Journal of Applied Analysis and Computation
Volume 13, Number 1, February 2023, 1–10
Website:http://www.jaac-online.com
DOI:10.11948/20210036

ON STRONGLY INDEFINITE SCHRÖDINGER EQUATIONS WITH NON-PERIODIC POTENTIAL*

Yue Wu[1] and Wei Chen[1,†]

Abstract This paper is concerned with the non-periodic superlinear Schrödinger equation $-\Delta u + V(x)u = f(x,u)$, $u \in H^1(\mathbb{R}^N)$. Here, the Shrödinger operator $-\Delta + V$ is strongly indefinite, that is, possesses a infinite dimensional negative space, which leads to more difficulty in verifying the compactness conditions. We prove the existence, as well as multiplicity provided $f(x,t)$ is odd in t, of solutions via variational methods.

Keywords Schrödinger equations, superlinear, variational methods, strongly indefinite functionals.

MSC(2010) 35J20, 35J60, 58E05.

1. Introduction and main results

In this paper, we consider the semilinear Schrödinger equation of the form

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geqslant 3$. In what follows, we denote by $S := -\Delta + V$ the Schrödinger operator, by $\sigma(S)$ the spectrum of S, by $\sigma_{\mathrm{pp}}(S)$ its pure point spectrum and by $\sigma_{\mathrm{ess}}(S)$ its essential spectrum. Set $F(x,t) := \int_0^t f(x,s)\mathrm{d}s$.

Problems of the form (1.1) have received growing attention in recent years. There is a great deal of literature on existence and multiplicity results. It fall broadly into three categories according to the location of 0 relative to $\sigma(S)$. For the case where $\inf \sigma(S) > 0$, the variational functional has the mountain pass geometry. There are many papers giving various assumptions on V and f; for related researches, see, e.g., [4, 6, 8, 10, 12–14, 21, 23] and references therein. For the case where 0 is a boundary point of a gap of $\sigma(S)$, precisely, $0 \in \sigma(S)$ and $(0,\xi) \cap \sigma(S) = \emptyset$ for some constant $\xi > 0$, Bartsch and Ding [3], Willem and Zou [22] obtained the existence and multiplicity results. For the case where 0 lies in a gap of $\sigma(S)$, that is,

$$\sup\left(\sigma(S) \cap (-\infty, 0)\right) < 0 < \inf\left(\sigma(S) \cap (0,\infty)\right),$$

†The corresponding author. Email: cwei1990@126.com(W. Chen)
1School of Mathematics and Statistics, Linyi University, Linyi 276100, China
*The authors were supported by National Natural Science Foundation of China (Nos. 12201277, 12271232, 11701251), the Natural Science Foundation of Shandong Province (Nos. ZR2022QA008, ZR2017BA015).

the variational functional is indefinite, then it has the linking structure. In particular, if $\sigma_{\text{ess}}(S) \cap (-\infty, 0] \neq \emptyset$, the problem is strongly indefinite. With periodic assumptions on V and f, Troestler and Willem [20] and Kryszewski and Szulkin [9] obtained the existence and multiplicity of solutions of (1.1) by establishing a new degree theory and a infinite dimensional linking theorem.

In this current paper, we do not assume any compactness conditions on the potential function V. It is well known that a main difficulty in studying (1.1) in $\mathbb{R}^N$ is the lack of compactness. This difficulty can be avoided for (1.1) in bounded domains or if the potential function V possesses some compactness conditions. For example, if $\lim_{|x|\to\infty} V(x) = \infty$ or u is radially symmetric, one can get some compactness embedding and then the Palais–Smale condition can be proved. Refer to [16] in this direction. In the strongly indefinite case, the Schrödinger operator S possesses a infinite dimensional negative space, which leads to more difficulty in verifying the compactness conditions. Moreover, the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is not compact. To overcome this, most papers consider the periodic problem. Under periodic assumptions, the variational functional is invariant under translations, so one can construct multi-bump solutions up to a suitable translation (see, for example, [1,3,7,9,11,17,19]). Without periodic assumptions, there are not many studies for now. Liu, Su and Weth [15] and Zhao, Zhao and Ding [24] obtained the existence and multiplicity of solutions of (1.1) with asymptotically linear nonlinearity.

