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SOLVABILITY OF FRACTIONAL FUNCTIONAL BOUNDARY-VALUE PROBLEMS WITH P-LAPLACIAN OPERATOR ON A HALF-LINE AT RESONANCE

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Abstract This paper aims to consider the existence of solutions for p-Laplacian functional boundary-value problems at resonance on a half-line with two dimensional kernel. By employing some operators which satisfies suitable conditions and the Re and Gen extension of coincidence degree theory, a new result on the existence of solutions is acquired.

Keywords P-Laplacian, functional boundary condition, half-line, fractional differential equations, resonance.

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1. Introduction

Boundary value problems on the half-line arise in various applications such as in the study of the unsteady flow of gas through a semi-infinite porous medium, in analyzing the heat transfer in radial flow between circle disks, in the study of plasma physics,in an analysis of the mass transfer on a rotating disk in a non-Newtonian fluid, etc. [1]. Su and Zhang [25] studied the following fractional differential equations on the half-line, using Schauder's fixed point theorem,

$$\begin{aligned}&D_{0^+}^{\alpha}u(t)=f(t,u(t),D_{0^+}^{\alpha-1}u(t)),t\in(0,+\infty),1<\alpha\le 2,\\&u(0)=0,\lim_{t\to\infty}D_{0^+}^{\alpha-1}u(t)=u_{\infty}.\end{aligned}$$

It's well-known that Leibenson [20] firstly introduced the p-Laplacian equation which is

$$(\phi_p(x'(t)))'=f(t,x(t),x'(t)),\tag{1.1}$$

where $\phi_p(s)=|s|^{p-2}s, p>1$ and can solve a famous mechanics problem that the turbulent flow in a porous medium. Recently, a growing number of scholars have devoted their attention to studying boundary value problems with p-Laplacian operator owing to their importance in theory and application of mathematics and physics. For some recent work on this branch of differential equations, see [1–15,18,

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19, 21, 26–28]) and the references therein. In [9], using the Ge and Ren extension of Mawhin coincidence theory [4], the author investigated the following p-Laplacian third order integral and m-point boundary value problem at resonance

$$\begin{cases} (\phi_p(u''(t)))' = \omega(t, u(t), u'(t), u''(t)), t \in (0,1); \\ \phi_p(u''(0)) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \phi_p(u''(t))dt, u''(1) = 0, u'(1) = \beta u'(\eta), \end{cases}$$

where the function $\omega : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $0 < \xi_1 < \xi_2 < \ldots < \xi_m < 1, \beta > 0, \alpha_i (1 \leq i \leq m) \in \mathbb{R}, \sum_{i=1}^{m} \alpha_i \xi_i = 1$ and $\eta \in (0,1)$.

Fractional differential equations appear naturally in various fields of science and engineering. This is due to the fact that the differential equations of arbitrary order provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials. For recent publication on fractional calculus and fractional differential, we refer the reader to see [17, 22, 23]. In [26], the author discussed the existence of solutions for the following multipoint boundary value problem of fractional p-Laplacian equation at resonance

$$\begin{cases} (\phi_p(D_{0^+}^{\alpha} x(t)))' + f(t, x(t), D_{0^+}^{\alpha-1} x(t),\ D_{0^+}^{\alpha} x(t)) = 0,\ 0 < t < +\infty, \\ x(0) = x'(0) = 0,\ \phi_p(D_{0^+}^{\alpha} x(+\infty)) = \sum_{i=1}^{n} \alpha_i \phi_p(D_{0^+}^{\alpha} x(\xi_i)), \end{cases}$$

where $1 < \alpha \leq 2$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < \ldots < \xi_n < +\infty, \alpha_i > 0, \Sigma_{i=1}^{n} \alpha_i = 1$.

Motivated by the above results, in this paper, we study boundary value problem

$$\begin{cases} D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha} u))(t) = \mu(t) f\big(t, u(t), D_{0^+}^{\alpha-(n-1)} u(t), D_{0^+}^{\alpha-(n-2)} u(t), \\ \qquad \ldots, D_{0^+}^{\alpha-1} u(t), D_{0^+}^{\alpha} u(t)\big), \\ t \in [0, +\infty), \\ D_{0^+}^{\alpha} u(0) = 0, D_{0^+}^{\alpha-2} u(0) = D_{0^+}^{\alpha-3} u(0) = \ldots = D_{0^+}^{\alpha-(n-2)} u(0) = 0, u(0) = 0, \\ \Gamma_1(u) = 0, \Gamma_2(u) = 0, \end{cases} \tag{1.2}$$

in infinite interval, where $0 < \beta \leq 1, n-1 < \alpha \leq n,\ n \geq 3,\ \varphi_p(s) = |s|^{p-2}s, p > 1, \Gamma_1, \Gamma_2$ are continuous linear functionals with the resonance condition: $\Gamma_1(t^{\alpha-1}) = \Gamma_2(t^{\alpha-n+1}) = \Gamma_1(t^{\alpha-n+1}) = \Gamma_2(t^{\alpha-1}) = 0$. Boundary value problem (1.2) is to be at resonance if $D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha} u))(t) = 0$ subject to boundary conditions has a non-trivial solution.

To the best of our knowledge, this is the first paper to study p-Laplacian functional boundary-value problems at resonance on a half-line with two dimensional kernel, which contains two-point boundary conditions, multi-point boundary conditions, differential boundary conditions, integral boundary conditions and integral differential boundary conditions that are commonly studied, and has a high degree of generality. And the main difficulties are that we have to construct suitable Banach spaces for the problem and establish operators P and Q. Of course, when

$n-1<\alpha\le n$, we need to be more careful in dealing with some details. It is worth noting that the operator Q is not a projector.

In this paper, we will always suppose that the following conditions hold.

(H_1) $\mu(t)\in L[0,+\infty)\cap C[0,+\infty)$, $\mu(t)\neq 0, t\in[0,+\infty)$, $\sup\limits_{t\in[0,+\infty)}|I^{\beta}_{0+}\mu|<+\infty$.

(H_2) $f(t,u_1,u_2,\ldots,u_{n+1})$ is continuous in a.e. $[0,+\infty)\times\mathbb{R}^{n+1}$.

2. Preliminaries

Definition 2.1 (see [4, 15]). Let X and Z be two Banach spaces with norms $\|\cdot\|_X,\|\cdot\|_Z$, respectively. A continuous operator $M\colon X\cap\operatorname{dom}M\to Z$ is said to be quasi-linear if

(i) $\operatorname{Im}M:=M(X\cap\operatorname{dom}M)$ is a closed subset of Z;

(ii) $\operatorname{Ker}M:=\{x\in X\cap\operatorname{dom}M : Mx=0\}$ is linearly homeomorphic to $\mathbb{R}^n, n<\infty$,

where $\operatorname{dom}M$ denotes the domain of the operator M.

In this paper, an operator $T:X\to Z$ is said to be bounded if $T(V)\subset Z$ is bounded for any bounded subset $V\subset X$.

Let $X_1=\operatorname{Ker}M$ and X_2 be the complement space of X_1 in X. Then $X=X_1\oplus X_2$. Let $P:X\to X_1$ be projector and $\Omega\subset X$ be an open and bounded set with the origin $\theta\in\Omega$.

Definition 2.2 (see [4]). Suppose that $N_\lambda:\overline{\Omega}\to Z,\lambda\in[0,1]$ is a continuous and bounded operator. Denote N_1 by N. Let $\Sigma_\lambda=\{x\in\overline{\Omega}:Mx=N_\lambda x\}$. N_λ is said to be M-quasi-compact in $\overline{\Omega}$ if there exists a vector subspace Z_1 of Z satisfying $\dim Z_1=\dim X_1$ and two operators Q and R such that for $\lambda\in[0,1]$,

(a) $\operatorname{Ker}Q=\operatorname{Im}M$,

(b) $QN_\lambda x=\theta,\lambda\in(0,1)\Leftrightarrow QNx=\theta$,

(c) $R(\cdot,0)$ is the zero operator and $R(\cdot,\lambda)|_{\Sigma_\lambda}=(I-P)|_{\Sigma_\lambda}$,

(d) $M[P+R(\cdot,\lambda)]=(I-Q)N_\lambda$,

where $Q:Z\to Z_1, QZ=Z_1$ is continuous, bounded and satisfies $Q(I-Q)=0$ and $R:\overline{\Omega}\times[0,1]\to X_2$ is continuous and compact with $Pu+R(x,\lambda)\in dom\ M, x\in\overline{\Omega},\lambda\in[0,1]$.

Theorem 2.1 (see [4, 15]). *Let X and Z be two Banach spaces with the norms $\|\cdot\|_X,\|\cdot\|_Z$, respectively,and $\Omega\subset X$ be an open and bounded nonempty set. Suppose*

$$M:X\cap\operatorname{dom}M\to Z$$

is a quasi-linear operator and that $N_\lambda:\overline{\Omega}\to Z,\lambda\in[0,1]$ is $M-$quasi-compact. In addition, if the following conditions hold:

(C') *$Mx\neq N_\lambda x$,$\forall x\in\operatorname{dom}M\ \cap\partial\Omega,\lambda\in(0,1)$;*

(C'') *$\deg(JQN,\Omega\cap\operatorname{Ker}M,0)\neq0$,*

then the abstract equation $Mx=Nx$ has at least one solution in $\operatorname{dom}M\cap\overline{\Omega}$, where $N=N_1, J:\operatorname{Im}Q\to\operatorname{Ker}M$ is a homeomorphism with $J(\theta)=\theta$ and deg is the Brouwer degree.

Theorem 2.2 (see [1]). *Let $M \subset X$. Then M is relatively compact if the following conditions hold:*

(*a*) *M is bounded in X;*

(*b*) *the functions belonging to M are equi-continuous on any compact interval of $\mathbb{R}^+$;*

(*c*) *the functions from M are equi-convergent at $+\infty$.*

The following definitions can be found in [17, 22, 23].

Definition 2.3. The Riemann-Liouville fractional integrals of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2.4. The fractional derivatives of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0_+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} y(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, and this derivative is called the right side is pointwise defined on $(0, +\infty)$.

Lemma 2.1 (see [22]). *Assume $f \in L(0, +\infty), q \geq p \geq 0$, then $D_{0+}^{p} I_{0+}^{q} f(t) = I_{0+}^{q-p} f(t)$.*

Lemma 2.2 (see [22]).

(1) $D_{0+}^{\alpha} u(t) = 0$ if and only if $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, where $c_i = \dfrac{(I_{0+}^{n-\alpha} u)^{(n-i)}(0)}{\Gamma(\alpha - i + 1)} \in \mathbb{R}, i = 1, 2, \ldots, n$, and $n = [\alpha] + 1$.

(2) Assume $\alpha > 0, \lambda > -1$, then

$$D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(n+\lambda-\alpha+1)} \frac{d^n}{dt^n} (t^{n+\lambda-\alpha}),$$

where $n = [\alpha] + 1$, and

$$I_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} (t^{\lambda+\alpha}).$$

Example. *$D_{0+}^{\alpha} t^{\alpha-i} = 0, i = 1, 2, \ldots, n.$*

Lemma 2.3 (see [23]). *If the fractional derivatives $D_{0+}^{\alpha} y(t)$ and $D_{0+}^{\alpha+m} y(t)$ exist, then $(D_{0+}^{\alpha} y(t))^{(m)} = D_{0+}^{\alpha+m} y(t)$, where $\alpha > 0, m$ is a positive integer.*

Lemma 2.4 (see [24]). *For any $u, v \geq 0$, then*
(1) $\varphi_p(u+v) \leq \varphi_p(u) + \varphi_p(v), 1 < p \leq 2$;
(2) $\varphi_p(u+v) \leq 2^{p-2}(\varphi_p(u) + \varphi_p(v)), p \geq 2$,
where $\varphi_p(s) = |s|^{p-2}s = s^{p-1}, s \geq 0$.

