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UNIFORM ISOCHRONOUS CENTER OF HIGHER-DEGREE POLYNOMIAL DIFFERENTIAL SYSTEMS*

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Abstract In this paper, we study the uniform isochronous center of a class of more general higher-degree of polynomial differential systems and give the necessary and sufficient conditions for the origin point to be a center. At the same time, we illustrate that under some restrictions, the composition conjecture about these differential systems is valid. As corollaries, the previous results can easily be derived from the current conclusion.

Keywords Composition conjecture, center condition, uniform isochronous center, higher-degree systems.

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1. Introduction

Consider differential systems of the form

$$\begin{cases} x' = -y + F(x,y), \\ y' = x + G(x,y), \end{cases} \tag{1.1}$$

where $F(x,y)$ and $G(x,y)$ are real polynomial functions of degree $n+1$ without constant and linear terms. If every orbit in a punctured neighbourhood of O is a nontrivial cycle then the origin point $O(0,0)$ is said to be a center. The center-focus problem consists in determining necessary and sufficient conditions on $F(x,y)$ and $G(x,y)$ such that system (1.1) has a center at the origin. This problem has attracted the attention of many authors. Up to now, only for quadratic systems and some special systems the center-focus problem has been solved e.g. [2, 15, 17, 19, 23–25]. But for the more higher degree polynomial differential systems, the corresponding results are very few.

In particular, if every cycle in a punctured neighbourhood of O has the same period then this origin point is said to be an isochronous center. The interest in the isochronous centers started in the XVII century with the works of [3, 8, 11, 16] and references therein. The isochronous phenomena appear in many physical problems [14]. Aside from its interest in physical applications, isochronicity is strictly related to the existence and uniqueness of solutions of some boundary value, bifurcation

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or perturbation problems. Moreover, isochronicity has a strong relationship to stability: a periodic solution of the central region is Liapunov stable if and only if the neighbouring periodic solutions have the same period. In literature [22], the authors have proved that if the system (1.1) has a center at $O(0,0)$, then this center is an uniform isochronous center if and only if doing a linear change of variables and a scaling of the time it can be written as the rigid system:

$$\begin{cases} x' = -y + xP(x,y), \\ y' = x + yP(x,y), \end{cases} \tag{1.2}$$

where $P(x,y) = \sum_{k=1}^{n} P_k(x,y)$, $P_k(x,y)\,(k = 1,2,...,n)$ are homogeneous polynomials in x and y of degree k.

In polar coordinates the system (1.2) becomes

$$\frac{dr}{d\theta} = r\sum_{k=1}^{n} P_k(\theta)r^k, \tag{1.3}$$

where $P_k(\theta) = P_k(\cos\theta, \sin\theta)$. By [2,5,23] we see that the system (1.2) has a center at $O(0,0)$ if and only if all the solutions $r(\theta)$ of equation (1.3) near $r = 0$ are periodic. In such case it is said that equation (1.3) has a center at $r = 0$.

Alwash and Lloyd [6,7,9] give the following simple sufficient condition for the Abel equation

$$\frac{dr}{d\theta} = r(R_1(\theta)r + R_2(\theta)r^2) \tag{1.4}$$

to have a center, where $R_1(\theta)$ and $R_2(\theta)$ are continuous 2π-periodic functions.

Theorem 1.1 ([7,9]). *If there exists a differentiable function $u(\theta)$ of period 2π such that*

$$R_1(\theta) = u'(\theta)\check{R}_1(u(\theta)), R_2(\theta) = u'(\theta)\check{R}_2(u(\theta))$$

for some continuous functions $\check{R}_1$ and $\check{R}_2$, then the Abel equation (1.4) has a center at $r = 0$.

The following statement presents a generalization of Theorem 1.1.

Theorem 1.2 ([6,28]). *If there exists a differentiable function $u(\theta)$ of period 2π such that*

$$P_i(\theta) = u'\check{P}_i(u),\ (i = 1,2,...,n)$$

for some continuous functions $\check{P}_i\,(i = 1,2,...,n)$, then the equation (1.3) has a center at $r = 0$.

The condition in Theorem 1.1 (or Theorem 1.2) is called the **Composition Condition**. When an Abel equation (1.2) (or (1.3)) has a center because its coefficients satisfy the composition condition we will say that this equation has a **Composition Center**. Obviously, the composition condition is the sufficient condition for $r = 0$ to be a center. A counterexample was presented in [9, 10] to demonstrate that composition condition is not a necessary condition of a center. Whether the composition condition is the necessary and sufficient conditions for $r = 0$ to be a center? This problem is called **Composition Conjecture**, which first appeared in [7]. What kind of differential system is this conjecture right? Studying this problem has attracted the interest of many scholars. In [15, 18, 21] the authors have proved that for some Abel differential equations the composition conjecture is valid. The authors of [2] studied the system (1.2) with $P = P_1 + P_m$ or $P = P_2 + P_{2m}$, m is an arbitrary natural number, having polynomial commutator and their centers are reversible. In [27,28] we used different methods to get the center conditions and demonstrate that the composition conjecture is correct for this system. In literatures [2,8,12,20,29] the authors using different methods study the system (1.2) with $P = P_1 + P_2 + P_3$, for which the composite conjecture is correct, and analyze its global behavior. In [26,30,31], we have prove that for system (1.2) with $P = P_1 + P_m + P_{2m+1}$, $(P_1 \neq 0)$, $(m = 2, 3, ...)$ the composition conjecture is valid under several restrictions conditions.

In this paper we will study when is the origin point of the more general higher degree polynomial differential system

$$\begin{cases} x' = -y + x(P_1(x,y) + P_m(x,y) + P_n(x,y)), \\ y' = x + y(P_1(x,y) + P_m(x,y) + P_n(x,y)) \end{cases} \tag{1.5}$$

to be an uniform isochronous center? Where $P_k(x,y) = \sum_{i+j=k} p_{ij}x^iy^j$, $p_{ij} \in R\,(k = 1, m, n)$, $p_{10}^2 + p_{01}^2 \neq 0$ and m, n are positive integers. We adopt some simple transformations and some computational skills to reduce the amount of calculation and obtain the center condition. As corollaries, from our new conclusions, it is not difficult to deduce the previous results of [26,30,31]. In addition, we prove that under several restrictions conditions the composition conjecture is correct for its corresponding periodic differential equation:

$$\frac{dr}{d\theta} = r(P_1(\cos\theta, \sin\theta)r + P_m(\cos\theta, \sin\theta)r^m + P_n(\cos\theta, \sin\theta)r^n).$$

In the following we denote

$$S := \sin\theta;\ P_k = P_k(\cos\theta, \sin\theta);\ \bar{P}_k = \int_0^\theta P_k(\cos\tau, \sin\tau)d\tau;$$
$$\overline{\bar{P}_i^{\,k} P_j} = \int_0^\theta (\int_0^\tau P_i(\cos t, \sin t)dt)^k P_j(\cos\tau, \sin\tau)d\tau;\ C_m^k = \frac{m!}{k!(m-k)!};$$

$[x]$ express as the integer part of x; $\sum_i^k(\cdot) = 0$, if $k < i$.

2. Main results

For the system (1.5), when $m = n$ or $m = 1$, $n > 1$ the center conditions has been obtained by [27]; When $m > 1$, $n = 2m + 1$ the center-focus problem has been discussed by [31]. In the following we will consider (1.5) with $3 < 2m + 1 \leq n$.

