Journal of Applied Analysis and Computation
Volume 12, Number 6, December 2022, 2600–2620
Website:http://www.jaac-online.com
DOI:10.11948/20220146

IMPROVEMENT OF THE SPECTRAL METHOD FOR SOLVING MULTI-TERM TIME-SPACE RIESZ-CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

H. Dehestani[1] and Y. Ordokhani[1,†]

Abstract In this paper, we study the numerical solution of multi-term time-space Riesz-Caputo fractional differential equations with the help of shifted Vieta-Lucas polynomials. To get the desired purpose, we introduce a new method for obtaining the operational matrices. The constructed method for finding the matrices influence directly in the accuracy of the methodology. Thus, the combination of shifted Vieta-Lucas polynomials properties with the operational matrices has reduced the problem to a system of algebraic equations. The proposed approach provides the approximate solutions to the problem which are convergent to the exact solution. Finally, we represent the accuracy and efficiency of the methodology by examining some examples and presenting the results in the form of graphs and tables.

Keywords Shifted Vieta-Lucas polynomials, Riesz fractional derivative, time-space fractional differential equations, error estimation.

MSC(2010) 65M70, 35Q99.

1. Introduction

In recent decades, fractional derivatives have been introduced as an excellent instrument for the description of memory and hereditary features of different materials. Hence, different types of fractional derivatives have been defined by many researchers such as Caputo [6], Riemann-Liouville [19], Riesz [20], fractal [8], and Atangana-Baleanu [1]. It should be mentioned that the proposed fractional derivative operators are used in the field of science and technology. For instance, biology, electrochemical process, viscoelastic materials, porous media, finance, and hydrology [13,18,26,28]. Also, various methods have been presented for solving the problems which contain the fractional derivative, see ([9,12,22,23]).

Among the fractional differential equations, fractional partial differential equations involving Riesz fractional derivative have attracted the attention of many mathematicians. On the other hand, it should be pointed out that the fractional differential equations containing Riesz fractional derivatives have appeared in signal analysis and the random walk of suspension flows and so on [14,15,25,29]. Hence, many useful numerical and analytical approaches have been described for solving

†The corresponding author. Email address:ordokhani@alzahra.ac.ir(Y. Ordokhani)
[1]Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran

the proposed problems. For example, variational iteration method [11], Jacobi tau approximation method [3], combination of the fractional predictor-corrector method with L1/L2 discrete schemes [27], improved matrix transform method [30], finite-difference scheme [31], Crank-Nicolson method [21], and so on.

The above-mentioned numerous works motivated us to introduce an efficient and accurate method to solve this class of problems. Therefore, we introduce a novel discretization method with the help of Vieta-Lucas polynomials. These polynomials have been applied in some research papers, which we point out below.

- Agarwal and El-Sayed [2] applied Vieta-Lucas polynomials with the spectral method for solving the fractional advection-dispersion equation.
- Heydari et al. [17] used proposed polynomials for the numerical solution of coupled nonlinear variable-order fractional Ginzburg-Landau equations.

For numerically solving multi-term time-space Riesz-Caputo fractional differential equations, we provide the algorithm based on the discretization method and Vieta-Lucas polynomials. The main characteristic of this method in solving the proposed problem is the operational matrices and the algorithm for calculating them. These matrices obtain with high accuracy, which properties of them are effective in the precision of results. In this work, we focus on solving the multi-term time-space Riesz-Caputo fractional differential equations with the following form:

$$\begin{aligned} K_\alpha \mathbf{D}_t^\alpha u(x,t) + K_\nu \mathbf{D}_t^\nu u(x,t) &= K_\beta \frac{\partial^\beta}{\partial |x|^\beta} u(x,t) + K_\gamma \frac{\partial^\gamma}{\partial |x|^\gamma} u(x,t) \\ &\quad + \mathcal{F}\left(x, t, u(x,t), \frac{\partial u(x,t)}{\partial t}, \frac{\partial^2 u(x,t)}{\partial t^2}\right), \quad (1.1) \\ 0 < \alpha, \nu \le 1, \quad 1 < \beta < 2,& \quad 0 < \gamma < 1, \end{aligned}$$

with the initial and boundary conditions

$$\begin{aligned} &u(x,0) = f_0(x), \quad \frac{\partial u(x,0)}{\partial t} = f_1(x), \quad 0 \le x \le 1, \\ &u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad 0 \le t \le 1, \end{aligned} \tag{1.2}$$

where K_α, K_ν, K_β and K_γ are constant, $f_0(x)$, $f_1(x)$, $g_0(t)$ and $g_1(t)$ are known functions, and $u(x,t)$ is an unknown function defined over the interval $\Omega = [0,1] \times [0,1]$. And also, $\mathbf{D}_t^\alpha$ denotes the Caputo fractional derivative operator of order α. Besides, $\frac{\partial^\beta}{\partial |x|^\beta}$ represent Riesz fractional derivative of order β, which is defined by the following formula [3]:

$$\frac{\partial^\beta}{\partial |x|^\beta} h(x) = \frac{1}{2\cos\frac{\pi\beta}{2}} \left(D_{*+}^\beta h(x) + D_{*-}^\beta h(x) \right), \quad \beta \ne 1, \tag{1.3}$$

wherein, D_{*+}^β and D_{*-}^β denote the left- and right-sided Riemann-Liouville fractional derivatives of order β, which defined by left- and right-sided Caputo fractional derivatives:

$$\begin{aligned} D_{*+}^\beta h(x) &= D_+^\beta h(x) + \sum_{i=0}^{\lceil \beta \rceil - 1} \frac{h^{(i)}(0)}{\Gamma(i+1-\beta)} x^{i-\beta}, \\ D_{*-}^\beta h(x) &= D_-^\beta h(x) + \sum_{i=0}^{\lceil \beta \rceil - 1} \frac{(-1)^i h^{(i)}(1)}{\Gamma(i+1-\beta)} (1-x)^{i-\beta}. \end{aligned} \tag{1.4}$$

This paper is divided into seven sections. Section 2 reviews the definition and properties of shifted Vieta-Lucas polynomials. In Section 3, we introduce the computational technique for constructing the operational matrix of fractional integration, modified operational matrices of Riesz fractional derivative and modified operational matrix of integration. The structure of the numerical scheme is presented in Section 4. The error analysis is formulated in Section 5. In Section 6, the performance of the approach is examined in several numerical examples. Finally, conclusion is contained in Section 7.

2. Shifted Vieta-Lucas polynomials

Here, we present the definition and properties of shifted Vieta-Lucas polynomials. The shifted Vieta-Lucas polynomials $\mathcal{VL}_m(x)$ on the interval $[0,1]$ are defined by the following recurrence formula [2]:

$$\mathcal{VL}_{m+1}(x) = (4x-2)\mathcal{VL}_m(x) - \mathcal{VL}_{m-1}(x), \quad m = 1, 2, \dots, \tag{2.1}$$

where $\mathcal{VL}_0(x) = 2$ and $\mathcal{VL}_1(x) = 4x - 2$. Also, the explicit analytical formula of shifted Vieta-Lucas polynomials of degree m is given as follows [2]:

$$\mathcal{VL}_m(x) = 2m \sum_{i=0}^{m} (-1)^i \frac{4^{m-i}\Gamma(2m-i)}{\Gamma(i+1)\Gamma(2m-2i+1)} x^{m-i}, \quad m = 2, 3, \dots. \tag{2.2}$$

The orthogonality property of these polynomials with respect to the weight function $w(x) = \frac{1}{\sqrt{x-x^2}}$ on the interval $[0,1]$ is

$$\langle \mathcal{VL}_m(x), \mathcal{VL}_n(x) \rangle = \int_0^1 \mathcal{VL}_m(x)\mathcal{VL}_n(x)w(x)dx = \begin{cases} 0, & n \neq m \neq 0, \\ 4\pi, & n = m = 0, \\ 2\pi, & n = m \neq 0. \end{cases} \tag{2.3}$$

A function $u(x,t)$ defined over the interval Ω may be expanded by shifted Vieta-Lucas polynomials with weight function $w(x,t) = \frac{1}{\sqrt{x-x^2}\sqrt{t-t^2}}$ as:

$$u(x,t) \simeq u_{M_1 M_2}(x,t) = \sum_{m=0}^{M_1} \sum_{n=0}^{M_2} c_{mn} \mathcal{VL}_m(x)\mathcal{VL}_n(t) = \mathcal{VL}^T(x) C \mathcal{VL}(t), \tag{2.4}$$

where

$$\mathcal{VL}(x) = [\mathcal{VL}_0(x), \mathcal{VL}_1(x), \dots, \mathcal{VL}_{M_1}(x)]^T, \mathcal{VL}(t) = [\mathcal{VL}_0(t), \mathcal{VL}_1(t), \dots, \mathcal{VL}_{M_2}(t)]^T,$$

and

$$C = [c_{mn}]_{(M_1+1)\times(M_2+1)}, \quad m = 0, 1, \dots, M_1, \quad n = 0, 1, \dots, M_2.$$

The coefficients are determined with the following relations:

$$c_{00} = \frac{1}{4\pi^2} \int_0^1 \int_0^1 w(x,t)u(x,t)dxdt,$$

$$c_{mn} = \frac{1}{4\pi^2} \int_0^1 \int_0^1 w(x,t)u(x,t)\mathcal{VL}_m(x)\mathcal{VL}_n(t)dxdt, m = 1, 2, \dots, M_1, n = 1, 2, \dots, M_2.$$

3. Fundamental relations for the operational matrices

This section presents the operational matrices, which we require them in further sections.