In the following, we will always assume that q satisfies $1/p+1/q=1$.

Take

$$X=\Big\{u|u, D_{0^+}^{\alpha-(n-1)}u, D_{0^+}^{\alpha-(n-2)}u,\ldots,D_{0^+}^{\alpha-1}u, D_{0^+}^{\alpha}u\in C[0,+\infty),\\ \sup_{t\in[0,+\infty)}\frac{|ut^{n-\alpha}|}{e^{(q+n-1)t}}<+\infty,\ \sup_{t\in[0,+\infty)}\frac{|D_{0^+}^{\alpha-i}u|}{e^{(q+i-1)t}}<+\infty,\\ \sup_{t\in[0,+\infty)}\frac{|D_{0^+}^{\alpha}u|}{(1+t^{\beta+1})^{q-1}}<+\infty, i=1,2,\ldots,n-1\Big\}$$

with the norm

$$\|u\|_X=\max\Big\{\Big\|\frac{ut^{n-\alpha}}{e^{(q+n-1)t}}\Big\|_\infty,\Big\|\frac{D_{0^+}^{\alpha-(n-1)}u}{e^{(q+n-2)t}}\Big\|_\infty,\\ \Big\|\frac{D_{0^+}^{\alpha-(n-2)}u}{e^{(q+n-3)t}}\Big\|_\infty,\ldots,\Big\|\frac{D_{0^+}^{\alpha-1}u}{e^{qt}}\Big\|_\infty,\Big\|\frac{D_{0^+}^{\alpha}u}{(1+t^{\beta+1})^{q-1}}\Big\|_\infty\Big\},$$

where $\|u\|_\infty=\sup\limits_{t\in[0,+\infty)}|u(t)|$.

Let $Y=\{y\in C[0,+\infty):\sup\limits_{t\in[0,+\infty)}|y(t)|<\infty\}$ be endowed with the following norm $\|y\|_Y=\sup\limits_{t\in[0,+\infty)}|y|=\|y\|_\infty$ and $Z=\{\mu y: y\in Y\}$ with norm $\|\mu y\|_Z=\|y\|_\infty$ and μ is introduced in (H_1).

Define operators $M: X\cap \mathrm{dom}M\to Z$, $N_\lambda: X\to Z$ and $T_i: Y\to\mathbb{R}, i=1,2$ by

$$Mu(t)=D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha}u))(t),\\ N_\lambda u(t)=\lambda\mu(t)f\big(t,u(t),D_{0^+}^{\alpha-(n-1)}u(t),\ldots,D_{0^+}^{\alpha-1}u(t),D_{0^+}^{\alpha}u(t)\big), t\in[0,+\infty),\lambda\in[0,1],$$

and

$$T_i(y)=\Gamma_i\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s(s-r)^{\beta-1}\mu(r)y(r)dr\Big)ds\Big), i=1,2, \tag{2.1}$$

where $\mathrm{dom}M=\Big\{u\in X\Big|\frac{D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha}u))}{\mu(t)}\in Y, D_{0^+}^{\alpha}u(0)=0, D_{0^+}^{\alpha-2}u(0)=D_{0^+}^{\alpha-3}u(0)=\ldots=D_{0^+}^{\alpha-(n-2)}u(0)=0, u(0)=0,\Gamma_1(u)=0,\Gamma_2(u)=0\Big\}$.

Then we can write the problem (1.2) as $Mu=Nu, u\in\mathrm{dom}M$.

Lemma 2.5. *$(X,\|\cdot\|_X),(Y,\|\cdot\|_Y)$ and $(Z,\|\cdot\|_Z)$ are Banach spaces.*

Proof. We know that $(Y,\|\cdot\|_Y)$ and $(Z,\|\cdot\|_Z)$ are Banach spaces. Clearly, $(X,\|\cdot\|_X)$ is a normed space. Let $\{u_k\}_{k=1}^\infty$ be a Cauchy sequence in the space $(X,\|\cdot\|_X)$. Then the sequences $\{\frac{u_kt^{n-\alpha}}{e^{(q+n-1)t}}\}_{k=1}^\infty$, $\{\frac{D_{0^+}^{\alpha-(n-1)}u_k}{e^{(q+n-2)t}}\}_{k=1}^\infty,\ldots,\{\frac{D_{0^+}^{\alpha-2}u_k}{e^{(q+1)t}}\}_{k=1}^\infty,\{\frac{D_{0^+}^{\alpha-1}u_k}{e^{qt}}\}_{k=1}^\infty$ and $\{\frac{D_{0^+}^{\alpha}u_k}{(1+t^{\beta+1})^{q-1}}\}_{k=1}^\infty$ are Cauchy sequences in the space $(Y,\|\cdot\|_Y)$. Assume that they converge to $\frac{ut^{n-\alpha}}{e^{(q+n-1)t}}$, $\frac{v_1}{e^{(q+n-2)t}},\ldots,\frac{v_{n-2}}{e^{(q+1)t}},\frac{v_{n-1}}{e^{qt}},\frac{v_n}{(1+t^{\beta+1})^{q-1}}$, respectively.

Obviously, $\lim\limits_{k\to\infty}u_k(t)=u(t),\lim\limits_{k\to\infty}D_{0^+}^{\alpha-(n-1)}u_k(t)=v_1(t),\ldots,\lim\limits_{k\to\infty}D_{0^+}^{\alpha-1}u_k(t)=v_{n-1}(t)$, and $\lim\limits_{k\to\infty}D_{0^+}^{\alpha}u_k(t)=v_n(t)$.

Now we prove that $v_1=D_{0^+}^{\alpha-(n-1)}u, v_2=D_{0^+}^{\alpha-(n-2)}u,\ldots,v_n=D_{0^+}^{\alpha}u$.

By Lemam 2.7 and (1) in Lemma 2.8, it is easy to deduce that

$$\begin{aligned}
&I_{0^+}^{\alpha}D_{0^+}^{\alpha}u_k = u_k + c_1^{(1)}t^{\alpha-1} + c_2^{(1)}t^{\alpha-2} + \ldots + c_n^{(1)}t^{\alpha-n},\\
&I_{0^+}^{\alpha-1}D_{0^+}^{\alpha-1}u_k = u_k + c_1^{(2)}t^{\alpha-2} + c_2^{(2)}t^{\alpha-3} + \ldots + c_{n-1}^{(2)}t^{\alpha-n},\\
&\vdots\\
&I_{0^+}^{\alpha-(n-1)}D_{0^+}^{\alpha-(n-1)}u_k = u_k + c_1^{(n)}t^{\alpha-n}.
\end{aligned}$$

By Definition 2.5, we have

$$\begin{aligned}
&I_{0^+}^{\alpha}D_{0^+}^{\alpha}u_k = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}D_{0^+}^{\alpha}u_k(s)ds,\\
&I_{0^+}^{\alpha-1}D_{0^+}^{\alpha-1}u_k = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}D_{0^+}^{\alpha-1}u_k(s)ds,\\
&\vdots\\
&I_{0^+}^{\alpha-(n-1)}D_{0^+}^{\alpha-(n-1)}u_k = \frac{1}{\Gamma(\alpha-n+1)}\int_0^t (t-s)^{\alpha-n}D_{0^+}^{\alpha-n+1}u_k(s)ds.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&u_k + \sum_{i=1}^{n} c_i^{(1)}t^{\alpha-i} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}D_{0^+}^{\alpha}u_k(s)ds,\\
&u_k + \sum_{i=1}^{n-1} c_i^{(2)}t^{\alpha-i-1} = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}D_{0^+}^{\alpha-1}u_k(s)ds,\\
&\vdots\\
&u_k + c_1^{(n)}t^{\alpha-n} = \frac{1}{\Gamma(\alpha-n+1)}\int_0^t (t-s)^{\alpha-n}D_{0^+}^{\alpha-n+1}u_k(s)ds.
\end{aligned}$$

Since $\{\frac{u_k t^{n-\alpha}}{e^{(q+n-1)t}}\}_{k=1}^{\infty}$, $\{\frac{D_{0^+}^{\alpha-(n-1)}u_k}{e^{(q+n-2)t}}\}_{k=1}^{\infty}$, $\ldots$, $\{\frac{D_{0^+}^{\alpha-2}u_k}{e^{(q+1)t}}\}_{k=1}^{\infty}$,$\{\frac{D_{0^+}^{\alpha-1}u_k}{e^{qt}}\}_{k=1}^{\infty}$ and $\{\frac{D_{0^+}^{\alpha}u_k}{(1+t^{\beta+1})^{q-1}}\}_{k=1}^{\infty}$ are bounded in the space $\{Y, \|\cdot\|_Y\}$, $\int_0^t (t-s)^{\alpha-i}e^{(q+i-2)s}ds \le e^{(q+i-2)t}\int_0^t (t-s)^{\alpha-i}ds, i = 2, 3, \ldots, n$, $\int_0^t (t-s)^{\alpha-i}(1+s^{\beta+1})^{q-1}ds \le (1+t^{\beta+1})^{q-1}\cdot \int_0^t (t-s)^{\alpha-1}ds$, by Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
&u + \sum_{i=1}^{n} c_i^{(1)}t^{\alpha-i} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_n(s)ds,\\
&u + \sum_{i=1}^{n-1} c_i^{(2)}t^{\alpha-i-1} = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}v_{n-1}(s)ds,\\
&\vdots\\
&u + c_1^{(n)}t^{\alpha-n} = \frac{1}{\Gamma(\alpha-n+1)}\int_0^t (t-s)^{\alpha-n}v_1(s)ds.
\end{aligned}$$

These mean that $D_{0+}^{\alpha}u = v_n, D_{0+}^{\alpha-1}u = v_{n-1}, \ldots, D_{0+}^{\alpha-n+1}u = v_1$. Consequently, we conclude that $\{X, \|\cdot\|_X\}$ is a Banach space, which completes the proof of Lemma 2.11. □

3. Main result

In order to obtain the main result, we will introduce the following assumptions:

(C_1) The functionals $\Gamma_1, \Gamma_2 : X \to \mathbb{R}$ are linear continuous and satisfy $\Gamma_2(t^{\alpha-1}) = \Gamma_2(t^{\alpha-n+1}) = \Gamma_1(t^{\alpha-1}) = \Gamma_1(t^{\alpha-n+1}) = 0$.

(C_2) There must exist $g(t) \in Y$ such that $T_2(g(t)) \neq T_1(g(t))$, where $T_i, i = 1, 2$ are shown in the following lemma.

Lemma 3.1 (see [16]). *For $y \in Y$, there is only one constant $c_i \in \mathbb{R}$ such that $T_i(y) = c_i$ with $|c_i| \leq \|y\|_\infty$, where $T_i(y - c_i) = 0, i = 1, 2$ and $T_i : Y \to \mathbb{R}, i = 1, 2$ is continuous, $T_i(c) = c$, $T_i(y + c) = T_i(y) + c$, $T_i(cy) = cT_i(y), i = 1, 2, c \in \mathbb{R},\ y \in Y$.*

Lemma 3.2. *The operator M is quasi-linear.*

Proof. It is easy to get that

$$\operatorname{Ker} M = \{at^{\alpha-1} + bt^{\alpha-n+1}, a, b \in \mathbb{R}\} := X_1.$$

For $u \in X \cap \operatorname{dom} M$, if $Mu = \mu y$, then

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q\Big(\frac{1}{\Gamma(\beta)} \int_0^s (s-r)^{\beta-1} \mu(r) y(r) dr\Big) ds + ct^{\alpha-1} + dt^{\alpha-n+1}, \tag{3.1}$$

where c and d are two arbitrary constants.

