11) respectively. Then,*

$$\|\tilde{p}_h - p_h\|_0 \le C\|\!|\tilde{\mathbf{u}}_h - \mathbf{u}_h|\!\|. \tag{4.5}$$

Proof. By (3.8) and (3.9), the equation (3.10) is equivalent to the following for any $\mathbf{v} \in V_h$,

$$(\nabla_w \mathbf{u}_h,\ \nabla_w \mathbf{v}) - (\nabla_w \cdot \mathbf{v},\ p_h) = (\mathbf{f},\ \mathbf{v}) \quad \forall E_h \mathbf{v} \in \tilde{V}_h. \tag{4.6}$$

The difference of (4.6) and (4.1) gives

$$(\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h),\ \nabla_w \mathbf{v}) - (\nabla_w \cdot \mathbf{v},\ \tilde{p}_h - p_h) = 0 \quad \forall E_h \mathbf{v} \in \tilde{V}_h. \tag{4.7}$$

It follows from (3.14),

$$\begin{aligned} \|\tilde{p}_h - p_h\|_0 &\le C \sup_{\mathbf{v} \in V_h} \frac{(\nabla_w \cdot \mathbf{v}, \tilde{p}_h - p_h)}{\|\!|\mathbf{v}|\!\|} \\ &= C \sup_{\mathbf{v} \in V_h} \frac{(\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h),\ \nabla_w \mathbf{v})}{\|\!|\mathbf{v}|\!\|} \\ &\le C\|\!|\tilde{\mathbf{u}}_h - \mathbf{u}_h|\!\|, \end{aligned} \tag{4.8}$$

which completes the proof. ☐

Lemma 4.3. *Let $\mathbf{u} \in [H^{k+3}(\Omega)]^d$ and Q_h be defined in (3.3). Then we have*

$$\|\!|Q_h \mathbf{u} - \Pi_k \mathbf{u}|\!\| \le Ch^{k+2}|\mathbf{u}|_{k+3}. \tag{4.9}$$

The proof of the above lemma can be found in [27]. It follows from (4.3) and (4.9),

$$\|\!|\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h|\!\| \le \|\!|\Pi_k \mathbf{u} - Q_h \mathbf{u}|\!\| + \|\!|Q_h \mathbf{u} - \tilde{\mathbf{u}}_h|\!\| \le Ch^{k+2}(|\mathbf{u}|_{k+3} + |p|_{k+2}). \tag{4.10}$$

Lemma 4.4. *Let $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \tilde{V}_h \times W_h$ and $(\mathbf{u}_h, p_h) \in V_h \times W_h$ be the solution of (4.1)-(4.2) and (3.10)-(3.11) respectively. Then,*

$$\|\!|\tilde{\mathbf{u}}_h - \mathbf{u}_h|\!\| \le Ch^{k+2}(|\mathbf{u}|_{k+3} + |p|_{k+2}). \tag{4.11}$$

Proof. The difference of (3.11) and (4.2) gives

$$(\nabla_w \cdot (\tilde{\mathbf{u}}_h - \mathbf{u}_h), w) = 0 \quad \forall w \in W_h. \tag{4.12}$$

Letting $w = \tilde{p}_h - p_h$ in (4.12) gives

$$(\nabla_w \cdot (\tilde{\mathbf{u}}_h - \mathbf{u}_h), \tilde{p}_h - p_h) = 0. \tag{4.13}$$

Using (4.13), we have

$$(\nabla_w \cdot (\Pi_k \mathbf{u} - \mathbf{u}_h), \tilde{p}_h - p_h) = (\nabla_w \cdot (\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h), \tilde{p}_h - p_h). \tag{4.14}$$

Letting $\mathbf{v} = \Pi_k \mathbf{u} - \mathbf{u}_h$ in (4.7) and using (4.14), we have

$$\begin{aligned} 0 &= (\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h),\ \nabla_w(\Pi_k \mathbf{u} - \mathbf{u}_h)) - (\nabla_w \cdot (\Pi_k \mathbf{u} - \mathbf{u}_h),\ \tilde{p}_h - p_h) \\ &= (\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h),\ \nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h)) + (\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h),\ \nabla_w(\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h)) \\ &\quad - (\nabla_w \cdot (\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h), \tilde{p}_h - p_h), \end{aligned} \tag{4.15}$$

which implies

$$|\!|\!|\tilde{\mathbf{u}}_h - \mathbf{u}_h|\!|\!|^2 = (\nabla_w \cdot (\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h), \tilde{p}_h - p_h) - (\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h),\ \nabla_w(\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h)).$$