As $p_{10}^2 + p_{01}^2 \neq 0$, taking $X = p_{10}x + p_{01}y$, $Y = -p_{01}x + p_{10}y$, then the system (1.5) becomes

$$\begin{cases} X' = -Y + X(X + \tilde{P}_m(X,Y) + \tilde{P}_n(X,Y)), \\ Y' = X + Y(X + \tilde{P}_m(X,Y) + \tilde{P}_n(X,Y)), \end{cases}$$

where $\tilde{P}_k(X,Y) = P_k(\frac{p_{10}X - p_{01}Y}{p_{10}^2 + p_{01}^2}, \frac{p_{10}Y + p_{01}X}{p_{10}^2 + p_{01}^2})$, $(k = m, n)$.

For convenience, in the following we will discuss the center-focus problem of the system (1.5) in the form of

$$\begin{cases} x' = -y + x(x + P_m(x,y) + P_n(x,y)), \\ y' = x + y(x + P_m(x,y) + P_n(x,y)), \end{cases} \tag{2.1}$$

where $3 < 2m+1 \leq n$, $P_k(x,y) = \sum_{i+j=k} p_{ij}x^iy^j$, $p_{i\,j} \in R$, $(k = m,n)$, m and n are positive integers.

In polar coordinates the system (2.1) becomes

$$\frac{dr}{d\theta} = r(r\cos\theta + P_m r^m + P_n r^n). \tag{2.2}$$

The origin point $(0,0)$ is a center of system (2.1) if and only if all the solutions of equation (2.2) near $r = 0$ are 2π periodic [2].

Lemma 2.1. *If*

$$\int_0^{2\pi} S^k P_m d\theta = 0\,(k = 0,1,2,..,m),$$

then

$$P_m = \cos\theta \sum_{i=1}^{l} \lambda_{2i} S^{2i-1}, m = 2l;\ P_m = \cos\theta \sum_{i=0}^{l} \lambda_{2i+1} S^{2i}, m = 2l+1, \tag{2.3}$$

where $\lambda_i\,(i = 0,1,2,...,m)$ *are real numbers and*

$$\lambda_m = \sum_{i=0}^{[\frac{m-1}{2}]} (-1)^i p_{2i+1\ m-1-2i}. \tag{2.4}$$

Proof. In the Lemma 3.4 of [27] taking $P_1 = \cos\theta$ which implies that the equality (2.3) is valid and

$$P_m(x,y) = \sum_{i+j=2l} p_{ij}x^iy^j = x\sum_{i=1}^{l} \lambda_{2i} y^{2i-1}(x^2+y^2)^{l-i},\ m = 2l;$$

$$P_m(x,y) = \sum_{i+j=2l+1} p_{ij}x^iy^j = x\sum_{i=0}^{l} \lambda_{2i+1} y^{2i}(x^2+y^2)^{l-i},\ m = 2l+1.$$

Equating the corresponding coefficients of the same power of x, y, we obtain

$$\lambda_{2l} = \sum_{i=0}^{l-1}(-1)^i p_{2i+1\,2(l-i)-1},\ m = 2l;$$

$$\lambda_{2l+1} = \sum_{i=0}^{l}(-1)^i p_{2i+1\,2(l-i)},\ m = 2l+1.$$

Therefore, the conclusion of the present lemma is valid. □

To keep the statement of the following theorem simple, we first give some expressions to be used below.

$$d_0 = e_0 = 1,\ d_k = C^k_{k+m-2},\ e_k = C^k_{k+n-2},\ (k = 1, 2, ...)$$

$$\delta_k = d_k \frac{\lambda_m}{m+k},\ (k = 0, 1, 2, ..., m-1),$$

$$\delta_{m+k} = \frac{\lambda_m}{2m+k}(d_{m+k} + \sum_{j=1}^{[\frac{k}{m}]+1} \sum_{i=0}^{m+k-jm} d_i C^j_{m+1+i} \times \sum_{i_1+i_2+...+i_j=m+k-jm-i} \delta_{i_1}\delta_{i_2}\cdots\delta_{i_j}), (k = 0, 1, 2, ..).$$

$$\mu_{l+m} = e_{m+l} + \sum_{i=0}^{l} \delta_i e_{l-i}(n-m-2i+l),\ (l = 0, 1, 2, ..., m-1), \tag{2.5}$$

$$\mu_{2m+l} = e_{2m+l} + \hat{\gamma}^1_{m+l} + \hat{\xi}^1_l + \hat{\eta}^1_{l-m},\ (l = 0, 1, 2, ..., n-2m), \tag{2.6}$$

$$\hat{\gamma}^1_{l+m} = \Gamma^1_{l+m}[1] - \Gamma^2_{l+m}[\frac{\lambda_m}{j_1 m + s_1 + i_1}],\ \hat{\gamma}^2_{l+m} = \Gamma^2_{l+m}[\frac{\lambda_m}{j_1 m + s_1 + i_1}],$$

$$\hat{\xi}^1_l = -\Upsilon_l[\frac{\lambda_m}{j_2 m + s_2 + i_2}] \cdot \hat{\gamma}^1_{l+m} - \Upsilon_l[\frac{\lambda_m}{m + j_2 m + s_2 + i_1 + i_2}] \cdot \hat{\gamma}^2_{l+m},$$

$$\hat{\xi}^2_l = \Upsilon_l[\frac{\lambda_m}{j_2 m + s_2 + i_2}] \cdot \hat{\gamma}^1_{l+m},\ \hat{\xi}^3_l = \Upsilon_l[\frac{\lambda_m}{m + j_2 m + s_2 + i_1 + i_2}] \cdot \hat{\gamma}^2_{l+m},$$

$$\hat{\eta}^1_{l-m} = -\Lambda_{l-m}[\frac{\lambda_m}{j_3 m + s_3 + i_3}] \cdot \hat{\xi}^1_l - \Lambda_{l-m}[\frac{\lambda_m}{j_2 m + j_3 m + s_2 + s_3 + i_2 + i_3}] \cdot \hat{\xi}^2_l - \Lambda_{l-m}[\frac{\lambda_m}{(1 + j_2 + j_3)m + s_2 + s_3 + i_1 + i_2 + i_3}] \cdot \hat{\xi}^3_l,$$

$$\Gamma^1_{l+m} := \sum_{j_1=1}^{2+[\frac{l}{m}]} \sum_{i_1=0}^{2m+l-j_1 m} e_{i_1} C^{j_1}_{n+1+i_1} \sum_{r_1+r_2+...+r_{j_1}=2m+l-j_1 m-i_1} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_1}},$$

$$\Gamma^2_{l+m} := \sum_{j_1=1}^{2+[\frac{l}{m}]} \sum_{i_1=0}^{2m+l-j_1 m} j_1 d_{i_1} C^{j_1}_{m+1+i_1} \sum_{s_1=0}^{l+2m-j_1 m-i_1} e_{l+2m-j_1 m-i_1-s_1} \times \sum_{r_1+r_2+...+r_{j_1-1}=s_1} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_1-1}},$$

$$\Upsilon_l := \sum_{j_2=1}^{1+[\frac{l}{m}]} j_2 \sum_{i_2=0}^{l+m-j_2 m} d_{i_2} C^{j_2}_{m+1+i_2} \sum_{s_2=0}^{l+m-j_2 m-i_2} \sum_{r_1+r_2+...+r_{j_2-1}=s_2} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_2-1}},$$

$$\Lambda_{l-m} := \sum_{j_3=1}^{[\frac{l}{m}]} j_3 \sum_{i_3=0}^{l-j_3 m} d_{i_3} C^{j_3}_{m+1+i_3} \sum_{s_3=0}^{l-j_3 m-i_3} \sum_{r_1+r_2+...+r_{j_3-1}=s_3} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_3-1}}.$$

Theorem 2.1. *Suppose that* $\prod_{k=m}^{n} \mu_k \neq 0$*, then* $r = 0$ *is a center of* (2.2)*, if and only if*

$$\int_0^{2\pi} S^i P_m d\theta = 0\,(i = 0, 1, 2, .., m), \tag{2.7}$$

$$\int_0^{2\pi} S^k P_n d\theta = 0\,(k = 0, 1, 2, .., n), \tag{2.8}$$

where $\mu_k\,(k=m,m+1,...,n)$ *are expressed by* (2.5) *and* (2.6).