In this paper, we consider the non-periodic superlinear problems. For the potential V, we assume

- (V_1) $V \in L^q_{\text{loc}}(\mathbb{R}^N)$, $V^- := \min\{V, 0\} \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$, for some $q \in [2, \infty) \cap (N/2, \infty)$;
- (V_2) $b := \sup\{(-\infty, 0) \cap \sigma_{\text{ess}}(S)\} < 0 < a := \inf\{(0, \infty) \cap \sigma_{\text{ess}}(S)\}$.

Assumption (V_1) ensures that the Schrödinger operator S is self-adjoint and semi-bounded on $L^2(\mathbb{R}^N)$ (see [18, Theorem A.2.7]). Since eigenvalues may also appear in gaps of the essential spectrum $\sigma_{\text{ess}}(S)$, (V_2) induces that $0 \in \sigma_{\text{pp}}(S)$ is possible. So (V_2) is a more general spectral assumption.

For the nonlinearity f, we assume

- (f_1) $f \in C^1(\mathbb{R}^N \times \mathbb{R})$ and there exist constants $p \in (2, 2^*)$ and $c > 0$ such that $|f(x,t)| \leqslant c(1 + |t|^{p-1})$ for $(x,t) \in (\mathbb{R}^N, \mathbb{R})$, where $2^* = 2N/(N-2)$;
- (f_2) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^N$;
- (f_3) there exist $\mu > 2$ such that $0 < \mu F(x,t) \leqslant t f(x,t)$ for $x \in \mathbb{R}^N$ and $t \neq 0$;
- (f_4) $f^* := \limsup_{|x|\to\infty} \sup_{|t|\leq r} \frac{f(x,t)}{t} < a$ for all $r > 0$.

Our main result is the following theorem.

Theorem 1.1. *Under assumptions (V_1)-(V_2) and (f_1)-(f_4), problem (1.1) possesses at least one nontrivial solution. Moreover, if $f(x,t)$ is odd in t, problem (1.1) possesses infinitely many solutions.*

2. Preliminaries

By virtue of (V_2), we know that 0 is at most an eigenvalue of finite multiplicity of S. Moreover, (V_2) induces the orthogonal decomposition

$$L^2 \equiv L^2(\mathbb{R}^N) = L^- \oplus L^0 \oplus L^+, \quad u = u^- + u^0 + u^+$$

according to the spectrum of S such that S is negative definite (resp. positive definite) in L^- (resp. in L^+) and $L^0 = \ker S$. Let $E = D(|S|^{1/2})$ be the Hilbert space equipped with the inner product

$$(u,v) = (|S|^{1/2}u, |S|^{1/2}v)_2 + (u^0, v^0)_2$$

and norm $\|u\| = (u,u)^{1/2}$, where $(\cdot,\cdot)_2$ denotes the inner product of L^2. We have the decomposition

$$E = E^- \oplus E^0 \oplus E^+, \quad \text{where } E^\pm = E \cap L^\pm \text{ and } E^0 = L^0.$$

It is orthogonal with respect to both $(\cdot,\cdot)$ and $(\cdot,\cdot)_2$. It is easy to see that E embeds continuously into $H^1(\mathbb{R}^N)$, and then continuously into $L^s(\mathbb{R}^N)$ for $s \in [2, 2^*]$ and compactly into $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $s \in [2, 2^*)$. Consequently, there exists a constant $\tau_s > 0$ such that

$$|u|_s \leqslant \tau_s \|u\|, \quad \forall u \in E, \tag{2.1}$$

where $|\cdot|_s$ denotes the L^s norm.

Define a functional $\Phi : E \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - \Psi(u), \text{ where } \Psi(u) = \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x. \tag{2.2}$$

Under our assumptions, it is easy to see that $\Phi \in C^1(E, \mathbb{R})$ and for $u, v \in E$,

$$\Phi'(u)v = (u^+, v^+) - (u^-, v^-) - \int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x.$$

It is well known that the critical points of Φ are solutions of problem (1.1).

We turn next recall some abstract critical point theorems, which will be used in the proof of our main result. Recall that a sequence $\{u_n\} \subset E$ is called a *(PS)$_c$ sequence*, if

$$\Phi(u_n) \to c \quad \text{and} \quad \|\Phi'(u_n)\| \to 0. \tag{2.3}$$

We say that Φ satisfies the *(PS) condition* if any (PS)$_c$ sequence of Φ contains a convergent subsequence for all $c \in \mathbb{R}$. Let $R > r > 0$ and let $\phi \in E^+ \setminus \{\mathbf{0}\}$ with $\|\phi\| = 1$. Define

$$M := \left\{u \in E^- \oplus E^0 \oplus \mathbb{R}^+\phi \mid \|u\| \leqslant R\right\}, \quad N := \left\{u \in E^+ \mid \|u\| = r\right\}. \tag{2.4}$$

Denote by ∂M the boundary of M.