Applying the resonance conditions $\Gamma_i(t^{\alpha-1}) = 0, \Gamma_i(t^{\alpha-n+1}) = 0, i = 1, 2$ to (3.1) and using the boundary value conditions $\Gamma_i(u) = 0, i = 1, 2$, one has

$$\Gamma_i(u(t)) = \Gamma_i\Big(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q\Big(\frac{1}{\Gamma(\beta)} \int_0^s (s-r)^{\beta-1} \mu(r) y(r) dr\Big) ds\Big) = 0, i = 1, 2.$$

That is,

$$\operatorname{Im} M \subseteq \{\mu y | y \in Y : T_i(y) = 0, i = 1, 2\}. \tag{3.2}$$

On the other hand, if $y \in Y, T_i(y) = 0, i = 1, 2$, take

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q\Big(\frac{1}{\Gamma(\beta)} \int_0^s (s-r)^{\beta-1} \mu(r) y(r) dr\Big) ds.$$

By a simple calculation, we get $u \in X \cap \operatorname{dom} M$ and $Mu = \mu y$. Thus $\operatorname{Im} M = \{\mu y | y \in Y, T_i y = 0\}$.

By the continuity of T, we get that $\operatorname{Im} M \subset Z$ is closed. So, M is quasi-linear. □

Proof of Lemma 3.1. Take a projector $P : X \to X_1$ and an operator $Q : Z \to Z_1$ as follows:

$$Pu(t) = \frac{(D_{0+}^{\alpha-1}u)(0)}{\Gamma(\alpha)} t^{\alpha-1} + \frac{(D_{0+}^{\alpha-n+1}u)(0)}{\Gamma(\alpha-n+2)} t^{\alpha-n+1},$$

$Q\mu y = \mu(Q_1 y + (Q_2 y)g(t))$, where

$$Q_1 y = \frac{T_2(g(t))T_1(y) - T_1(g(t))T_2(y)}{T_2(g(t)) - T_1(g(t))}, \quad Q_2 y = \frac{T_2(y) - T_1(y)}{T_2(g(t)) - T_1(g(t))},$$

$Z_1 = \{a\mu + b\mu_1 | \mu_1 = \mu g\}$ and $g(t)$ is introduced in condition (C_2),

By Lemma 3.1 and condition (C_2), we carefully check that

$$\begin{aligned}
& Q_1(Q_1 y + (Q_2 y)g(t)) \\
=& \frac{T_2(g(t))T_1\big(Q_1 y + (Q_2 y)g(t)\big) - T_1(g(t))T_2\big(Q_1 y + (Q_2 y)g(t)\big)}{T_2(g(t)) - T_1(g(t))} \\
=& \frac{T_2(g(t))T_1(Q_1 y) + T_2(g(t))T_1(g(t)Q_2 y - T_1(g(t))T_2(Q_1 y) - T_1(g(t))T_2(g(t))Q_2 y}{T_2(g(t)) - T_1(g(t))} \\
=& \frac{T_2(g(t))T_1(Q_1 y) - T_1(g(t))T_2(Q_1 y)}{T_2(g(t)) - T_1(g(t))} = Q_1 y, \\
& Q_2(Q_1 y + (Q_2 y)g(t)) \\
=& \frac{T_2\big(Q_1 y + (Q_2 y)g(t)\big) - T_1\big(Q_1 y + (Q_2 y)g(t)\big)}{T_2(g(t)) - T_1(g(t))} \\
=& \frac{T_2(Q_1 y) + T_2(g(t))Q_2 y - T_1(Q_1 y) - T_1(g(t))Q_2 y}{T_2(g(t)) - T_1(g(t))} = Q_2 y,
\end{aligned}$$

so,

$$\begin{aligned}
Q^2(\mu y) &= Q(\mu Q_1 y + \mu(Q_2 y)g(t)) \\
&= \mu Q_1(Q_1 y + (Q_2 y)g(t)) + \mu Q_2(Q_1 y + (Q_2 y)g(t))g(t) \\
&= \mu Q_1 y + \mu(Q_2 y)g(t) = Q\mu y.
\end{aligned}$$

Obviously, $QZ := Z_1$, $Q(I-Q) = 0$, $\dim Z_1 = \dim X_1$ and $KerQ = ImM$. It follows from Lemma 3.1 and $g(t) \in Y$ that $Q : Z \to Z_1$ is continuous and bounded.

For $u \in X$, set $u = u - Pu + Pu$. It is easy to check that

$$\begin{aligned}
P^2 u(t) = P(Pu(t)) &= \frac{(D_{0^+}^{\alpha-1} Pu)(0)}{\Gamma(\alpha)} t^{\alpha-1} + \frac{(D_{0^+}^{\alpha-n+1} Pu)(0)}{\Gamma(\alpha-n+2)} t^{\alpha-n+1} \\
&= \frac{\frac{(D_{0^+}^{\alpha-1} u)(0)}{\Gamma(\alpha)}\Gamma(\alpha)}{\Gamma(\alpha)} t^{\alpha-1} + \frac{\frac{(D_{0^+}^{\alpha-n+1} u)(0)}{\Gamma(\alpha-n+2)}\Gamma(\alpha-n+2)}{\Gamma(\alpha-n+2)} t^{\alpha-n+1} \\
&= \frac{(D_{0^+}^{\alpha-1} u)(0)}{\Gamma(\alpha)} t^{\alpha-1} + \frac{(D_{0^+}^{\alpha-n+1} u)(0)}{\Gamma(\alpha-n+2)} t^{\alpha-n+1} = Pu(t), u \in X,
\end{aligned}$$

and it is also elementary to confirm the identity $ImP = KerM$ and $ImP \bigcap KerP = \{0\}$. So, $X = KerM \oplus KerP$. □

Define an operator R as

$$R(u,\lambda)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q\Big(\frac{1}{\Gamma(\beta)} \int_0^s (s-r)^{\beta-1} (I-Q) N_\lambda u(r) dr\Big) ds,$$

$(u,\lambda) \in X \times [0,1]$.

Lemma 3.3. *$R : \overline{\Omega}\times[0,1] \to X_2$ is continuous and compact, $Pu+R(u,\lambda) \in domM$, $u \in \overline{\Omega}$, $\lambda \in [0,1]$, where $X_2 = \{u \in X : D_{0^+}^{\alpha-1}R(u,\lambda)(0) = 0, D_{0^+}^{\alpha-n+1}R(u,\lambda)(0) = 0\}, \Omega \subset X$ is an open bounded set.*

Proof. Firstly, we prove that $R : \overline{\Omega}\times[0,1] \to X_2$ and $Pu+R(u,\lambda) \in \mathrm{dom}M, u \in \overline{\Omega}, \lambda \in [0,1]$. Considering the continuity of Q and f, we can easily check that

$$R(u,\lambda)(t), D_{0^+}^{\alpha-n+1}R(u,\lambda)(t), \dots, D_{0^+}^{\alpha}R(u,\lambda)(t) \in C[0,+\infty).$$

By the continuity of f, $|T_i y| \le \|y\|_\infty$ and $\sup\limits_{t\in[0,+\infty)} |I_{0^+}^{\beta}\mu| < +\infty$, we get that for any $r > 0$, there exists a constant $M_r > 0$ such that if $\frac{|u_1 t^{n-\alpha}|}{e^{(q+n-1)t}} \le r, \frac{|u_i|}{e^{(q+n-i)t}} \le r, \frac{|u_{n+1}|}{(1+t^{\beta+1})^{q-1}} \le r, i = 2,3,\dots,n, t \in [0,+\infty)$, then $f(t,u_1,u_2,\dots,u_{n+1}) \le M_r$, for any $\varepsilon>0$, there exists $\delta>0$ such that $|f(t,u_1,u_2,\dots,u_{n+1}) - f(t,v_1,v_2,\dots,v_{n+1})| < \varepsilon$ for $t \in [0,+\infty)$,$u_i, v_i \in \mathbb{R}, i = 1,2,\dots,n+1$, satisfying $\frac{|(u_1-v_1)t^{n-\alpha}|}{e^{(q+n-1)t}} < \delta$, $\frac{|u_j - v_j|}{e^{(q+n-j)t}} < \delta$, $\frac{|u_{n+1}-v_{n+1}|}{(1+t^{\beta+1})^{q-1}} < \delta$, and $\frac{|u_1 t^{n-\alpha}|}{e^{(q+n-1)t}} \le r$, $\frac{|v_1 t^{n-\alpha}|}{e^{(q+n-1)t}} \le r$, $\frac{|u_j|}{e^{(q+n-j)t}} \le r$, $\frac{|v_j|}{e^{(q+n-j)t}} \le r, j = 2,3,\dots,n$, $\frac{|u_{n+1}|}{(1+t^{\beta+1})^{q-1}} \le r$, $\frac{|v_{n+1}|}{(1+t^{\beta+1})^{q-1}} \le r$, and there exists a constant $C > 0$ such that $|I_{0^+}^{\beta}(I-Q)N_\lambda u| \le C$ in $\overline{\Omega}$ for all $\lambda \in [0,1]$, which implies that

$$\begin{aligned}
&|\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u)(t)| \le C^{q-1},\\
&\Big|\frac{R(u,\lambda)(t)t^{n-\alpha}}{e^{(q+n-1)t}}\Big| = \Big|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}t^{n-\alpha}}{e^{(q+n-1)t}}\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u)(s)ds\Big|\\
&\qquad\qquad \le \frac{1}{\Gamma(\alpha+1)}\frac{t^n}{e^{(q+n-1)t}}C^{q-1} \le C^{q-1},\\
&\Big|\frac{D_{0^+}^{\alpha-i}R(u,\lambda)(t)}{e^{(q+i-1)t}}\Big| = \Big|\frac{1}{(i-1)!}\int_0^t \frac{(t-s)^{i-1}}{e^{(q+i-1)t}}\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u)(s)ds\Big|\\
&\qquad\qquad \le \frac{1}{i!}\frac{t^i}{e^{(q+i-1)t}}C^{q-1} \le C^{q-1}, i = 1,2,\dots,n-1,
\end{aligned}$$

and

$$\begin{aligned}
\Big|\frac{D_{0^+}^{\alpha}R(u,\lambda)(t)}{(1+t^{\beta+1})^{q-1}}\Big| &= \Big|\frac{\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u)(t)}{(1+t^{\beta+1})^{q-1}}\Big|\\
&\le \frac{1}{(1+t^{\beta+1})^{q-1}}C^{q-1} \le C^{q-1}, u \in X.
\end{aligned}$$

Therefore, $R(u,\lambda) \in X$. It is clear that

$$D_{0^+}^{\alpha-1}R(u,\lambda)(0) = \lim_{t\to 0}\int_0^t \varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u(s))ds = 0,$$

and

$$D_{0^+}^{\alpha-n+1}R(u,\lambda)(0) = \lim_{t\to 0}\int_0^t \frac{(t-s)^{n-2}}{(n-2)!}\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u(s))ds = 0.$$

Thus $R(u,\lambda)\in X_2$. Clearly, $R(u,\lambda)+Pu\in X$. It follows from $D^{\beta}_{0^+}\varphi_p\big(D^{\alpha}_{0^+}(R(u,\lambda)(t)+Pu(t))\big)=\lambda\mu(t)f(t,u(t),D^{\alpha-n+1}_{0^+}u(t),\dots,D^{\alpha}_{0^+}u(t))$ and (H_2) that

$$\frac{D^{\beta}_{0^+}\varphi_p\big(D^{\alpha}_{0^+}(R(u,\lambda)(t)+Pu(t))\big)}{\mu(t)}=\lambda f(t,u(t),D^{\alpha-n+1}_{0^+}u(t),\dots,D^{\alpha}_{0^+}u(t))\in Y,$$
$$D^{\alpha}_{0^+}(R(u,\lambda)+Pu)(0)=0, D^{\alpha-2}_{0^+}(R(u,\lambda)+Pu)(0)=0,\dots,$$
$$D^{\alpha-(n-2)}_{0^+}(R(u,\lambda)+Pu)(0)=0,\ (R(u,\lambda)+Pu)(0)=0.$$

Meanwhile, note that $(I-Q)N_\lambda u\in \mathrm{Ker}Q=\mathrm{Im}M$, we obtain that

$$\begin{aligned}\Gamma_1(R(u,\lambda)+Pu)=&\Gamma_1\big(I^{\alpha}_{0^+}\varphi_q(I^{\beta}_{0^+}(I-Q)N_\lambda u(t))\big)+\frac{D^{\alpha-1}_{0^+}u(0)}{\Gamma(\alpha)}\Gamma_1(t^{\alpha-1})\\&+\frac{D^{\alpha-n+1}_{0^+}u(0)}{\Gamma(\alpha-n+2)}\Gamma_1(t^{\alpha-n+1})=0,\end{aligned}$$

similarly, $\Gamma_2(R(u,\lambda)+Pu)=0$. So, $R(u,\lambda)+Pu\in\mathrm{dom}M$.

