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# SPECTRA OF GRAPH OPERATIONS BASED ON SPLITTING GRAPH*

Zhiqin Lu[1], Xiaoling Ma[1,†] and Minshao Zhang[1]

**Abstract** The splitting graph $SP(G)$ of a graph $G$ is the graph obtained from $G$ by taking a new vertex $u'$ for each $u \in V(G)$ and joining $u'$ to all vertices of $G$ adjacent to $u$. For a connected regular graph $G_1$ and an arbitrary regular graph $G_2$, we determine the adjacency (respectively, Laplacian and signless Laplacian) spectra of two types of graph operations on $G_1$ and $G_2$ involving the $SP$-graph of $G_1$. Moreover, applying these results we construct some non-regular simultaneous cospectral graphs for the adjacency, Laplacian and signless Laplacian matrices, and compute the Kirchhoff index and the number of spanning trees of the newly constructed graphs.

**Keywords** Adjacency spectrum, Laplacian spectrum, signless Laplacian spectrum, cospectral graphs, Kirchhoff index, the number of spanning trees.

**MSC(2010)** 05C50.

## 1. Introduction

All graphs considered in this paper are undirected and simple. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of $G$, denoted by $A(G)$, is the $n \times n$ matrix whose $(i, j)$-entry is 1 if $v_i$ and $v_j$ are adjacent in $G$ and 0 otherwise. Let $d_i = d_G(v_i)$ be the degree of vertex $v_i$ in $G$, and $D(G)$ be the diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$. The *Laplacian matrix* and the *signless Laplacian matrix* of $G$ are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. For an $n \times n$ matrix $M$, we denote the characteristic polynomial $\det(xI_n - M)$ of $M$ by $f_M(x)$, where $I_n$ is the identity matrix of order $n$. In particular, for a graph $G$, $f_{A(G)}(x)$ (respectively, $f_{L(G)}(x)$ and $f_{Q(G)}(x)$) is the adjacency (respectively, Laplacian and signless Laplacian) *characteristic polynomial* of $G$, and its roots are the adjacency (respectively, Laplacian and signless Laplacian) eigenvalues of $G$. Denote the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively, by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G),$$
$$\mu_1(G) < \mu_2(G) \leq \cdots \leq \mu_n(G),$$
$$\nu_1(G) \geq \nu_2(G) \geq \cdots \geq \nu_n(G).$$

[†]The corresponding author. Email: mxling2018@163.com(X. Ma)
[1]College of Mathematics and System Sciences, Xinjiang University, 830046, Urumqi, China
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Note that $\mu_1(G) = 0$. The collection of eigenvalues of $A(G)$ (respectively, $L(G)$, $Q(G)$) together with their multiplicities are called the *$A$-spectrum* (respectively, *$L$-spectrum*, *$Q$-spectrum*) of $G$. Two graphs are said to be *$A$-cospectral* (respectively, *$L$-cospectral*, *$Q$-cospectral*) if they have the same *$A$-spectrum* (respectively, *$L$-spectrum*, *$Q$-spectrum*).

Some principal properties and structure of a graph can be explored through the information about its various spectra. Hence, the investigation of spectra of graphs becomes crucial in the field of graph theory. In addition, the spectral graph theoretic techniques have been used in a great many of fields such as quantum physics, chemistry, computer science, etc. [3,7–9]. In recent years, several researchers studied the spectral properties of graphs which are constructed by graph operations. Some well-known graph operations in this direction are the disjoint union, the Cartesian product, the Kronocker product, the strong product, the lexicographic product, the corona, the edge corona, the neighbourhood corona etc. For the results on the spectra of these graphs, see [1,2,8,11,15,21].

The *join* [13] of two graphs is their disjoint union together with all the edges that connect all the vertices of the first graph with all the vertices of the second graph. Recently, many researchers provided several variants of join operations of graphs and investigated their spectral properties. Cardoso et. al. [5] introduced a generalized join of regular graph families and characterized adjacency and Laplacian spectra of the operation of graphs. In [16], Indulal defined the subdivision-vertex(edge) joins of graphs and obtained adjacency spectra of subdivision-vertex(edge) joins for two regular graphs in terms of their spectra. Later, Liu and Zhang [18] determined the spectra, (signless)Laplacian spectra of the subdivision-vertex(edge) joins for a regular graph and arbitrary graph. As applications, they constructed infinitely many pairs of cospectral graphs and gave the number of spanning trees and the Kirchhoff index of the subdivision-vertex(edge) joins. Then in [19], the $R$-vertex(edge) joins of graphs are defined and the Laplacian spectra of $R$-vertex(edge) joins are formulated. As applications, the resistance distances and Kirchhoff index of $R$-vertex(edge) joins are computed. Recently, DAS and Panigrahi [10] derived the $Q$-vertex(edge) joins of graphs and determined the full adjacency, Laplacian and normalized Laplacian spectrum of the $Q$-vertex(edge) joins of a connected regular graph with an arbitrary regular graph in terms of the corresponding eigenvalues. Butler [4] constructed non-regular bipartite graphs which are cospectral with respect to both the adjacency and normalized Laplacian matrices, which means they constructed simultaneous cospectral graphs for the adjacency and normalized Laplacian matrices, and then asked for existence of non-regular graphs which are cospectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian.

Motivated by these researches, we give two types of graph operations based on the splitting graph [22]. The *splitting graph* $SP(G)$ of a graph $G$ is the graph obtained from $G$ by taking a new vertex $u'$ for each $u \in V(G)$ and joining $u'$ to all vertices of $G$ adjacent to $u$. We call it the $SP$-graph of $G$. The set of such new vertices is denoted by $S(G)$, i.e., $S(G) = V(SP(G)) \setminus V(G)$. Now we define splitting $V$-vertex join and splitting $S$-vertex join of graphs which are given below.

**Definition 1.1.** Let $G_1$ and $G_2$ be two vertex-disjoint graphs with number of vertices $n_1$ and $n_2$, respectively. Then

(i) The splitting $V$-vertex join of $G_1$ and $G_2$, denoted by $G_1 \veebar G_2$, is the graph obtained from $SP(G_1)$ and $G_2$ by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$.

(ii) The splitting $S$-vertex join of $G_1$ and $G_2$, denoted by $G_1 \barwedge G_2$, is the graph obtained from $SP(G_1)$ and $G_2$ by joining each vertex of $S(G_1)$ with every vertex of $V(G_2)$.

Note that if $G_i$ has $n_i$ vertices and $m_i$ edges for $i = 1, 2$, then $G_1 \veebar G_2$ has $2n_1 + n_2$ vertices and $3m_1 + n_1 n_2 + m_2$ edges, $G_1 \barwedge G_2$ has $2n_1 + n_2$ vertices and $3m_1 + n_1 n_2 + m_2$ edges.

**Example 1.1.** Let $G_1$ and $G_2$ be the path $P_4$ and the complete graph $K_2$, respectively. Two graphs $P_4 \veebar K_2$ and $P_4 \barwedge K_2$ are given in Figure 1.