3.1. Operational matrix of fractional integration

Herein, we introduced the procedure of obtaining the elements of the operational matrix of Riemann-Liouville fractional integration. Accordingly, suppose

$$\mathbf{I}_t^{\alpha}\mathcal{VL}(t)=\frac{t^{\alpha}}{2\pi}\mathbf{\Phi}^{\alpha}\mathcal{VL}(t), \tag{3.1}$$

where $\mathbf{\Phi}^{\alpha}$ denotes the fractional integral operational matrix. To calculate the elements of this matrix, for $\mathcal{VL}_0(t)$ and $\mathcal{VL}_1(t)$, we have

$$\begin{aligned}
\mathbf{I}_t^{\alpha}\mathcal{VL}_0(t)&=\frac{2}{\Gamma(1+\alpha)}t^{\alpha}=\frac{t^{\alpha}}{2\pi}\frac{2\pi}{\Gamma(1+\alpha)}\mathcal{VL}_0(t),\\
\mathbf{I}_t^{\alpha}\mathcal{VL}_1(t)&=\frac{4}{\Gamma(2+\alpha)}t^{1+\alpha}-\frac{2}{\Gamma(1+\alpha)}t^{\alpha}=t^{\alpha}\left(\frac{4}{\Gamma(2+\alpha)}t-\frac{2}{\Gamma(1+\alpha)}\right)\\
&=\frac{t^{\alpha}}{2\pi}\left(\frac{2\pi}{\Gamma(2+\alpha)}\mathcal{VL}_1(t)+2\pi\left[\frac{\Gamma(1+\alpha)-\Gamma(2+\alpha)}{\Gamma(2+\alpha)\Gamma(1+\alpha)}\right]\mathcal{VL}_0(t)\right).
\end{aligned} \tag{3.2}$$

Also, we use Eq. (2.2) and Riemann-Liouville fractional integral operator as follows:

$$\begin{aligned}
\mathbf{I}_t^{\alpha}\mathcal{VL}_n(t)&=\mathbf{I}_t^{\alpha}\left(2n\sum_{i=0}^{n}(-1)^i\frac{4^{n-i}\Gamma(2n-i)}{\Gamma(i+1)\Gamma(2n-2i+1)}t^{n-i}\right)\\
&=2n\sum_{i=0}^{n}(-1)^i\frac{4^{n-i}\Gamma(2n-i)}{\Gamma(i+1)\Gamma(2n-2i+1)}\frac{\Gamma(n-i+1)}{\Gamma(n-i+1+\alpha)}t^{n-i+\alpha}\\
&=t^{\alpha}\sum_{i=0}^{n}b_{i,n}^{\alpha}t^{n-i},
\end{aligned} \tag{3.3}$$

where

$$b_{i,n}^{\alpha}=(-1)^i\frac{2n4^{n-i}\Gamma(2n-i)}{\Gamma(i+1)\Gamma(2n-2i+1)}\frac{\Gamma(n-i+1)}{\Gamma(n-i+1+\alpha)}.$$

Next, expanding t^{n-i} by the shifted Vieta-Lucas polynomials on the interval $[0,1]$ yields:

$$\begin{aligned}
t^{n-i}&=\sum_{j=0}^{M_2}c_j\mathcal{VL}_j(t)\\
&=\left(\frac{1}{2\pi}\int_0^1 w(t)t^{n-i}dt\right)\mathcal{VL}_0(t)+\sum_{j=1}^{M_2}\left(\frac{1}{2\pi}\int_0^1 w(t)t^{n-i}\mathcal{VL}_j(t)dt\right)\mathcal{VL}_j(t)\\
&=\frac{1}{2\pi}\left[\int_0^1 w(t)t^{n-i}dt,\int_0^1 w(t)t^{n-i}\mathcal{VL}_1(t)dt,\ldots,\int_0^1 w(t)t^{n-i}\mathcal{VL}_{M_2}(t)dt\right]\mathcal{VL}(t).
\end{aligned} \tag{3.4}$$

Therefore, from Eqs. (3.3) and (3.4), we conclude

$$\mathbf{I}_t^{\alpha}\mathcal{VL}_n(t) = \frac{t^{\alpha}}{2\pi}\left[\sum_{i=0}^{n} b_{i,n}^{\alpha}\int_0^1 w(t)t^{n-i}dt, \sum_{i=0}^{n} b_{i,n}^{\alpha}\int_0^1 w(t)t^{n-i}\mathcal{VL}_1(t)dt,\right.$$
$$\left. \dots, \quad \sum_{i=0}^{n} b_{i,n}^{\alpha}\int_0^1 w(t)t^{n-i}\mathcal{VL}_{M_2}(t)dt\right]\mathcal{VL}(t), \tag{3.5}$$

Consequently, in view of Eq. (3.5) the general form of an operational matrix of fractional integration is obtained:

$$\mathbf{\Phi}^{\alpha} = \begin{bmatrix} \frac{2\pi}{\Gamma(1+\alpha)} & 0 & \dots & 0 \\ 2\pi\left[\frac{\Gamma(1+\alpha)-\Gamma(2+\alpha)}{\Gamma(2+\alpha)\Gamma(1+\alpha)}\right] & \frac{2\pi}{\Gamma(2+\alpha)} & \dots & 0 \\ \sum_{i=0}^{2} b_{i,2}^{\alpha}\int_0^1 w(t)t^{2-i}dt & \sum_{i=0}^{2} b_{i,2}^{\alpha}\int_0^1 w(t)t^{2-i}\mathcal{VL}_1(t)dt & \dots & \sum_{i=0}^{2} b_{i,2}^{\alpha}\int_0^1 w(t)t^{2-i}\mathcal{VL}_{M_2}(t)dt \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{M_2} b_{i,M_2}^{\alpha}\int_0^1 w(t)t^{M_2-i}dt & \sum_{i=0}^{M_2} b_{i,M_2}^{\alpha}\int_0^1 w(t)t^{M_2-i}\mathcal{VL}_1(t)dt & \dots & \sum_{i=0}^{M_2} b_{i,M_2}^{\alpha}\int_0^1 w(t)t^{M_2-i}\mathcal{VL}_{M_2}(t)dt \end{bmatrix}.$$

3.2. Modified operational matrices of Riesz fractional derivative

Here, the Riesz fractional derivative of the vector $\mathcal{VL}(x)$ is defined in Eq. (2.4) can be obtained as

$$\frac{\partial^{\gamma}}{\partial|x|^{\gamma}}\mathcal{VL}(x) \simeq x^{-\gamma}\mathbf{\Lambda}\mathcal{VL}(x) + (1-x)^{-\gamma}\mathbf{\Delta}\mathcal{VL}(x), \quad 0<\gamma<1, \tag{3.6}$$

where $\mathbf{\Lambda}$ and $\mathbf{\Delta}$ are called modified operational matrices of Riesz fractional derivative. To provide the proposed matrices, we continue the approximation process for the function $\mathcal{VL}_0(x)$, $\mathcal{VL}_1(x)$ and $\mathcal{VL}_m(x)$, $m=2,3,\dots,M_1$, separately. Thus, we get