Secondly, we show that R is continuous.

Since Ω is bounded, there exists a constant $r>0$ such that $\|u\|_X\le r, u\in\overline{\Omega}$. By $(H_1),(H_2)$ and boundedness of Q, there exist two constants M_r and C_r such that

$$|f(t,u(t),D^{\alpha-n+1}_{0^+}u(t),\dots,D^{\alpha}_{0^+}u(t))|\le M_r, |I^{\beta}_{0^+}(I-Q)N_\lambda u|\le C_r, u\in\overline{\Omega}, t\in[0,+\infty).$$

By the uniform continuity of $\varphi_q(x)$ in $[-C_r,\max\{1,C_r\}]$, we obtain that for any $\varepsilon>0$, there exists a constant $\delta_\varepsilon>0$ such that $|\varphi_q(x_1)-\varphi_q(x_2)|<\varepsilon, |x_1-x_2|\le\delta_\varepsilon, x_1,x_2\in[-C_r,\max\{1,C_r\}]$.

For $\eta=\dfrac{\delta_\varepsilon}{\sup\limits_{t\in[0,+\infty)}|I^{\beta}_{0^+}\mu|}$, by (H_2) and continuity of Q, there exists a constant δ_η such that if $u,v\in\overline{\Omega}, \|u-v\|_X<\delta_\eta$, then

$$\begin{aligned}&|(I-Q)N_\lambda u-(I-Q)N_\lambda v|\\=&\big|\lambda\mu(t)\big(F(u)-F(v)+Q_1(F(v))-Q_1(F(u))+\big(Q_2(F(v))-Q_2(F(u))\big)g(t)\big)\big|\\\le&|\mu(t)|\eta,\quad t\in[0,+\infty),\end{aligned}$$

where $F(u)=f\big(t,u(t),D^{\alpha-(n-1)}_{0^+}u(t),\dots,D^{\alpha-1}_{0^+}u(t),D^{\alpha}_{0^+}u(t)\big)$. So, we have

$$\begin{aligned}&\Big|\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}(I-Q)N_\lambda u(s)ds-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}(I-Q)N_\lambda v(s)ds\Big|\\=&\Big|\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\Big((I-Q)N_\lambda u-(I-Q)N_\lambda v\Big)(s)ds\Big|\\\le&\sup_{t\in[0,+\infty)}|I^{\beta}_{0^+}\mu|\eta=\delta_\varepsilon, \|u-v\|_X<\delta_\eta.\end{aligned}$$

Take $\delta=\min\{\delta_\varepsilon,\delta_\eta\}$. For $u,v\in\overline{\Omega},\lambda_1,\lambda_2\in[0,1]$, if $\|u-v\|_X<\delta, |\lambda_1-\lambda_2|<\delta$, then

$$\Big|\frac{D^{\alpha}_{0^+}R(u,\lambda_1)(t)-D^{\alpha}_{0^+}R(v,\lambda_2)(t)}{(1+t^{\beta+1})^{q-1}}\Big|$$

$$
\begin{aligned}
=&\Big|\varphi_q(\frac{I_{0^+}^{\beta}(I-Q)N_{\lambda_1}u}{1+t^{\beta+1}})-\varphi_q(\frac{I_{0^+}^{\beta}(I-Q)N_{\lambda_2}v}{1+t^{\beta+1}})\Big|\\
=&\Big|\frac{\varphi_q(\lambda_1)\varphi_q(I_{0^+}^{\beta}(I-Q)(\mu F(u)))-\varphi_q(\lambda_2)\varphi_q(I_{0^+}^{\beta}(I-Q)(\mu F(v)))}{(1+t^{\beta+1})^{q-1}}\Big|\\
\leq&\Big|\varphi_q(\lambda_1)\varphi_q(I_{0^+}^{\beta}(I-Q)(\mu F(u))-\varphi_q(\lambda_1)\varphi_q(I_{0^+}^{\beta}(I-Q)(\mu F(v))\\
&+\varphi_q(\lambda_1)\varphi_q(I_{0^+}^{\beta}(I-Q)(\mu F(v))-\varphi_q(\lambda_2)\varphi_q(I_{0^+}^{\beta}(I-Q)(\mu F(v))\Big|\\
\leq&\Big|\varphi_q(\lambda_1)\varepsilon+(\varphi_q(\lambda_1)-\varphi_q(\lambda_2))\varphi_q(C_r)\Big|\\
\leq&\big(1+\varphi_q(C_r)\big)\varepsilon, t\in[0,+\infty),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\frac{D_{0^+}^{\alpha-i}R(u,\lambda_1)(t)-D_{0^+}^{\alpha-i}R(v,\lambda_2)(t)}{e^{(q+i-1)t}}\Big|\\
=&\Big|\frac{\int_0^t(t-s)^{i-1}\varphi_q(I_{0^+}^{\beta}(I-Q)N_{\lambda_1}u)(s)ds-\int_0^t(t-s)^{i-1}\varphi_q(I_{0^+}^{\beta}(I-Q)N_{\lambda_2}v)(s)ds}{(i-1)!e^{(q+i-1)t}}\Big|\\
=&\Big|\frac{\int_0^t(t-s)^{i-1}\Big(\varphi_q(I_{0^+}^{\beta}(I-Q)N_{\lambda_1}u)-\varphi_q(I_{0^+}^{\beta}(I-Q)N_{\lambda_2}v)\Big)(s)ds}{(i-1)!e^{(q+i-1)t}}\Big|\\
\leq&\Big|\frac{(1+t^{\beta+1})^{q-1}}{i!e^{(q+i-1)t}}\frac{\varphi_q(I_{0^+}^{\beta}(I-Q)N_{\lambda_1}u)-\varphi_q(I_{0^+}^{\beta}(I-Q)N_{\lambda_2}v)}{(1+t^{\beta+1})^{q-1}}\Big|\\
\leq&\Big\|\frac{D_{0^+}^{\alpha}R(u,\lambda_1)}{(1+t^{\beta+1})^{q-1}}-\frac{D_{0^+}^{\alpha}R(v,\lambda_2)}{(1+t^{\beta+1})^{q-1}}\Big\|_{\infty}, t\in[0,+\infty).
\end{aligned}
$$

These, together with

$$
\begin{aligned}
&\Big|\frac{R(u,\lambda_1)(t)t^{n-\alpha}}{e^{(q+n-1)t}}-\frac{R(v,\lambda_2)(t)t^{n-\alpha}}{e^{(q+n-1)t}}\Big|\\
=&\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}t^{n-\alpha}}{e^{(q+n-1)t}}\Big[\varphi_q(I_{0^+}^{\beta}(I-Q)N_{\lambda_1}u)-\varphi_q(I_{0^+}^{\beta}(I-Q)N_{\lambda_2}v)\Big](s)ds\\
\leq&\frac{t^n(1+t^{\beta+1})^{q-1}}{\Gamma(\alpha+1)e^{(q+n-1)t}}\Big\|\frac{D_{0^+}^{\alpha}R(u,\lambda_1)}{(1+t^{\beta+1})^{q-1}}-\frac{D_{0^+}^{\alpha}R(v,\lambda_2)}{(1+t^{\beta+1})^{q-1}}\Big\|_{\infty}\\
\leq&\Big\|\frac{D_{0^+}^{\alpha}R(u,\lambda_1)}{(1+t^{\beta+1})^{q-1}}-\frac{D_{0^+}^{\alpha}R(v,\lambda_2)}{(1+t^{\beta+1})^{q-1}}\Big\|_{\infty},
\end{aligned}
$$

mean that $R:\overline{\Omega}\times[0,1]\to X_2\cap\mathrm{dom}M$ is continuous.

We will prove that $R:\overline{\Omega}\times[0,1]\to X_2\cap\mathrm{dom}M$ is compact.

It is easy to get that $\Big\{\frac{R(u,\lambda)(t)t^{n-\alpha}}{e^{(q+n-1)t}}:u\in\overline{\Omega},\lambda\in[0,1]\Big\}$, $\Big\{\frac{D_{0^+}^{\alpha-i}R(u,\lambda)(t)}{e^{(q+i-1)t}}, i=1,2,\dots,n-1:u\in\overline{\Omega},\lambda\in[0,1]\Big\}$ and $\Big\{\frac{D_{0^+}^{\alpha}R(u,\lambda)(t)}{(1+t^{\beta+1})^{q-1}}:u\in\overline{\Omega},\lambda\in[0,1]\Big\}$ are bounded.

For any $T>0, t_1,t_2\in[0,T], t_1<t_2, u\in\overline{\Omega},\lambda\in[0,1]$, we have

$$
\Big|\frac{R(u,\lambda)(t_1){t_1}^{n-\alpha}}{e^{(q+n-1)t_1}}-\frac{R(u,\lambda)(t_2){t_2}^{n-\alpha}}{e^{(q+n-1)t_2}}\Big|
$$

$$
\begin{aligned}
&= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}{t_1}^{n-\alpha}}{e^{(q+n-1)t_1}}\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}{t_2}^{n-\alpha}}{e^{(q+n-1)t_2}}\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds\Big| \\
&\le \Big|\frac{{t_1}^{n-\alpha}}{e^{(q+n-1)t_1}}-\frac{{t_2}^{n-\alpha}}{e^{(q+n-1)t_2}}\Big|\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds\Big| \\
&\quad +\Big|\frac{{t_2}^{n-\alpha}}{e^{(q+n-1)t_2}}\Big|\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds\Big| \\
&\quad +\Big|\frac{{t_2}^{n-\alpha}}{e^{(q+n-1)t_2}}\Big|\Big|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds\Big| \\
&\le |\frac{{t_1}^{n-\alpha}}{e^{(q+n-1)t_1}}-\frac{{t_2}^{n-\alpha}}{e^{(q+n-1)t_2}}\Big|\frac{T^\alpha}{\Gamma(\alpha+1)}\varphi_q(C_r) \\
&\quad +\frac{{t_2}^{n-\alpha}}{e^{(q+n-1)t_2}}\frac{(t_2-t_1)^\alpha+{t_2}^\alpha-{t_1}^\alpha}{\Gamma(\alpha+1)}\varphi_q(C_r)+\frac{{t_2}^{n-\alpha}}{e^{(q+n-1)t_2}}\frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}\varphi_q(C_r).
\end{aligned}
$$

Since $\frac{t^{n-\alpha}}{e^{(q+n-1)t}}, t^\alpha$ and t are uniformly continuous on $[0,T]$, we get that $\Big\{\frac{R(u,\lambda)(t)t^{n-\alpha}}{e^{(q+n-1)t}} : u\in\overline{\Omega}, \lambda\in[0,1]\Big\}$ is equicontinuous on $[0,T]$.