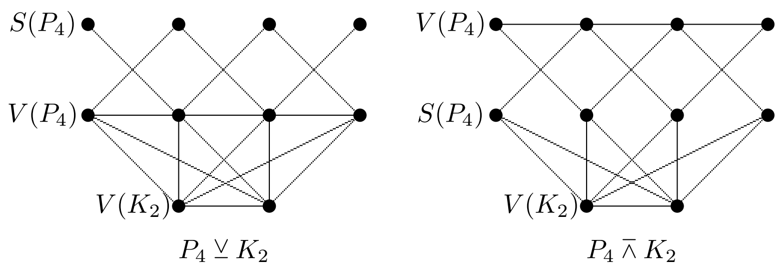


**Figure 1.** The splitting V-vertex join of $P_4 \veebar K_2$ and the splitting $S$-vertex join of $P_4 \barwedge K_2$.

The main goal of this paper is the determination of the adjacency, Laplacian and signless Laplacian spectra of splitting $V$-vertex join and splitting $S$-vertex join for a connected regular graph $G_1$ and an arbitrary regular graph $G_2$ in terms of the corresponding eigenvalues of $G_1$ and $G_2$. Moreover, applying these results we construct some non-regular simultaneous cospectral graphs for the adjacency, Laplacian and signless Laplacian matrices, and compute the number of spanning trees and the Kirchhoff index of the newly constructed graphs.

## 2. Preliminaries

In this section we provide some useful results which have an important role in the proof of the main results. Throughout the paper for any integers $k$, $n_1$ and $n_2$ , $I_k$ denotes the identity matrix of size $k$, $\mathbf{1}_k$ denotes the column vector of size $k$ whose all entries are 1 and $J_{n_1 \times n_2}$ denotes $n_1 \times n_2$ matrix whose all entries are 1.

**Lemma 2.1** (Schur Complements, [14]). *Let $M_1$, $M_2$, $M_3$, $M_4$ be respectively $p \times p$, $p \times q$, $q \times p$, $q \times q$ matrices with $M_1$ and $M_4$ invertible. Then*

$$\det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_1) \cdot \det(M_4 - M_3 M_1^{-1} M_2)$$
$$= \det(M_4) \cdot \det(M_1 - M_2 M_4^{-1} M_3),$$

*where $M_1 - M_2 M_4^{-1} M_3$ and $M_4 - M_3 M_1^{-1} M_2$ are called the Schur complements of $M_4$ and $M_1$, respectively.*

**Lemma 2.2** ( [8]). *Let $A$ be an $n \times n$ real matrix, and $J_{n\times n}$ denote the $n \times n$ matrix with all entries equal to one. Then*

$$\det(A + \alpha J_{n\times n}) = \det(A) + \alpha \mathbf{1}_n^T adj(A)\mathbf{1}_n,$$

*where $\alpha$ is an real number and $adj(A)$ is the adjugate matrix of $A$.*

**Lemma 2.3** (Lemma 3, [10]). *For any real numbers $c, d > 0$, we have*

$$(cI_n - dJ_{n\times n})^{-1} = \frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}.$$

One famous resistance distance-based parameter call the *Kirchhoff index* of G is defined as:

$$Kf(G) = \sum_{i<j} r_{ij},$$

where $r_{ij}$ denotes the resistance distance between two arbitrary vertices $v_i$ and $v_j$ in electrical networks with replacing every edge by a unit resistor.

Gutman and Mohar [12] and Zhu et al. [23] derived independently the Kirchhoff index of a graph in terms of Laplacian spectrum as follows:

**Lemma 2.4.** *Let $G$ be an $n$-vertex connected graph. Then*

$$Kf(G) = n\sum_{i=2}^{n} \frac{1}{\mu_i(G)}, \tag{2.1}$$

*where $0 = \mu_1(G) < \mu_2(G) \leq \cdots \leq \mu_n(G)$ are the eigenvalues of $L(G)$.*

**Lemma 2.5** ( [7]). *Suppose $G$ be a connected graph with $n$ vertices, $\{0 = \mu_1(G) < \mu_2(G) \leq \cdots \leq \mu_n(G)\}$ is the spectrum on the Laplacian matrix of $G$. Then the number of spanning trees of $G$ is*

$$\tau(G) = \frac{\prod_{i=2}^{n} \mu_i(G)}{n}.$$

The $M$-coronal $\Gamma_M(x)$ of an $n \times n$ matrix $M$ is defined [6, 20] to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is,

$$\Gamma_M(x) = \mathbf{1}_n^T(xI_n - M)^{-1}\mathbf{1}_n. \tag{2.2}$$

**Lemma 2.6** (Proposition 2, [6]). *If $M$ is an $n \times n$ matrix with each row sum equal to a constant $t$, then $\Gamma_M(x) = \frac{n}{x-t}$.*

# 3. Spectra of splitting V-vertex join

In this section, we will first concentrate on the $A$-spectra, $L$-spectra, $Q$-spectra of splitting $V$-vertex join for two regular graphs and then consider some applications of related results.

Let $G_i$ be an $r_i$-regular graph on $n_i$ vertices for $i = 1, 2$. By Definition1.1, the vertices of $G_1 \veebar G_2$ are partitioned by $V(G_1) \cup S(G_1) \cup V(G_2)$, where $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $S(G_1) = \{v_1', v_2', \ldots, v_{n_1}'\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$.

Obviously, the degrees of the vertices of $G_1 \veebar G_2$ are:

$$\begin{aligned}
&d_{G_1 \veebar G_2}(v_i) = 2r_1 + n_2, && i = 1, \ldots, n_1,\\
&d_{G_1 \veebar G_2}(v_i') = r_1, && i = 1, \ldots, n_1,\\
&d_{G_1 \veebar G_2}(u_j) = r_2 + n_1, && j = 1, \ldots, n_2.
\end{aligned}$$

## 3.1. A-spectra of splitting V-vertex join

The adjacency matrix and the degree diagonal matrix of $G_1 \veebar G_2$ can be expressed in the form of block-matrix according to the ordering of $V(G_1)$, $S(G_1)$ and $V(G_2)$ as follows:

$$A(G_1 \veebar G_2) = \begin{pmatrix} A(G_1) & A(G_1) & J_{n_1\times n_2} \\ A(G_1) & O_{n_1\times n_1} & O_{n_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times n_1} & A(G_2) \end{pmatrix}, \tag{3.1}$$

$$D(G_1 \veebar G_2) = \begin{pmatrix} (2r_1+n_2)I_{n_1} & O_{n_1\times n_1} & O_{n_1\times n_2} \\ O_{n_1\times n_1} & r_1 I_{n_1} & O_{n_1\times n_2} \\ O_{n_2\times n_1} & O_{n_2\times n_1} & (r_2+n_1)I_{n_2} \end{pmatrix}. \tag{3.2}$$

**Theorem 3.1.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices. Then the adjacency spectrum of $G_1 \veebar G_2$ consists of:*

*(i)* $\lambda_j(G_2)$ *for each* $j = 2, 3, \ldots, n_2$*;*

*(ii) two roots of the equation* $x^2 - \lambda_i(G_1)x - \lambda_i^2(G_1) = 0$ *for each* $i = 2, 3, \ldots, n_1$*;*

*(iii) three roots of the equation* $x^3 - (r_1 + r_2)x^2 - (r_1^2 + n_1 n_2 - r_1 r_2)x + r_1^2 r_2 = 0$.