Proof. Taking $\rho=\frac{r}{1+rS}$, the equation (2.2) becomes

$$\frac{d\rho}{d\theta}=\rho^{m+1}(1+rS)^{m-1}P_m+\rho^{n+1}(1+rS)^{n-1}P_n. \tag{2.9}$$

Applying the Langrange-B$\ddot{u}$rman formula [1] we have

$$(1+rS)^k=1+\sum_{j=1}^{\infty}C_{j+k-1}^{k-1}\rho^jS^j,$$

thus, the equation (2.9) can be written as

$$\frac{d\rho}{d\theta}=P_m\sum_{k=0}^{\infty}d_k\rho^{m+k+1}S^k+P_n\sum_{k=0}^{\infty}e_k\rho^{n+k+1}S^k, \tag{2.10}$$

where $d_0=e_0=1$, $d_k=C_{k+m-2}^k$, $e_k=C_{k+n-2}^k$, $(k=1,2,......)$. Therefore, the $r=0$ is a center of (2.2), if and only if all the solutions of equation (2.10) near $\rho=0$ are $2\pi-$ periodic [2].

Let $\rho(\theta,c)$ be the solution of (2.10) such that $\rho(0,c)=c\,(0<c\ll 1)$. We write

$$\rho(\theta,c)=c\sum_{k=0}^{\infty}a_k(\theta)c^n,$$

where $a_0(0)=1$ and $a_k(0)=0$ for $k\geq 1$. The $\rho=0$ is a center of (2.10) if and only if $\rho(\theta+2\pi,c)=\rho(\theta,c)$, i.e., $a_0(2\pi)=1$, $a_k(2\pi)=0\,(k=1,2,3,...)$ [7].

Substituting $\rho(\theta,c)$ into (2.10) we obtain

$$c\sum_{i=0}^{\infty}a_i'(\theta)c^n=P_m\sum_{k=0}^{\infty}d_kS^k(c\sum_{i=0}^{\infty}a_i(\theta)c^i)^{m+k+1}+P_n\sum_{k=0}^{\infty}e_kS^k(c\sum_{i=0}^{\infty}a_i(\theta)c^i)^{n+k+1}. \tag{2.11}$$

Equating the corresponding coefficients of c^k of (2.11) yields

$$a_0(\theta)=1,\ a_i(\theta)=0,\ (i=1,2,...,m-1).$$

Rewriting $\rho(\theta,c)$ as following

$$\rho=c(1+c^mh),\ h=\sum_{i=0}^{\infty}h_i(\theta)c^i, h_i(0)=0,\ (i=0,1,2....).$$

Substituting it into (2.10) we get

$$\begin{aligned}\sum_{k=0}^{\infty}h_k'(\theta)c^k=&P_m\sum_{k=0}^{\infty}d_kc^kS^k\sum_{j=0}^{m+1+k}C_{m+1+k}^jh^jc^{mj}\\&+P_n\sum_{k=0}^{\infty}e_kc^{n-m+k}S^k\sum_{j=0}^{n+k+1}C_{n+k+1}^jh^jc^{mj},\ h_k(0)=0\,(k=0,1,2,...).\end{aligned} \tag{2.12}$$

Equating the corresponding coefficients of c^k of the equation (2.12) we obtain

$$h_k' = d_k S^k P_m, h_k(0) = 0\,(k = 0, 1, 2, ..., m-1),$$
$$h_m' = P_m C_{m+1}^1 h_0 + P_m d_m S^m,\ h_m(0) = 0.$$

Solving these equations we get

$$h_k = d_k \overline{S^k P_m}, (k = 0, 1, 2, ..., m-1),$$
$$h_m = d_m \overline{S^m P_m} + \alpha_0,\ \alpha_0 = \frac{m+1}{2} \bar{P}_m^2.$$

As $d_k \neq 0 (k = 0, 1, 2....)$, from $h_k(2\pi) = 0\,(k = 0, 1, 2, ..., m)$ follow that

$$\int_0^{2\pi} S^k P_m d\theta = 0\,(k = 0, 1, 2, ..., m),$$

i.e., the condition (2.7) is a necessary condition for $\rho = 0$ to be a center. By Lemma 2.1 which implies that

$$P_m = \cos\theta \sum_{k=1}^{m} \lambda_k S^{k-1},\ \bar{P}_m = \int_0^{\theta} P_m d\theta = \sum_{k=1}^{m} \frac{\lambda_k}{k} S^k, \tag{2.13}$$

where $\lambda_k (k = 1, 2, ..., m)$ are real numbers and λ_m is expressed by (2.4). Thus h_k is a polynomial in S of degree $m + k$ and

$$h_k = \delta_k S^{m+k} + ..., (k = 0, 1, 2, ..., m) \tag{2.14}$$
$$\delta_k = d_k \frac{\lambda_m}{m+k},\ (k = 0, 1, 2, ..., m-1),\ \delta_m = \frac{\lambda_m}{2m}(d_m + C_{m+1}^1 d_0 \delta_0),$$

here and below the parts omitted indicate the parts with a lower degree than the first item.

Equating the corresponding coefficients of c^{m+k} of the equation (2.12) we obtain

$$h_{m+k}' = P_m d_{m+k} S^{m+k} + P_m \sum_{j=1}^{[\frac{k}{m}]+1} \sum_{i=0}^{m+k-jm} d_i S^i C_{m+1+i}^j \times \sum_{i_1+i_2+...+i_j = m+k-jm-i} h_{i_1} h_{i_2} \cdots h_{i_j}, \quad (k = 1, 2, ..., n-2m-1).$$

Solving these equations we get

$$h_{m+k} = d_{m+k} \overline{S^{m+k} P_m} + \alpha_k,\ (k = 1, 2, ..., n-2m-1),$$

where

$$\alpha_k = \sum_{j=1}^{[\frac{k}{m}]+1} \sum_{i=0}^{m+k-jm} d_i C_{m+1+i}^j \sum_{i_1+i_2+...+i_j = m+k-jm-i} \overline{S^i h_{i_1} h_{i_2} \cdots h_{i_j} P_m}. \tag{2.15}$$

From this we see that h_{m+k} is a polynomial in S of degree $2m + k$ and

$$h_{m+k} = \delta_{m+k} \sin^{2m+k}\theta + ...,\ (k = 1, 2, ..., n-2m-1), \tag{2.16}$$

where

$$\delta_{m+k}=\frac{\lambda_m}{2m+k}(d_{m+k}+\sum_{j=1}^{[\frac{k}{m}]+1}\sum_{i=0}^{m+k-jm}d_iC_{m+1+i}^{j}\sum_{i_1+i_2+...+i_j=m+k-jm-i}\delta_{i_1}\delta_{i_2}\cdots\delta_{i_j}). \tag{2.17}$$