It follows from the above equation, (4.5) and (4.10),

$$\begin{aligned} |\!|\!|\tilde{\mathbf{u}}_h - \mathbf{u}_h|\!|\!|^2 &\le \|\nabla_w \cdot (\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h)\|_0 \|\tilde{p}_h - p_h\|_0 \\ &\quad + \|\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h)\|_0 \|\nabla_w(\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h))\|_0 \\ &\le C |\!|\!|\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h|\!|\!| |\!|\!|\tilde{\mathbf{u}}_h - \mathbf{u}_h|\!|\!| \\ &\le Ch^{k+2}(|\mathbf{u}|_{k+3} + |p|_{k+2}) |\!|\!|\tilde{\mathbf{u}}_h - \mathbf{u}_h|\!|\!|, \end{aligned} \tag{4.16}$$

which yields

$$|\!|\!|\tilde{\mathbf{u}}_h - \mathbf{u}_h|\!|\!| \le Ch^{k+2}(|\mathbf{u}|_{k+3} + |p|_{k+2}).$$

We have completed the proof. □

Theorem 4.1. *Let $(\mathbf{u}_h, p_h) \in V_h \times W_h$ be the solution of (3.10)-(3.11). Then, the following error estimate holds true*

$$|\!|\!|\Pi_k \mathbf{u} - \mathbf{u}_h|\!|\!| + \|\Pi_{k+1} p - p_h\|_0 \le Ch^{k+2}(|\mathbf{u}|_{k+3} + |p|_{k+2}). \tag{4.17}$$

Proof. Using (4.10) and (4.11), we have

$$\begin{aligned} |\!|\!|\Pi_k \mathbf{u} - \mathbf{u}_h|\!|\!| &\le |\!|\!|\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_h|\!|\!| + |\!|\!|\tilde{\mathbf{u}}_h - \mathbf{u}_h|\!|\!| \\ &\le Ch^{k+2}(|\mathbf{u}|_{k+3} + |p|_{k+2}). \end{aligned} \tag{4.18}$$

It follows from (4.4), (4.5) and (4.11),

$$\begin{aligned} \|\Pi_{k+1} p - p_h\|_0 &\le \|\Pi_{k+1} p - \tilde{p}_h\|_0 + \|\tilde{p}_h - p_h\|_0 \\ &\le \|\Pi_{k+1} p - \tilde{p}_h\|_0 + C |\!|\!|\tilde{\mathbf{u}}_h - \mathbf{u}_h|\!|\!| \\ &\le Ch^{k+2}(|\mathbf{u}|_{k+3} + |p|_{k+2}). \end{aligned} \tag{4.19}$$

The proof of the theorem is completed. □

5. Error Estimates in L^2 Norm

In this section, order two superconvergence for velocity in the L^2 norm will be studied. Consider the dual problem of seeking $(\boldsymbol{\psi}, \xi)$ such that

$$-\Delta\boldsymbol{\psi} + \nabla\xi = \tilde{\mathbf{u}}_0 - \mathbf{u}_h \quad \text{in } \Omega, \tag{5.1}$$

$$\nabla\cdot\boldsymbol{\psi} = 0 \quad \text{in } \Omega, \tag{5.2}$$

$$\boldsymbol{\psi} = 0 \quad \text{on } \partial\Omega. \tag{5.3}$$

Recall $\tilde{\mathbf{u}}_h = \{\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_b\}$ and $\mathbf{u}_h$ are the solution for (4.1)-(4.2) and (3.10)-(3.11) respectively. Assume the following a priori estimate holds true:

$$\|\boldsymbol{\psi}\|_2 + \|\xi\|_1 \le C\|\tilde{\mathbf{u}}_0 - \mathbf{u}_h\|_0. \tag{5.4}$$

Theorem 5.1. *Let $(\mathbf{u}_h, p_h) \in V_h \times W_h$ be the solution of (3.10)-(3.11). Then,*

$$\|\Pi_k\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^{k+3}(|\mathbf{u}|_{k+3} + |p|_{k+2}). \tag{5.5}$$