**Proof.** According to (3.1), the adjacency characteristic polynomial of $G_1 \veebar G_2$ is

$$\begin{aligned}
f_{A(G_1 \veebar G_2)}(x) &= \det\big(xI_{2n_1+n_2} - A(G_1 \veebar G_2)\big)\\
&= \det \left(\begin{array}{cc|c} xI_{n_1} - A(G_1) & -A(G_1) & -J_{n_1\times n_2} \\ -A(G_1) & xI_{n_1} & O_{n_1\times n_2} \\ \hline -J_{n_2\times n_1} & O_{n_2\times n_1} & xI_{n_2} - A(G_2) \end{array}\right)\\
&= \det\big(xI_{n_2} - A(G_2)\big)\det(S) = \prod_{j=1}^{n_2}\big(x - \lambda_j(G_2)\big)\det(S),
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
S = &\begin{pmatrix} xI_{n_1} - A(G_1) & -A(G_1) \\ -A(G_1) & xI_{n_1} \end{pmatrix}\\
&- \begin{pmatrix} -J_{n_1\times n_2} \\ O_{n_1\times n_2} \end{pmatrix} \big(xI_{n_2} - A(G_2)\big)^{-1} \begin{pmatrix} -J_{n_2\times n_1} & O_{n_2\times n_1} \end{pmatrix}
\end{aligned}$$

is the Schur complement of $xI_{n_2} - A(G_2)$ obtained by applying Lemma 2.1. By using (2.2), we have

$$S = \begin{pmatrix} xI_{n_1} - A(G_1) & -A(G_1) \\ -A(G_1) & xI_{n_1} \end{pmatrix} - \begin{pmatrix} -I_{n_1\times n_2} \\ O_{n_1\times n_2} \end{pmatrix} \Gamma_{A(G_2)}(x) J_{n_2\times n_2} \begin{pmatrix} -I_{n_2\times n_1} & O_{n_2\times n_1} \end{pmatrix}$$
$$= \begin{pmatrix} xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x) J_{n_1\times n_1} & -A(G_1) \\ -A(G_1) & xI_{n_1} \end{pmatrix}.$$

Thus, based on Lemma 2.1 again, we have

$$\begin{aligned}
\det(S) &= \det(xI_{n_1}) \det\big(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x) J_{n_1\times n_1} - \frac{1}{x}A^2(G_1)\big) \\
&= x^{n_1}\Big(\det\big(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(G_1)\big) - \Gamma_{A(G_2)}(x)\mathbf{1}_{n_1}^T adj\big(xI_{n_1} - A(G_1) \\
&\quad - \frac{1}{x}A^2(G_1)\big)\mathbf{1}_{n_1}\Big) \\
&= x^{n_1}\det\big(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(G_1)\big) \times \Big[1 - \Gamma_{A(G_2)}(x)\mathbf{1}_{n_1}^T\big(xI_{n_1} - A(G_1) \\
&\quad - \frac{1}{x}A^2(G_1)\big)^{-1}\mathbf{1}_{n_1}\Big] \\
&= x^{n_1}\det\big(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(G_1)\big) \times \Big[1 - \Gamma_{A(G_2)}(x)\Gamma_{A(G_1)+\frac{1}{x}A^2(G_1)}(x)\Big] \\
&= x^{n_1}\det\big(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(G_1)\big) \times \Big[1 - \frac{n_1 n_2}{(x-r_2)(x-r_1-\frac{r_1^2}{x})}\Big] \\
&= x^{n_1}\prod_{i=1}^{n_1}\Big(x - \lambda_i(G_1) - \frac{1}{x}\lambda_i^2(G_1)\Big) \times \Big[1 - \frac{n_1 n_2}{(x-r_2)(x-r_1-\frac{r_1^2}{x})}\Big]. \qquad (3.4)
\end{aligned}$$

Combining with (3.3) and (3.4), this gives

$$\begin{aligned}
f_{A(G_1 \veebar G_2)}(x) =& x^{n_1}\prod_{j=1}^{n_2}\Big(x - \lambda_j(G_2)\Big)\prod_{i=1}^{n_1}\Big(x - \lambda_i(G_1) - \frac{1}{x}\lambda_i^2(G_1)\Big) \\
&\times \Big[1 - \frac{n_1 n_2}{(x-r_2)(x-r_1-\frac{r_1^2}{x})}\Big] \\
=& \prod_{j=2}^{n_2}\Big(x - \lambda_j(G_2)\Big)\prod_{i=2}^{n_1}\big(x^2 - \lambda_i(G_1)x - \lambda_i^2(G_1)\big) \\
&\times \big(x^3 - (r_1+r_2)x^2 - (r_1^2 + n_1 n_2 - r_1 r_2)x + r_1^2 r_2\big).
\end{aligned}$$

The desired results holds. □

## 3.2. L-spectra of splitting V-vertex join

In this subsection, the spectrum of Laplacian matrix of $G_1 \veebar G_2$ is detrmined in the following theorem. Then explicit closed formula of Kirchhoff index and the number of spanning trees of $G_1 \veebar G_2$ are derived in terms of the Laplacian spectrum.

**Theorem 3.2.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices. Then the Laplacian spectrum of $G_1 \veebar G_2$ consists of:*

*(i)* $n_1 + \mu_j(G_2)$ *for each* $j = 2, 3, \ldots, n_2$;

*(ii)* *two roots of the equation* $x^2-(2r_1+n_2+\mu_i(G_1))x+n_2r_1-\mu_i^2(G_1)+3r_1\mu_i(G_1) = 0$ *for each* $i = 2, 3, \ldots, n_1$;

*(iii)* *three roots of the equation* $x^3 - (2r_1 + n_1 + n_2)x^2 + (2r_1n_1 + r_1n_2)x = 0$.

**Proof.** Combining with (3.1) and (3.2), we get the Laplacian matrix of $G_1 \veebar G_2$ as follows:

$$\begin{aligned}
L(G_1 \veebar G_2) &= D(G_1 \veebar G_2) - A(G_1 \veebar G_2)\\
&= \begin{pmatrix} (2r_1 + n_2)I_{n_1} - A(G_1) & -A(G_1) & -J_{n_1\times n_2}\\ -A(G_1) & r_1 I_{n_1} & O_{n_1\times n_2}\\ -J_{n_2\times n_1} & O_{n_2\times n_1} & (r_2 + n_1)I_{n_2} - A(G_2) \end{pmatrix}\\
&= \begin{pmatrix} (r_1 + n_2)I_{n_1} + L(G_1) & -A(G_1) & -J_{n_1\times n_2}\\ -A(G_1) & r_1 I_{n_1} & O_{n_1\times n_2}\\ -J_{n_2\times n_1} & O_{n_2\times n_1} & n_1 I_{n_2} + L(G_2) \end{pmatrix}.
\end{aligned}$$