Equating the corresponding coefficients of c^{n-m+l} of the equation (2.12) we obtain

$$\begin{aligned}h'_{n-m+l}=&P_ne_lS^l+d_{n-m+l}S^{n-m+l}P_m\\&+P_m\sum_{j=1}^{[\frac{n-m+l}{m}]}\sum_{i=0}^{n-m+l-jm}d_iS^iC_{m+1+i}^{j}\sum_{i_1+i_2+...+i_j=n-m+l-jm-i}h_{i_1}h_{i_2}\cdots h_{i_j},\\&(l=0,1,2,...,m-1),\end{aligned}$$

solving these equations we get

$$h_{n-m+l}=e_l\overline{S^lP_n}+\beta_l(S),\ (l=0,1,2,...,m-1), \tag{2.18}$$

where

$$\beta_l(S)=d_{n-m+l}\overline{S^{n-m+l}P_m}+\alpha_{n-2m+l}=\delta_{n-m+l}S^{n+l}+...,$$

α_{n-2m+l} and δ_{n-m+l} are obtained respectively by (2.15) and (2.17) taking $k=n-2m+l$. As $e_l\neq 0$, by (2.18) we see that if $h_{n-m+l}(2\pi)=0$ then

$$\int_0^{2\pi}S^lP_nd\theta=0,\ (l=0,1,2,...,m-1). \tag{2.19}$$

Equating the corresponding coefficients of c^{n+l} of equation (2.12) we obtain

$$\begin{aligned}h'_{n+l}=&P_ne_{m+l}S^{m+l}+P_n\sum_{i=0}^{l}e_iS^iC_{n+1+i}^{1}h_{l-i}+d_{n+l}P_mS^{n+l}\\&+P_m\sum_{j=1}^{[\frac{n+l}{m}]}\sum_{i=0}^{n+l-jm}d_iS^iC_{m+1+i}^{j}\sum_{i_1+i_2+...+i_j=n+l-jm-i}h_{i_1}h_{i_2}\cdots h_{i_j},\\&(l=0,1,2,...,m-1),\end{aligned}$$

solving these equations we obtain

$$h_{n+l}=e_{m+l}\overline{S^{m+l}P_n}+\beta_{m+l}+\gamma_l,\ (l=0,1,2,...,m-1),$$

where

$$\gamma'_l=P_n\sum_{i=0}^{l}e_iS^iC_{n+1+i}^{1}d_{l-i}\overline{S^{l-i}P_m}+P_m\sum_{i=0}^{l}d_iS^iC_{m+1+i}^{1}e_{l-i}\overline{S^{l-i}P_n},$$

$$\begin{aligned}\gamma_l=&\sum_{i=0}^{l}d_ie_{l-i}C_{m+1+i}^{1}\overline{S^iP_m}\ \overline{S^{l-i}P_n}+\sum_{i=0}^{l}d_ie_{l-i}(C_{n+1+l-i}^{1}-C_{m+1+i}^{1})\overline{\overline{S^iP_m}S^{l-i}P_n}\\=&\hat{\gamma}_l^1\overline{S^{m+l}P_n}+\hat{\gamma}_l^2S^{i+m}\overline{S^{l-i}P_n},\end{aligned}$$

in which

$$\hat{\gamma}_l^1=\sum_{i=0}^{l}d_ie_{l-i}\frac{n-m+l-2i}{m+i}\lambda_m,\ \hat{\gamma}_l^2=\sum_{i=0}^{l}d_ie_{l-i}C_{m+1+i}^1\frac{1}{m+i}\lambda_m. \tag{2.20}$$

$$\begin{aligned}\beta_{m+l}=&(d_{n+l}+\sum_{j=1}^{[\frac{n+l}{m}]}\sum_{i=0}^{n+l-jm}d_iC_{m+1+i}^j\sum_{i_1+i_2+...+i_j=n+l-jm-i}\delta_{i_1}\delta_{i_2}\cdots\delta_{i_j})\overline{S^{n+l}P_m}+...\\ =&\ \delta_{n+l}S^{n+m+l}+....,\end{aligned}$$

where δ_{n+l} can be obtained by (2.17) taking $k=n-m+l$. Thus, if $h_{n+l}(2\pi)=0$, then $\mu_{l+m}\int_0^{2\pi}S^{m+l}P_nd\theta=0$, where

$$\mu_{l+m}=e_{m+l}+\sum_{i=0}^{l}\delta_ie_{l-i}(n-m-2i+l),\ (l=0,1,2,...,m-1).$$

By the assumptions, we have $\mu_{l+m}\neq 0$, thus

$$\int_0^{2\pi}S^{m+l}P_nd\theta=0,\ (l=0,1,2,...,m-1). \tag{2.21}$$

Equating the corresponding coefficients of c^{n+m+l} of equation (2.12) we obtain

$$\begin{aligned}h'_{n+m+l}=&P_ne_{2m+l}S^{2m+l}+P_n\sum_{j=1}^{2+[\frac{l}{m}]}\sum_{i=0}^{2m+l-jm}e_iS^iC_{n+1+i}^j\\ &\times\sum_{i_1+i_2+...+i_j=2m+l-jm-i}h_{i_1}h_{i_2}\cdots h_{i_j}+d_{n+m+l}S^{n+m+l}P_m\\ &+P_m\sum_{j=1}^{1+[\frac{n+l}{m}]}\sum_{i=0}^{n+l+m-jm}d_iS^iC_{m+1+i}^j\sum_{i_1+i_2+...+i_j=n+l+m-jm-i}h_{i_1}h_{i_2}\cdots h_{i_j},\\ &(l=0,1,2,...,n-2m).\end{aligned}$$

Rewriting the above equations as the form of

$$\begin{aligned}h'_{n+m+l}=&P_ne_{2m+l}S^{2m+l}+\sum_{j=1}^{2+[\frac{l}{m}]}I_j+d_{n+m+l}S^{n+m+l}P_m+\sum_{j=1}^{1+[\frac{n+l}{m}]}K_j,\\ &(l=0,1,2,...,n-2m),\end{aligned} \tag{2.22}$$

where

$$I_j=P_n\sum_{i=0}^{2m+l-jm}e_iS^iC_{n+1+i}^j\sum_{i_1+i_2+...+i_j=2m+l-jm-i}h_{i_1}h_{i_2}\cdots h_{i_j}.$$

When $l=n-2m$,

$$I_1=C_{n+1}^1P_n\bar{P}_n+\sum_{i=0}^{m+l}e_iC_{n+1+i}^1\delta_{m+l-i}S^{2m+l}P_n+...;$$

When $l < n - 2m$,

$$I_1 = \sum_{i=0}^{m+l} e_i C_{n+1+i}^1 \, \delta_{m+l-i} S^{2m+l} P_n + \dots .$$

For $j > 1$,

$$I_j = \sum_{i=0}^{2m+l-jm} e_i C_{n+1+i}^j \sum_{i_1+i_2+\dots+i_j=2m+l-jm-i} \delta_{i_1}\delta_{i_2}\cdots\delta_{i_j} S^{2m+l} P_n + \dots ,$$

where the parts omitted represent these items of $S^k P_n$, $(k < 2m + l)$.