Proof. Let $(\boldsymbol{\psi}_h, \xi_h) \in \tilde{V}_h \times W_h$ be the solution of WG method defined in (4.1)-(4.2) for the problem (5.1)-(5.3) such that for any $(\mathbf{v}, w) \in \tilde{V}_h \times W_h$

$$(\nabla_w\boldsymbol{\psi}_h,\ \nabla_w\mathbf{v}) - (\nabla_w\cdot\mathbf{v},\ \xi_h) = (\tilde{\mathbf{u}}_0 - \mathbf{u}_h,\ \mathbf{v}_0), \tag{5.6}$$

$$(\nabla_w\cdot\boldsymbol{\psi}_h,\ w) = 0. \tag{5.7}$$

Letting $\mathbf{v} = \tilde{\mathbf{u}}_h - E_h\mathbf{u}_h$ in (5.6) and using (3.8)-(3.9) and (4.12) give

$$\begin{aligned}\|\tilde{\mathbf{u}}_0 - \mathbf{u}_h\|_0^2 &= (\nabla_w\boldsymbol{\psi}_h,\ \nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h)) - (\nabla_w\cdot(\tilde{\mathbf{u}}_h - \mathbf{u}_h),\ \xi_h) \\ &= (\nabla_w\boldsymbol{\psi}_h,\ \nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h)).\end{aligned} \tag{5.8}$$

Let $\mathbf{v} = \Pi_k\boldsymbol{\psi}$ in (4.7) and we obtain

$$(\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h),\ \nabla_w\Pi_k\boldsymbol{\psi}) - (\nabla_w\cdot\Pi_k\boldsymbol{\psi},\ \tilde{p}_h - p_h) = 0. \tag{5.9}$$

It follows from (5.8), (5.9) and (5.2),

$$\begin{aligned}\|\tilde{\mathbf{u}}_0 - \mathbf{u}_h\|_0^2 &= (\nabla_w(\boldsymbol{\psi}_h - \Pi_k\boldsymbol{\psi}),\ \nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h)) + (\nabla_w\cdot\Pi_k\boldsymbol{\psi},\ \tilde{p}_h - p_h) \\ &= (\nabla_w(\boldsymbol{\psi}_h - \Pi_k\boldsymbol{\psi}),\ \nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h)) + (\nabla_w\cdot(\Pi_k\boldsymbol{\psi} - \boldsymbol{\psi}),\ \tilde{p}_h - p_h).\end{aligned} \tag{5.10}$$

By (5.10), (4.5) and (4.11), we have

$$\begin{aligned}\|\tilde{\mathbf{u}}_0 - \mathbf{u}_h\|_0^2 &\le \|\nabla_w(\boldsymbol{\psi}_h - \Pi_k\boldsymbol{\psi})\|_0\|\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h))\|_0 \\ &\quad + \|\nabla_w\cdot(\Pi_k\boldsymbol{\psi} - \boldsymbol{\psi})\|_0\|\tilde{p}_h - p_h\|_0 \\ &\le Ch^{k+3}(|\mathbf{u}|_{k+3} + |p|_{k+2})(\|\boldsymbol{\psi}\|_2 + \|\xi\|_1).\end{aligned} \tag{5.11}$$

By (5.11) and (5.4), we have

$$\|\tilde{\mathbf{u}}_0 - \mathbf{u}_h\|_0 \le Ch^{k+3}(|\mathbf{u}|_{k+3} + |p|_{k+2}). \tag{5.12}$$

The triangle inequality, (4.4) and (5.12) imply

$$\|\Pi_k\mathbf{u} - \mathbf{u}_h\|_0 \le \|\Pi_k\mathbf{u} - \tilde{\mathbf{u}}_0\|_0 + \|\tilde{\mathbf{u}}_0 - \mathbf{u}_h\|_0 \le Ch^{k+3}(|\mathbf{u}|_{k+3} + |p|_{k+2}),$$

which completes the proof. $\square$

6. Two-order lifted solution

As the P_k conforming discontinuous Galerkin solution $\mathbf{u}_h$ is two-order superconvergent in both L^2 norm and H^1-like norm, we define a local post-process, which lifts such a P_k solution to an optimal-order P_{k+2} solution.