This leads the Laplacian characteristic polynomial of $G_1 \veebar G_2$ below

$$\begin{aligned}
f_{L(G_1\veebar G_2)}(x) &= \det\big(xI_{2n_1+n_2} - L(G_1 \veebar G_2)\big)\\
&= \left(\begin{array}{cc:c} (x - r_1 - n_2)I_{n_1} - L(G_1) & A(G_1) & J_{n_1\times n_2}\\ A(G_1) & (x - r_1)I_{n_1} & O_{n_1\times n_2}\\ \hdashline J_{n_2\times n_1} & O_{n_2\times n_1} & (x - n_1)I_{n_2} - L(G_2) \end{array}\right)\\
&= \det\big((x - n_1)I_{n_2} - L(G_2)\big)\det(S) = \prod_{j=1}^{n_2}\big(x - n_1 - \mu_j(G_2)\big)\det(S),
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
S &= \begin{pmatrix} (x - r_1 - n_2)I_{n_1} - L(G_1) & A(G_1)\\ A(G_1) & (x - r_1)I_{n_1} \end{pmatrix}\\
&\quad - \begin{pmatrix} J_{n_1\times n_2}\\ O_{n_1\times n_2} \end{pmatrix}\big((x - n_1)I_{n_2} - L(G_2)\big)^{-1}\begin{pmatrix} J_{n_2\times n_1} & O_{n_2\times n_1} \end{pmatrix}\\
&= \begin{pmatrix} (x - r_1 - n_2)I_{n_1} - L(G_1) & A(G_1)\\ A(G_1) & (x - r_1)I_{n_1} \end{pmatrix}\\
&\quad - \begin{pmatrix} I_{n_1\times n_2}\\ O_{n_1\times n_2} \end{pmatrix}\mathbf{1}_{n_2}\big(\mathbf{1}_{n_2}^T\big((x - n_1)I_{n_2} - L(G_2)\big)^{-1}\mathbf{1}_{n_2}\big)\mathbf{1}_{n_2}^T\begin{pmatrix} I_{n_2\times n_1} & O_{n_2\times n_1} \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} (x-r_1-n_2)I_{n_1}-L(G_1) & A(G_1) \\ A(G_1) & (x-r_1)I_{n_1} \end{pmatrix} \\
&\quad - \begin{pmatrix} I_{n_1\times n_2} \\ O_{n_1\times n_2} \end{pmatrix} \Gamma_{L(G_2)}(x-n_1)J_{n_2\times n_2} \begin{pmatrix} I_{n_2\times n_1} & O_{n_2\times n_1} \end{pmatrix} \\
&= \begin{pmatrix} (x-r_1-n_2)I_{n_1}-L(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1} & A(G_1) \\ A(G_1) & (x-r_1)I_{n_1} \end{pmatrix}
\end{aligned}
$$

is the Schur complement of $(x-n_1)I_{n_2}-L(G_2)$. And in a similar way, we get

$$
\begin{aligned}
\det(S) =& \det\big((x-r_1)I_{n_1}\big)\det\big((x-r_1-n_2)I_{n_1}-L(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1} \\
&-\frac{1}{x-r_1}A^2(G_1)\big) \\
=&(x-r_1)^{n_1}\Big(\det\big((x-r_1-n_2)I_{n_1}-L(G_1)-\frac{1}{x-r_1}A^2(G_1)\big) \\
&-\Gamma_{L(G_2)}(x-n_1)\mathbf{1}_{n_1}^T adj\big((x-r_1-n_2)I_{n_1}-L(G_1)-\frac{1}{x-r_1}A^2(G_1)\big)\mathbf{1}_{n_1}\Big) \\
=&(x-r_1)^{n_1}\det\big((x-r_1-n_2)I_{n_1}-L(G_1)-\frac{1}{x-r_1}A^2(G_1)\big) \\
&\times\Big[1-\Gamma_{L(G_2)}(x-n_1)\mathbf{1}_{n_1}^T\big((x-r_1-n_2)I_{n_1}-L(G_1)-\frac{1}{x-r_1}A^2(G_1)\big)^{-1}\mathbf{1}_{n_1}\Big] \\
=&(x-r_1)^{n_1}\det\big((x-r_1-n_2)I_{n_1}-L(G_1)-\frac{1}{x-r_1}A^2(G_1)\big) \\
&\times\Big[1-\Gamma_{L(G_2)}(x-n_1)\Gamma_{L(G_1)+\frac{1}{x-r_1}A^2(G_1)}(x-r_1-n_2)\Big] \\
=&(x-r_1)^{n_1}\det\big((x-r_1-n_2)I_{n_1}-L(G_1)-\frac{1}{x-r_1}A^2(G_1)\big) \\
&\times\Big[1-\frac{n_1n_2}{(x-n_1)(x-r_1-n_2-\frac{r_1^2}{x-r_1})}\Big] \\
=&(x-r_1)^{n_1}\prod_{i=1}^{n_1}\Big(x-r_1-n_2-\mu_i(G_1)-\frac{1}{x-r_1}\big(r_1-\mu_i(G_1)\big)^2\Big) \\
&\times\Big[1-\frac{n_1n_2}{(x-n_1)(x-r_1-n_2-\frac{r_1^2}{x-r_1})}\Big].
\end{aligned}
\tag{3.6}
$$

Substituting (3.6) to (3.5), we obtain

$$
\begin{aligned}
f_{L(G_1\veebar G_2)}(x) =&(x-r_1)^{n_1}\times\Big[1-\frac{n_1n_2}{(x-n_1)(x-r_1-n_2-\frac{r_1^2}{x-r_1})}\Big] \\
&\prod_{j=1}^{n_2}\big(x-n_1-\mu_j(G_2)\big)\prod_{i=1}^{n_1}\Big(x-r_1-n_2-\mu_i(G_1)-\frac{1}{x-r_1}\big(r_1-\mu_i(G_1)\big)^2\Big) \\
=&\prod_{j=2}^{n_2}\big(x-n_1-\mu_j(G_2)\big)\prod_{i=2}^{n_1}\Big(x^2-(2r_1+n_2+\mu_i(G_1))x+n_2r_1-\mu_i^2(G_1) \\
&+3r_1\mu_i(G_1)\Big)\times\big(x^3-(2r_1+n_1+n_2)x^2+(2r_1n_1+r_1n_2)x\big).
\end{aligned}
$$

Evidently, the result yields. $\square$

In the subsequent section, we will compute the Kirchhoff index and the number of spanning trees of the splitting $V$-vertex join for a connected regular graph $G_1$ and an arbitrary regular graph $G_2$ as applications of Theorem 3.2.

**Corollary 3.1.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices. Then*

$$\begin{aligned}Kf(G_1 \veebar G_2) =&(2n_1 + n_2)\Big(\frac{2r_1 + n_1 + n_2}{2r_1n_1 + r_1n_2} + \sum_{i=2}^{n_1} \frac{2r_1 + n_2 + \mu_i(G_1)}{n_2r_1 + 3r_1\mu_i(G_1) - \mu_i^2(G_1)}\\ &+ \sum_{j=2}^{n_2} \frac{1}{n_1 + \mu_j(G_2)}\Big).\end{aligned}$$

**Proof.** According to (2.1), we know $Kf(G) = n\sum_{i=2}^{n} \frac{1}{\mu_i(G)}$, where $\mu_i$ is the Laplacian eigenvalues of $G$. So the Kirchhoff index $Kf(G_1 \veebar G_2)$ can be computed below. First, applying Theorem 3.2 $(i)$, we obtain

$$\mu_i(G_1 \veebar G_2) = n_1 + \mu_j(G_2), \tag{3.7}$$

where $j = 2, 3, \ldots, n_2$. So

$$\frac{1}{\mu_i(G_1 \veebar G_2)} = \frac{1}{n_1 + \mu_j(G_2)}.$$

Next, using Theorem 3.2 $(ii)$, we have the Laplacian eigenvalues of $G_1 \veebar G_2$ consists of two roots $\beta_1$ , $\beta_2$ of the equation

$$x^2 - (2r_1 + n_2 + \mu_i(G_1))x + n_2r_1 - \mu_i^2(G_1) + 3r_1\mu_i(G_1) = 0 \tag{3.8}$$

for each eigenvalue $\mu_i(G_1)$, where $i = 2, 3, \ldots, n_1$. Hence, by Vieta Theorem, we have

$$\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{\beta_1 + \beta_2}{\beta_1\beta_2} = \frac{2r_1 + n_2 + \mu_i(G_1)}{n_2r_1 + 3r_1\mu_i(G_1) - \mu_i^2(G_1)}, \quad \text{for } i = 2, 3, \ldots, n_1.$$