Rewriting

$$\sum_{j=1}^{1+[\frac{l+n}{m}]} K_j = \sum_{j=1}^{2+[\frac{l}{m}]} K_j + \sum_{j=3+[\frac{l}{m}]}^{1+[\frac{l+n}{m}]} K_j = L_1 + L_2.$$

As $2 + \frac{l}{m} < j = 3 + [\frac{l}{m}] \leq 3 + \frac{l}{m}$, $n + l + m - jm - i < n - m - i$, thus by the above we get

$$\begin{aligned}
L_2 &= \sum_{j=3+[\frac{l}{m}]}^{1+[\frac{l+n}{m}]} K_j \\
&= \sum_{j=3+[\frac{l}{m}]}^{1+[\frac{n+l}{m}]} \sum_{i=0}^{n+l+m-jm} d_i S^i C_{m+1+i}^j \sum_{i_1+i_2+\dots+i_j=n+l+m-jm-i} h_{i_1}h_{i_2}\cdots h_{i_j} P_m \\
&= \sum_{j=3+[\frac{l}{m}]}^{1+[\frac{n+l}{m}]} \sum_{i=0}^{n+l+m-jm} d_i C_{m+1+i}^j \sum_{i_1+i_2+\dots+i_j=n+l+m-jm-i} \delta_{i_1}\delta_{i_2}\cdots\delta_{i_j} S^{l+m+n} P_m + \dots; \\
L_1 &= \sum_{j=1}^{2+[\frac{l}{m}]} K_j \\
&= \sum_{j=1}^{2+[\frac{l}{m}]} \sum_{i=0}^{n+l+m-jm} d_i S^i C_{m+1+i}^j \sum_{i_1+i_2+\dots+i_j=n+l+m-jm-i} h_{i_1}h_{i_2}\cdots h_{i_j} P_m \\
&= \sum_{j=1}^{2+[\frac{l}{m}]} P_m \sum_{i=0}^{l+2m-jm} d_i S^i C_{m+1+i}^j \sum_{i_1+i_2+\dots+i_j=n+l+m-jm-i} h_{i_1}h_{i_2}\cdots h_{i_j} \\
&\quad + \sum_{j=1}^{2+[\frac{l}{m}]} P_m \sum_{i=l+2m-jm+1}^{n+l+m-jm} d_i S^i C_{m+1+i}^j \sum_{i_1+i_2+\dots+i_j=n+l+m-jm-i} h_{i_1}h_{i_2}\cdots h_{i_j} \\
&= K_{j_1} + K_{j_2},
\end{aligned}$$

where

$$K_{j_2} = \sum_{j=1}^{2+[\frac{l}{m}]} \sum_{i=l+2m-jm+1}^{n+l+m-jm} d_i C_{m+1+i}^j \sum_{i_1+i_2+\dots+i_j=n+l+m-jm-i} \delta_{i_1}\delta_{i_2}\cdots\delta_{i_j} S^{n+l+m} P_m + \dots;$$

$$\begin{aligned}K_{j_1} = &\sum_{j=1}^{2+[\frac{l}{m}]} jP_m \sum_{i=0}^{l+2m-jm} d_i S^i C_{m+1+i}^j \\ &\times (\sum_{s=0}^{l+2m-jm-i} (e_{l+2m-jm-i-s}\overline{S^{l+2m-jm-i-s}\,P_n} + \beta_{2m+l-jm-i-s}) \\ &\times \sum_{s=0}^{l+m-jm-i} \gamma_{l+m-jm-i-s} + \sum_{s=0}^{l-jm-i} \xi_{l-jm-i-s}) \\ &\times \sum_{i_1+i_2+\ldots+i_{i_{j-1}}=s} h_{i_1}h_{i_2}\cdots h_{i_{j-1}} \\ = &\sum_{j=1}^{2+[\frac{l}{m}]} j \sum_{i=0}^{l+2m-jm} d_i C_{m+1+i}^j \\ &\times (\sum_{s=0}^{l+2m-jm-i} (e_{l+2m-jm-i-s}\overline{S^{l+2m-jm-i-s}\,P_n} + \beta_{2m+l-jm-i-s}) \\ &\times \sum_{s=0}^{l+m-jm-i} \gamma_{l+m-jm-i-s} + \sum_{s=0}^{l-jm-i} \xi_{l-jm-i-s}) \\ &\times \sum_{i_1+i_2+\ldots+i_{i_{j-1}}=s} \delta_{i_1}\delta_{i_2}\cdots\delta_{i_{j-1}}\, S^{(j-1)m+s}\,P_m.\end{aligned}$$

Remark: In the above formula and below, when $j = 1$,

$$\begin{aligned}&\sum_{s=0}^{l+2m-jm-i} e_{l+2m-jm-i-s} \sum_{i_1+i_2+\ldots+i_{i_{j-1}}=s} \delta_{i_1}\delta_{i_2}\cdots\delta_{i_{j-1}} \\ =&e_{l+2m-jm-i-s}|_{j=1,s=0} = e_{l+m-i}.\end{aligned}$$

Solving the equation (2.22) we get

$$h_{m+n+l} = e_{2m+l}\overline{S^{2m+l}P_n} + \beta_{2m+l} + \gamma_{m+l} + \xi_l + \eta_{l-m},\ (l = 0, 1, 2, ..., n-2m-1), \tag{2.23}$$

$$h_{2n-m} = e_n\overline{S^nP_n} + \beta_n + \gamma_{n-m} + \xi_{n-2m} + \eta_{n-3m} + \frac{1}{2}C_{n+1}^1\bar{P}_n^2, \tag{2.24}$$

where $\eta_i = 0\,(i < 0)$, β_{2m+l}, γ_{m+l}, ξ_l and η_{l-m} are the solutions, respectively, of the following equations.

$$\begin{aligned}\gamma'_{m+l} = &\sum_{j_1=1}^{2+[\frac{l}{m}]} \sum_{i_1=0}^{2m+l-j_1m} (e_{i_1}C_{n+1+i_1}^{j_1} \sum_{r_1+r_2+\ldots+r_{j_1}=2m+l-j_1m-i_1} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_1}} S^{2m+l}P_n \\ &+ j_1 d_{i_1} C_{m+1+i_1}^{j_1} \sum_{s_1=0}^{l+2m-j_1m-i_1} e_{l+2m-j_1m-i_1-s_1} \sum_{r_1+r_2+\ldots+r_{j_1-1}=s} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_1-1}} \\ &\times S^{(j_1-1)m+s_1+i_1}P_m\,\overline{S^{l+2m-j_1m-i_1-s_1}P_n}) \\ =&\Gamma_{l+m}^1[S^{2m+l}P_n] + \Gamma_{l+m}^2[S^{(j_1-1)m+s_1+i_1}P_m\overline{S^{l+2m-j_1m-i_1-s_1}P_n}],\end{aligned} \tag{2.25}$$

where

$$\Gamma^1_{l+m} := \sum_{j_1=1}^{2+[\frac{l}{m}]} \sum_{i_1=0}^{2m+l-j_1m} e_{i_1} C^{j_1}_{n+1+i_1} \sum_{r_1+r_2+\ldots+r_{j_1}=2m+l-j_1m-i_1} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_1}},$$

$$\Gamma^2_{l+m} := \sum_{j_1=1}^{2+[\frac{l}{m}]} \sum_{i_1=0}^{2m+l-j_1m} j_1 d_{i_1} C^{j_1}_{m+1+i_1} \sum_{s_1=0}^{l+2m-j_1m-i_1} e_{l+2m-j_1m-i_1-s_1} \times \sum_{r_1+r_2+\ldots+r_{j_1-1}=s_1} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_1-1}}.$$