$$
\begin{aligned}
&\Big|\frac{D^\alpha_{0^+}R(u,\lambda)(t_1)}{(1+{t_1}^{\beta+1})^{q-1}}-\frac{D^\alpha_{0^+}R(u,\lambda)(t_2)}{(1+{t_2}^{\beta+1})^{q-1}}\Big| \\
=&\Big|\varphi_q\big(\frac{I^\beta_{0^+}(I-Q)N_\lambda u(t_1)}{1+{t_1}^{\beta+1}}\big)-\varphi_q\big(\frac{I^\beta_{0^+}(I-Q)N_\lambda u(t_2)}{1+{t_2}^{\beta+1}}\big)\Big| \\
=&\Big|\varphi_q\big(\frac{I^\beta_{0^+}(I-Q)N_\lambda u(t_1)}{1+{t_1}^{\beta+1}}\big)-\varphi_q\big(\frac{I^\beta_{0^+}(I-Q)N_\lambda u(t_1)}{1+{t_2}^{\beta+1}}\big) \\
&+\varphi_q\big(\frac{I^\beta_{0^+}(I-Q)N_\lambda u(t_1)}{1+{t_2}^{\beta+1}}\big)-\varphi_q\big(\frac{I^\beta_{0^+}(I-Q)N_\lambda u(t_2)}{1+{t_2}^{\beta+1}}\big)\Big| \\
\le&|\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u(t_1))\big(\frac{1}{(1+{t_1}^{\beta+1})^{q-1}}-\frac{1}{(1+{t_2}^{\beta+1})^{q-1}}\big)| \\
&+\frac{1}{(1+{t_2}^{\beta+1})^{q-1}}\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u(t_1)-\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u(t_2))|.
\end{aligned}
$$

Take $G(t)=I^\beta_{0^+}(I-Q)N_\lambda u(t)$, we have $|G(t)|\le C_r, u\in\overline{\Omega}, t\in[0,T]$, and $|I-Q)N_\lambda u|\le C$ in $\overline{\Omega}$ for all $\lambda\in[0,1]$. Then

$$
\begin{aligned}
|G(t_2)-G(t_1)| &= \Big|\frac{1}{\Gamma(\beta)}\int_0^{t_1}(t_1-s)^{\beta-1}(I-Q)N_\lambda u(s)ds \\
&\quad -\frac{1}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}(I-Q)N_\lambda u(s)ds\Big| \\
&\le \Big|\frac{1}{\Gamma(\beta)}\int_0^{t_1}\big[(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}\big](I-Q)N_\lambda u(s)ds\Big| \\
&\quad +\Big|\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}(I-Q)N_\lambda u(s)ds\Big| \\
&\le \Big|\frac{1}{\Gamma(\beta)}\int_0^{t_1}\big[(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}\big]|I-Q)N_\lambda u|ds\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\Big|\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}|I-Q)N_\lambda u|ds\Big|\\
&\le \frac{C}{\Gamma(\beta+1)}\big(|{t_2}^{\beta}-{t_1}^{\beta}|+|(t_1-t_2)^{\beta}|\big)+\frac{C}{\Gamma(\beta+1)}|(t_1-t_2)^{\beta}|.
\end{aligned}
$$

It follows from the uniform continuity of $t, t^\beta, \frac{1}{1+t^{\beta+1}}$ on $[0,T]$ and $\varphi_q(\cdot)$ in $[-C_r, \max\{1, C_r\}]$ that $\Big\{\dfrac{D^\alpha_{0^+}R(u,\lambda)(t)}{(1+t^{\beta+1})^{q-1}} : u\in\overline{\Omega}, \lambda\in[0,1]\Big\}$ is equicontinuous on $[0,T]$. Furthermore,

$$
\begin{aligned}
&\Big|\frac{D^{\alpha-i}_{0^+}R(u,\lambda)(t_1)}{e^{(q+i-1)t_1}}-\frac{D^{\alpha-i}_{0^+}R(u,\lambda)(t_2)}{e^{(q+i-1)t_2}}\Big|\\
&=\Big|\frac{1}{(i-1)!e^{(q+i-1)t_1}}\int_0^{t_1}(t_1-s)^{i-1}\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds\\
&\quad-\frac{1}{(i-1)!e^{(q+i-1)t_2}}\int_0^{t_2}(t_2-s)^{i-1}\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds\Big|\\
&\le\Big|\frac{1}{(i-1)!}\big(\frac{1}{e^{(q+i-1)t_1}}-\frac{1}{e^{(q+i-1)t_2}}\big)\int_0^{t_1}(t_1-s)^{i-1}\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds\Big|\\
&\quad+\Big|\frac{1}{(i-1)!e^{(q+i-1)t_2}}\int_0^{t_1}\big[(t_1-s)^{i-1}-(t_2-s)^{i-1}\big]\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds\Big|\\
&\quad+\Big|\frac{1}{(i-1)!e^{(q+i-1)t_2}}\int_{t_1}^{t_2}(t_2-s)^{i-1}\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds\Big|\\
&\le\frac{\varphi_q(C_r){t_1}^i}{i!}\big(\frac{1}{e^{(q+i-1)t_1}}-\frac{1}{e^{(q+i-1)t_2}}\big)+\frac{\varphi_q(C_r)}{i!e^{(q+i-1)t_2}}\big({t_2}^i-{t_1}^i+(t_2-t_1)^i\big)\\
&\quad+\frac{\varphi_q(C_r)}{i!e^{(q+i-1)t_2}}(t_2-t_1)^i.
\end{aligned}
$$

Considering the uniform continuity of $\dfrac{1}{e^{(q+i-1)t}}, t^i$ and $t, i=1,2,\dots,n-1$ on $[0,T]$, we hold that $\Big\{\dfrac{D^{\alpha-i}_{0^+}R(u,\lambda)(t)}{e^{(q+i-1)t}}, i=1,2,\dots,n-1 : u\in\overline{\Omega}, \lambda\in[0,1]\Big\}$ are equicontinuous on $[0,T]$.

For any $\varepsilon>0$, there exists a constant $T_1>0$ such that for any $t>T_1$,

$$
\begin{aligned}
&\Big|\frac{t^{n-\alpha}[(t-T_1)^\alpha-t^\alpha]}{e^{(q+n-1)t}}\Big|<\frac{\varepsilon}{4}{\varphi_q}^{-1}(C_r),\Big|\frac{t^{n-\alpha}(t-T_1)^\alpha}{e^{(q+n-1)t}}\Big|<\frac{\varepsilon}{4}{\varphi_q}^{-1}(C_r),\\
&\Big|\frac{(t-T_1)^i-t^i}{e^{(q+i-1)t}}\Big|<\frac{\varepsilon}{4}{\varphi_q}^{-1}(C_r),\Big|\frac{(t-T_1)^i}{e^{(q+i-1)t}}\Big|<\frac{\varepsilon}{4}{\varphi_q}^{-1}(C_r), i=1,2,\dots,n-1.
\end{aligned}
$$

Obviously, there exists a constant $T>T_1$ such that for any $t>T$

$$
\frac{(t-T)^\beta-t^\beta}{1+t^{\beta+1}}<\frac{\delta_\varepsilon}{4C}\Gamma(\beta+1), \frac{(t-T)^\beta}{1+t^{\beta+1}}<\frac{\delta_\varepsilon}{4C}\Gamma(\beta+1).
$$

For any $t_2>t_1>T$, we have

$$
\begin{aligned}
&\Big|\frac{R(u,\lambda)(t_1){t_1}^{n-\alpha}}{e^{(q+n-1)t_1}}-\frac{R(u,\lambda)(t_2){t_2}^{n-\alpha}}{e^{(q+n-1)t_2}}\Big|\\
&=\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}{t_1}^{n-\alpha}}{e^{(q+n-1)t_1}}\varphi_q(I^\beta_{0^+}(I-Q)N_\lambda u)(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-1} {t_2}^{n-\alpha}}{e^{(q+n-1)t_2}} \varphi_q(I^{\beta}_{0^+}(I-Q)N_\lambda u)(s)ds\Big| \\
\le& \frac{{t_1}^{n-\alpha}}{e^{(q+n-1)t_1}} \Big| \frac{1}{\Gamma(\alpha)} \int_0^{T_1} (t_1-s)^{\alpha-1} \varphi_q(I^{\beta}_{0^+}(I-Q)N_\lambda u)(s)ds\Big| \\
&+ \frac{{t_1}^{n-\alpha}}{e^{(q+n-1)t_1}} \Big| \frac{1}{\Gamma(\alpha)} \int_{T_1}^{t_1} (t_1-s)^{\alpha-1} \varphi_q(I^{\beta}_{0^+}(I-Q)N_\lambda u)(s)ds\Big| \\
&+ \frac{{t_2}^{n-\alpha}}{e^{(q+n-1)t_2}} \Big| \frac{1}{\Gamma(\alpha)} \int_0^{T_1} (t_2-s)^{\alpha-1} \varphi_q(I^{\beta}_{0^+}(I-Q)N_\lambda u)(s)ds\Big| \\
&+ \frac{{t_2}^{n-\alpha}}{e^{(q+n-1)t_2}} \Big| \frac{1}{\Gamma(\alpha)} \int_{T_1}^{t_2} (t_2-s)^{\alpha-1} \varphi_q(I^{\beta}_{0^+}(I-Q)N_\lambda u)(s)ds\Big| \\
\le& \Big[\frac{{t_1}^{n-\alpha}|(t_1-T_1)^\alpha - {t_1}^\alpha|}{e^{(q+n-1)t_1}} + \frac{{t_1}^{n-\alpha}(t_1-T_1)^\alpha}{e^{(q+n-1)t_1}} \\
&+ \frac{{t_2}^{n-\alpha}|(t_2-T_1)^\alpha - {t_2}^\alpha|}{e^{(q+n-1)t_2}} + \frac{{t_2}^{n-\alpha}(t_2-T_1)^\alpha}{e^{(q+n-1)t_2}} \Big] \varphi_q(C_r) < \varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
&\Big| \frac{D^{\alpha}_{0^+} R(u,\lambda)(t_1)}{(1+{t_1}^{\beta+1})^{q-1}} - \frac{D^{\alpha}_{0^+} R(u,\lambda)(t_2)}{(1+{t_2}^{\beta+1})^{q-1}} \Big| \\
=& \Big| \varphi_q\Big(\frac{I^{\beta}_{0^+}(I-Q)N_\lambda u(t_1)}{1+{t_1}^{\beta+1}} \Big) - \varphi_q\Big(\frac{I^{\beta}_{0^+}(I-Q)N_\lambda u(t_2)}{1+{t_2}^{\beta+1}} \Big) \Big|.
\end{aligned}
$$