And then, by Theorem 3.2 $(iii)$, we have $\mu_1(G_1 \veebar G_2) = 0$. Then the other two roots of the equation

$$x^3 - (2r_1 + n_1 + n_2)x^2 + (2r_1n_1 + r_1n_2)x = 0 \tag{3.9}$$

expressed as $x_1$,$x_2$. Due to Vieta Theorem, we get

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1 + x_2}{x_1x_2} = \frac{2r_1 + n_1 + n_2}{2n_1r_1 + r_1n_2}.$$

Note that $|V(G_1 \veebar G_2)| = 2n_1 + n_2$. Therefore, the Kirchhoff index of $G_1 \veebar G_2$ is related as

$$\begin{aligned}Kf(G_1 \veebar G_2) =&(2n_1 + n_2)\Big(\frac{2r_1 + n_1 + n_2}{2r_1n_1 + r_1n_2} + \sum_{i=2}^{n_1} \frac{2r_1 + n_2 + \mu_i(G_1)}{n_2r_1 + 3r_1\mu_i(G_1) - \mu_i^2(G_1)}\\ &+ \sum_{j=2}^{n_2} \frac{1}{n_1 + \mu_j(G_2)}\Big).\end{aligned}$$

$\square$

**Corollary 3.2.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices. Then*

$$\tau(G_1 \veebar G_2) = r_1 \prod_{i=2}^{n_1} \big(n_2 r_1 + 3r_1\mu_i(G_1) - \mu_i^2(G_1)\big) \cdot \prod_{j=2}^{n_2} \big(n_1 + \mu_j(G_2)\big).$$

**Proof.** Using Lemma 2.5, we know

$$\tau(G) = \frac{\prod_{i=2}^{n} \mu_i(G)}{n}. \tag{3.10}$$

In order to get the result, we consider the Laplacian eigenvalues of $G_1 \veebar G_2$ in the following way:

From (3.8), we have

$$\beta_1\beta_2 = n_2 r_1 + 3r_1\mu_i(G_1) - \mu_i^2(G_1), \tag{3.11}$$

where $i = 2, 3, \ldots, n_1$.

From (3.9), we have

$$x_1 x_2 = 2n_1 r_1 + r_1 n_2. \tag{3.12}$$

Substituting (3.7), (3.11) and (3.12) to (3.10), we obtain

$$\begin{aligned}
\tau(G_1 \veebar G_2) =& \frac{1}{2n_1 + n_2}\Big((2r_1 n_1 + r_1 n_2) \prod_{i=2}^{n_1} \big(n_2 r_1 + 3r_1\mu_i(G_1) - \mu_i^2(G_1)\big) \\
& \prod_{j=2}^{n_2} \big(n_1 + \mu_j(G_2)\big)\Big) \\
=& r_1 \prod_{i=2}^{n_1} \big(n_2 r_1 + 3r_1\mu_i(G_1) - \mu_i^2(G_1)\big) \cdot \prod_{j=2}^{n_2} \big(n_1 + \mu_j(G_2)\big).
\end{aligned}$$

□

## 3.3. Q-spectra of splitting V-vertex join

At this place, we now focus on determining the signless Laplacian spectra of $G_1 \veebar G_2$ in terms of the signless Laplacian spectra of $G_1$ and $G_2$.

**Theorem 3.3.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices. Then the signless Laplacian spectrum of $G_1 \veebar G_2$ consists of:*

*(i)* $n_1 + \nu_j(G_2)$ *for each* $j = 2, 3, \ldots, n_2$;

*(ii)* *two roots of the equation* $x^2 - (2r_1 + n_2 + \nu_i(G_1))x + n_2 r_1 - \nu_i^2(G_1) + 3r_1\nu_i(G_1) = 0$ *for each* $i = 2, 3, \ldots, n_1$;

*(iii)* *three roots of the equation* $x^3 - (4r_1 + n_1 + n_2 + 2r_2)x^2 + (2r_1^2 + 8r_1 r_2 + 4r_1 n_1 + 2r_2 n_2 + r_1 n_2)x - -2r_1^2 n_1 - 4r_1^2 r_2 - 2r_1 r_2 n_2 = 0$.

**Proof.** Combining with (3.1) and (3.2), the signless Laplacian matrix of $G_1 \veebar G_2$ can be expressed as:

$$Q(G_1 \veebar G_2) = D(G_1 \veebar G_2) + A(G_1 \veebar G_2)$$

$$= \begin{pmatrix} (2r_1+n_2)I_{n_1}+A(G_1) & A(G_1) & J_{n_1\times n_2} \\ A(G_1) & r_1 I_{n_1} & O_{n_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times n_1} & (r_2+n_1)I_{n_2}+A(G_2) \end{pmatrix}$$

$$= \begin{pmatrix} (r_1+n_2)I_{n_1}+Q(G_1) & A(G_1) & J_{n_1\times n_2} \\ A(G_1) & r_1 I_{n_1} & O_{n_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times n_1} & n_1 I_{n_2}+Q(G_2) \end{pmatrix}.$$

Thus, the signless Laplacian characteristic polynomial of $G_1 \veebar G_2$ is

$$\begin{aligned} f_{Q(G_1\veebar G_2)}(x) &= \det\big(xI_{2n_1+n_2} - Q(G_1\veebar G_2)\big) \\ &= \left(\begin{array}{cc:c} (x-r_1-n_2)I_{n_1}-Q(G_1) & -A(G_1) & -J_{n_1\times n_2} \\ -A(G_1) & (x-r_1)I_{n_1} & O_{n_1\times n_2} \\ \hdashline -J_{n_2\times n_1} & O_{n_2\times n_1} & (x-n_1)I_{n_2}-Q(G_2) \end{array}\right) \\ &= \det\big((x-n_1)I_{n_2}-Q(G_2)\big)\det(S) = \prod_{j=1}^{n_2}\big(x-n_1-\nu_j(G_2)\big)\det(S), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} S &= \begin{pmatrix} (x-r_1-n_2)I_{n_1}-Q(G_1) & -A(G_1) \\ -A(G_1) & (x-r_1)I_{n_1} \end{pmatrix} \\ &\quad - \begin{pmatrix} -J_{n_1\times n_2} \\ O_{n_1\times n_2} \end{pmatrix} \big((x-n_1)I_{n_2}-Q(G_2)\big)^{-1} \begin{pmatrix} -J_{n_2\times n_1} & O_{n_2\times n_1} \end{pmatrix} \\ &= \begin{pmatrix} (x-r_1-n_2)I_{n_1}-Q(G_1) & -A(G_1) \\ -A(G_1) & (x-r_1)I_{n_1} \end{pmatrix} \\ &\quad - \begin{pmatrix} -I_{n_1\times n_2} \\ O_{n_1\times n_2} \end{pmatrix} \mathbf{1}_{n_2}\big(\mathbf{1}_{n_2}^T\big((x-n_1)I_{n_2}-Q(G_2)\big)^{-1}\mathbf{1}_{n_2}\big)\mathbf{1}_{n_2}^T \begin{pmatrix} -I_{n_2\times n_1} & O_{n_2\times n_1} \end{pmatrix} \\ &= \begin{pmatrix} (x-r_1-n_2)I_{n_1}-Q(G_1) & -A(G_1) \\ -A(G_1) & (x-r_1)I_{n_1} \end{pmatrix} \\ &\quad - \begin{pmatrix} -I_{n_1\times n_2} \\ O_{n_1\times n_2} \end{pmatrix} \Gamma_{Q(G_2)}(x-n_1)J_{n_2\times n_2} \begin{pmatrix} -I_{n_2\times n_1} & O_{n_2\times n_1} \end{pmatrix} \\ &= \begin{pmatrix} (x-r_1-n_2)I_{n_1}-Q(G_1)-\Gamma_{Q(G_2)}(x-n_1)J_{n_1\times n_1} & -A(G_1) \\ -A(G_1) & (x-r_1)I_{n_1} \end{pmatrix}, \end{aligned}$$