Solving the equation (2.25) we obtain

$$\gamma_{m+l} = \hat{\gamma}^1_{l+m}\overline{S^{2l+m}P_n} + \hat{\gamma}^2_{i+m} S^{j_1m+s_1+i_1}\,\overline{S^{l+2m-j_1m-s_1-i_1}\,P_n}, \tag{2.26}$$

where

$$\hat{\gamma}^1_{l+m} = \Gamma^1_{l+m} - \Gamma^2_{l+m}\frac{\lambda_m}{j_1m+s_1+i_1},\ \hat{\gamma}^2_{l+m} = \Gamma^2_{l+m}\frac{\lambda_m}{j_1m+s_1+i_1}.$$

$$\xi'_l = \sum_{j_2=1}^{1+[\frac{l}{m}]} j_2 \sum_{i_2=0}^{l+m-j_2m} d_{i_2} C^{j_2}_{m+1+i_2} \sum_{s_2=0}^{l+m-j_2m-i_2} \gamma_{l+m-j_2m-i_2-s_2} \sum_{r_1+r_2+\ldots+r_{j_2-1}=s_2} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_2-1}} S^{(j_2-1)m+s_2+i_2} P_m = \Upsilon_l[S^{(j_2-1)m+s_2+i_2}P_m\gamma_{l+m-j_2m-i_2-s_2}],$$

here

$$\Upsilon_l := \sum_{j_2=1}^{1+[\frac{l}{m}]} j_2 \sum_{i_2=0}^{l+m-j_2m} d_{i_2} C^{j_2}_{m+1+i_2} \sum_{s_2=0}^{l+m-j_2m-i_2} \sum_{r_1+r_2+\ldots+r_{j_2-1}=s_2} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_2-1}}.$$

Solving this equation we get

$$\begin{aligned}\xi_l =& \hat{\xi}^1_l\overline{S^{2m+l}P_n} + \hat{\xi}^2_l S^{j_2m+s_2+i_2}\overline{S^{2m+l-j_2m-s_2-i_2}P_n} \\ &+ \hat{\xi}^3_l S^{j_1m+j_2m+s_1+s_2+i_1+i_2}\overline{S^{2m+l-j_1m-j_2m-s_1-s_2-i_1-i_2}P_n},\end{aligned} \tag{2.27}$$

where

$$\hat{\xi}^1_l = -\hat{\xi}^2_l - \hat{\xi}^3_l,\quad \hat{\xi}^2_l = \Upsilon_l\frac{\lambda_m}{j_2m+s_2+i_2}\hat{\gamma}^1_{l+m-j_2m-i_2-s_2},$$

$$\hat{\xi}^3_l = \Upsilon_l\hat{\gamma}^2_{l+m-j_2m-i_2-s_2}\frac{\lambda_m}{j_1m+j_2m+s_1+s_2+i_1+i_2},$$

$\hat{\gamma}^j_{l+m-j_2m-i_2-s_2}$ $(j=1,2)$ are expressed by (2.20) replace l with $l+m-j_2m-i_2-s_2$.

$$\eta'_{l-m} = \Lambda_{l-m}[S^{(j_3-1)m+s_3+i_3}P_m\xi_{l-j_3m-i_3-s_3}],\ (l=m,m+1,...,n-2m)$$

$$\Lambda_{l-m} := \sum_{j_3=1}^{[\frac{l}{m}]} j_3 \sum_{i_3=0}^{l-j_3m} d_{i_3} C^{j_3}_{m+1+i_3} \sum_{s_3=0}^{l-j_3m-i_3} \sum_{r_1+r_2+\ldots+r_{j_3-1}=s_3} \delta_{r_1}\delta_{r_2}\cdots\delta_{r_{j_3-1}}.$$

Solving this equation we get

$$\eta_{l-m} = \hat{\eta}^1_{l-m}\overline{S^{2m+l}P_n} + \hat{\eta}^2_{l-m}S^{j_3m+s_3+i_3}\overline{S^{l+2m-j_3m-i_3-s_3}P_n}$$

$$+\hat{\eta}^3_{l-m}S^{j_2m+j_3m+s_2+s_3+i_2+i_3}\overline{S^{2m+l-j_3m-j_2m-s_2-s_3-i_2-i_3}P_n}$$
$$+\hat{\eta}^4_{l-m}S^{(j_1+j_2+j_3)m+s_1+s_2+s_3+i_1+i_2+i_3}\overline{S^{2m+l-(j_1+j_2+j_3)m-s_1-s_2-s_3-i_1-i_2-i_3}P_n}, \tag{2.28}$$

here

$$\hat{\eta}^2_{l-m}=\Lambda_{l-m}\frac{\lambda_m}{j_3m+s_3+i_3}\hat{\xi}^1_{l-j_3m-i_3-s_3},$$
$$\hat{\eta}^3_{l-m}=\Lambda_{l-m}\hat{\xi}^2_{l-j_3m-i_3-s_3}\frac{\lambda_m}{j_2m+j_3m+s_2+s_3+i_2+i_3},$$
$$\hat{\eta}^4_{l-m}=\Lambda_{l-m}\hat{\xi}^3_{l-j_3m-i_3-s_3}\frac{\lambda_m}{(j_1+j_2+j_3)m+s_1+s_2+s_3+i_1+i_2+i_3},$$
$$\hat{\eta}^1_{l-m}+\hat{\eta}^2_{l-m}+\hat{\eta}^3_{l-m}+\hat{\eta}^4_{l-m}=0.$$
$$\beta'_{2m+l}=d_{n+m+l}S^{n+m+l}P_m+K_{j2}+L_2$$
$$+\sum_{j=1}^{2+[\frac{l}{m}]}\sum_{i=0}^{2m+l-jm}jd_iC^j_{m+1+i}\sum_{s=0}^{l+2m-jm-i}\beta_{l+2m-jm-i-s}$$
$$\times\sum_{i_1+i_2+...+i_{i_{j-1}}=s}\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{j-1}}S^{(j-1)m+s+i}P_m,$$

which implies that β_{2m+l} is a function on $\sin\theta$, so it is 2π periodic. Thus, from (2.23) to (2.28) we deduce that if $h_{n+m+l}(2\pi)=0$, then

$$\mu_{2m+l}\int_0^{2\pi}S^{2m+l}P_nd\theta=0,\ (l=0,1,2,...,n-2m),$$

where $\mu_{2m+l}=e_{2m+l}+\hat{\gamma}^1_{m+l}+\hat{\xi}^1_l+\hat{\eta}^1_{l-m}$, by the assumptions we see that $\mu_{2m+l}\neq 0$, thus

$$\int_0^{2\pi}S^{2m+l}P_nd\theta=0,\ (l=0,1,2,...,n-2m). \tag{2.29}$$

By (2.19) and (2.21) and (2.29) we see that under the assumptions of the present theorem, the (2.7) and (2.8) are the necessary conditions for $\rho=0$ to be a center of (2.10). Therefore, the necessity has been proved. On the other hand, by Lemma 2.1, if the conditions (2.7) and (2.8) are satisfied, then $P_m=\cos\theta\sum_{i=1}^m\lambda_i\sin^{i-1}\theta$, $P_n=\cos\theta\sum_{i=1}^n\varsigma_i\sin^{i-1}\theta$, λ_i, ς_i are real numbers. By Theorem 1.2 we see that $\rho=0$ is a center and composition center of (2.10), thus the origin point is an uniform isochronous center of system (2.1), this means that the sufficiency is proved.