$$\begin{aligned}\frac{\partial^{\gamma}}{\partial|x|^{\gamma}}\mathcal{VL}_0(x) &= \frac{1}{2\cos\frac{\pi\gamma}{2}}\left(\frac{2}{\Gamma(1-\gamma)}x^{-\gamma} + \frac{2}{\Gamma(1-\gamma)}(1-x)^{-\gamma}\right) \\ &= x^{-\gamma}\frac{1}{2\cos\frac{\pi\gamma}{2}\Gamma(1-\gamma)}\mathcal{VL}_0(x) + (1-x)^{-\gamma}\frac{1}{2\cos\frac{\pi\gamma}{2}\Gamma(1-\gamma)}\mathcal{VL}_0(x) \\ &= x^{-\gamma}\left[\tfrac{1}{2\cos\frac{\pi\gamma}{2}\Gamma(1-\gamma)}\ 0\ 0 \dots\ 0\right]\mathcal{VL}(x) \\ &\quad +(1-x)^{-\gamma}\left[\tfrac{1}{2\cos\frac{\pi\gamma}{2}\Gamma(1-\gamma)}\ 0\ 0 \dots\ 0\right]\mathcal{VL}(x), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\frac{\partial^{\gamma}}{\partial|x|^{\gamma}}\mathcal{VL}_1(x) \\ &= \frac{1}{2\cos\frac{\pi\gamma}{2}}\left(\left[D_{+}^{\gamma}\left[4x-2\right] + \frac{-2}{\Gamma(1-\gamma)}x^{-\gamma}\right]\right. \\ &\quad \left. + \left[D_{-}^{\gamma}\left[4x-2\right] + \frac{-4}{\Gamma(2-\gamma)}(1-x)^{1-\gamma} + \frac{-2}{\Gamma(1-\gamma)}(1-x)^{1-\gamma}\right]\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\cos\frac{\pi\gamma}{2}} \left(\left[\frac{4}{\Gamma(2-\gamma)} x^{1-\gamma} + \frac{-2}{\Gamma(1-\gamma)} x^{-\gamma} \right] \right. \\
&\quad \left. + \left[\frac{-4}{\Gamma(2-\gamma)} (1-x)^{1-\gamma} + \frac{-4}{\Gamma(1-\gamma)} (1-x)^{-\gamma} + \frac{-2}{\Gamma(1-\gamma)} (1-x)^{-\gamma} \right] \right) \\
&= \frac{1}{2\cos\frac{\pi\gamma}{2}} \left(x^{-\gamma} \left[\frac{4}{\Gamma(2-\gamma)} x + \frac{-2}{\Gamma(1-\gamma)} \right] + (1-x)^{-\gamma} \left[\frac{4(x-1)}{\Gamma(2-\gamma)} + \frac{-6}{\Gamma(1-\gamma)} \right] \right) \\
&= \frac{1}{2\cos\frac{\pi\gamma}{2}} \left(x^{-\gamma} \left[\frac{1}{\Gamma(2-\gamma)} \mathcal{VL}_1(x) + \frac{\Gamma(1-\gamma)-\Gamma(2-\gamma)}{\Gamma(1-\gamma)\Gamma(2-\gamma)} \mathcal{VL}_0(x) \right] \right. \\
&\quad \left. + (1-x)^{-\gamma} \left[\frac{1}{\Gamma(2-\gamma)} \mathcal{VL}_1(x) - \frac{\Gamma(1-\gamma)+3\Gamma(2-\gamma)}{\Gamma(2-\gamma)\Gamma(1-\gamma)} \mathcal{VL}_0(x) \right] \right) \\
&= x^{-\gamma} \left[\tfrac{\Gamma(1-\gamma)-\Gamma(2-\gamma)}{2\cos\frac{\pi\gamma}{2}\Gamma(1-\gamma)\Gamma(2-\gamma)} \;\; \tfrac{1}{2\cos\frac{\pi\gamma}{2}\Gamma(2-\gamma)} \;\; 0 \ldots \; 0 \right] \\
&\quad + (1-x)^{-\gamma} \left[-\tfrac{\Gamma(1-\gamma)+3\Gamma(2-\gamma)}{2\cos\frac{\pi\gamma}{2}\Gamma(2-\gamma)\Gamma(1-\gamma)} \;\; \tfrac{1}{2\cos\frac{\pi\gamma}{2}\Gamma(2-\gamma)} \;\; 0 \ldots \; 0 \right].
\end{aligned} \tag{3.8}
$$

Also, for $m = 2, 3, \ldots, M_1$, we get

$$
\begin{aligned}
\frac{\partial^\gamma}{\partial |x|^\gamma} \mathcal{VL}_m(x) &= \frac{1}{2\cos\frac{\pi\gamma}{2}} \left\{ D^\gamma_{*+} \mathcal{VL}_m(x) + D^\gamma_{*-} \mathcal{VL}_m(x) \right\} \\
&= \frac{1}{2\cos\frac{\pi\gamma}{2}} \left\{ \sum_{i=0}^{m} \mu_i \left(D^\gamma_{*+} \left[x^{m-i} \right] + D^\gamma_{*-} \left[x^{m-i} \right] \right) \right\},
\end{aligned} \tag{3.9}
$$

where

$$
\mu_i = 2m(-1)^i \frac{4^{m-i}\Gamma(2m-i)}{\Gamma(i+1)\Gamma(2m-2i+1)}.
$$

On the other hand, the right-sided Riemann-Liouville fractional derivative of x^{m-i} is approximated with respect to shifted Vieta-Lucas polynomials as follows:

$$
\begin{aligned}
D^\gamma_{*+} \left[x^{m-i} \right] &= D^\gamma_{+} x^{m-i} + \sum_{j=0}^{\lceil \gamma \rceil - 1} \frac{(x^{m-i})^{(j)}\big|_{x=0}}{\Gamma(j+1-\gamma)} x^{j-\gamma} \\
&= \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{(\xi^{m-i})'}{(x-\xi)^\gamma} d\xi + \sum_{j=0}^{\lceil \gamma \rceil - 1} \frac{(x^{m-i})^{(j)}\big|_{x=0}}{\Gamma(j+1-\gamma)} x^{j-\gamma} \\
&\simeq x^{-\gamma} \sum_{j=0}^{M_1} a_j \mathcal{VL}_j(x).
\end{aligned} \tag{3.10}
$$

Similarly, for the left-sided Riemann-Liouville fractional derivative of x^{m-i}, we obtain

$$
\begin{aligned}
D^\gamma_{*-} \left[x^{m-i} \right] &= D^\gamma_{-} x^{m-i} + \sum_{j=0}^{\lceil \gamma \rceil - 1} \frac{(-1)^i (x^{m-i})^{(j)}\big|_{x=1}}{\Gamma(j+1-\gamma)} (1-x)^{j-\gamma} \\
&= \frac{-1}{\Gamma(1-\gamma)} \int_x^1 \frac{(\xi^{m-i})'}{(\xi-x)^\gamma} d\xi + \sum_{j=0}^{\lceil \gamma \rceil - 1} \frac{(-1)^i (x^{m-i})^{(j)}\big|_{x=1}}{\Gamma(j+1-\gamma)} (1-x)^{j-\gamma} \\
&\simeq (1-x)^{-\gamma} \sum_{j=0}^{M_1} b_j \mathcal{VL}_j(x).
\end{aligned} \tag{3.11}
$$

Therefore, from Eqs. (3.9)-(3.11), the Riesz fractional derivative of $\mathcal{VL}_m(x)$ is calculated as follows:

$$\begin{aligned}
&\frac{\partial^\gamma}{\partial|x|^\gamma}\mathcal{VL}_m(x)\\
&\simeq \frac{1}{2\cos\frac{\pi\gamma}{2}}\left\{\sum_{i=0}^{m}\mu_i\left(x^{-\gamma}\sum_{j=0}^{M_1}a_j\mathcal{VL}_j(x)+(1-x)^{-\gamma}\sum_{j=0}^{M_1}b_j\mathcal{VL}_j(x)\right)\right\}\\
&= x^{-\gamma}\sum_{j=0}^{M_1}\left(\sum_{i=0}^{m}\frac{\mu_i a_j}{2\cos\frac{\pi\gamma}{2}}\right)\mathcal{VL}_j(x)+(1-x)^{-\gamma}\sum_{j=0}^{M_1}\left(\sum_{i=0}^{m}\frac{\mu_i b_j}{2\cos\frac{\pi\gamma}{2}}\right)\mathcal{VL}_j(x)\\
&= x^{-\gamma}\sum_{j=0}^{M_1}\lambda_{ijm}^{\gamma}\mathcal{VL}_j(x)+(1-x)^{-\gamma}\sum_{j=0}^{M_1}\delta_{ijm}^{\gamma}\mathcal{VL}_j(x). \qquad (3.12)
\end{aligned}$$

Consequently, we get each element of the Riesz fractional derivative of shifted Vieta-Lucas vector as follows:

$$\mathbf{\Lambda}=\begin{bmatrix}
\frac{1}{2\cos\frac{\pi\gamma}{2}\Gamma(1-\gamma)} & 0 & 0 & \dots & 0\\
\frac{\Gamma(1-\gamma)-\Gamma(2-\gamma)}{2\cos\frac{\pi\gamma}{2}\Gamma(1-\gamma)\Gamma(2-\gamma)} & \frac{1}{2\cos\frac{\pi\gamma}{2}\Gamma(2-\gamma)} & 0 & \dots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\lambda_{i0M_1}^{\gamma} & \lambda_{i1M_1}^{\gamma} & \lambda_{i2M_1}^{\gamma} & \dots & \lambda_{iM_1M_1}^{\gamma}
\end{bmatrix},$$

and

$$\mathbf{\Delta}=\begin{bmatrix}
\frac{1}{2\cos\frac{\pi\gamma}{2}\Gamma(1-\gamma)} & 0 & 0 & \dots & 0\\
-\frac{\Gamma(1-\gamma)+3\Gamma(2-\gamma)}{2\cos\frac{\pi\gamma}{2}\Gamma(2-\gamma)\Gamma(1-\gamma)} & \frac{1}{2\cos\frac{\pi\gamma}{2}\Gamma(2-\gamma)} & 0 & \dots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\delta_{i0M_1}^{\gamma} & \delta_{i1M_1}^{\gamma} & \delta_{i2M_1}^{\gamma} & \dots & \delta_{iM_1M_1}^{\gamma}
\end{bmatrix}.$$

3.3. Modified operational matrix of integration

In this section, we present the process of calculating the components of the modified operational matrix of integration. For this purpose, we will have:

$$\begin{aligned}
&\int_0^t \mathcal{VL}_0(\eta)d\eta = 2t = \frac{1}{2}\mathcal{VL}_1(t)+\frac{1}{2}\mathcal{VL}_0(t),\\
&\int_0^t \mathcal{VL}_1(\eta)d\eta = 2t^2-2t = \frac{1}{8}\mathcal{VL}_2(t)+\frac{1}{8}\mathcal{VL}_0(t).
\end{aligned} \qquad (3.13)$$