is the Schur complement of $(x-n_1)I_{n_2}-Q(G_2)$ obtained by applying Lemma 2.1. Similarly, we have

$$\begin{aligned}
\det(S) =&\det\big((x-r_1)I_{n_1}\big)\det\big((x-r_1-n_2)I_{n_1}-Q(G_1)-\Gamma_{Q(G_2)}(x-n_1)J_{n_1\times n_1}\\
&-\frac{1}{x-r_1}A^2(G_1)\big)\\
=&(x-r_1)^{n_1}\Big(\det\big((x-r_1-n_2)I_{n_1}-Q(G_1)-\frac{1}{x-r_1}A^2(G_1)\big)\\
&-\Gamma_{Q(G_2)}(x-n_1)\mathbf{1}_{n_1}^T adj\big((x-r_1-n_2)I_{n_1}-Q(G_1)-\frac{1}{x-r_1}A^2(G_1)\big)\mathbf{1}_{n_1}\Big)\\
=&(x-r_1)^{n_1}\det\big((x-r_1-n_2)I_{n_1}-Q(G_1)-\frac{1}{x-r_1}A^2(G_1)\big)\\
&\times\Big[1-\Gamma_{Q(G_2)}(x-n_1)\mathbf{1}_{n_1}^T\big((x-r_1-n_2)I_{n_1}-Q(G_1)-\frac{1}{x-r_1}A^2(G_1)\big)^{-1}\mathbf{1}_{n_1}\Big]\\
=&(x-r_1)^{n_1}\det\big((x-r_1-n_2)I_{n_1}-Q(G_1)-\frac{1}{x-r_1}A^2(G_1)\big)\\
&\times\Big[1-\Gamma_{Q(G_2)}(x-n_1)\Gamma_{Q(G_1)+\frac{1}{x-r_1}A^2(G_1)}(x-r_1-n_2)\Big]\\
=&(x-r_1)^{n_1}\det\big((x-r_1-n_2)I_{n_1}-Q(G_1)-\frac{1}{x-r_1}A^2(G_1)\big)\\
&\times\Big[1-\frac{n_1n_2}{(x-n_1-2r_2)(x-3r_1-n_2-\frac{r_1^2}{x-r_1})}\Big]\\
=&(x-r_1)^{n_1}\prod_{i=1}^{n_1}\Big(x-r_1-n_2-\nu_i(G_1)-\frac{1}{x-r_1}\big(\nu_i(G_1)-r_1\big)^2\Big)\\
&\times\Big[1-\frac{n_1n_2}{(x-n_1-2r_2)(x-3r_1-n_2-\frac{r_1^2}{x-r_1})}\Big]. \qquad (3.14)
\end{aligned}$$

Substituting (3.14) to (3.13), we obtain

$$\begin{aligned}
&f_{Q(G_1\veebar G_2)}(x)\\
=&(x-r_1)^{n_1}\Big[1-\frac{n_1n_2}{(x-n_1-2r_2)(x-3r_1-n_2-\frac{r_1^2}{x-r_1})}\Big]\\
&\prod_{j=1}^{n_2}\Big(x-n_1-\nu_j(G_2)\Big)\prod_{i=1}^{n_1}\Big(x-r_1-n_2-\nu_i(G_1)-\frac{1}{x-r_1}\big(\nu_i(G_1)-r_1\big)^2\Big)\\
=&\prod_{j=2}^{n_2}\Big(x-n_1-\nu_j(G_2)\Big)\prod_{i=2}^{n_1}\Big(x^2-(2r_1+n_2+\nu_i(G_1))x+n_2r_1-\nu_i^2(G_1)\\
&+3r_1\nu_i(G_1)\Big)\times\big(x^3-(4r_1+n_1+n_2+2r_2)x^2+(2r_1^2+8r_1r_2+4r_1n_1\\
&+2r_2n_2+r_1n_2)x-2r_1^2n_1-4r_1^2r_2-2r_1r_2n_2\big).
\end{aligned}$$

Obviously, we get the result. □

# 4. Spectra of splitting S-vertex join

In this section, we will consider the $A$-spectra, $L$-spectra, $Q$-spectra and some applications of splitting $S$-vertex join for two regular graphs.

Let $G_i$ be an $r_i$-regular graph on $n_i$ vertices for $i = 1, 2$. Similarly, the vertices of $G_1 \barwedge G_2$ are partitioned by $V(G_1) \cup S(G_1) \cup V(G_2)$, where $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $S(G_1) = \{v'_1, v'_2, \ldots, v'_{n_1}\}$, $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$.

Evidently, the degrees of the vertices of $G_1 \barwedge G_2$ are:

$$\begin{aligned}
&d_{G_1 \barwedge G_2}(v_i) = 2r_1, && i = 1, \ldots, n_1,\\
&d_{G_1 \barwedge G_2}(v'_i) = r_1 + n_2, && i = 1, \ldots, n_1,\\
&d_{G_1 \barwedge G_2}(u_j) = r_2 + n_1, && j = 1, \ldots, n_2.
\end{aligned}$$

## 4.1. A-spectra of splitting S-vertex join

The adjacency matrix and the degree diagonal matrix of $G_1 \barwedge G_2$ can be represented in the form of block-matrix on the basis of the ordering of $V(G_1)$, $S(G_1)$ and $V(G_2)$ as below:

$$A(G_1 \barwedge G_2) = \begin{pmatrix} A(G_1) & A(G_1) & O_{n_1 \times n_2} \\ A(G_1) & O_{n_1 \times n_1} & J_{n_1 \times n_2} \\ O_{n_2 \times n_1} & J_{n_2 \times n_1} & A(G_2) \end{pmatrix}. \tag{4.1}$$

$$D(G_1 \barwedge G_2) = \begin{pmatrix} 2r_1 I_{n_1} & O_{n_1 \times n_1} & O_{n_1 \times n_2} \\ O_{n_1 \times n_1} & (r_1 + n_2) I_{n_1} & O_{n_1 \times n_2} \\ O_{n_2 \times n_1} & O_{n_2 \times n_1} & (r_2 + n_1) I_{n_2} \end{pmatrix}. \tag{4.2}$$

**Theorem 4.1.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices. Then the adjacency spectrum of $G_1 \barwedge G_2$ is composed of:*

*(i)* $\lambda_j(G_2)$ *for each* $j = 2, 3, \ldots, n_2$;

*(ii)* *two roots of the equation* $x^2 - \lambda_i(G_1)x - \lambda_i^2(G_1) = 0$ *for each* $i = 2, 3, \ldots, n_1$;

*(iii)* *three roots of the equation* $x^3 - (r_1 + r_2)x^2 - (r_1^2 + n_1 n_2 - r_1 r_2)x + r_1^2 r_2 + n_1 n_2 r_1 = 0$.