Proposition 2.1 ([9, Theorem 3.4])**.** *Assume that $\Psi \in C^1(E, \mathbb{R})$ is bounded from below, weakly sequentially lower semicontinuous and Ψ' is weakly sequentially continuous. Let Φ be a functional on E of the form (2.2) satisfying the (PS) condition. If*

$$\inf_N \Phi > \sup_{\partial M} \Phi,$$

then Φ has a nontrivial critical point.

In the proof of the multiplicity result, we will use the following theorem, which is a generalization of the classical fountain theorem of Bartsch [2] (see also [21, Theorem 3.6]). Denote by $\{e_j\}$ a total orthonormal sequence in E^+. For $k \in \mathbb{N}$, let

$$Y_k = \left(E^- \oplus E^0\right) \oplus \operatorname{span}\{e_1, \ldots e_k\}, \qquad Z_k = \overline{\operatorname{span}\{e_k, e_{k+1}, \ldots\}}. \tag{2.5}$$

Proposition 2.2 ([5, Theorem 12]). *Assume that $\Psi \in C^1(E, \mathbb{R})$ is even, bounded from below, weakly sequentially lower semicontinuous and Ψ' is weakly sequentially continuous. Let Φ be a functional on E of the form (2.2) satisfying the (PS) condition. If there exists $\rho_k > r_k > 0$ such that*

(A_1) $\alpha_k = \inf_{u \in Z_k, \|u\|=r_k} \Phi(u) \to \infty$, *as* $k \to \infty$;

(A_2) $\beta_k = \sup_{u \in Y_k, \|u\|=\rho_k} \Phi(u) \leqslant 0$,

then Φ has a sequence of critical points $\{u_k\}$ such that $\Phi(u_k) \to \infty$.

3. Proof of Theorem 1.1

Lemma 3.1. *Suppose that (V_1)-(V_2) and (f_1)-(f_2) are satisfied, then there exists some $r > 0$ such that* $\inf \Phi(\partial B_r(\mathbf{0}) \cap E^+) > 0$.

Proof. It follows from (f_1) and (f_2) that, for given $\varepsilon > 0$, there is some constant $c_\varepsilon > 0$ such that
$$|F(x,t)| \leqslant \varepsilon |t|^2 + c_\varepsilon |t|^p \tag{3.1}$$
and
$$|f(x,t)| \leqslant \varepsilon |t| + c_\varepsilon |t|^{p-1}. \tag{3.2}$$
Then for $u \in E^+$ we have
$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\|u\|^2 - \varepsilon |u|_2^2 - c_\varepsilon |u|_p^p \\ &\geqslant \left(\frac{1}{2} - \varepsilon\tau_2\right)\|u\|^2 - \tau_p c_\varepsilon \|u\|^p,\end{aligned}$$
where τ_2 and τ_p are constants in (2.1). Let $\varepsilon = \frac{1}{4\tau_2}$. Since $p > 2$, we can fix some r small enough such that
$$\inf_{u \in E^+, \|u\|=r} \Phi(u) > 0.$$
The proof is completed. □

Lemma 3.2. *Suppose that (V_1)-(V_2) and (f_1)-(f_3) are satisfied, then, for any $k \in \mathbb{N}^+$, there exists some $R > r > 0$ such that* $\sup \Phi(Y_k \backslash B_R(\mathbf{0})) \leqslant 0$, *where Y_k is the subspace of E given in (2.5) and r is the constant given by Lemma 3.1.*

Proof. If not, there exists a sequence $\{u_n\} \subset Q$ with $\|u_n\| \to \infty$ such that $\Phi(u_n) \geqslant 0$ for all n. Set $v_n = \|u_n\|^{-1} u_n = v_n^0 + v_n^- + \lambda_n \phi$, then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, $v_n^- \rightharpoonup v^-$ in E^-, $v_n^0 \to v^0$ in E^0 and $\lambda_n \to \lambda$ in $\mathbb{R}^+$. Then
$$0 \leqslant \frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2}\lambda_n^2 - \frac{1}{2}\left\|v_n^-\right\|^2 - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2}\mathrm{d}x \leqslant \frac{1}{2}\lambda_n^2 - \frac{1}{2}\left\|v_n^-\right\|^2,$$
since $F(x,t) \geqslant 0$ by (f_3). This implies that
$$\frac{1}{2}\lambda_n^2 \geqslant \frac{1}{2}\left\|v_n^-\right\|^2.$$

If $v \equiv \mathbf{0}$, then $v_n^0 \to \mathbf{0}$ and $\lambda_n \to 0$. Hence the above inequality induces that $\|v_n^-\| \to 0$. Therefore, $v_n \to \mathbf{0}$ in E, which contradicts $\|v_n\| = 1$.