In summary, the proof is complete. □

Remark 2.1. Duo to the definite integral from 0 to 2π of an odd degree polynomial in $\sin\theta$ and $\cos\theta$ is equal to zero, thus, the assumptions of Theorem 2.1 can be changed to $\prod_{k=m}^n\mu_k\neq 0$ and the equalities (2.7) and (2.8) are valid when k and n, i and m, are positive integers with the same parity.

Corollary 2.1. *If $\lambda_m=0$, then $r=0$ is a center of* (2.2)*, if and only if ,* (2.7) *and* (2.8) *are satisfied.*

Proof. By the proof of Theorem 2.1 we see that if $\lambda_m=0$, then $\delta_k=0$, $(k=0,1,2,...)$ and $\mu_k=e_k=C^k_{k+n-2}\neq 0$, $(k=m,m+1,...,n)$. Therefore, by the above theorem implies that the result of the present corollary is valid. □

In Theorem 2.1 taking $n=2m+1$ we deduce the following corollary.

Corollary 2.2. ($\dot{1}$). *If $m=2k, k$ is a positive integer, $n=2m+1$ and $\prod_{i=k}^{2k}\mu_{2i+1}\neq 0$, then $r=0$ is a center of (2.2), if and only if*

$$\int_0^{2\pi} S^{2i}P_m d\theta=0,\ (i=0,1,2,...,k),\ \int_0^{2\pi} S^{2j+1}P_n d\theta=0,\ (j=0,1,2,...,2k).$$

($\dot{2}$). *If $m=2k+1, k$ is a positive integer, $n=2m+1$ and $\prod_{i=k}^{2k+1}\mu_{2i+1}\neq 0$, then $r=0$ is a center of (2.2), if and only if*

$$\int_0^{2\pi} S^{2i+1}P_m d\theta=0,\ (i=0,1,...,k),\ \int_0^{2\pi} S^{2j+1}P_n d\theta=0,\ (j=0,1,2,...,2k+1).$$

Where

$$\mu_{l+m}=e_{m+l}+\sum_{i=0}^{l}\delta_i e_{l-i}(m+1+l-2i),\ (l=0,1,2,...,m-1);$$

$$\mu_{2m+1}=e_{2m+1}+\lambda_m\sum_{i=1}^{m+1}\frac{d_{m+1-i}}{2m+1-i}2ie_i+4\lambda_m^2\frac{m^2+m+1}{m}$$

$$d_0=e_0=1,\ d_k=C_{k+m-2}^k,\ e_k=C_{k+2m-1}^k,\ (k=1,2,3,......);$$

$$\delta_k=d_k\frac{\lambda_m}{m+k},\ (k=0,1,2,...,m-1);\ \delta_m=\frac{\lambda_m}{2m}(d_m+C_{m+1}^1 d_0\delta_0);$$

$$\delta_{m+1}=\frac{\lambda_m}{2m+1}(d_{m+1}+\sum_{i=0}^{1}d_iC_{m+1+i}^1\delta_{1-i});\ \lambda_m=\sum_{i=0}^{[\frac{m-1}{2}]}(-1)^i p_{2i+1\ m-1-2i}.$$

Proof. In (2.24) taking $n=2m+1$ and $l=n-2m=1$ we get

$$\begin{aligned}\hat{\gamma}_{m+l}^1=&\sum_{i=0}^{m+1}e_i\delta_{m+1-i}C_{2m+2+i}^1+\sum_{i=0}^{1}e_iC_{2m+2+i}^2\sum_{i_1+i_2=1-i}\delta_{i_1}\delta_{i_2}\\&-\sum_{i=0}^{m+1}\frac{\lambda_m d_i}{m+i}e_{m+1-i}C_{m+1+i}^1-2d_0C_{m+1}^2(\delta_0 e_1\frac{\lambda_m}{2m}+\delta_1 e_0\frac{\lambda_m}{2m+1})\\&-2d_1C_{m+2}^2\delta_0 e_0\frac{\lambda_m}{2m+1},\\ \hat{\xi}_1^1=&-d_0C_{m+1}^1\lambda_m^2\sum_{i=0}^{1}d_ie_{1-i}(\frac{m+1+i}{(m+i)(2m+i)}+\frac{m+2-2i}{m(m+i)})\\&-d_0e_0d_1C_{m+2}^1\lambda_m^2\frac{3m+2}{m(2m+1)}.\end{aligned}$$

Then

$$\begin{aligned}\mu_{2m+1}=&e_{2m+1}+\hat{\gamma}_{m+1}^1+\hat{\xi}_1^1\\=&e_{2m+1}+\sum_{i=0}^{m+1}e_i\delta_{m+1-i}C_{2m+2+i}^1+\sum_{i=0}^{1}e_iC_{2m+2+i}^2\sum_{i_1+i_2=1-i}\delta_{i_1}\delta_{i_2}\\&-\sum_{i=0}^{m+1}\frac{\lambda_m d_i}{m+i}e_{m+1-i}C_{m+1+i}^1-2d_0C_{m+1}^2(\delta_0 e_1\frac{\lambda_m}{2m}+\delta_1 e_0\frac{\lambda_m}{2m+1})\end{aligned}$$

$$
\begin{aligned}
& -2d_1C_{m+2}^2\delta_0e_0\frac{\lambda_m}{2m+1} \\
& -d_0C_{m+1}^1\lambda_m^2\sum_{i=0}^{1}d_ie_{1-i}\left(\frac{m+1+i}{(m+i)(2m+i)}+\frac{m+2-2i}{m(m+i)}\right) \\
& -d_0e_0d_1C_{m+2}^1\lambda_m^2\frac{3m+2}{m(2m+1)} \\
=& e_{2m+1}+\sum_{i=0}^{m+1}\left(C_{2m+2+i}^1e_i\delta_{m+1-i}-\frac{\lambda_md_i}{m+i}C_{m+1+i}^1e_{m+1-i}\right)+\frac{(5+5m-4m^2)}{m(2m+1)}\lambda_m^2 \\
=& e_{2m+1}+\lambda_m\sum_{i=0}^{m+1}\left(\frac{d_{m+1-i}}{2m+1-i}C_{2m+2+i}^1e_i-\frac{d_i}{m+i}C_{m+1+i}^1e_{m+1-i}\right) \\
& +4\lambda_m^2\frac{m^2+m+1}{m} \\
=& e_{2m+1}+\lambda_m\sum_{i=1}^{m+1}\frac{d_{m+1-i}}{2m+1-i}2ie_i+4\lambda_m^2\frac{m^2+m+1}{m}.
\end{aligned}
$$

Thus, using Theorem 2.1 we see that the conclusion of the present corollary is valid. □

Remark 2.2. Although, in the literature [31] the center conditions have been obtained. Obviously, the statement of this corollary is much more concise and beautiful than the Theorem 3.1 of [31].