On each element T, we solve a local problem that finds $\hat{\mathbf{u}}_h \in \Pi_{T\in\mathcal{T}_h} P_{k+2}(T)$ by

$$(\nabla\hat{\mathbf{u}}_h - \nabla_w\mathbf{u}_h, \nabla\mathbf{v})_T = 0 \quad \forall\mathbf{v}\in[P_{k+2}(T)]^2\setminus[P_0(T)]^2, \tag{6.1}$$

$$(\hat{\mathbf{u}}_h - \mathbf{u}_h, \mathbf{v})_T = 0 \qquad \forall\mathbf{v}\in[P_0(T)]^2, \tag{6.2}$$

where $\mathbf{u}_h$ is the CDG solution in (3.10). We prove briefly that (6.1)–(6.2) well define a solution. Letting $\mathbf{u}_h = \mathbf{0}$ in (6.1), we get that $\|\nabla\hat{\mathbf{u}}_h\|_0^2 = 0$ and $\hat{\mathbf{u}}_h$ is a constant vector on each T. By (6.2), the constant is zero. Thus (6.1)-(6.2) has a unique solution. As the linear system is square and finite dimensional, the uniqueness implies the existence of solution.

Theorem 6.1. *Let $\mathbf{u}\in H_0^1(\Omega)\cap H^{k+3}(\Omega)$ be the exact solution of* (1.1). *Let $\hat{\mathbf{u}}_h \in \Pi_{T\in\mathcal{T}_h} P_{k+2}(T)$ be the locally lifted solution of* (6.1)–(6.2)*.There exists a constant C such that*

$$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_0 \le Ch^{k+3}|\mathbf{u}|_{k+3}. \tag{6.3}$$

Proof. (6.2) means that

$$\Pi_0\hat{\mathbf{u}}_h = \Pi_0\mathbf{u}_h,$$

where Π_k denotes the elementwise L^2 projection to $[P_k]^2$ space. We separate the error into two parts.

$$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_0 \le \|\Pi_0(\mathbf{u}-\hat{\mathbf{u}}_h)\|_0 + \|(I-\Pi_0)(\mathbf{u}-\hat{\mathbf{u}}_h)\|_0.$$

For the P_0 error, by (5.5), we have

$$\|\Pi_0(\mathbf{u}-\hat{\mathbf{u}}_h)\|_0 = \|\Pi_0(\Pi_k\mathbf{u}-\mathbf{u}_h)\|_0 \le \|\Pi_k\mathbf{u}-\mathbf{u}_h\|_0 \le Ch^{k+3}|\mathbf{u}|_{k+3}.$$

For the P_0-orthogonal error, we separate it further into two.

$$\begin{aligned}\|(I-\Pi_0)(\mathbf{u}-\hat{\mathbf{u}}_h)\|_0 &\le Ch\|\nabla(\mathbf{u}-\hat{\mathbf{u}}_h)\|_0\\ &\le Ch\|\nabla(\mathbf{u}-\Pi_{k+2}\mathbf{u})\|_0 + Ch\|\nabla(\Pi_{k+2}\mathbf{u}-\hat{\mathbf{u}}_h)\|_0\\ &\le Ch^{k+3}|\mathbf{u}|_{k+3} + Ch\|\nabla(\Pi_{k+2}\mathbf{u}-\hat{\mathbf{u}}_h)\|_0.\end{aligned}$$

In [1], it is proved that

$$\nabla_w(Q_h\mathbf{u}) = \Pi_{k+1}\nabla\mathbf{u}. \tag{6.4}$$

By (6.4), (6.1) and (4.9),

$$\begin{aligned}&\|\nabla(\Pi_{k+2}\mathbf{u}-\hat{\mathbf{u}}_h)\|_0^2\\ &= (\nabla(\Pi_{k+2}\mathbf{u}-\mathbf{u}), \nabla(\Pi_{k+2}\mathbf{u}-\hat{\mathbf{u}}_h)) + (\nabla\mathbf{u}-\Pi_{k+1}\nabla\mathbf{u}, \nabla(\Pi_{k+2}\mathbf{u}-\hat{\mathbf{u}}_h))\\ &\quad + (\nabla_w Q_h\mathbf{u}-\nabla_w\mathbf{u}_h, \nabla(\Pi_{k+2}\mathbf{u}-\hat{\mathbf{u}}_h))\\ &\le \Big(\|\nabla(\Pi_{k+2}\mathbf{u}-\mathbf{u})\|_0 + \|\nabla\mathbf{u}-\Pi_{k+1}\nabla\mathbf{u}\|_0 + \|\nabla_w(Q_h\mathbf{u}-\mathbf{u}_h)\|_0\Big)\\ &\quad \times\|\nabla(\Pi_{k+2}\mathbf{u}-\hat{\mathbf{u}}_h)\|_0\\ &\le Ch^{k+2}|\mathbf{u}|_{k+3}\|\nabla(\Pi_{k+2}\mathbf{u}-\hat{\mathbf{u}}_h)\|_0.\end{aligned}$$

Combining above three inequalities, (6.3) is proved.