If $v \neq \mathbf{0}$, then the set $\Theta := \left\{ x \in \mathbb{R}^N \,\middle|\, v(x) \neq 0 \right\}$ has positive measure. Hence

$$u_n(x) = v_n(x) \|u_n\| \to \infty, \quad \forall x \in \Theta.$$

By (f_3), for any $\delta > 0$ there exists $c_\delta > 0$ such that

$$F(x,t) \geqslant c_\delta |t|^\mu - \delta |t|^2.$$

Then it follows from the Fatou's lemma that

$$\begin{aligned}
\frac{1}{2} - \frac{c+o(1)}{\|u_n\|^2} &= \frac{\|u_n^+\|^2 - \|u_n^-\|^2}{2\|u_n\|^2} - \frac{\Phi(u_n)}{\|u_n\|^2} \\
&= \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|} \mathrm{d}x \geqslant \int_\Theta \frac{F(x,u_n)}{u_n^2} v_n^2 \mathrm{d}x \\
&\geqslant \int_\Theta \left(c_\delta |u_n|^{\mu-2} - \delta \right) v_n^2 \mathrm{d}x \to \infty. \qquad (3.3)
\end{aligned}$$

This is a contradiction. The proof is completed. □

Lemma 3.3. *Suppose that* (V_1)*-*(V_2) *and* (f_1)*-*(f_3) *are satisfied, then the* $(PS)_c$ *sequence of* Φ *is bounded for any* $c \in \mathbb{R}$*.*

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence of Φ. Suppose by contradiction that $\{u_n\}$ is unbounded. Passing to a subsequence, we may assume that

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\| \to 0 \text{ and } \quad \|u_n\| \to \infty. \qquad (3.4)$$

Let $v_n = \|u_n\|^{-1} u_n$, then $\|v_n\| = 1$. Up to a subsequence, we assume that $v_n \rightharpoonup v$ in E, $v_n^\pm \rightharpoonup v^\pm$ in $E^\pm$ and $v_n^0 \to v^0 \in E^0$. Let $\theta \in C_0^\infty(\mathbb{R}, \mathbb{R})$ be such that $0 \leqslant \theta \leqslant 1$, $\theta(t) = 1$ for $|t| \geqslant 2$ and $\theta(t) = 0$ for $|t| \leqslant 1$. Define

$$f_1(x,t) = \theta(t) f(x,t), \quad f_2(x,t) = (1-\theta(t)) f(x,t) \quad \text{and} \quad s = p/(p-1).$$

In view of (f_1) and (f_2), we have that

$$\begin{aligned}
&c_1 |f_1(x,u)|^s \leqslant |u|^{(p-1)(s-1)} |f_1(x,u)| = u f_1(x,u), \\
&c_1 |f_2(x,u)|^2 \leqslant |u| |f_2(x,u)| = u f_2(x,u),
\end{aligned}$$

for some $c_1 > 0$. For n sufficient large, we obtain from (3.4) that

$$\begin{aligned}
c + 1 + \|u_n\| &\geqslant \Phi(u_n) - \frac{1}{2} \langle \Phi'(u_n), u_n \rangle \\
&= \int_{\mathbb{R}^N} \left(\frac{1}{2} u_n f(x,u_n) - F(x,u_n) \right) \mathrm{d}x \\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^N} u_n f(x,u_n) \mathrm{d}x \\
&\geqslant c_1 \left(\frac{1}{2} - \frac{1}{\mu} \right) \left(|f_1(x,u_n)|_s^s + |f_2(x,u_n)|_2^2 \right).
\end{aligned}$$