Now, by considering the analytical form of shifted Vieta-Lucas polynomials for $m=2,3,\dots,M_2-1$, we obtain

$$\int_0^t \mathcal{VL}_m(\eta)d\eta = 2m\sum_{i=0}^{m}(-1)^i\frac{4^{m-i}\Gamma(2m-i)}{\Gamma(i+1)\Gamma(2m-2i+1)(m-i+1)}t^{m-i+1}$$

$$= \sum_{i=0}^{m} \theta_{m,i} t^{m-i+1}, \; \theta_{m,i} = \frac{2m(-1)^i 4^{m-i}\Gamma(2m-i)}{\Gamma(i+1)\Gamma(2m-2i+1)(m-i+1)}. \tag{3.14}$$

Then, by expanding t^{m-i+1} by shifted Vieta-Lucas polynomials, we have

$$\begin{aligned} t^{m-i+1} =& \frac{1}{2\pi}\left(\int_0^1 \frac{1}{\sqrt{t-t^2}} t^{m-i+1} dt\right)\mathcal{VL}_0(t) \\ &+ \sum_{j=1}^{M_2} \frac{1}{2\pi}\left(\int_0^1 \frac{1}{\sqrt{t-t^2}} t^{m-i+1}\mathcal{VL}_j(t)dt\right)\mathcal{VL}_j(t). \end{aligned} \tag{3.15}$$

Hence, by substituting the above approximation in Eq. (3.14), the following relation is achieved:

$$\begin{aligned} \int_0^t \mathcal{VL}_m(\eta)d\eta =& \sum_{i=0}^{m} \theta_{m,i}\left[\frac{1}{2\pi}\left(\int_0^1 \frac{1}{\sqrt{t-t^2}} t^{m-i+1} dt\right)\mathcal{VL}_0(t)\right. \\ &\left. + \sum_{j=1}^{M_2} \frac{1}{2\pi}\left(\int_0^1 \frac{1}{\sqrt{t-t^2}} t^{m-i+1}\mathcal{VL}_j(t)dt\right)\mathcal{VL}_j(t)\right] \\ =& \xi_{m,i,0}\mathcal{VL}_0(t) + \sum_{j=1}^{M_2} \xi_{m,i,j}\mathcal{VL}_j(t). \end{aligned} \tag{3.16}$$

In addition, to avoid errors in the approximation process, we follow the following procedure for $m = M_2$:

$$\begin{aligned} \int_0^t \mathcal{VL}_{M_2}(\eta)d\eta =& \sum_{i=1}^{M_2} \theta_{M_2,i} t^{M_2-i+1} + \theta_{M_2,0} t^{M_2+1} \\ =& \sum_{i=1}^{M_2} \theta_{M_2,i}\left[\frac{1}{2\pi}\left(\int_0^1 \frac{1}{\sqrt{t-t^2}} t^{M_2-i+1} dt\right)\mathcal{VL}_0(t)\right. \\ &\left. + \sum_{j=1}^{M_2} \frac{1}{2\pi}\left(\int_0^1 \frac{1}{\sqrt{t-t^2}} t^{M_2-i+1}\mathcal{VL}_j(t)dt\right)\mathcal{VL}_j(t)\right] + \theta_{M_2,0} t^{M_2+1} \\ =& \bar{\xi}_{M_2,i,0}\mathcal{VL}_0(t) + \sum_{j=1}^{M_2} \bar{\xi}_{M_2,i,j}\mathcal{VL}_j(t) + \theta_{M_2,0} t^{M_2+1}. \end{aligned} \tag{3.17}$$

As a result, the general form of modified operational matrix of integration and complement vector is obtained as follows:

$$\int_0^t \mathcal{VL}(\eta)d\eta = \mathbf{\Upsilon}\mathcal{VL}(t) + \mathbf{\Theta}(t), \tag{3.18}$$

where

$$\mathbf{\Upsilon} = \begin{bmatrix} \xi_{0,i,0} & \xi_{0,i,1} & \cdots & \xi_{0,i,M_2} \\ \xi_{1,i,0} & \xi_{1,i,1} & \cdots & \xi_{1,i,M_2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\xi}_{M_2,i,0} & \bar{\xi}_{M_2,i,1} & \cdots & \bar{\xi}_{M_2,i,M_2} \end{bmatrix}, \quad \mathbf{\Theta}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \theta_{M_2,0} t^{M_2+1} \end{bmatrix},$$

are the modified operational matrix of integration and complement vector, respectively.

4. The structure of the numerical method

To begin the process of calculating the approximate solution of the presented problem in Eq. (1.1), we consider

$$\frac{\partial^2 u(x,t)}{\partial t^2} \simeq \mathcal{VL}^T(x) U \mathcal{VL}(t), \tag{4.1}$$

where

$$U = [u_{mn}]_{(M_1+1)\times(M_2+1)}, \quad m = 0,1,\ldots,M_1, \quad n = 0,1,\ldots,M_2.$$

With the help of the modified operational matrix of integration and integrating the above relation concerning with variable t, we deduce

$$\frac{\partial u(x,t)}{\partial t} \simeq \mathcal{VL}^T(x) U \left(\mathbf{\Upsilon}\mathcal{VL}(t) + \mathbf{\Theta}(t)\right) + f_1(x). \tag{4.2}$$

By repeating the above procedure for Eq. (4.2), we will have:

$$u(x,t) \simeq \mathcal{VL}^T(x) U \left(\mathbf{\Upsilon}^2\mathcal{VL}(t) + \mathbf{\Upsilon\Theta}(t) + \int_0^t \mathbf{\Theta}(\eta)d\eta\right) + tf_1(x) + f_0(x). \tag{4.3}$$

Then, by using the modified operational matrices of Riesz fractional derivative and Eq. (4.3) the following relation is achieved:

$$\begin{aligned}\frac{\partial^\gamma}{\partial|x|^\gamma}u(x,t) \simeq &\left(x^{-\gamma}\mathbf{\Lambda}\mathcal{VL}(x) + (1-x)^{-\gamma}\mathbf{\Delta}\mathcal{VL}(x)\right) U \left(\mathbf{\Upsilon}^2\mathcal{VL}(t) + \mathbf{\Upsilon\Theta}(t) + \int_0^t \mathbf{\Theta}(\eta)d\eta\right)\\ &+t\frac{\partial^\gamma}{\partial|x|^\gamma}f_1(x) + \frac{\partial^\gamma}{\partial|x|^\gamma}f_0(x). \end{aligned}\tag{4.4}$$

It should be noted that $\frac{\partial^\beta}{\partial|x|^\beta}u(x,t)$ is calculated similarly to the above equation. Also, according to Eqs. (3.2) and (4.2) the Caputo derivative of an unknown function $u(x,t)$ can be expanded by shifted Vieta-Lucas polynomials as follows:

$$\begin{aligned}\mathbf{D}_t^\alpha u(x,t) &\simeq \mathbf{I}_t^{1-\alpha}\left(\frac{\partial u(x,t)}{\partial t}\right) = \mathbf{I}_t^{1-\alpha}\left(\mathcal{VL}^T(x) U \left(\mathbf{\Upsilon}\mathcal{VL}(t) + \mathbf{\Theta}(t)\right) + f_1(x)\right)\\ &= \mathcal{VL}^T(x) U \left(\frac{t^{1-\alpha}}{2\pi}\mathbf{\Upsilon\Phi}^{1-\alpha}\mathcal{VL}(t) + \mathbf{I}_t^{1-\alpha}\mathbf{\Theta}(t)\right) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}f_1(x). \end{aligned}\tag{4.5}$$

Similarly, $\mathbf{D}_t^\nu u(x,t)$ is also calculated. Therefore, by substituting the calculated relations in Eq. (1.1), the problem transfer to the following equation:

$$\begin{aligned}&\mathbf{Y}(x,t)\\ =&K_\alpha\left(\mathcal{VL}^T(x) U \left(\frac{t^{1-\alpha}}{2\pi}\mathbf{\Upsilon\Phi}^{1-\alpha}\mathcal{VL}(t) + \mathbf{I}_t^{1-\alpha}\mathbf{\Theta}(t)\right) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}f_1(x)\right)\\ &+ K_\nu\left(\mathcal{VL}^T(x) U \left(\frac{t^{1-\nu}}{2\pi}\mathbf{\Upsilon\Phi}^{1-\nu}\mathcal{VL}(t) + \mathbf{I}_t^{1-\nu}\mathbf{\Theta}(t)\right) + \frac{t^{1-\nu}}{\Gamma(2-\nu)}f_1(x)\right)\end{aligned}$$