$\square$

7. Numerical Experiments

Consider problem (1.1)–(1.3) with $\Omega = (0,1)^2$. The source term and the boundary value $\mathbf{g}$ are chosen so that the exact solution is

$$\begin{cases} \mathbf{u}(x,y) = \begin{pmatrix} \partial g/\partial y \\ -\partial g/\partial x \end{pmatrix}, \\ p(x,y) = \dfrac{\partial^2 g}{\partial x \partial y}, \end{cases} \quad \text{where} \quad g = (x-x^2)^2(y-y^2)^2.$$

In the first computation, we use uniform triangular grids shown in Figure 2. In Table 1, we list the errors and the orders of convergence of the CDG $[P_4]^2$-C^0P_5 finite element method. That is, the velocity is approximated by the discontinuous P^4 polynomials and the pressure is approximated by the continuous P_5 polynomials. We can see that two orders of superconvergence is achieved for the velocity in both norms. The pressure solution converges at the optimal order. Further, we can see in the second part of Table 1 that the P_4 CDG solution is lifted to a P_6 solution which converges at the seventh order, two orders higher than that of $\mathbf{u}_h$.

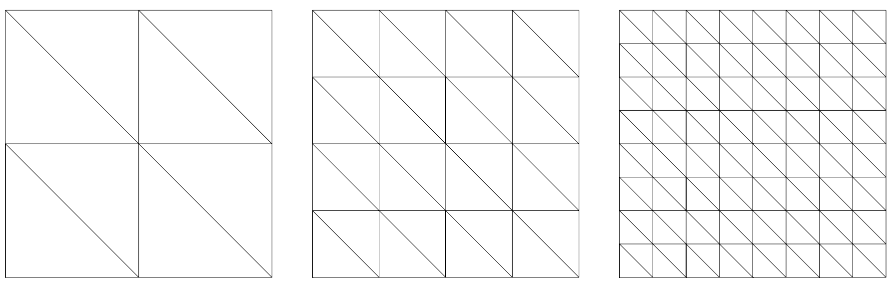

Figure 2. The first three uniform triangular grids for the computation in Tables 1–5.

Table 1. Error profiles by the CDG $[P_4]^2$-C^0P_5 finite element on grids shown in Figure 2.

Grid	$\|\Pi_4\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\!\|\!\|\Pi_4\mathbf{u}-\mathbf{u}_h\|\!\|\!\|$	rate	$\|p-p_h\|_0$	rate
1	0.480E-03	0.0	0.137E-01	0.0	0.136E-01	0.0
2	0.508E-05	6.6	0.259E-03	5.7	0.288E-03	5.6
3	0.458E-07	6.8	0.437E-05	5.9	0.494E-05	5.9
4	0.381E-09	6.9	0.706E-07	6.0	0.796E-07	6.0
	$\|\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_0$	rate		
1	0.518E-03	0.0	0.498E-03	0.0		
2	0.942E-05	5.8	0.523E-05	6.6		
3	0.264E-06	5.2	0.470E-07	6.8		
4	0.825E-08	5.0	0.390E-09	6.9		

In Tables 2–4, we list the results of the CDG $[P_3]^2$-C^0P_4, CDG $[P_2]^2$-C^0P_3 and CDG $[P_1]^2$-C^0P_2 finite element methods, respectively, on the uniform triangulations shown in Figure 2. We can see, as the theory predicts, that two orders of superconvergence is achieved for the velocity in both norms and in all cases. Further, as the theory predicts, the lifted P_{k+2} velocity solution converges at the optimal order.

Table 2. Error profiles by the CDG $[P_3]^2$-C^0P_4 finite element on grids shown in Figure 2.