It follows that

$$|f_1(x,u)|_s \leqslant c_2 \|u_n\|^{1/s},$$
$$|f_2(x,u)|_2 \leqslant c_2 \|u_n\|^{1/2},$$

for n large enough and some $c_2 > 0$. Therefore, by (2.1), (3.4) and the Hölder's inequality, we have

$$\begin{aligned}\|u_n^+\|^2 &= \Phi'(u_n)u_n^+ + \int_{\mathbb{R}^N} f(x,u_n)u_n^+ \mathrm{d}x \\ &\leqslant \|u_n^+\| + |u_n^+|_p \, |f_1(x,u_n)|_s + |u_n^+|_2 \, |f_2(x,u_n)|_2 \\ &\leqslant \|u_n^+\| + c_2\tau_p \left\|u_n^+\right\| \|u_n\|^{1/s} + c_2\tau_2 \left\|u_n^+\right\| \|u_n\|^{1/2}\end{aligned}$$

for n large enough. Note that $1/s < 1$. Hence we can deduce that

$$\left\|v_n^+\right\|^2 = \frac{\|u_n^+\|^2}{\|u_n\|^2} \to 0 \quad \text{as } n \to \infty.$$

By the similar argument, we also have

$$\left\|v_n^-\right\|^2 = \frac{\|u_n^-\|^2}{\|u_n\|^2} \to 0 \quad \text{as } n \to \infty.$$

Consequently,

$$\left\|v_n^0\right\|^2 = 1 - \left\|v_n^-\right\|^2 - \left\|v_n^+\right\|^2 \to 1 \quad \text{as } n \to \infty.$$

Therefore $\left\|v^0\right\| = 1$, $v^0 \neq \mathbf{0}$, and then $v \neq \mathbf{0}$. So the set $\Theta := \left\{x \in \mathbb{R}^N \,\middle|\, v(x) \neq 0\right\}$ has positive measure. Hence

$$u_n(x) = v_n(x) \|u_n\| \to \infty, \quad \forall x \in \Theta.$$

By (f_3), for any $\delta > 0$ there exists $c_\delta > 0$ such that

$$F(x,t) \geqslant c_\delta |t|^\mu - \delta |t|^2.$$

Then it follows from the Fatou's lemma that

$$\begin{aligned}o(1) = \frac{\|v_n^+\|^2 - \|v_n^-\|^2}{2} - \frac{\Phi(u_n)}{\|u_n\|^2} &= \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|}\mathrm{d}x \\ \geqslant \int_\Theta \frac{F(x,u_n)}{u_n^2} v_n^2 \mathrm{d}x &\geqslant \int_\Theta \left(c_\delta |u_n|^{\mu-2} - \delta\right) v_n^2 \mathrm{d}x \to \infty.\end{aligned}$$

This is impossible. Therefore $\{u_n\}$ is bounded in E. □

Lemma 3.4. *Suppose that (V_1)-(V_2) and (f_1)-(f_4) are satisfied, then Φ satisfies the (PS) condition.*

Proof. Suppose $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence of Φ. By Lemma 3.3, $\{u_n\}$ is bounded in E. Hence we can assume, passing to a subsequence, that $u_n \rightharpoonup u$ in E. Set

$w_n = u_n - u$, then $w_n \rightharpoonup \mathbf{0}$ and $|w_n|_s \to 0$ in $L^s_{\rm loc}(\mathbb{R}^N)$ for $s \in [2, 2^*)$. In order to establish strong convergence, it suffices to show $\|w_n\| \to 0$. We claim that

$$\frac{f(x,u_n)P_{1,3}w_n}{u_n} \rightharpoonup \mathbf{0} \quad \text{in } L^2(\mathbb{R}^N). \tag{3.5}$$

Indeed, for $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{f(x,u_n)P_{1,3}w_n}{u_n}\varphi \mathrm{d}x &\leqslant \int_{\mathbb{R}^N} \left(1 + c_3|u_n|^{p-1}\right)|P_{1,3}w_n|\,|\varphi|\,\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^N} |P_{1,3}w_n||\varphi|\mathrm{d}x + c_3\,|\varphi|_\infty \int_{\mathbb{R}^N} |u_n|^{p-1}|P_{1,3}w_n|\mathrm{d}x \\ &\leqslant |P_{1,3}w_n|_{L^2(\mathrm{supp}\varphi)}|\varphi|_2 + c_3\,|\varphi|_\infty\,|u_n|_p^{p-1}|P_{1,3}w_n|_{L^p(\mathrm{supp}\varphi)} \\ &\to 0\end{aligned}$$

for some $c_3 > 0$. Therefore, (3.5) holds.