$$
\begin{aligned}
&- K_\beta \left(\left(x^{-\beta}\boldsymbol{\Lambda}\mathcal{V}\mathcal{L}(x) + (1-x)^{-\beta}\boldsymbol{\Delta}\mathcal{V}\mathcal{L}(x)\right) U \left(\boldsymbol{\Upsilon}^2\mathcal{V}\mathcal{L}(t) + \boldsymbol{\Upsilon}\boldsymbol{\Theta}(t) + \int_0^t \boldsymbol{\Theta}(\eta)d\eta \right) \right.\\
&\left. + \, t\frac{\partial^\beta}{\partial|x|^\beta} f_1(x) + \frac{\partial^\beta}{\partial|x|^\beta} f_0(x) \right)\\
&- K_\gamma \left(\left(x^{-\gamma}\boldsymbol{\Lambda}\mathcal{V}\mathcal{L}(x) + (1-x)^{-\gamma}\boldsymbol{\Delta}\mathcal{V}\mathcal{L}(x)\right) U \left(\boldsymbol{\Upsilon}^2\mathcal{V}\mathcal{L}(t) + \boldsymbol{\Upsilon}\boldsymbol{\Theta}(t) + \int_0^t \boldsymbol{\Theta}(\eta)d\eta \right) \right.\\
&\left. + \, t\frac{\partial^\gamma}{\partial|x|^\gamma} f_1(x) + \frac{\partial^\gamma}{\partial|x|^\gamma} f_0(x) \right)\\
&- \mathcal{F} \begin{pmatrix} x, t, \mathcal{V}\mathcal{L}^T(x) U \left(\boldsymbol{\Upsilon}^2\mathcal{V}\mathcal{L}(t) + \boldsymbol{\Upsilon}\boldsymbol{\Theta}(t) + \int_0^t \boldsymbol{\Theta}(\eta)d\eta \right) + t f_1(x) + f_0(x), \\ \mathcal{V}\mathcal{L}^T(x) U \left(\boldsymbol{\Upsilon}\mathcal{V}\mathcal{L}(t) + \boldsymbol{\Theta}(t) \right) + f_1(x), \\ \mathcal{V}\mathcal{L}^T(x) U \mathcal{V}\mathcal{L}(t) \end{pmatrix} = 0,
\end{aligned} \tag{4.6}
$$

In addition, from Eq. (4.3) and the boundary conditions, we obtain

$$
\begin{aligned}
W_0(t) &= u(0,t) - g_0(t)\\
&= \mathcal{V}\mathcal{L}^T(0) U \left(\boldsymbol{\Upsilon}^2\mathcal{V}\mathcal{L}(t) + \boldsymbol{\Upsilon}\boldsymbol{\Theta}(t) + \int_0^t \boldsymbol{\Theta}(\eta)d\eta \right) + t f_1(0) + f_0(0) - g_0(t),
\end{aligned} \tag{4.7}
$$

and

$$
\begin{aligned}
W_1(t) &= u(1,t) - g_1(t)\\
&= \mathcal{V}\mathcal{L}^T(1) U \left(\boldsymbol{\Upsilon}^2\mathcal{V}\mathcal{L}(t) + \boldsymbol{\Upsilon}\boldsymbol{\Theta}(t) + \int_0^t \boldsymbol{\Theta}(\eta)d\eta \right) + t f_1(1) + f_0(1) - g_1(t).
\end{aligned} \tag{4.8}
$$

To obtain unknown matrix U, we substitute the Newton–Cotes nodes [10] in Eqs. (4.6)-(4.8):

$$
\begin{cases} \mathbf{Y}(x_i, t_j) = 0, \; i = 1, 2, \ldots, M_1 + 1, \; j = 1, 2, \ldots, M_2 + 1, \\ W_0(t_j) = 0, \quad j = 1, 2, \ldots, M_2 + 1, \\ W_1(t_j) = 0, \quad j = 1, 2, \ldots, M_2 + 1. \end{cases} \tag{4.9}
$$

To get the goal, we replace $W_0(t_j)$ and $W_1(t_j) = 0, \; j = 1, 2, \ldots, M_2 + 1$ in the last rows of a system of an equation $\mathbf{Y}(x_i, t_j), \; i = 1, 2, \ldots, M_1 + 1, \; j = 1, 2, \ldots, M_2 + 1$. Hence, we get the approximation solution of the multi-term time-space Riesz-Caputo fractional differential equations.

5. Error analysis

Herein, we present the error analysis of the best approximation of Sobolev space. To simplify the theorem process, we consider $M_1 = M_2 = M$.

Lemma 5.1. *Assume that $u_{M_1M_2}$ mentioned in Eq. (2.4) is the best approximation of u according to the shifted Vieta-Lucas polynomials on the interval Ω. Then, the error bound is obtained as follows:*

$$\|u - u_{M_1M_2}\|_{L^2(\Omega)} \leq CM^{1-\mu}|u|_{H^{\mu;M}(\Omega)}, \tag{5.1}$$

and for $1 \leq l \leq \mu$,

$$\|u - u_{M_1M_2}\|_{H^l(\Omega)} \leq CM^{\epsilon(l)-\mu}|u|_{H^{\mu;M}(\Omega)}. \tag{5.2}$$

Here, C depends on μ and

$$\epsilon(l) = \begin{cases} 0, & l = 0, \\ 2l - \frac{1}{2}, & l > 0. \end{cases}$$

Proof. The desired results are obtained directly in view of the best approximation concept and theorems in [5]. □

From the results of the above lemma, it can be deduced that

$$\|\frac{\partial u}{\partial x} - \frac{\partial u_{M_1M_2}}{\partial x}\|_{L^2(\Omega)} \leq \|u - u_{M_1M_2}\|_{H^l(\Omega)} \leq CM^{\epsilon(l)-\mu}|u|_{H^{\mu;M}(\Omega)}. \tag{5.3}$$

Theorem 5.1. *Assume that the suppositions of the previous lemma are established. Then, the error bound of Riesz fractional derivative of order $0 < \gamma \leq 1$ defined in Eq. (1.3) is yielded as follows:*

$$\begin{aligned}
&\left\|\frac{\partial^\gamma}{\partial|x|^\gamma}u - \frac{\partial^\gamma}{\partial|x|^\gamma}u_{M_1M_2}\right\|_{L^2(\Omega)} \\
&= \frac{1}{2\cos\frac{\pi\gamma}{2}}\left(\frac{2C}{\Gamma(2-\gamma)}M^{\epsilon(l)-\mu}|u|_{H^{\mu;M}(\Omega)} + \frac{1}{\Gamma(1-\gamma)}CM^{1-\mu}|\ u|_{x=0}\ |_{H^{\mu;M}([0,1])}\right. \\
&\quad \left. + \frac{1}{\Gamma(1-\gamma)}CM^{1-\mu}|\ u|_{x=1}\ |_{H^{\mu;M}([0,1])}\right).
\end{aligned} \tag{5.4}$$

Proof. According to Riesz fractional derivative definition, we have

$$\begin{aligned}
&\|\frac{\partial^\gamma}{\partial|x|^\gamma}u - \frac{\partial^\gamma}{\partial|x|^\gamma}u_{M_1M_2}\|_{L^2(\Omega)} \\
&= \frac{1}{2\cos\frac{\pi\gamma}{2}}\left\|D^\gamma_{*+}(u - u_{M_1M_2}) + D^\gamma_{*-}(u - u_{M_1M_2})\right\|_{L^2(\Omega)} \\
&\leq \frac{1}{2\cos\frac{\pi\gamma}{2}}\left(\left\|D^\gamma_+(u(x,t) - u_{M_1M_2}(x,t))\right\|_{L^2(\Omega)} + \frac{1}{\Gamma(1-\gamma)}\|u(0,t) - u_{M_1M_2}(0,t)\|_{L^2(\Omega)}\right. \\
&\quad \left. + \left\|D^\gamma_-(u(x,t) - u_{M_1M_2}(x,t))\right\|_{L^2(\Omega)} + \frac{1}{\Gamma(1-\gamma)}\|u(1,t) - u_{M_1M_2}(1,t)\|_{L^2(\Omega)}\right).
\end{aligned} \tag{5.5}$$

Also, from left- and right-sided Caputo fractional derivatives, we obtain

$$\left\|D^\gamma_+(u(x,t) - u_{M_1M_2}(x,t))\right\|_{L^2(\Omega)} = \frac{1}{\Gamma(1-\gamma)}\int_0^x \frac{1}{(x-\xi)^\gamma}d\xi\left\|\frac{\partial u}{\partial x} - \frac{\partial u_{M_1M_2}}{\partial x}\right\|_{L^2(\Omega)}$$

$$= \frac{x^{1-\gamma}}{\Gamma(2-\gamma)} \left\| \frac{\partial u}{\partial x} - \frac{\partial u_{M_1 M_2}}{\partial x} \right\|_{L^2(\Omega)}$$
$$\leq \frac{C}{\Gamma(2-\gamma)} M^{\epsilon(l)-\mu} |u|_{H^{\mu;M}(\Omega)}, \tag{5.6}$$

and

$$\left\| D_-^{\gamma} \left(u(x,t) - u_{M_1 M_2}(x,t) \right) \right\|_{L^2(\Omega)} = \frac{1}{\Gamma(1-\gamma)} \int_x^1 \frac{1}{(\xi - x)^{\gamma}} d\xi \left\| \frac{\partial u}{\partial x} - \frac{\partial u_{M_1 M_2}}{\partial x} \right\|_{L^2(\Omega)}$$
$$= \frac{(1-x)^{1-\gamma}}{\Gamma(2-\gamma)} \left\| \frac{\partial u}{\partial x} - \frac{\partial u_{M_1 M_2}}{\partial x} \right\|_{L^2(\Omega)}$$
$$\leq \frac{C}{\Gamma(2-\gamma)} M^{\epsilon(l)-\mu} |u|_{H^{\mu;M}(\Omega)}. \tag{5.7}$$

Therefore, by applying the results of the previous lemma and, Eqs. (5.6) and (5.7), we get

$$\left\| \frac{\partial^{\gamma}}{\partial |x|^{\gamma}} u - \frac{\partial^{\gamma}}{\partial |x|^{\gamma}} u_{M_1 M_2} \right\|_{L^2(\Omega)}$$
$$\leq \frac{1}{2 \cos \frac{\pi \gamma}{2}} \left(\frac{2C}{\Gamma(2-\gamma)} M^{\epsilon(l)-\mu} |u|_{H^{\mu;M}(\Omega)} + \frac{1}{\Gamma(1-\gamma)} C M^{1-\mu} \left| \, u|_{x=0} \right|_{H^{\mu;M}([0,1])} \right.$$
$$\left. + \frac{1}{\Gamma(1-\gamma)} C M^{1-\mu} \left| \, u|_{x=1} \right|_{H^{\mu;M}([0,1])} \right). \tag{5.8}$$

□

6. Numerical results

In this section, we perform the methodology in some numerical experiments to illustrate its efficiency and accuracy.