**Proof.** Based on (4.1), we know the adjacency characteristic polynomial of $G_1 \barwedge G_2$ is

$$\begin{aligned}
f_{A(G_1 \barwedge G_2)}(x) &= \det\left(xI_{2n_1+n_2} - A(G_1 \barwedge G_2)\right)\\
&= \left(\begin{array}{cc|c} xI_{n_1} - A(G_1) & -A(G_1) & O_{n_1 \times n_2} \\ -A(G_1) & xI_{n_1} & -J_{n_1 \times n_2} \\ \hline O_{n_2 \times n_1} & -J_{n_2 \times n_1} & xI_{n_2} - A(G_2) \end{array}\right)\\
&= \det\left(xI_{n_2} - A(G_2)\right)\det(S) = \prod_{j=1}^{n_2}\left(x - \lambda_j(G_2)\right)\det(S).
\end{aligned} \tag{4.3}$$

where

$$S = \begin{pmatrix} xI_{n_1} - A(G_1) & -A(G_1) \\ -A(G_1) & xI_{n_1} \end{pmatrix} - \begin{pmatrix} O_{n_1\times n_2} \\ -J_{n_1\times n_2} \end{pmatrix} \big(xI_{n_2} - A(G_2)\big)^{-1} \Big( O_{n_2\times n_1} \ -J_{n_2\times n_1} \Big)$$
$$= \begin{pmatrix} xI_{n_1} - A(G_1) & -A(G_1) \\ -A(G_1) & xI_{n_1} - \Gamma_{A(G_2)}(x) J_{n_1\times n_1} \end{pmatrix}.$$

is the Schur complement of $xI_{n_2} - A(G_2)$. Thus, using Lemma 2.1, we get

$$\begin{aligned}
\det(S) =& \det\big(xI_{n_1} - \Gamma_{A(G_2)}(x) J_{n_1\times n_1}\big) \det\Big(xI_{n_1} - A(G_1) - A(G_1)\big(xI_{n_1} \\
& - \Gamma_{A(G_2)}(x) J_{n_1\times n_1}\big)^{-1} A(G_1)\Big) \\
=& x^{n_1}\big(1 - \frac{\Gamma_{A(G_2)}(x) n_1}{x}\big) \det\Big(xI_{n_1} - A(G_1) - A(G_1)\big(\frac{1}{x} I(G_1) \\
& + \frac{\Gamma_{A(G_2)}(x)}{x\big(x - n_1\Gamma_{A(G_2)}(x)\big)} J_{n_1\times n_1}\big) A(G_1)\Big) \\
=& x^{n_1}\big(1 - \frac{\Gamma_{A(G_2)}(x) n_1}{x}\big) \det\big(xI_{n_1} - A(G_1) - \frac{1}{x} A^2(G_1) \\
& - \frac{\Gamma_{A(G_2)}(x)}{x\big(x - n_1\Gamma_{A(G_2)}(x)\big)} A(G_1) J_{n_1\times n_1} A(G_1)\big) \\
=& x^{n_1}\big(1 - \frac{\Gamma_{A(G_2)}(x) n_1}{x}\big) \det\Big(xI_{n_1} - A(G_1) - \frac{1}{x} A^2(G_1) \\
& - \frac{r_1^2 \Gamma_{A(G_2)}(x)}{x\big(x - n_1\Gamma_{A(G_2)}(x)\big)} J_{n_1\times n_1}\Big) \\
=& x^{n_1}\big(1 - \frac{\Gamma_{A(G_2)}(x) n_1}{x}\big)\Big( \det\big(xI_{n_1} - A(G_1) - \frac{1}{x} A^2(G_1)\big) \\
& - \frac{r_1^2 \Gamma_{A(G_2)}(x)}{x\big(x - n_1\Gamma_{A(G_2)}(x)\big)} \mathbf{1}_{n_1}^T adj\big(xI_{n_1} - A(G_1) - \frac{1}{x} A^2(G_1)\big)\mathbf{1}_{n_1}\Big) \\
=& x^{n_1}\big(1 - \frac{\Gamma_{A(G_2)}(x) n_1}{x}\big) \det\big(xI_{n_1} - A(G_1) - \frac{1}{x} A^2(G_1)\big) \\
& \times \Big[1 - \frac{r_1^2 \Gamma_{A(G_2)}(x)}{x\big(x - n_1\Gamma_{A(G_2)}(x)\big)} \mathbf{1}_{n_1}^T \big(xI_{n_1} - A(G_1) - \frac{1}{x} A^2(G_1)\big)^{-1}\mathbf{1}_{n_1}\Big] \\
=& x^{n_1}\big(1 - \frac{\Gamma_{A(G_2)}(x) n_1}{x}\big) \det\big(xI_{n_1} - A(G_1) - \frac{1}{x} A^2(G_1)\big) \\
& \times \Big[1 - \frac{r_1^2 \Gamma_{A(G_2)}(x) \Gamma_{A(G_1) + \frac{1}{x} A^2(G_1)}(x)}{x\big(x - n_1\Gamma_{A(G_2)}(x)\big)}\Big] \\
=& x^{n_1}\big(1 - \frac{n_1 n_2}{x(x - r_2)}\big) \prod_{i=1}^{n_1} \big(x - \lambda_i(G_1) - \frac{1}{x}\lambda_i^2(G_1)\big) \\
& \times \Big[1 - \frac{n_1 n_2 r_1^2}{x(x - r_2)(x - \frac{n_1 n_2}{x - r_2})(x - r_1 - \frac{r_1^2}{x})}\Big].
\end{aligned} \tag{4.4}$$

Substituting (4.4) to (4.3), we have

$$f_{A(G_1 \barwedge G_2)}(x) = x^{n_1}\big(1 - \frac{n_1 n_2}{x(x - r_2)}\big) \prod_{i=1}^{n_1} \big(x - \lambda_i(G_1) - \frac{1}{x}\lambda_i^2(G_1)\big) \prod_{j=1}^{n_2} \big(x - \lambda_j(G_2)\big)$$

$$\times\left[1-\frac{n_1n_2r_1^2}{x(x-r_2)(x-\frac{n_1n_2}{x-r_2})(x-r_1-\frac{r_1^2}{x})}\right]$$
$$=\prod_{j=2}^{n_2}\Big(x-\lambda_j(G_2)\Big)\prod_{i=2}^{n_1}\big(x^2-\lambda_i(G_1)x-\lambda_i^2(G_1)\big)$$
$$\times\big(x^3-(r_1+r_2)x^2-(r_1^2+n_1n_2-r_1r_2)x+r_1^2r_2+n_1n_2r_1\big).$$

□

## 4.2. L-spectra of splitting S-vertex join

Combining with (4.1) and (4.2), the Laplacian matrix of $G_1\,\overline{\barwedge}\,G_2$ can be written as:

$$\begin{aligned}L(G_1\,\overline{\barwedge}\,G_2)&=D(G_1\,\overline{\barwedge}\,G_2)-A(G_1\,\overline{\barwedge}\,G_2)\\&=\begin{pmatrix}2r_1I_{n_1}-A(G_1) & -A(G_1) & O_{n_1\times n_2}\\ -A(G_1) & (r_1+n_2)I_{n_1} & -J_{n_1\times n_2}\\ O_{n_2\times n_1} & -J_{n_2\times n_1} & (r_2+n_1)I_{n_2}-A(G_2)\end{pmatrix}\\&=\begin{pmatrix}r_1I_{n_1}+L(G_1) & -A(G_1) & O_{n_1\times n_2}\\ -A(G_1) & (r_1+n_2)I_{n_1} & -J_{n_1\times n_2}\\ O_{n_2\times n_1} & -J_{n_2\times n_1} & n_1I_{n_2}+L(G_2)\end{pmatrix}.\end{aligned}\tag{4.5}$$

As a summarize of the above analysis, we present the following theorem which gives the Laplacian spectrum of $G_1\,\overline{\barwedge}\,G_2$ in terms of the Laplacian spectra of $G_1$ and $G_2$.