We next adopt an argument of Liu, Su and Weth [15]. Fix some $\eta > 0$ small enough such that $(a-\eta, a) \cap \sigma(S) = \emptyset$, $(b, b+\eta) \cap \sigma(S) = \emptyset$ and $f^* < a - \eta$. Let P_1 be the projection associated with $(a-\eta, \infty)$, P_2 associated with $[b+\eta, a-\eta]$ and P_3 associated with $(-\infty, b+\eta)$. Then $P_1u^+ = P_1u$, $P_1u^- = \mathbf{0}$, $P_3u^- = P_3u$ and $P_3u^+ = \mathbf{0}$. Moreover, in view of (V_2), we have

$$(a-\eta)\,|P_1w_n|_2^2 \leqslant \|P_1w_n\|^2 \quad \text{and} \quad -(b+\eta)\,|P_3w_n|_2^2 \leqslant \|P_3w_n\|^2. \tag{3.6}$$

Also note that, since $w_n \rightharpoonup \mathbf{0}$ and the projection P_2 has finite range, we obtain that $P_2w_n \to \mathbf{0}$.

We now prove that $\|P_1w_n\| \to 0$ as $n \to \infty$. Since

$$\Phi'(u_n)P_1w_n = (P_1u_n, P_1w_n) - \int_{\mathbb{R}^N} f(x,u_n)P_1w_n\mathrm{d}x \to 0,$$

we have

$$\begin{aligned}0 &\leqslant \limsup_{n\to\infty} \|P_1w_n\|^2 \\ &= \limsup_{n\to\infty}\,(P_1u_n, P_1w_n) = \limsup_{n\to\infty} \int_{\mathbb{R}^N} f(x,u_n)P_1w_n\mathrm{d}x.\end{aligned} \tag{3.7}$$

Let $\varepsilon > 0$ be arbitrary. For $r \geqslant 1$, it follows from $(\mathbf{f_1})$ that

$$\begin{aligned}\int_{|u_n|\geqslant r} f(x,u_n)P_1w_n\mathrm{d}x &\leqslant 2c\int_{|u_n|\geqslant r} |u_n|^{p-1}\,|P_1w_n|\,\mathrm{d}x \\ &\leqslant 2cr^{p-2^*}\int_{|u_n|\geqslant r} |u_n|^{2^*-1}\,|P_1w_n|\,\mathrm{d}x \\ &\leqslant 2cr^{p-2^*}\,|u_n|_{2^*}^{2^*-1}\,|P_1w_n|_{2^*}.\end{aligned}$$

Since $p < 2^*$, we may fix r large enough such that

$$\int_{|u_n|\geqslant r} f(x,u_n)P_1w_n\mathrm{d}x < \frac{\varepsilon}{3} \tag{3.8}$$

for all n. By (f_4) and (3.6), there exists $R>0$ such that

$$\begin{aligned}\int_{\substack{|x|\geqslant R\\|u_n|\leqslant r}}\frac{f(x,u_n)}{u_n}\left(P_1w_n\right)^2\mathrm{d}x &\leqslant |P_1w_n|_2^2\sup_{|t|\leqslant r,|x|\geqslant R}\frac{f(x,t)}{t}\\ &\leqslant f^*\left|P_1w_n\right|_2^2\leqslant\frac{f^*}{a-\eta}\left\|P_1w_n\right\|^2\end{aligned}$$

for all n. Moreover, by (3.5), we have that

$$\int_{\substack{|x|\geqslant R\\|u_n|\leqslant r}}\frac{f(x,u_n)P_1w_n}{u_n}u\mathrm{d}x<\frac{\varepsilon}{3}$$

for n large enough. Therefore,

$$\begin{aligned}\int_{\substack{|x|\geqslant R\\|u_n|\leqslant r}}f(x,u_n)P_1w_n\mathrm{d}x &= \int_{\substack{|x|\geqslant R\\|u_n|\leqslant r}}\frac{f(x,u_n)}{u_n}\left(P_1w_n\right)^2\mathrm{d}x+\int_{\substack{|x|\geqslant R\\|u_n|\leqslant r}}\frac{f(x,u_n)}{u_n}uP_1w_n\mathrm{d}x\\ &<\frac{f^*}{a-\eta}\left\|P_1w_n\right\|^2+\frac{\varepsilon}{3}\end{aligned}\tag{3.9}$$