Let $G_1(t) = \dfrac{I^{\beta}_{0^+}(I-Q)N_\lambda u}{1+t^{\beta+1}}$, one has

$$
\begin{aligned}
|G_1(t_2) - G_1(t_1)| =& \Big| \frac{1}{\Gamma(\beta)(1+{t_1}^{\beta+1})} \int_0^{t_1} (t_1-s)^{\beta-1} (I-Q)N_\lambda u(s)ds \\
&- \frac{1}{\Gamma(\beta)(1+{t_2}^{\beta+1})} \int_0^{t_2} (t_2-s)^{\beta-1} (I-Q)N_\lambda u(s)ds \Big| \\
\le& \Big| \frac{1}{\Gamma(\beta)(1+{t_1}^{\beta+1})} \int_0^{T} (t_1-s)^{\beta-1} \|I-Q)N_\lambda u\|_Z ds \Big| \\
&+ \Big| \frac{1}{\Gamma(\beta)(1+{t_1}^{\beta+1})} \int_T^{t_1} (t_1-s)^{\beta-1} |I-Q)N_\lambda u| ds \Big| \\
&+ \Big| \frac{1}{\Gamma(\beta)(1+{t_2}^{\beta+1})} \int_0^{T} (t_2-s)^{\beta-1} |I-Q)N_\lambda u| ds \Big| \\
&+ \Big| \frac{1}{\Gamma(\beta)(1+{t_2}^{\beta+1})} \int_T^{t_2} (t_2-s)^{\beta-1} |I-Q)N_\lambda u| ds \Big| \\
\le& \frac{C}{\Gamma(\beta+1)} \Big[\frac{|(t_1-T)^\beta - {t_1}^\beta|}{1+{t_1}^{\beta+1}} + \frac{(t_1-T)^\beta}{1+{t_1}^{\beta+1}} \\
&+ \frac{|(t_2-T)^\beta - {t_2}^\beta|}{1+{t_2}^{\beta+1}} + \frac{(t_2-T)^\beta}{1+{t_2}^{\beta+1}} \Big] < \delta_\varepsilon,
\end{aligned}
$$

and

$$
\Big| \frac{D^{\alpha-i}_{0^+} R(u,\lambda)(t_1)}{e^{(q+i-1)t_1}} - \frac{D^{\alpha-i}_{0^+} R(u,\lambda)(t_2)}{e^{(q+i-1)t_2}} \Big|
$$

$$
\begin{aligned}
&= \Big| \frac{1}{(i-1)!e^{(q+i-1)t_1}} \int_0^{t_1} (t_1-s)^{i-1}\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u)(s)ds \\
&\quad - \frac{1}{(i-1)!e^{(q+i-1)t_2}} \int_0^{t_2} (t_2-s)^{i-1}\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u)(s)ds \Big| \\
&\le \Big| \frac{1}{(i-1)!e^{(q+i-1)t_1}} \int_0^{T_1} (t_1-s)^{i-1}\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u)(s)ds \Big| \\
&\quad + \Big| \frac{1}{(i-1)!e^{(q+i-1)t_1}} \int_{T_1}^{t_1} (t_1-s)^{i-1}\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u)(s)ds \Big| \\
&\quad + \Big| \frac{1}{(i-1)!e^{(q+i-1)t_2}} \int_0^{T_1} (t_2-s)^{i-1}\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u)(s)ds \Big| \\
&\quad + \Big| \frac{1}{(i-1)!e^{(q+i-1)t_2}} \int_{T_1}^{t_2} (t_2-s)^{i-1}\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u)(s)ds \Big| \\
&\le \Big[\frac{|(t_1-T_1)^i - {t_1}^i|}{e^{(q+i-1)t_1}} + \frac{(t_1-T_1)^i}{e^{(q+i-1)t_1}} + \frac{|(t_2-T_1)^i - {t_2}^i|}{e^{(q+i-1)t_2}} + \frac{(t_2-T_1)^i}{e^{(q+i-1)t_2}} \Big] \varphi_q(C_r) < \varepsilon.
\end{aligned}
$$

By Theorem 2.4, we get that $\{R(u,\lambda) | u \in \overline{\Omega}, \lambda \in [0,1]\}$ is relatively compact. The proof is completed. □

Now, we will show that N_λ is M-quasi-compact in $\overline{\Omega}$, where $\Omega \subset X$ is an open and bounded set with $\theta \in \Omega$.

Lemma 3.4. *Assume that $\Omega \subset X$ is an open and bounded set. Then N_λ is M-quasi-compact in $\overline{\Omega}$.*

Proof. It is clear that $ImP = KerM$, $dimKerM = dimImQ$, $QN_\lambda u = 0, \lambda \in (0,1) \Leftrightarrow Q_i(\lambda f) = 0 \Leftrightarrow T_i(\lambda f) = 0 \Leftrightarrow \lambda T_i(f) = 0 \Leftrightarrow QNu = 0, i = 1,2, i.e.,$ Definition 2.2(a) and (b) are satisfied.

For $u \in \Sigma_\lambda = \{u \in \overline{\Omega}, Mu = N_\lambda u\}$, we have $QN_\lambda u = 0$ and $N_\lambda u = D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha}u))$. It follows from $D_{0^+}^{\alpha}u(0) = D_{0^+}^{\alpha}R(u,\lambda)(0) = 0$ that

$$
\begin{aligned}
R(u,\lambda)(t) &= I_{0^+}^{\alpha}(\varphi_q(I_{0^+}^{\beta}(I-Q)N_\lambda u(t))) = I_{0^+}^{\alpha}(\varphi_q(I_{0^+}^{\beta}N_\lambda u(t))) \\
&= I_{0^+}^{\alpha}(\varphi_q(I_{0^+}^{\beta}(D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha}u))))) = I_{0^+}^{\alpha}(D_{0^+}^{\alpha}u) \\
&= u(t) - \frac{D_{0^+}^{\alpha-1}u(0)}{\Gamma(\alpha)}t^{\alpha-1} - \frac{D_{0^+}^{\alpha-n+1}u(0)}{\Gamma(\alpha-n+2)}t^{\alpha-n+1} \\
&= u(t) - Pu(t).
\end{aligned}
$$

Clearly, $R(.,0) = 0$. Thus, definition 2.2(c) is satisfied.

For $u \in \overline{\Omega}$, we have $M[Pu + R(u,\lambda)](t) = (I-Q)N_\lambda u(t)$. So, Definition 2.2($d$) is satisfied.

Considering (H_2), $\|N_\lambda(u) - N_\lambda(v)\|_Z \le \sup_{t\in[0,+\infty)} |f(t,u(t),\ldots,D_{0^+}^{\alpha}u) - f(t,v(t),\ldots,D_{0^+}^{\alpha}v)|$, $u,v \in \overline{\Omega}$, we can obtain that N_λ is continuous and bounded in $\overline{\Omega}$.

These, together with Lemma 3.3, mean that N_λ is M-quasi-compact in $\overline{\Omega}$. The proof is completed. □

Theorem 3.1. *Suppose that $(H_1), (H_2), (C_1), (C_2)$ and the following conditions hold:*

(H_3) *There exist constants $M_i > 0, i = 1,2$ such that if $|D_{0^+}^{\alpha-1}u(t)| > M_1, \forall t \in$*

$[0, +\infty)$, *then*

$$T_1(f(t, u(t), \ldots, D^{\alpha}_{0^+}u(t))) \neq 0,$$

or if $|D^{\alpha-n+1}_{0^+}u(t)| > M_2, \forall t \in [0, +\infty)$ *then*

$$T_2(f(t, u(t), \ldots, D^{\alpha}_{0^+}u(t))) \neq 0.$$

(H_4) *There exist nonnegative functions* $a_i(t), i = 1, 2, \ldots, n+1, b(t)$ *with* $a_1(t)\mu(t) \cdot \dfrac{e^{(q+n-1)(p-1)t}}{t^{(n-\alpha)(p-1)}}$, $a_j(t)\mu(t)e^{(q+j-1)(p-1)t}, j = 2, \ldots, n, a_{n+1}(t)\mu(t)(1+t^{\beta+1})$ *and* $b(t)\mu(t) \in Y$ *such that*

$$|f(t, u_1, u_2, \ldots, u_{n+1})| \leq \sum_{i=1}^{n+1} a_i(t)|\varphi_p(|u_i|)| + b(t), a.e., t \in [0, +\infty),$$

with

$$\frac{2^{q-2}\varphi_q\Big(\|a_1\mu(t)\frac{e^{(q+n-1)(p-1)t}}{t^{(n-\alpha)(p-1)}}\|_\infty + \sum\limits_{i=1}^{n-1}\|a_{i+1}\mu(t)e^{(q+i-1)(p-1)t}\|_\infty + \|a_{n+1}\mu(t)(1+t^{\beta+1})\|_\infty\Big)}{\varphi_q(\Gamma(\beta+1))} < 1,$$

if $1 < p \leq 2$*;*

$$\frac{\varphi_q\big(2^{p-2}(\|a_1\mu(t)\frac{e^{(q+n-1)(p-1)t}}{t^{(n-\alpha)(p-1)}}\|_\infty + 2^{p-2}\sum\limits_{i=1}^{n-1}\|a_{i+1}\mu(t)e^{(q+i-1)(p-1)t}\|_\infty + \|a_{n+1}\mu(t)(1+t^{\beta+1})\|_\infty\big)}{\varphi_q(\Gamma(\beta+1))} < 1,$$

if $p \geq 2$*;*
(H_5) *There exist constants* $M_j > 0, j = 3, 4$ *such that either for each* $(a, b) \in \mathbb{R}^2$*:*

$$aT_1\big(f(t, at^{\alpha-1} + bt^{\alpha-n+1}, \frac{a\Gamma(\alpha)}{\Gamma(n-1)}t^{n-2} + b\Gamma(\alpha-n+2), \ldots, a\Gamma(\alpha), 0)\big) > 0, \quad \text{for } |a| > M_3, \tag{3.3}$$

and

$$bT_2\big(f(t, at^{\alpha-1} + bt^{\alpha-n+1}, \frac{a\Gamma(\alpha)}{\Gamma(n-1)}t^{n-2} + b\Gamma(\alpha-n+2), \ldots, a\Gamma(\alpha), 0)\big) > 0, \quad \text{for } |b| > M_4, \tag{3.4}$$

or $(a, b) \in \mathbb{R}^2$*:*

$$aT_1\big(f(t, at^{\alpha-1} + bt^{\alpha-n+1}, \frac{a\Gamma(\alpha)}{\Gamma(n-1)}t^{n-2} + b\Gamma(\alpha-n+2), \ldots, a\Gamma(\alpha), 0)\big) < 0, \quad \text{for } |a| > M_3, \tag{3.5}$$

and

$$bT_2\big(f(t, at^{\alpha-1} + bt^{\alpha-n+1}, \frac{a\Gamma(\alpha)}{\Gamma(n-1)}t^{n-2} + b\Gamma(\alpha-n+2), \ldots, a\Gamma(\alpha), 0)\big) < 0, \quad \text{for } |b| > M_4\ . \tag{3.6}$$

Then the boundary value problem (1.2) *has at least one solution.*

In order to prove Theorem 3.5, we show two lemmas.

Lemma 3.5. *Assume* $(C_1),(C_2)$ *and* (H_1)–(H_4) *hold. Then the set* $\Omega_1 = \{u \in domM : Mu = N_\lambda u, \lambda \in (0,1)\}$ *is bounded.*

Proof. For $u \in \Omega_1$, we have $QN_\lambda u = 0, i.e., T_1(f) = 0, T_2(f) = 0$. By (H_3), there exist constants $t_i \in [0,+\infty), i = 1,2$ such that $|D_{0^+}^{\alpha-1}u(t_1)| \le M_1$ and $|D_{0^+}^{\alpha-n+1}u(t_2)| \le M_2$.