Grid	$\|\Pi_3\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\|\|\Pi_3\mathbf{u}-\mathbf{u}_h\|\|\|$	rate	$\|p-p_h\|_0$	rate
1	0.430E-03	0.0	0.110E-01	0.0	0.950E-02	0.0
2	0.402E-04	3.4	0.201E-02	2.5	0.179E-02	2.4
3	0.900E-06	5.5	0.781E-04	4.7	0.634E-04	4.8
4	0.164E-07	5.8	0.270E-05	4.9	0.194E-05	5.0
5	0.276E-09	5.9	0.885E-07	4.9	0.582E-07	5.1
	$\|\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_0$	rate		
1	0.564E-03	0.0	0.458E-03	0.0		
2	0.576E-04	3.3	0.424E-04	3.4		
3	0.283E-05	4.3	0.929E-06	5.5		
4	0.171E-06	4.0	0.168E-07	5.8		
5	0.107E-07	4.0	0.282E-09	5.9		

Table 3. Error profiles by the CDG $[P_2]^2$-C^0P_3 finite element on grids shown in Figure 2.

Grid	$\|\Pi_2\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\|\|\Pi_2\mathbf{u}-\mathbf{u}_h\|\|\|$	rate	$\|p-p_h\|_0$	rate
1	0.213E-02	0.0	0.350E-01	0.0	0.202E-01	0.0
2	0.145E-03	3.9	0.540E-02	2.7	0.481E-02	2.1
3	0.794E-05	4.2	0.587E-03	3.2	0.378E-03	3.7
4	0.284E-06	4.8	0.452E-04	3.7	0.254E-04	3.9
5	0.943E-08	4.9	0.311E-05	3.9	0.166E-05	3.9
	$\|\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_0$	rate		
1	0.303E-02	0.0	0.223E-02	0.0		
2	0.268E-03	3.5	0.164E-03	3.8		
3	0.269E-04	3.3	0.857E-05	4.3		
4	0.323E-05	3.1	0.304E-06	4.8		
5	0.403E-06	3.0	0.100E-07	4.9		

Table 4. Error profiles by the CDG $[P_1]^2$-C^0P_2 finite element on grids shown in Figure 2.

Grid	$\|\Pi_1\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\|\|\Pi_1\mathbf{u}-\mathbf{u}_h\|\|\|$	rate	$\|p-p_h\|_0$	rate
4	0.326E-05	4.1	0.462E-03	2.7	0.313E-03	2.7
5	0.190E-06	4.1	0.619E-04	2.9	0.431E-04	2.9
6	0.114E-07	4.1	0.798E-05	3.0	0.565E-05	2.9
	$\|\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_0$	rate		
4	0.448E-04	2.2	0.383E-05	4.0		
5	0.111E-04	2.0	0.235E-06	4.0		
6	0.277E-05	2.0	0.145E-07	4.0		

The last computation on the uniform triangular grids is by the CDG $[P_0]^2$-C^0P_1 finite element. This element is not covered by our theory. In Table 5, we can see that the discrete pressure solution converges at the optimal order. The discrete velocity $\mathbf{u}_h$ (piecewise constant vector) converges at two orders above the optimal order in H^1-like norm, and at one order above the optimal order in L^2-norm. After local lifting, we can get an optimal order P_1 solution $\hat{\mathbf{u}}_h$.

Table 5. Error profiles by the CDG $[P_0]^2$-C^0P_1 finite element on grids shown in Figure 2.

Grid	$\|\Pi_0\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\|\|\Pi_0\mathbf{u}-\mathbf{u}_h\|\|\|$	rate	$\|p-p_h\|_0$	rate
5	0.137E-04	2.3	0.513E-03	2.0	0.327E-03	2.0
6	0.314E-05	2.1	0.129E-03	2.0	0.844E-04	2.0
7	0.765E-06	2.0	0.326E-04	2.0	0.214E-04	2.0
	$\|\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_0$	rate		
5	0.397E-03	1.0	0.142E-04	2.4		
6	0.198E-03	1.0	0.318E-05	2.2		
7	0.992E-04	1.0	0.767E-06	2.0		

We next use a family of perturbed triangular grids, shown as in Figure 3, to compute the same example. We can see, on non-uniform grids, that all the methods still have a two-order superconvergence for the velocity. And the P_k velocity is lifted to a P_{k+2} optimal-order solution.

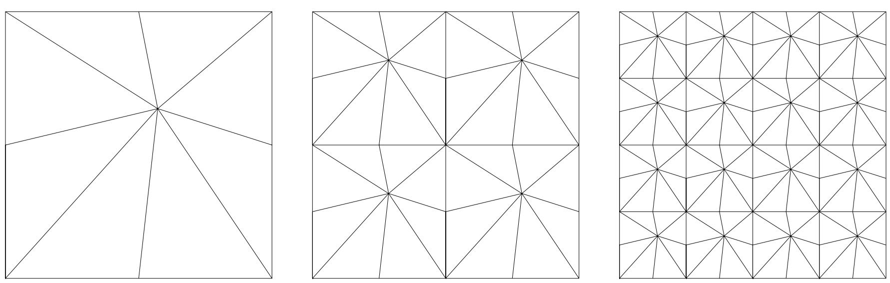

Figure 3. The first three perturbed triangular grids for the computation in Tables 6–8.