for n large enough. Finally, since $P_1w_n\to\mathbf{0}$ in $L^s(B_R(\mathbf{0}))$ for $s\in[2,2^*)$, it follows from (3.2) that

$$\begin{aligned}\int_{\substack{|x|\leqslant R\\|u_n|\leqslant r}}f(x,u_n)P_1w_n\mathrm{d}x &\leqslant \int_{\substack{|x|\leqslant R\\|u_n|\leqslant r}}|u_n|\left|P_1w_n\right|\mathrm{d}x+c_4\int_{\substack{|x|\leqslant R\\|u_n|\leqslant r}}|u_n|^{p-1}\left|P_1w_n\right|\mathrm{d}x\\ &\leqslant |u_n|_2\left|P_1w_n\right|_{L^2(B_R(\mathbf{0}))}+c_4\left|u_n\right|_p^{p-1}\left|P_1w_n\right|_{L^p(B_R(\mathbf{0}))}\\ &<\frac{\varepsilon}{3}\end{aligned}\tag{3.10}$$

for n large enough. Combining (3.7)-(3.10) we conclude that

$$0\leqslant\limsup_{n\to\infty}\left(1-\frac{f^*}{a-\eta}\right)\left\|P_1w_n\right\|^2\leqslant\varepsilon.$$

Consequently, it follows from the arbitrariness of ε that $\|P_1w_n\|\to 0$ as $n\to\infty$.

The same goes for $\|P_3w_n\|\to 0$. Since

$$\begin{aligned}o(1) &= -\Phi'(u_n)P_3w_n\\ &= o(1)+\left\|P_3w_n\right\|^2+\int_{\mathbb{R}^N}f(x,u_n)P_3w_n\mathrm{d}x\\ &= o(1)+\left\|P_3w_n\right\|^2+\int_{\mathbb{R}^N}\frac{f(x,u_n)}{u_n}\left(P_3w_n\right)^2\mathrm{d}x+\int_{\mathbb{R}^N}\frac{f(x,u_n)P_3w_n}{u_n}u\mathrm{d}x\\ &\geqslant o(1)+\left\|P_3w_n\right\|^2,\end{aligned}$$

we obtain that $\|P_3w_n\|\to 0$ as $n\to\infty$. Therefore $\|w_n\|\to 0$ and $u_n\to u$. The lemma is proved. □

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. *(Existence)* Assumption (f_3) implies $\Psi(u)\geqslant 0$ for all $u\in E$. Since f is subcritical, it is easy to check that Ψ is weakly sequentially lower semicontinuous and Ψ' is weakly sequentially continuous. Lemmas 3.1 (with $k=1$, $e_1=\phi$) and 3.2 implies that there exist $R>r>0$ such that

$$\inf_N \Phi > 0 \geqslant \sup_{\partial M} \Phi.$$

According to Lemmas 3.3 and 3.4, Φ satisfies the (PS) condition. Therefore, by using Proposition 2.1, we have that Φ possesses at least one nontrivial critical point.

(Multiplicity) Since $f(x,t)$ is odd in t, Φ is an even functional. To use Proposition 2.2, it suffices to verify (A_1) and (A_2).

Let Y_k and Z_k defined as in (2.5). By Lemma 3.2, (A_2) holds. Define $\ell_k := \sup_{u\in Z_k, \|u\|=1} |u|_p$. Note that $Z_k \subset E^+$. Therefore, by (2.1) and (3.1) with $\varepsilon = 1/4\tau_2^2$, we have

$$\begin{aligned}
\Phi(u) &\geqslant \frac{1}{2}\|u\|^2 - \frac{1}{4\tau_2^2}|u|_2^2 - c_5|u|_p^p \\
&\geqslant \frac{1}{2}\|u\|^2 - \frac{1}{4\tau_2^2}|u|_2^2 - c_5\ell_k^p\|u\|^p \\
&\geqslant \frac{1}{4}\|u\|^2 - c_5\ell_k^p\|u\|^p .
\end{aligned}$$

Let $r_k = (2pc_5\ell_k^p)^{1/(2-p)}$. Then, for $u \in Z_k$ with $\|u\| = r_k$, we have

$$\Phi(u) \geqslant \frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)(2pc_5\ell_k^p)^{1/(2-p)} .$$

Since $\ell_k \to 0$ as $k\to\infty$ by [21, Lemma 3.8] and $p>2$, it follows that

$$\beta_k = \inf_{u\in Z_k, \|u\|=r_k} \Phi(u) \to \infty.$$

Hence (A_1) is satisfied. Therefore, Φ has infinitely many critical points, which are solutions of problem (1.1). Theorem 1.1 is proved. □

References

[1] S. Alama and Y. Y. Li, *On "multibump" bound states for certain semilinear elliptic equations*, Indiana Univ. Math. J., 1992, 41(4), 983–1026.