Example 6.1. Consider the two-term time-space Riesz-Caputo fractional diffusion equation [27]:

$$\mathbf{D}_t^{\alpha} u(x,t) + \mathbf{D}_t^{\frac{\alpha}{2}} u(x,t) = \frac{\partial^{\beta}}{\partial |x|^{\beta}} u(x,t) + \mathcal{F}(x,t), \quad 0 < \alpha \leq 1, \quad 1 < \beta < 2,$$

and the functions of initial and boundary conditions are:

$$f_0(x) = 1 - x^2, \quad g_0(t) = 1 + t^2, \quad g_1(t) = 0,$$

where $\mathcal{F}(x,t) = 2(1-x^2)\left[\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{2-\frac{\alpha}{2}}}{\Gamma(3-\frac{\alpha}{2})}\right] - \frac{\partial^{\beta}}{\partial |x|^{\beta}}\left(1-x^2\right)(1+t^2)$. The exact solution of this problem is $u(x,t) = (1+t^2)(1-x^2)$. By taking to account the proposed method and choosing $M_1 = 3, M_2 = 2$ and $\alpha = 0.95, \beta = 1.5$ each elements of unknown coefficient matrix is obtained as follows:

$u_{00} = 0.1562500000000005$, $u_{01} = 0.15625000000000072$,
$u_{02} = 3.7532818666491457 \times 10^{-16}$, $u_{10} = -0.12499999999999926$,

$$u_{11} = -0.12499999999999888, \quad u_{12} = 6.251283672055747 \times 10^{-16}$$
$$u_{20} = -0.03125000000000044, \quad u_{21} = -0.03125000000000065,$$
$$u_{22} = -3.4564265639028146 \times 10^{-16}, \quad u_{30} = -5.539929339399769 \times 10^{-16},$$
$$u_{31} = -8.519335708842823 \times 10^{-16}, \quad u_{32} = -4.227387224902173 \times 10^{-16}.$$

Thus, the approximate solution is achieved as follows:

$$\begin{aligned} u(x,t) = &1.290066742032 \times 10^{-14} tx^2 - 1.88257900554 \times 10^{-15} t \\ &-3.83683279502 \times 10^{-15} xt^2 - 1.59741549498 \times 10^{-14} x^3 \\ &-3.83349228724321 \times 10^{-14} xt^3 - 1.00000000000011 x^2 t^2 \\ &+1.073947288418 \times 10^{-13} x^3 t^2 + 1.86947385903 \times 10^{-13} x^2 t^3 \\ &-1.442948172766 \times 10^{-13} x^3 t^3 + 7.076129782827 \times 10^{-15} xt \\ &+1.0000000000000061 t^2 - 1.84217722070105 \times 10^{-15} t^3 - x^2 + 1. \end{aligned}$$

Also, the absolute error corresponding to $M_1 = 3, M_2 = 2$ with $\alpha = 0.8, \beta = 1.25$ at different times is demonstrated in Figure 1.

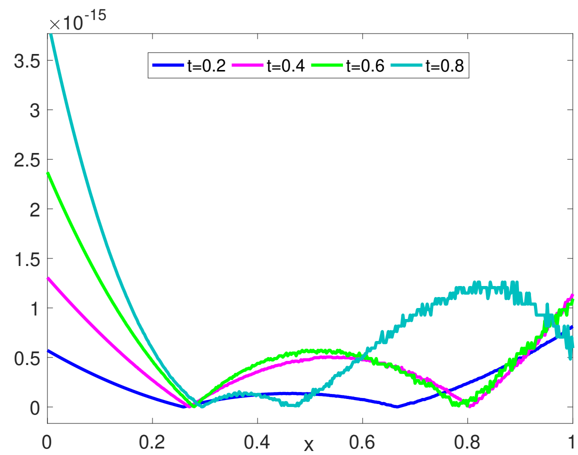

Figure 1. The absolute error for $M_1 = 3, M_2 = 2$ with $\alpha = 0.8$ and $\beta = 1.25$ at different times of Example 6.1.

Example 6.2. Consider the space Riesz fractional reaction dispersion equation [16,31]:

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \frac{\partial^\beta}{\partial |x|^\beta} u(x,t) + \mathcal{F}(x,t), \quad 0 < \alpha \leq 1, \quad 1 < \beta < 2,$$

and the functions of initial and boundary conditions are:

$$f_0(x) = x^2(1-x)^2, \quad g_0(t) = 0, \quad g_1(t) = 0,$$

where $\mathcal{F}(x,t) = (5+t)(t+1)^3 x^2 (1-x)^2 - \frac{\partial^\beta}{\partial |x|^\beta} \left(x^2(1-x)^2\right)(1+t)^4$. The exact solution of this problem is $u(x,t) = x^2(1-x)^2(1+t)^4$. The corresponding numerical results are presented in Table 1 and Figure 2. In Table 1, the obtained absolute error from the presented method is compared with methods in [16,31]. Further, the absolute error for different choices of β is plotted in Figure 2. It should be noted that the presented results are verified the accuracy of the approach.

Table 1. The absolute error for $M_1 = 5, M_2 = 4$ with $\beta = 1.4$ at $t = 0.02$ of Example 6.2.

x	Present method	Ref. [16]	Ref. [31]
0.1	7.4079×10^{-16}	2.6866×10^{-4}	2.5678×10^{-5}
0.2	6.9987×10^{-16}	6.4025×10^{-5}	1.6114×10^{-5}
0.3	2.2809×10^{-17}	9.2992×10^{-5}	1.1691×10^{-5}
0.4	1.1989×10^{-15}	1.8279×10^{-4}	9.6881×10^{-6}
0.5	2.5096×10^{-15}	2.1176×10^{-4}	9.1036×10^{-6}
0.6	3.5964×10^{-15}	1.8279×10^{-4}	9.6881×10^{-6}
0.7	4.1112×10^{-15}	9.2992×10^{-5}	1.1691×10^{-5}
0.8	3.7674×10^{-15}	6.4025×10^{-5}	1.6114×10^{-5}
0.9	2.3899×10^{-15}	2.6866×10^{-4}	2.5678×10^{-5}

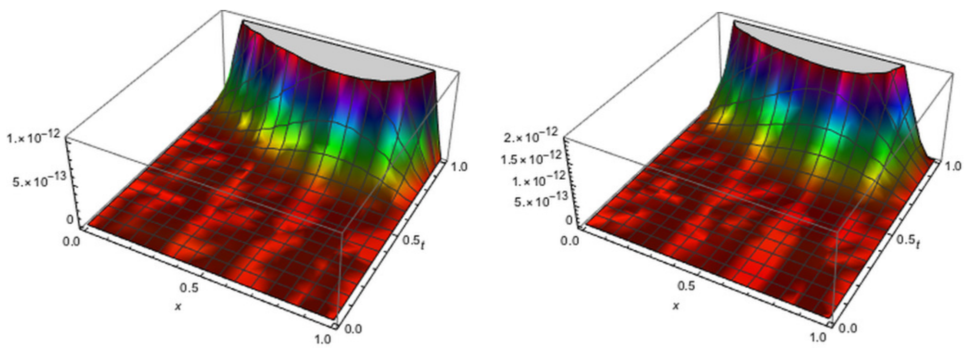

Figure 2. The absolute error for $\beta = 1.5$ (left) and $\beta = 1.9$ (right) with $M_1 = 5, M_2 = 4$ of Example 6.2.

Example 6.3. Consider the time-space Riesz-Caputo reaction dispersion equation [31]:

$$\mathbf{D}_t^{\alpha} u(x,t) = -u(x,t) + \frac{\partial^{\beta}}{\partial |x|^{\beta}} u(x,t) + \mathcal{F}(x,t), \quad 0 < \alpha \leq 1, \quad 1 < \beta < 2,$$

and the functions of initial and boundary conditions are:

$$f_0(x) = x^2(L-x)^2, \quad g_0(t) = 0, \quad g_1(t) = 0, \quad 0 \leq x \leq L, \quad 0 \leq t \leq T,$$

where $\mathcal{F}(x,t) = -\frac{\partial^{\beta}}{\partial |x|^{\beta}} \left(x^2(L-x)^2 \right) \exp(-t)$. The exact solution of this problem, when $\alpha = 1$ is $u(x,t) = x^2(L-x)^2 \exp(-t)$. The absolute error for different values of M_2 is provided in Table 2. The results imply that the proposed numerical algorithm is more accurate than methods in [3, 31]. Moreover, the absolute error for diverse values of α and the approximate solution for diverse values of M_2 are indicated in Figures 3 and 4, respectively. As the α value approaches 1, the approximate solutions tend to the available exact solution.