Since $u(t) = I_{0^+}^{\alpha-n+1}D_{0^+}^{\alpha-n+1}u(t) + ct^{\alpha-n}$ and $u(0) = 0$, then

$$\begin{aligned} |\frac{u(t)t^{n-\alpha}}{e^{(q+n-1)t}}| &= |\frac{1}{\Gamma(\alpha-n+1)}\int_0^t \frac{(t-s)^{\alpha-n}t^{n-\alpha}e^{(q+n-2)s}}{e^{(q+n-1)t}}\frac{D_{0^+}^{\alpha-n+1}u(s)}{e^{(q+n-2)s}}ds| \\ &\le \frac{1}{\Gamma(\alpha-n+2)}\Big\|\frac{D_{0^+}^{\alpha-n+1}u}{e^{(q+n-2)t}}\Big\|_\infty. \end{aligned} \tag{3.7}$$

Considering $D_{0^+}^{\alpha-(n-2)}u(0) = D_{0^+}^{\alpha-(n-3)}u(0) = \ldots = D_{0^+}^{\alpha-2}u(0) = 0$ and Lemma 2.9,

$$|\frac{D_{0^+}^{\alpha-i}u(t)}{e^{(q+i-1)t}}| = \int_0^t |\frac{e^{(q+i-2)s}}{e^{(q+i-1)t}}\frac{D_{0^+}^{\alpha-(i-1)}u(s)}{e^{(q+i-2)s}}ds| \le \|\frac{D_{0^+}^{\alpha-(i-1)}u(s)}{e^{(q+i-2)t}}\|_\infty, i = 2,3,\ldots,n-2.$$

That is,

$$\|\frac{D_{0^+}^{\alpha-(n-2)}u}{e^{(q+n-3)t}}\|_\infty \le \|\frac{D_{0^+}^{\alpha-(n-3)}u}{e^{(q+n-4)t}}\|_\infty \le \cdots \le \|\frac{D_{0^+}^{\alpha-2}u}{e^{(q+1)t}}\|_\infty \le \|\frac{D_{0^+}^{\alpha-1}u}{e^{qt}}\|_\infty. \tag{3.8}$$

It follows from $D_{0^+}^{\alpha-1}u(t) = D_{0^+}^{\alpha-1}u(t_1) + \int_{t_1}^t D_{0^+}^{\alpha}u(s)ds$ and $D_{0^+}^{\alpha-n+1}u(t) = D_{0^+}^{\alpha-n+1}u(t_2) + \int_{t_2}^t D_{0^+}^{\alpha-n+2}u(s)ds$ that

$$\|\frac{D_{0^+}^{\alpha-n+1}u}{e^{(q+n-2)t}}\|_\infty \le M_2 + \|\frac{D_{0^+}^{\alpha-n+2}u}{e^{(q+n-3)t}}\|_\infty \tag{3.9}$$

and

$$\|\frac{D_{0^+}^{\alpha-1}u}{e^{qt}}\|_\infty \le M_1 + \|\frac{D_{0^+}^{\alpha}u}{(1+t^{\beta+1})^{q-1}}\|_\infty. \tag{3.10}$$

By $Mu = N_\lambda u$,(H_4) and $D_{0^+}^{\alpha}u(0) = 0$, then

$$D_{0^+}^{\alpha}u(t) = \varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\lambda\mu(s)f(s,u(s),D_{0^+}^{\alpha-(n-1)}u(s),\ldots,D_{0^+}^{\alpha}u(s)))ds\Big),$$

which is combined with (H_4), and we can check that

$$\begin{aligned} &|\frac{D_{0^+}^{\alpha}u(t)}{(1+t^{\beta+1})^{q-1}}| \\ =&\varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^t \frac{(t-s)^{\beta-1}}{1+t^{\beta+1}}\lambda\mu(s)f(s,u(s),D_{0^+}^{\alpha-(n-1)}u(s),\ldots,D_{0^+}^{\alpha}u(s)))ds\Big) \\ \le&\varphi_q\Big[\frac{1}{\Gamma(\beta+1)}\Big(\|\frac{a_1(t)\mu(t)e^{(q+n-1)(p-1)t}}{t^{(n-\alpha)(p-1)}}\|_\infty\varphi_p(\|\frac{ut^{n-\alpha}}{e^{(q+n-1)t}}\|_\infty) \\ &+\sum_{i=1}^{n-1}\|a_{i+1}(t)\mu(t)e^{(q+i-1)(p-1)t}\|_\infty\varphi_p(\|\frac{D_{0^+}^{\alpha-i}u}{e^{(q+i-1)t}}\|_\infty) \end{aligned}$$

$$+\|a_{n+1}(t)\mu(t)(1+t^{\beta+1})\|_\infty\varphi_p(\|\frac{D_{0^+}^\alpha u}{(1+t^{\beta+1})^{q-1}}\|_\infty)+\|b(t)\mu(t)\|_\infty\Big)\Big].$$

If $1<p\le 2$, then

$$\begin{aligned}
&|\frac{D_{0^+}^\alpha u(t)}{(1+t^{\beta+1})^{q-1}}|\\
\le&\frac{1}{(\Gamma(\beta+1))^{q-1}}\varphi_q\Big[\|\frac{a_1(t)\mu(t)(e^{(q+n-1)(p-1)t})}{t^{(n-\alpha)(p-1)}}\|_\infty\varphi_p(M_1+M_2)\\
&+\|b(t)\mu(t)\|_\infty+\sum_{i=1}^{n-2}\|a_{i+1}(t)\mu(t)(e^{(q+i-1)(p-1)t})\|_\infty\varphi_p(M_1)\\
&+\|a_n(t)\mu(t)(e^{(q+n-2)(p-1)t})\|_\infty\varphi_p(M_1+M_2)\\
&+\Big(\|\frac{a_1(t)\mu(t)e^{(q+n-1)(p-1)t}}{t^{(n-\alpha)(p-1)}}\|_\infty+\sum_{i=1}^{n-1}\|a_{i+1}(t)\mu(t)e^{(q+i-1)(p-1)t}\|_\infty\\
&+\|a_{n+1}(t)\mu(t)(1+t^{\beta+1})\|_\infty\Big)\varphi_p(\|\frac{D_{0^+}^\alpha u}{(1+t^{\beta+1})^{q-1}}\|_\infty)\Big]\\
\le&\frac{1}{(\Gamma(\beta+1))^{q-1}}2^{q-2}\Big[\varphi_q\Big(\|\frac{a_1(t)\mu(t)(e^{(q+n-1)(p-1)t})}{t^{(n-\alpha)(p-1)}}\|_\infty\varphi_p(M_1+M_2)+\|b(t)\mu(t)\|_\infty\\
&+\sum_{i=1}^{n-2}\|a_{i+1}(t)\mu(t)e^{(q+i-1)(p-1)t}\|_\infty\varphi_p(M_1)+\|a_n(t)\mu(t)(e^{(q+n-2)(p-1)t})\|_\infty\\
&\times\varphi_p(M_1+M_2)\Big)+\varphi_q\Big(\|\frac{a_1(t)\mu(t)(e^{(q+n-1)(p-1)t})}{t^{(n-\alpha)(p-1)}}\|_\infty\\
&+\sum_{i=1}^{n-1}\|a_{i+1}(t)\mu(t)e^{(q+i-1)(p-1)t}\|_\infty+\|a_{n+1}(t)\mu(t)(1+t^{\beta+1})\|_\infty\Big)\|\frac{D_{0^+}^\alpha u}{(1+t^{\beta+1})^{q-1}}\|_\infty\Big].
\end{aligned}$$

If $p\ge 2$, then

$$\begin{aligned}
|\frac{D_{0^+}^\alpha u(t)}{(1+t^{\beta+1})^{q-1}}|\le&\frac{1}{(\Gamma(\beta+1))^{q-1}}\Big[\varphi_q\big(2^{p-2}\|\frac{a_1(t)\mu(t)e^{(q+n-1)(p-1)t}}{t^{(n-\alpha)(p-1)}}\|_\infty\varphi_p(M_1+M_2)\\
&+\|b(t)\mu(t)\|_\infty+2^{p-2}\sum_{i=1}^{n-2}\|a_{i+1}(t)\mu(t)e^{(q+i-1)(p-1)t}\|_\infty\varphi_p(M_1)\\
&+2^{p-2}\|a_n(t)\mu(t)e^{(q+n-2)(p-1)t}\|_\infty\varphi_p(M_1+M_2)\big)\\
&+\varphi_q\Big(2^{p-2}\|\frac{a_1(t)\mu(t)e^{(q+n-1)(p-1)t}}{t^{(n-\alpha)(p-1)}}\|_\infty\\
&+2^{p-2}\sum_{i=1}^{n-1}\|a_{i+1}(t)\mu(t)e^{(q+i-1)(p-1)t}\|_\infty\\
&+\|a_{n+1}(t)\mu(t)(1+t^{\beta+1})\|_\infty\Big)\|\frac{D_{0^+}^\alpha u}{(1+t^{\beta+1})^{q-1}}\|_\infty\Big].
\end{aligned}$$

These, together with (3.7) $\sim$ (3.10), mean that Ω_1 is bounded in X. □

Lemma 3.6. *Assume that* $(C_1),(C_2),(H_1)\sim(H_3)$ *and* (H_5) *hold. Then the set* $\Omega_2=\{u\in KerM: QNu=0\}$ *and* $\Omega_3=\{u\in KerM:\rho\delta Iu+(1-\delta)JQNu=$

$0, \delta \in [0,1]\}$ *are bounded, where* $J : \operatorname{Im} Q \to \operatorname{Ker} M$ *is a homeomorphism with*

$$J\Big(\mu(t)\frac{T_2(g(t))a-T_1(g(t))b}{T_2(g(t))-T_1(g(t))}+\mu(t)\frac{b-a}{T_2(g(t))-T_1(g(t))}g(t)\Big)=at^{\alpha-1}+bt^{\alpha-n+1}, a, b\in\mathbb{R},$$

$$\rho = \begin{cases} 1, & if\ (3.3)\ and\ (3.4)\ hold; \\ -1. & if\ (3.5)\ and\ (3.6)\ hold. \end{cases} \tag{3.11}$$

Proof. If $u_1 \in \Omega_2$, then $u_1(t) = ct^{\alpha-1} + dt^{\alpha-n+1}$ and

$$T_i\big(f(t, ct^{\alpha-1} + dt^{\alpha-n+1}, \frac{c\Gamma(\alpha)}{\Gamma(n-1)}t^{n-2} + d\Gamma(\alpha-n+2), \dots, c\Gamma(\alpha), 0)\big) = 0, i = 1, 2.$$

By (H_5), we get

$$\begin{aligned} &|\frac{u_1(t)t^{n-\alpha}}{e^{(q+n-1)t}}| = |\frac{ct^{n-1}+d}{e^{(q+n-1)t}}| \le M_3 + M_4, \\ &|\frac{D_{0^+}^{\alpha-i}u_1(t)}{e^{(q+i-1)t}}| = \Big|\frac{a\Gamma(\alpha)}{\Gamma(i)}\frac{t^{i-1}}{e^{(q+i-1)t}} + d\Gamma(\alpha-n+2)\Big| \le M_3\Gamma(\alpha) + M_4\Gamma(\alpha-n+2), \\ &i = 1, 2, \dots, n-1, \end{aligned}$$

and $|\frac{D_{0^+}^{\alpha}u_1(t)}{(1+t^{\beta+1})^{q-1}}| = 0$. These mean that Ω_2 is bounded.

For $u_2 \in \Omega_3, u_2 = at^{\alpha-1} + bt^{\alpha-n+1}$ and $\rho\delta I u_2 + (1-\delta)JQNu_2 = 0$.

If $\delta = 1$, then $u_2 = 0$. If $\delta = 0$, then $T_i\big(f(t, at^{\alpha-1} + bt^{\alpha-n+1}, \frac{a\Gamma(\alpha)}{\Gamma(n-1)}t^{n-2} + b\Gamma(\alpha-n+2), \dots, a\Gamma(\alpha), 0)\big) = 0$, and $\|u_2\|_X \le \max\{M_3 + M_4, M_3\Gamma(\alpha) + M_4\Gamma(\alpha-n+2)\} < +\infty$.