[2] T. Bartsch, *Infinitely many solutions of a symmetric Dirichlet problem*, Nonlinear Anal., 1993, 20(10), 1205–1216.

[3] T. Bartsch and Y. Ding, *On a nonlinear Schrödinger equation with periodic potential*, Math. Ann., 1999, 313(1), 15–37.

[4] T. Bartsch, Z. Liu and T. Weth, *Sign changing solutions of superlinear Schrödinger equations*, Comm. Partial Differential Equations, 2004, 29(1–2), 25–42.

[5] C. Joël Batkam and F. Colin, *Generalized fountain theorem and applications to strongly indefinite semilinear problems*, J. Math. Anal. Appl., 2013, 405(2), 438–452.

[6] W. Chen, Z. Fu and Y. Wu, *Positive solutions for nonlinear Schrödinger – Kirchhoff equations in* $\mathbb{R}^3$, Appl. Math. Lett., 2020, 104, 106274.

[7] Y. Ding and C. Lee, *Multiple solutions of Schrödinger equations with indefinite linear part and super or asymptotically linear terms*, J. Differential Equations, 2006, 222(1), 137–163.

[8] L. Jeanjean and K. Tanaka, *A positive solution for an asymptotically linear elliptic problem on $\mathbb{R}^N$ autonomous at infinity*, ESAIM Control Optim. Calc. Var., 2002, 7, 597–614 (electronic).

[9] W. Kryszewski and A. Szulkin, *Generalized linking theorem with an application to a semilinear Schrödinger equation*, Adv. Differential Equations, 1998, 3(3), 441–472.

[10] Q. Li, X. Du and Z. Zhao, *Existence of sign-changing solutions for nonlocal Kirchhoff-Schrödinger-type equations in R^3*, J. Math. Anal. Appl., 2019, 477(1), 174–186.

[11] S. Liu, *On superlinear Schrödinger equations with periodic potential*, Calc. Var. Partial Differential Equations, 2012, 45(1–2), 1–9.

[12] S. Liu and S. Mosconi, *On the schrödinger-poisson system with indefinite potential and 3-sublinear nonlinearity*, J. Differential Equations, 2020, 269(1), 689–712.

[13] S. Liu and Z. Zhao, *Solutions for fourth order elliptic equations on r^n involving $u\delta(u^2)$ and sign-changing potentials*, J. Differential Equations, 2019, 267(3), 1581–1599.

[14] S. Liu and J. Zhou, *Standing waves for quasilinear Schrödinger equations with indefinite potentials*, J. Differential Equations, 2018, 256(9), 3970–3987.

[15] Z. Liu, J. Su and T. Weth, *Compactness results for Schrödinger equations with asymptotically linear terms*, J. Differential Equations, 2006, 231(2), 501–512.

[16] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys., 1992, 43(2), 270–291.

[17] Z. Shen and S. Liu, *On asymptotically linear elliptic equations in $\mathbb{R}^N$*, J. Math. Anal. Appl., 2012, 392(1), 83–88.

[18] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.), 1982, 7(3), 447–526.

[19] A. Szulkin and T. Weth, *Ground state solutions for some indefinite variational problems*, J. Funct. Anal., 2009, 257(12), 3802–3822.

[20] C. Troestler and M. Willem, *Nontrivial solution of a semilinear Schrödinger equation*, Comm. Partial Differential Equations, 1996, 21(9–10), 1431–1449.

[21] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser Boston Inc., Boston, MA, 1996.

[22] M. Willem and W. Zou, *On a Schrödinger equation with periodic potential and spectrum point zero*, Indiana Univ. Math. J., 2003, 52(1), 109–132.

[23] V. Coti Zelati and P. H. Rabinowitz, *Homoclinic type solutions for a semilinear elliptic PDE on $\mathbf{R}^n$*, Comm. Pure Appl. Math., 1992, 45(10), 1217–1269.

[24] F. Zhao, L. Zhao and Y. Ding, *Existence and multiplicity of solutions for a non-periodic Schrödinger equation*, Nonlinear Anal., 2008, 69(11), 3671–3678.