Example 6.4. Consider the Riesz space-fractional telegraph equation [7]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + 20 \frac{\partial u(x,t)}{\partial t} + 25 u(x,t) = \frac{\partial^{\beta}}{\partial |x|^{\beta}} u(x,t) + \mathcal{F}(x,t), \quad 1 < \beta < 2,$$

Table 2. Comparison of the absolute error for diverse values of M_2 with $\beta = 1.2$, $L = 2$ and $M_1 = 5$ at $t = 0.1$ of Example 6.3.

x	Present method		Ref. [31]	Ref. [3]
	$M_2 = 5$	$M_2 = 8$	$\tau = \frac{1}{2000}, h = \frac{1}{200}$	$M = N = 9$
0.2	1.0975×10^{-8}	1.9309×10^{-14}	1.61×10^{-3}	1.54×10^{-12}
0.4	1.8831×10^{-8}	1.3589×10^{-14}	1.42×10^{-3}	1.44×10^{-10}
0.6	1.8831×10^{-8}	1.8285×10^{-15}	8.75×10^{-4}	1.63×10^{-10}
0.8	1.0975×10^{-8}	1.0739×10^{-14}	7.50×10^{-4}	1.30×10^{-10}
1.0	3.2936×10^{-19}	4.0862×10^{-17}	7.13×10^{-4}	1.89×10^{-10}
1.2	4.6237×10^{-9}	1.0937×10^{-13}	7.50×10^{-4}	2.48×10^{-10}
1.4	1.1311×10^{-8}	4.7688×10^{-13}	8.75×10^{-4}	1.73×10^{-10}
1.6	6.6749×10^{-8}	1.3782×10^{-12}	1.42×10^{-3}	5.11×10^{-11}
1.8	1.8536×10^{-7}	3.2384×10^{-12}	1.61×10^{-3}	3.43×10^{-11}

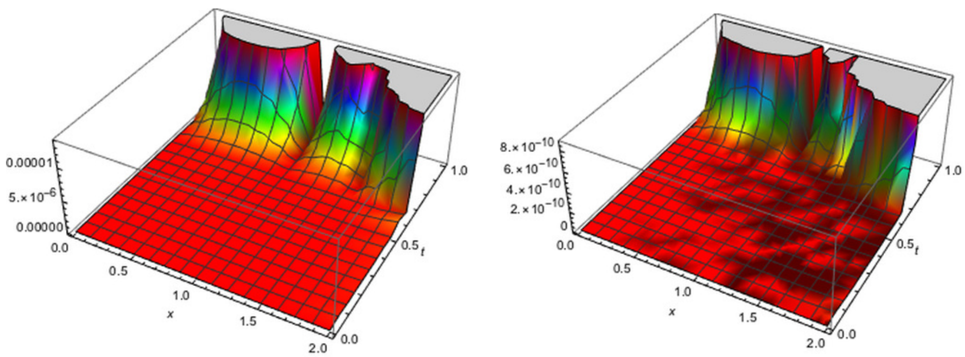

Figure 3. The absolute error for $M_2 = 5$ (left) and $M_2 = 8$ (right) with $M_1 = 5$, $\beta = 1.6$ and $L = 2$ of Example 6.3.

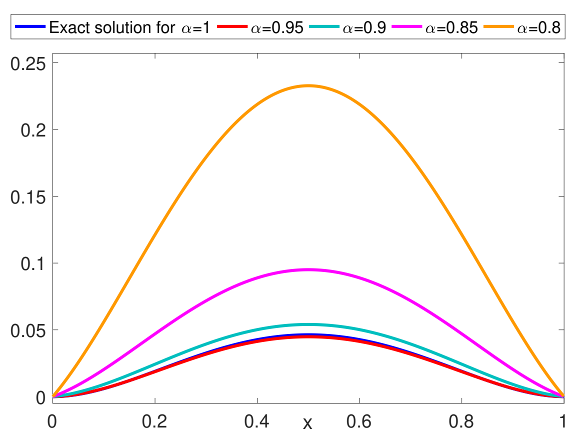

Figure 4. Approximate solution for diverse values of α with $M_2 = M_1 = 5$, $\beta = 1.3$ and $L = 1$ at $t = 0.8$ of Example 6.3.

Table 3. The absolute error for diverse values of M_2 and β with $M_1 = 5$ of Example 6.4.

$x = t$	$\beta = 1.3$		$\beta = 1.7$	
	$M_2 = 5$	$M_2 = 7$	$M_2 = 5$	$M_2 = 7$
0	0	0	0	0
0.1	1.4304×10^{-10}	3.5181×10^{-14}	3.9043×10^{-10}	1.7491×10^{-13}
0.2	9.8926×10^{-12}	2.3299×10^{-14}	2.4020×10^{-11}	1.8031×10^{-13}
0.3	6.1076×10^{-10}	1.9324×10^{-14}	1.2869×10^{-9}	1.0161×10^{-13}
0.4	8.7530×10^{-10}	3.5475×10^{-15}	1.7676×10^{-9}	2.8315×10^{-14}
0.5	3.9064×10^{-10}	1.0229×10^{-14}	7.6969×10^{-10}	4.0169×10^{-13}
0.6	5.8111×10^{-9}	2.3892×10^{-14}	1.1725×10^{-8}	1.6415×10^{-12}
0.7	2.0005×10^{-8}	5.3486×10^{-13}	4.2337×10^{-8}	7.6319×10^{-12}
0.8	1.4894×10^{-7}	7.7358×10^{-12}	3.4441×10^{-7}	8.1505×10^{-11}
0.9	2.9177×10^{-7}	7.3386×10^{-11}	7.9823×10^{-7}	5.9778×10^{-10}
1.0	4.8066×10^{-18}	2.9225×10^{-16}	1.0266×10^{-16}	1.9155×10^{-17}

and the functions of initial and boundary conditions are:

$$\begin{aligned} &f_0(x) = 0, \quad f_1(x) = x^2(1-x)^2, \quad 0 \le x \le 1, \\ &g_0(t) = 0, \quad g_1(t) = 0, \quad 0 \le t \le 1, \end{aligned}$$

where $\mathcal{F}(x,t) = x^2(1-x)^2[24\sin(t)+20\cos(t)] - \frac{\partial^\beta}{\partial|x|^\beta}\left(x^2(1-x)^2\right)\sin(t)$. The exact solution of this problem is $u(x,t) = x^2(1-x)^2\sin(t)$. In Table 3, we give the absolute error for various values of M_2 and β with $M_1 = 5$. Also, Figure 5 shows the curves of approximate solution and exact solution at the diverse times with $M_1 = M_2 = 5$, $\beta = 1.9$. From this table and figure, we find that the approximate solutions have excellent agreement with the exact solution.

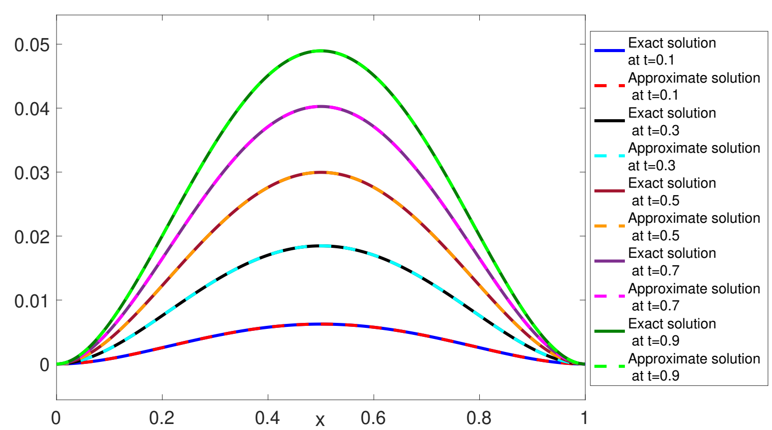

Figure 5. Approximate solution at diverse times $t = 0.1, 0.3, 0.5, 0.7, 0.9$ with $M_1 = M_2 = 5$ and $\beta = 1.9$ of Example 6.4.