For convenience, denote

$$f(t, at^{\alpha-1} + bt^{\alpha-n+1}, \frac{a\Gamma(\alpha)}{\Gamma(n-1)}t^{n-2} + b\Gamma(\alpha-n+2), \dots, a\Gamma(\alpha), 0) = f_{a,b}.$$

If $\delta \in (0,1)$, we can have

$$\rho\delta(at^{\alpha-1} + bt^{\alpha-n+1}) + (1-\delta)\big(T_1(f_{a,b})t^{\alpha-1} + T_2(f_{a,b})t^{\alpha-n+1}\big) = 0,$$

which implies

$$\rho\delta at^{\alpha-1} + (1-\delta)T_1(f_{a,b})t^{\alpha-1} = 0 \text{ and } \rho\delta bt^{\alpha-n+1} + (1-\delta)T_2(f_{a,b})t^{\alpha-n+1} = 0.$$

From condition (H_5) and (3.11), if $|a| > M_3$, $|b| > M_4$, we have

$$\begin{cases} \rho\delta a^2 = -(1-\delta)aT_1(f_{a,b}), \\ \rho\delta b^2 = -(1-\delta)bT_2(f_{a,b}), \end{cases}$$

which is a contradiction. So, $|a| \le M_3$, $|b| \le M_4$, i.e, Ω_3 is also bounded. □

The proof of Theorem 3.5. By Lemmas 3.6 and 3.7, $Mu \neq N_\lambda u$ for $u \in \partial\Omega \cap \operatorname{dom} M, \lambda \in (0,1)$ and $QNu \neq 0, u \in \operatorname{Ker} M \cap \partial\Omega$. So, (C') of Theorem 2.3 holds.

Now, we will prove the (C'') of Theorem 2.3.

Choose R_0 large enough such that $\Omega = \{u \in X : \|u\|_X < R_0\} \supset \overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3}$ and $R_0 \geq \max\{M_3 + M_4, M_3\Gamma(\alpha) + M_4\Gamma(\alpha - n + 2)\} + 1$.

Let $H(u, \delta) = \rho\delta I u + (1-\delta)JQNu, \delta \in [0, 1], u \in \operatorname{Ker} M \cap \partial\Omega$, noticing $\Omega_3 \subset \Omega$, we know $H(u, \delta) \neq 0, u \in \operatorname{Ker} M \cap \partial\Omega, \delta \in [0, 1]$. Thus, by invariance of degree under a homotopy, we get that

$$\begin{aligned}
\deg(JQN, \Omega \cap \operatorname{Ker} M, 0) &= \deg\big(H(\cdot, 0), \Omega \cap \operatorname{Ker} M, 0\big) \\
&= \deg\big(H(\cdot, 1), \Omega \cap \operatorname{Ker} M, 0\big) \\
&= \deg\big(\rho I, \Omega \cap \operatorname{Ker} M, 0\big) \neq 0.
\end{aligned}$$

Therefore, the condition (C'') of Theorem 2.3 holds. *i.e.*, the proof of Theorem 3.5 is completed. □

4. The example

Now, we illustrate Theorem 3.5 by the following example. Consider the functional boundary value problem

$$\begin{cases}
D_{0^+}^{\frac{1}{2}}(\varphi_p(D_{0^+}^{\frac{5}{2}}u))(t) = \mu(t)f(t, u(t), D_{0^+}^{\frac{1}{2}}u(t), D_{0^+}^{\frac{3}{2}}u(t), D_{0^+}^{\frac{5}{2}}u(t)) = 0, \\
t \in [0, +\infty) \\
D_{0^+}^{\frac{5}{2}}u(0) = 0, u(0) = 0, \\
\Gamma_1(u) = \frac{5}{2}u(1) = 0, \\
\Gamma_2(u) = (I_0^{\frac{1}{2}} D_{0^+}^{\frac{5}{2}}u)(\infty) - (I_0^{\frac{1}{2}} D_{0^+}^{\frac{5}{2}}u)(1) = 0,
\end{cases}$$

where $p = 2$,$\mu(t) = \dfrac{e^{-t}}{1 + t^{\frac{1}{2}}} \in L[0, +\infty) \cap C[0, +\infty), \sup\limits_{t \in [0, +\infty)} |I_0^{\frac{1}{2}}\mu| < +\infty$, and

$$\begin{aligned}
&f(t, u(t), D_{0^+}^{\frac{1}{2}}u(t), D_{0^+}^{\frac{3}{2}}u(t), D_{0^+}^{\frac{5}{2}}u(t)) \\
&= \begin{cases}
e^{-t} + \dfrac{t^{\frac{1}{2}}}{5e^{3t}} \sin u(t) + \frac{1}{5e^t} \sin D_{0^+}^{\frac{1}{2}}u(t) + \dfrac{1}{5e^{2t}} D_{0^+}^{\frac{3}{2}}u(t) + \frac{1}{5} \sin D_{0^+}^{\frac{5}{2}}u(t), t \in [0, 1], \\
e^{-t} + \dfrac{t^{\frac{1}{2}}}{5e^{3t}} \sin u(t) + \frac{1}{5e^t} D_{0^+}^{\frac{1}{2}}u(0) + \dfrac{1}{5e^{2t}} \sin D_{0^+}^{\frac{3}{2}}u(t) + \frac{1}{5} \sin D_{0^+}^{\frac{5}{2}}u(t), t \in (1, +\infty).
\end{cases}
\end{aligned}$$

Then the functional problem is at resonance with $\Gamma_1(t^{\frac{1}{2}}) = \Gamma_1(t^{\frac{3}{2}}) = \Gamma_2(t^{\frac{1}{2}}) = \Gamma_2(t^{\frac{3}{2}}) = 0, KerM = \{at^{\frac{3}{2}} + b^{\frac{1}{2}} | a, b \in \mathbb{R}\}$.In this case,$a_1(t) = \dfrac{1}{5}e^{-3t}t^{\frac{1}{2}t}, a_2(t) = \dfrac{1}{5}e^{-t}, a_3(t) = \dfrac{1}{5}e^{-2t}, a_4(t) = \dfrac{1}{5}$.

It is easy to check that

$$\begin{aligned}
&a_1(t)\mu(t)\frac{e^{(q+n-1)(p-1)t}}{t^{(n-\alpha)(p-1)}} = \frac{1}{5(1 + t^{\frac{1}{2}})}, \quad a_2(t)\mu(t)e^{(q+1)(p-1)t} = \frac{1}{5(1 + t^{\frac{1}{2}})}, \\
&a_3(t)\mu(t)e^{(q+2)(p-1)t} = \frac{1}{5(1 + t^{\frac{1}{2}})}, \quad a_4(t)\mu(t)(1 + t^{\beta+1}) = \frac{e^{-t}(1 + t^{\frac{3}{2}})}{5(1 + t^{\frac{1}{2}})},
\end{aligned}$$

and $b(t)\mu(t) = \dfrac{e^{-2t}}{1+t^{\frac{1}{2}}} \in Y$, moreover,

$$\frac{2^{q-2}\varphi_q\Big(\|a_1\mu(t)\frac{e^{(q+n-1)(p-1)t}}{t^{(n-\alpha)(p-1)}}\|_\infty + \sum_{i=1}^{n-1}\|a_{i+1}\mu(t)e^{(q+i-1)(p-1)t}\|_\infty + \|a_{n+1}\mu(t)(1+t^{\beta+1})\|_\infty\Big)}{\varphi_q(\Gamma(\beta+1))}$$

$=\dfrac{8}{5\sqrt{\pi}} \approx 0.903 < 1.$

Clearly,

$$\begin{aligned}&|f(t,u(t),D_{0^+}^{\frac{1}{2}}u(t),D_{0^+}^{\frac{3}{2}}u(t),D_{0^+}^{\frac{5}{2}}u(t))|\\ \leq& a_1(t)|u(t)| + a_2(t)|D_{0^+}^{\frac{1}{2}}u(t)| + a_3(t)|D_{0^+}^{\frac{3}{2}}u(t)| + a_4(t)|D_{0^+}^{\frac{5}{2}}u(t)| + e^{-t},\\ &a.e., t\in[0,+\infty),\end{aligned}$$

hence (H_4) holds.

For convenience, introduce $F(t) = f(t,u(t),D_{0^+}^{\frac{1}{2}}u(t),D_{0^+}^{\frac{3}{2}}u(t),D_{0^+}^{\frac{5}{2}}u(t))$. Hence

$$T_1(F(t)) = \Gamma_1(I_0^{\frac{5}{2}}(I_0^{\frac{1}{2}}\mu F(t))) = I_0^{\frac{1}{2}}\mu F(1) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (1-s)^{-\frac{1}{2}}\frac{e^{-s}}{1+s^{\frac{1}{2}}}F(s)ds,$$

and

$$\begin{aligned}T_2(F(t)) &= \Gamma_2(I_0^{\frac{5}{2}}(I_0^{\frac{1}{2}}\mu F(t)))\\ &= \int_0^\infty (1-s)^{-\frac{1}{2}}\frac{e^{-s}}{1+s^{\frac{1}{2}}}F(s)ds - \int_0^1 (1-s)^{-\frac{1}{2}}\frac{e^{-s}}{1+s^{\frac{1}{2}}}F(s)ds\\ &= \int_1^\infty (1-s)^{-\frac{1}{2}}\frac{e^{-s}}{1+s^{\frac{1}{2}}}F(s)ds.\end{aligned}$$

If $D_{0^+}^{\frac{3}{2}}u(t) > M_1 = 9$, then $F(t) > -\frac{3}{5} + \frac{M_1}{5} = \frac{6}{5} > 0$, and If $D_{0^+}^{\frac{3}{2}}u(t) < -M_1 = -9$, then $F(t) < \frac{8}{5} - \frac{M_1}{5} = -\frac{1}{5} < 0$.

Similarly, If $D_{0^+}^{\frac{1}{2}}u(t) > M_2 = 9$, then $F(t) > 0$, and If $D_{0^+}^{\frac{1}{2}}u(t) < -M_2 = -10$, then $F(t) < 0$. Thus,(H_3) holds.

Finally, for $u(t) = at^{\frac{3}{2}} + b^{\frac{1}{2}}$, one choose $|a| > M_3 = 45, |b| > M_4 = 25$,

$$\begin{aligned}&aT_1(f(t,at^{\frac{3}{2}}+b^{\frac{1}{2}},a\Gamma(\frac{5}{2})t+b\Gamma(\frac{3}{2}),a\Gamma(\frac{5}{2}),0))\\ =&\frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (1-s)^{-\frac{1}{2}}\frac{e^{-s}}{1+s^{\frac{1}{2}}}af(s,as^{\frac{3}{2}}+b^{\frac{1}{2}},a\Gamma(\frac{5}{2})s+b\Gamma(\frac{3}{2}),a\Gamma(\frac{5}{2}),0)ds > 0,\end{aligned}$$

and

$$\begin{aligned}&bT_2(f(t,at^{\frac{3}{2}}+b^{\frac{1}{2}},a\Gamma(\frac{5}{2})t+b\Gamma(\frac{3}{2}),a\Gamma(\frac{5}{2}),0))\\ =&\int_1^\infty (1-s)^{-\frac{1}{2}}\frac{e^{-s}}{1+s^{\frac{1}{2}}}bf(s,as^{\frac{3}{2}}+b^{\frac{1}{2}},a\Gamma(\frac{5}{2})s+b\Gamma(\frac{3}{2}),a\Gamma(\frac{5}{2}),0)ds > 0.\end{aligned}$$

Since

$$af(s,as^{\frac{3}{2}}+b^{\frac{1}{2}},a\Gamma(\frac{5}{2})s+b\Gamma(\frac{3}{2}),a\Gamma(\frac{5}{2}),0) > -|a| - \frac{3}{5}|a| + \frac{\Gamma(\frac{5}{2})}{5e^2}a^2$$

$$= -\frac{8}{5}|a| + \frac{\Gamma(\frac{5}{2})}{5e^2}a^2 > 0, t \in [0,1],$$

and

$$bf(s, as^{\frac{3}{2}} + b^{\frac{1}{2}}, a\Gamma(\frac{5}{2})s + b\Gamma(\frac{3}{2}), a\Gamma(\frac{5}{2}), 0) > -|b| - \frac{3}{5}|b| + \frac{\Gamma(\frac{3}{2})}{5e}b^2$$
$$= -\frac{8}{5}|b| + \frac{\Gamma(\frac{3}{2})}{5e}b^2 > 0, t \in (1, +\infty),$$

then condition (H_5) is satisfied. It follows from Theorem 3.5 that there must be at least one solution in X.

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