Corollary 2.3. *If* $(30+13\lambda_2)(84+113\lambda_2+21\lambda_2^2)\neq 0$, *then the origin point of system*

$$
\begin{cases}
x' = -y + x(x + P_2(x,y) + P_5(x,y)), \\
y' = x + y(x + P_2(x,y) + P_5(x,y))
\end{cases}
$$

is a center if and only if

$$
\int_0^{2\pi} S^{2i}P_2d\theta = 0,\ (i=0,1),\ \int_0^{2\pi} S^{2j+1}P_5d\theta = 0,\ (j=0,1,2)
$$

where $\lambda_2 = p_{11}$.

Proof. When $m=2$ and $n=5$, by (2.4) we have $\lambda_2 = p_{11}$. Using Corollary 2.2 and calculating we get

$$
\mu_3 = \frac{2}{3}(30+13\lambda_2),\ \mu_5 = 2(84+113\lambda_2+21\lambda_2^2),
$$

thus the present corollary is valid. □

Corollary 2.4. *If* $(42+\lambda_3)(105+23\lambda_3)(2970+2109\lambda_3+65\lambda_3^2)\neq 0$, *then the origin point of system*

$$
\begin{cases}
x' = -y + x(x + P_3(x,y) + P_7(x,y)), \\
y' = x + y(x + P_3(x,y) + P_7(x,y))
\end{cases}
$$

is a center if and only if

$$\int_0^{2\pi} S^{2i+1}P_3 d\theta = 0,\ (i = 0, 1),\ \int_0^{2\pi} S^{2j+1}P_7 d\theta = 0,\ (j = 0, 1, 2, 3)$$

where $\lambda_3 = p_{12} - p_{30}$.

Proof. When $m = 3$, by (2.4) we get $\lambda_3 = p_{12} - p_{30}$. In Corollary 2.2 taking $m = 3$, $n = 7$ and calculating we get

$$\mu_3 = \frac{4}{3}(42 + \lambda_3),\ \mu_5 = \frac{12}{5}(105 + 23\lambda_3),\ \mu_7 = \frac{4}{15}(2970 + 2109\lambda_3 + 65\lambda_3^2).$$

By Corollary 2.2 which implies that the present corollary is valid. □

Remark 2.3. In Corollary 2.3 taking $\lambda_2 = 2\mu$ which implies the Theorem 3.1 of [26]. In Corollary 2.4 taking $\lambda_3 = 3\mu$ we deduce that the Theorem 3.1 of [26] is valid.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, New York, Dover, 1965.

[2] A. Algaba and M. Reyes, *Computing center conditions for vector fields with constant angular speed*, J. Comput. Appl. Math., 2003, 154(1), 143–159.

[3] A. Algaba, M. Reyes and A. Bravo, *Geometry of the uniformly isochronous centers with polynomial commutator*, Differ. Equ. Dyn. Syst., 2002, 10(3–4), 257–275.

[4] A. Algaba and M. Reyes, *Centers with degenerate infinity and their commutators*, (English summary) J. Math. Anal. Appl., 2003, 278(1), 109–124.

[5] M. A. M. Alwash, *Periodic solutions of Abel differential equations*, J. Math. Anal. Appl., 2007, 329, 1161–1169.

[6] M. A. M. Alwash, *Composition conditions for two-dimensional polynomial systems*, Differ. Equ. Appl., 2013, 5(1), 1–12.

[7] M. A. M. Alwash and N. G. Lloyd, *Non-autonomous equations related to polynomial two-dimensional systems*, Proc. Roy. Soc. Edinburgh, 1987, 105, 129–152.

[8] M. A. M. Alwash, *Uniformly isochronous quartic systems*, Qual. Theory Dyn. Syst., 2020, 19(2), 1–13.

[9] M. A. M. Alwash, *On a condition for a center of cubic non-autonomous equations*, Proc. Roy. Soc. Edinburgh Sect. A, 1989, 113, 289–291.

[10] M. A. M. Alwash, *The composition conjecture for Abel equation*, Expo. Math., 2009, 27, 241–250.

[11] J. C. Artés, J. Itikawa and J. Llibre, *Uniform isochronous cubic and quartic centers*, J. Comput. Appl. Math., 2017, 313, 448–453.

[12] J. Chavarriga, I. A. García and J. Giné, *On integrability of differential equations defined by the sum of homogeneous vector fields with degenerate infinity*, Int. J. Bifur. Chaos Appl. Sci. Engrg., 2001, 11(3), 711–722.

[13] J. Chavarriga, I. A. García and J. Giné, *The nondegenerate center problem in certain families of planar differential systems*, Int. J. Bifur. Chaos Appl. Sci. Engrg., 2009, 19(1), 435–443.

[14] A. G. Choudhury and P. Guha, *On commuting vector fields and Darboux functions for planar differential equations*, Lobachevskii J. Math., 2013, 34, 212–226.

[15] A. Cima, A. Gasull and F. Manosas, *Centers for trigonometric Abel equations*, Qual. Theory Dyn. Syst., 2012, 11, 19–37.

[16] R. Conti, *Uniformly isochronous centers of polynomial systems in R^2*, in: Lecture Notes in Pure and Appl. Math., 1994, 152, 21–31.

[17] J. Devlin, N. G. Lloyd and J. M. Pearson, *Cubic systems and Abel equations*, J. Differ. Equ., 1989, 147, 435–454.

[18] J. Giné, M. Grau and X. Santallusia, *The center problem and composition condition for Abel differential equations*, Expo. Math., 2016, 34(2), 210–222.

[19] J. Giné, *On some open problems in planar differential systems and Hilbert's 16th problem*, Chaos Solitons Fractals, 2007, 31(5), 1118–1134.

[20] J. Giné and X. Santallusia, *On the Poincaré-Lyapunov constants and the Poincaré series*, Appl. Math. (Warsaw), 2001, 28(1), 17–30.

[21] J. Giné, M. Grau and X. Santallusia, *Composition conditions in the trigonometric Abel equation*, J. Appl. Anal. Comput., 2013, 3(2), 133–144.

[22] J. Itikawa and J. Llibre, *Phase portraits of uniform isochronous quartic centers*, J. Comput. Appl. Math., 2015, 287, 98–114.

[23] N. G. Lloyd and J. M. Pearson, *Computing centre conditions for certain cubic systems*, J. Comput. Appl. Math., 1992, 40, 323–336.

[24] E. P. Volokitin, *Center conditions for a simple class of quintic systems*, Int. J. Math. Math. Sci., 2002, 29(11), 625–632.

[25] E. P. Volokitin, *Centering conditions for planar septic systems*, Electron. J. Differ. Equ., 2003, 34, 1–7.

[26] Y. Yan and Z. Zhou, *Research on the composition center of a class of rigid differential systems*, J. Appl. Anal. Comput., 2020, 10(6), 2506–2520.

[27] Z. Zhou, *The Poincare center-focus problem for a class of higher order polynomial differential systems*, Math. Meth. Appl. Sci., 2019, 5475, 1–15.

[28] Z. Zhou, *On the center-focus problem for a family of high-order polynomial differential systems*, J. Dyn. Differ. Equ., 2020, 32(1), 393–418.

[29] Z. Zhou and V. G. Romanovski, *The center problem and the composition condition for a family of quartic differential systems*, Electron. J. Qual. Theory. Differ. Equ., 2018, 15, 1–17.

[30] Z. Zhou and Y. Yan, *On the composition center for a class of rigid system*, Bull. Braz. Math. Soc., New Series, 2020, 51, 139–155.

[31] Z. Zhou and Y. Yan, *On the composition conjecture for a class of rigid systems*, Electron. J. Qual. Theory. Differ. Equ. 2019, 95, 1–22.