Table 6. Error profiles by the CDG $[P_1]^2$-C^0P_2 element on non-uniform grids shown in Figure 3.

Grid	$\|\Pi_1\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\|\|\Pi_1\mathbf{u}-\mathbf{u}_h\|\|\|$	rate	$\|p-p_h\|_0$	rate
4	0.348E-05	4.2	0.473E-03	2.8	0.296E-03	2.8
5	0.200E-06	4.1	0.629E-04	2.9	0.389E-04	2.9
6	0.119E-07	4.1	0.808E-05	3.0	0.499E-05	3.0
	$\|\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_0$	rate		
4	0.508E-04	2.1	0.398E-05	4.1		
5	0.126E-04	2.0	0.240E-06	4.1		
6	0.315E-05	2.0	0.147E-07	4.0		

Table 7. Error profiles by the CDG $[P_2]^2$-C^0P_3 element on non-uniform grids shown in Figure 3.

Grid	$\|\Pi_2\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\|\|\Pi_2\mathbf{u}-\mathbf{u}_h\|\|\|$	rate	$\|p-p_h\|_0$	rate
3	0.218E-06	5.0	0.161E-04	4.1	0.167E-04	4.0
4	0.713E-08	4.9	0.101E-05	4.0	0.103E-05	4.0
5	0.230E-09	5.0	0.640E-07	4.0	0.633E-07	4.0
	$\|\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_0$	rate		
3	0.316E-04	2.8	0.656E-06	4.9		
4	0.405E-05	3.0	0.209E-07	5.0		
5	0.509E-06	3.0	0.659E-09	5.0		

Table 8. Error profiles by the CDG $[P_3]^2$-C^0P_4 element on non-uniform grids shown in Figure 3.

Grid	$\|\Pi_3\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\|\|\Pi_3\mathbf{u}-\mathbf{u}_h\|\|\|$	rate	$\|p-p_h\|_0$	rate
3	0.930E-08	6.0	0.825E-06	5.1	0.826E-06	5.0
4	0.166E-09	5.8	0.253E-07	5.0	0.257E-07	5.0
5	0.233E-11	6.2	0.792E-09	5.0	0.797E-09	5.0
	$\|\mathbf{u}-\mathbf{u}_h\|_0$	rate	$\|\mathbf{u}-\hat{\mathbf{u}}_h\|_0$	rate		
3	0.393E-05	3.9	0.350E-07	6.0		
4	0.246E-06	4.0	0.553E-09	6.0		
5	0.154E-07	4.0	0.860E-11	6.0		

References

[1] A. Al-Twaeel, S. Hussian and X. Wang, *A stabilizer free weak Galerkin finite element method for parabolic equation*, J. Comput. Appl. Math., 2021, 392, 113373.

[2] A. AL-Taweel, X. Wang, X. Ye and S. Zhang, *A stabilizer free weak Galerkin method with supercloseness of order two*, Numer. Meth. PDE, 2021, 37, 1012–1029.

[3] I. Babuška, *The finite element method with Lagrangian multiplier*, Numer. Math., 1973, 20, 179–192.

[4] F. Brezzi, *On the existence, uniqueness, and approximation of saddle point problems arising from Lagrangian multipliers*, RAIRO, Anal. Numér., 1974, 2, 129–151.

[5] G. Chen, M. Feng and X. Xie, *A robust WG finite element method for convection–diffusion–reaction equations*, J. Comput. Appl. Math., 2017, 315, 107–125.

[6] W. Chen, F. Wang and Y. Wang, *Weak Galerkin method for the coupled Darcy–Stokes flow*, IMA J. Numer. Anal., 2016, 36, 897–921.

[7] B. Deka and P. Roy, *Weak galerkin finite element methods for parabolic interface problems with nonhomogeneous jump conditions*, Numer. Funct. Anal. Optim., 2019, 40, 250–279.

[8] V. Girault and P. Raviart, *Finite Element Methods for the Navier-Stokes Equations: Theory and Algorithms,* Springer-Verlag, Berlin, 1986.