Example 6.5. Consider the two-term time-space Riesz-Caputo fractional diffusion equation [4,24]:

$$\mathbf{D}_t^{\alpha} u(x,t) + \mathbf{D}_t^{\nu} u(x,t) = \frac{\partial^{\gamma}}{\partial |x|^{\gamma}} u(x,t) + 2\frac{\partial^{\beta}}{\partial |x|^{\beta}} u(x,t) + \mathcal{F}(x,t),$$
$$0 < \alpha, \nu \leq 1, \quad 1 < \beta < 2, \quad 0 < \gamma < 1,$$

and the functions of initial and boundary conditions are:

$$f_0(x) = 100(x^2 - x^3), \quad 0 \leq x \leq 1,$$
$$g_0(t) = 0, \quad g_1(t) = 0, \quad 0 \leq t \leq 1,$$

where $\mathcal{F}(x,t) = 200(x^2 - x^3)\left[\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{2-\nu}}{\Gamma(3-\nu)}\right] - \frac{\partial^{\gamma}}{\partial |x|^{\gamma}}(x^2 - x^3)(t^2+1) - 2\frac{\partial^{\beta}}{\partial |x|^{\beta}}(x^2 - x^3)(t^2+1)$. The exact solution of this problem is $u(x,t) = 100(x^2 - x^3)(t^2+1)$. In view of the provided method, for $\alpha = 0.5, \nu = 0.2, \gamma = 0.6$ and $\beta = 1.6$, we gain the components of unknown coefficient matrix as follows:

$$\begin{aligned}
&u_{00} = 1.562500000000047, \quad u_{01} = 1.5625000000000724,\\
&u_{02} = 3.5653996759458474 \times 10^{-14}, \quad u_{10} = 0.7812500000002389,\\
&u_{11} = 0.7812500000003679, \quad u_{12} = 1.7873113172385974 \times 10^{-13},\\
&u_{20} = -1.5625000000000475, \quad u_{21} = -1.5625000000000735,\\
&u_{22} = -3.622133569723602 \times 10^{-14}, \quad u_{30} = -0.7812500000002389,\\
&u_{31} = -0.7812500000003676, \quad u_{32} = -1.7876700670933926 \times 10^{-13}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
u(x,t) = {} & 1.8709470794584 \times 10^{-15} t + 9.2899466243422 \times 10^{-12} t x^2\\
& + 1.9957661757488 \times 10^{-11} x t^2 - 6.4117962438763 \times 10^{-12} t x^3\\
& - 2.7419447165253 \times 10^{-12} x t^3 + 99.9999999999355695 x^2 t^2\\
& - 99.9999999999555234 x^3 t^2 + 8.8437820122351 \times 10^{-11} x^2 t^3\\
& - 6.10191382901211 \times 10^{-11} x^3 t^3 - 2.8811019156088 \times 10^{-12} t x\\
& + 2.7302635087176 \times 10^{-15} t^2 - 5.6689488245123 \times 10^{-15} t^3\\
& + 100 x^2 - 100 x^3.
\end{aligned}$$

Also, the comparison of the L_2-error obtained by the present method with Haar wavelet method [24] and finite element multigrid method [4] are listed in Table 4. Furthermore, the curve of error for $\alpha = 0.75, \nu = 0.35, \gamma = 0.45, \beta = 1.5$ and $M_1 = 3, M_2 = 2$ is demonstrated in Figure 6.

Table 4. Comparison of L_2-error obtained by present method for $\alpha = 0.5, \nu = 0.2, \gamma = 0.6$ and $\beta = 1.6$ with methods [4,24] at $t = 0.5$ of Example 6.5.

	Present method $M_1 = 3, M_2 = 2$	Haar wavelet method [24] $m = 128$	Finite element multigrid method [4] $m = 128$
L_2-error	3.3271×10^{-14}	4.7412×10^{-3}	7.5249×10^{-4}

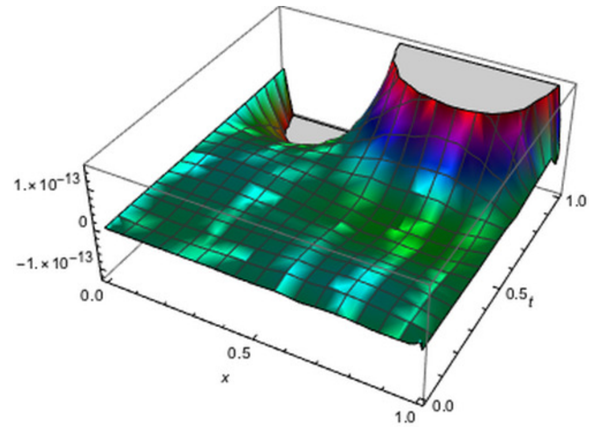

Figure 6. Error for $\alpha = 0.75, \nu = 0.35, \gamma = 0.45, \beta = 1.5$ and $M_1 = 3, M_2 = 2$ of Example 6.5.

Example 6.6. Consider the nonlinear Riesz space fractional Fisher equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^\beta}{\partial |x|^\beta} u(x,t) + u(x,t)(1 - u(x,t)) + \mathcal{F}(x,t), \quad 1 < \beta < 2,$$

with the initial and boundary conditions

$$f_0(x) = x^2 - 2, \quad 0 \le x \le 1,$$
$$g_0(t) = -2\exp(t), \quad g_1(t) = -\exp(t), \quad 0 \le t \le 1,$$

where $\mathcal{F}(x,t) = \exp(2t)x^4 - 4\exp(2t)x^2 + 4\exp(2t) - \frac{\partial^\beta}{\partial |x|^\beta}\left(x^2 - 2\right)\exp(t)$. The exact solution of this problem is $u(x,t) = (x^2 - 2)\exp(t)$. Using the approach described in the previous section, the obtained results for different values of M_2 and β are shown in Table 5. Also, to represent the behaviour of the approximate solution and absolute error in the whole domain of the problem, we plot the curves of them for $\beta = 1.3$ and $M_1 = 3, M_2 = 8$ in Figure 7.

Table 5. The absolute error for different choices of M_2 and β with $M_1 = 3$ of Example 6.6.

$x = t$	$\beta = 1.65$			$\beta = 1.95$		
	$M_2 = 3$	$M_2 = 5$	$M_2 = 8$	$M_2 = 3$	$M_2 = 5$	$M_2 = 8$
0	0	0	0	0	0	0
0.1	1.6713×10^{-5}	2.1969×10^{-8}	2.7818×10^{-13}	1.5040×10^{-5}	1.8181×10^{-8}	8.2218×10^{-14}
0.2	3.2477×10^{-6}	3.0399×10^{-8}	7.4007×10^{-14}	2.3606×10^{-6}	2.7375×10^{-8}	4.7707×10^{-13}
0.3	2.0439×10^{-5}	1.7843×10^{-8}	1.6225×10^{-12}	1.8016×10^{-5}	1.5839×10^{-8}	4.8730×10^{-13}
0.4	2.5843×10^{-5}	4.0959×10^{-8}	1.3417×10^{-12}	2.3324×10^{-5}	3.6688×10^{-8}	4.7951×10^{-13}
0.5	6.8892×10^{-5}	1.0971×10^{-7}	1.2954×10^{-11}	6.4443×10^{-5}	1.0273×10^{-7}	1.6801×10^{-12}
0.6	3.6878×10^{-4}	1.8240×10^{-6}	7.3849×10^{-10}	3.5393×10^{-4}	1.7161×10^{-6}	8.5641×10^{-11}
0.7	9.1229×10^{-4}	9.9876×10^{-6}	1.2069×10^{-8}	8.9504×10^{-4}	9.5198×10^{-6}	1.6416×10^{-9}
0.8	1.5366×10^{-3}	2.8951×10^{-5}	7.2231×10^{-8}	1.5361×10^{-3}	2.8010×10^{-5}	1.1105×10^{-8}
0.9	1.7119×10^{-3}	4.7001×10^{-5}	2.0936×10^{-7}	1.7383×10^{-3}	4.6194×10^{-5}	3.4784×10^{-8}
1.0	3.5562×10^{-4}	1.1466×10^{-6}	7.9297×10^{-11}	3.5562×10^{-4}	1.1466×10^{-6}	7.9288×10^{-11}

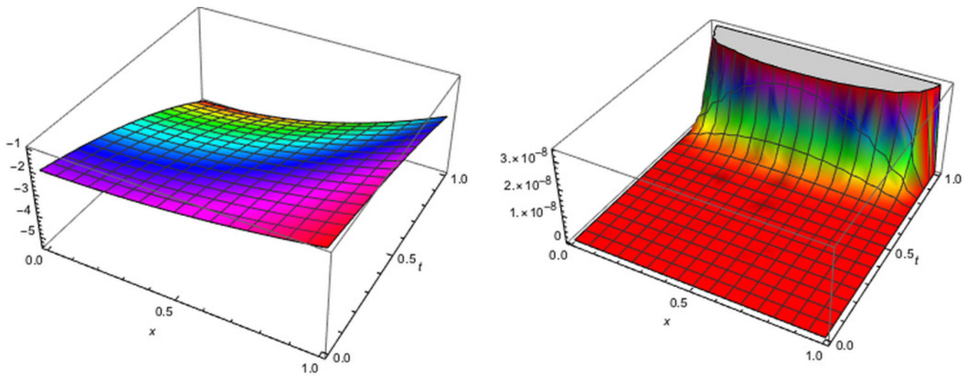

Figure 7. The approximate solution (left) and the absolute error (right) for $\beta = 1.3$ and $M_1 = 3, M_2 = 8$ of Example 6.6.

7. Conclusions

This paper develops the spectral method in view of shifted Vieta-Lucas polynomials for solving multi-term time-space Riesz-Caputo fractional differential equations. To achieve the aim, we use the operational matrix of fractional integration, modified operational matrices of Riesz fractional derivative and modified operational matrix of integration. In the overall algorithm of the method, the provided operational matrices play an efficient role. So that, the accuracy of operational matrices reflect in the process of numerical technique. Following that, the approximate solution is obtained with high accuracy. At last, we test some problems to confirm the methodology.

Acknowledgements

We express our sincere thanks to the anonymous referees for their valuable suggestions that improved the final manuscript.

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