[9] Q. Guan, M. Gunzburger and W. Zhao, *Weak-Galerkin finite element methods for a second-order elliptic variational inequality*, Comput. Methods Appl. Mech. and Engrg., 2018, 337, 677–688

[10] M. D. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows, A Guide to Theory, Practice and Algorithms,* Academic, San Diego, 1989.

[11] Q. Hu, Y. He and K. Wang, *Weak galerkin method for the helmholtz equation with DTN boundary condition*, Int. J of Numer. Anal. Model., 2020, 17, 643–661.

[12] G. Li, Y. Chen and Y. Huang, *A new weak Galerkin finite element scheme for general second-order elliptic problems*, J. Comput. Appl. Math., 2018, 344, 701–715.

[13] R. Lin, X. Ye, S. Zhang and P. Zhu, *A weak Galerkin finite element method for singularly perturbed convection-diffusion-reaction problems*, SIAM J. Num. Anal., 2018, 56, 1482–1497.

[14] J. Liu, S. Tavener, Z. Wang, *Lowest-order weak Galerkin finite element method for Darcy flow on convex polygonal meshes*, SIAM J. Sci. Comput., 2018, 40, 1229–1252.

[15] X. Liu, J. Li and Z. Chen, *A weak Galerkin finite element method for the Oseen equations*, Adv. Comput. Math., 2016, 42, 1473–1490.

[16] L. Mu, X. Ye and S. Zhang, *A stabilizer free, pressure robust and superconvergence weak Galerkin finite element method for the Stokes Equations on polytopal mesh*, SIAM J. Sci. Comput., 2021, 43, A2614–A2637.

[17] W. Qi and L. Song, *Weak Galerkin method with implicit θ-schemes for second-order parabolic problems*, Appl. Math. and Comp., 2020, 336, 124731.

[18] S. Shields, J. Li and E.A. Machorro, *Weak Galerkin methods for time-dependent Maxwell's equations*, Comput. Math. Appl., 2017, 74, 2106–2124.

[19] M. Sun and H. Rui, *A coupling of weak Galerkin and mixed finite element methods for poroelasticity*, Comput. & Math. with Appl., 2017, 73, 804–823.

[20] S. Toprakseven, *A weak Galerkin finite element method for time fractional reaction-diffusion-convection problems with variable coefficients*, Appl. Numer. Math., 2021, 168, 1–12.

[21] S. Xie and P. Zhu, *Superconvergence of a WG method for the Stokes equations with continuous pressure*, preprint.

[22] J. Wang, Q. Zhai, R. Zhang and S. Zhang, *A weak Galerkin finite element scheme for the Cahn-Hilliard equation*, Math. Comp., 2019, 88, 211–235.

[23] C. Wang and J. Wang, *Discretization of div–curl systems by weak Galerkin finite element methods on polyhedral partitions*, J. Sci. Comput., 2016, 68, 1144–1171.

[24] J. Wang and X. Ye, *A weak Galerkin finite element method for second-order elliptic problems*, J. Comput. Appl. Math., 2013, 241, 103–115.

[25] X. Ye and S. Zhang, *A stabilizer free weak Galerkin method for the biharmonic equation on polytopal meshes*, SIAM J. Numerical Analysis, 2020, 58, 2572–2588,

[26] X. Ye and S. Zhang, *A conforming discontinuous Galerkin finite element method*, Int. J. Numer. Anal. Model., 2020, 17(1), 110–117.

[27] X. Ye and S. Zhang, *Constructing a CDG element with order two superconvergence for second order problem on rectangular mesh*, preprint.

[28] H. Zhang, Y. Zou, S. Chai and H. Yue, *Weak Galerkin method with* $(r, r-1, r-1)$*-order finite elements for second order parabolic equations*, Appl. Math. and Comp., 2016, 275, 24–40.

[29] T. Zhang and T. Lin, *A posteriori error estimate for a modified weak Galerkin method solving elliptic problems*, Numer. Meth. PDE, 2017, 33, 381–398.

[30] A. Zhu, T. Xu and Q. Xu, *Weak G alerkin finite element methods for linear parabolic integro-differential equations*, Numer. Meth. of PDE, 2016, 32, 1357–1377.

[31] S. Zhou, F. Gao, B. Li, and Z. Sun, *Weak galerkin finite element method with second-order accuracy in time for parabolic problems*, Appl. Math. Lett., 2019, 90, 118–123.