**Theorem 4.2.** *For $i=1,2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices. Then the Laplacian spectrum of $G_1\,\overline{\barwedge}\,G_2$ consists of:*

*(i)* *$n_1+\mu_j(G_2)$ for each $j=2,3,\ldots,n_2$;*

*(ii)* *two roots of the equation $x^2-\big(2r_1+n_2+\mu_i(G_1)\big)x+n_2r_1-\mu_i^2(G_1)+3r_1\mu_i(G_1)+n_2\mu_i(G_1)=0$ for each $i=2,3,\ldots,n_1$;*

*(iii)* *three roots of the equation $x^3-(r_1+n_1+n_2+2)x^2+(2n_1+2n_2+r_1n_1)x=0$.*

**Proof.** By (4.5), the Laplacian characteristic polynomial of $G_1\,\overline{\barwedge}\,G_2$ is

$$\begin{aligned}f_{L(G_1\overline{\barwedge}G_2)}(x)&=\det\big(xI_{2n_1+n_2}-L(G_1\,\overline{\barwedge}\,G_2)\big)\\&=\left(\begin{array}{cc|c}(x-r_1)I_{n_1}-L(G_1) & A(G_1) & O_{n_1\times n_2}\\ A(G_1) & (x-r_1-n_2)I_{n_1} & J_{n_1\times n_2}\\ \hline O_{n_2\times n_1} & J_{n_2\times n_1} & (x-n_1)I_{n_2}-L(G_2)\end{array}\right)\\&=\det\big((x-n_1)I_{n_2}-L(G_2)\big)\det(S)=\prod_{j=1}^{n_2}\big(x-n_1-\mu_j(G_2)\big)\det(S),\end{aligned}$$

where

$$
\begin{aligned}
S =& \begin{pmatrix} (x-r_1)I_{n_1}-L(G_1) & A(G_1) \\ A(G_1) & (x-r_1-n_2)I_{n_1} \end{pmatrix} \\
&- \begin{pmatrix} O_{n_1\times n_2} \\ J_{n_1\times n_2} \end{pmatrix} \big((x-n_1)I_{n_2}-L(G_2)\big)^{-1} \begin{pmatrix} O_{n_2\times n_1} & J_{n_2\times n_1} \end{pmatrix} \\
=& \begin{pmatrix} (x-r_1)I_{n_1}-L(G_1) & A(G_1) \\ A(G_1) & (x-r_1-n_2)I_{n_1}-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1} \end{pmatrix}
\end{aligned}
$$

is the Schur complement of $(x-n_1)I_{n_2}-L(G_2)$. So, due to Lemma 2.1 again, we get

$$
\begin{aligned}
&\det(S) \\
=&\det\big((x-r_1-n_2)I_{n_1}-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1}\big)\det\Big((x-r_1)I_{n_1}-L(G_1) \\
&-A(G_1)\big((x-r_1-n_2)I_{n_1}-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1}\big)^{-1}A(G_1)\Big) \\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{L(G_2)}(x-n_1)}{x-r_1-n_2}\big)\det\Bigg((x-r_1)I_{n_1}-L(G_1)-A(G_1) \\
&\Big(\frac{1}{x-r_1-n_2}I_{n_1}+\frac{\Gamma_{L(G_2)}(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1)\big)}J_{n_1\times n_1}\Big)A(G_1)\Bigg) \\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{L(G_2)}(x-n_1)}{x-r_1-n_2}\big)\det\Big((x-r_1)I_{n_1}-L(G_1)-\frac{1}{x-r_1-n_2}A^2(G_1) \\
&-\frac{\Gamma_{L(G_2)}(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1)\big)}A(G_1)J_{n_1\times n_1}A(G_1)\Big) \\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{L(G_2)}(x-n_1)}{x-r_1-n_2}\big)\det\Big((x-r_1)I_{n_1}-L(G_1) \\
&-\frac{1}{x-r_1-n_2}A^2(G_1)-\frac{r_1^2\Gamma_{L(G_2)}(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1)\big)}J_{n_1\times n_1}\Big) \\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{L(G_2)}(x-n_1)}{x-r_1-n_2}\big)\Big(\det\big((x-r_1)I_{n_1}-L(G_1) \\
&-\frac{1}{x-r_1-n_2}A^2(G_1)\big)-\frac{r_1^2\Gamma_{L(G_2)}(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1)\big)} \\
&\mathbf{1}_{n_1}^T adj\big((x-r_1)I_{n_1}-L(G_1)-\frac{1}{x-r_1-n_2}A^2(G_1)\big)\mathbf{1}_{n_1}\Big) \\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{L(G_2)}(x-n_1)}{x-r_1-n_2}\big)\det\big((x-r_1)I_{n_1}-L(G_1)-\frac{1}{x-r_1-n_2}A^2(G_1)\big) \\
&\times\Big[1-\frac{r_1^2\Gamma_{L(G_2)}(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1)\big)} \\
&\mathbf{1}_{n_1}^T\big((x-r_1)I_{n_1}-L(G_1)-\frac{1}{x-r_1-n_2}A^2(G_1)\big)^{-1}\mathbf{1}_{n_1}\Big] \\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{L(G_2)}(x-n_1)}{x-r_1-n_2}\big)\det\Big((x-r_1)I_{n_1}-L(G_1)-\frac{1}{x-r_1-n_2}\big(r_1I_{n_1} \\
&-L(G_1)\big)^2\Big)\times\Big[1-\frac{r_1^2\Gamma_{L(G_2)}\Gamma_{L(G_1)+\frac{1}{x-r_1-n_2}A^2(G_1)}(x-r_1)(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1)\big)}\Big]
\end{aligned}
$$

$$
\begin{aligned}
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1n_2}{(x-n_1)(x-r_1-n_2)}\big)\\
&\times\prod_{i=1}^{n_1}\Big(x-r_1-\mu_i(G_1)-\frac{1}{x-r_1-n_2}\big(r_1-\mu_i(G_1)\big)^2\Big)\\
&\times\Bigg[1-\frac{n_1n_2r_1^2}{(x-n_1)(x-r_1-n_2)(x-r_1-n_2-\frac{n_1n_2}{x-n_1})(x-r_1-\frac{r_1^2}{x-r_1-n_2})}\Bigg].
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
&f_{L(G_1\bar{\wedge}G_2)}(x)\\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1n_2}{(x-n_1)(x-r_1-n_2)}\big)\prod_{j=1}^{n_2}\big(x-n_1-\mu_j(G_2)\big)\\
&\prod_{i=1}^{n_1}\Big(x-r_1-\mu_i(G_1)-\frac{1}{x-r_1-n_2}\big(r_1-\mu_i(G_1)\big)^2\Big)\\
&\times\Bigg[1-\frac{n_1n_2r_1^2}{(x-n_1)(x-r_1-n_2)(x-r_1-n_2-\frac{n_1n_2}{x-n_1})(x-r_1-\frac{r_1^2}{x-r_1-n_2})}\Bigg]\\
=&\prod_{i=2}^{n_1}\Big(x^2-\big(2r_1+n_2+\mu_i(G_1)\big)x+n_2r_1-\mu_i^2(G_1)+3r_1\mu_i(G_1)\\
&+n_2\mu_i(G_1)\Big)\times\big(x^3-(r_1+n_1+n_2+2)x^2+(2n_1+2n_2+r_1n_1)x\big)\\
&\prod_{j=2}^{n_2}\big(x-n_1-\mu_j(G_2)\big).
\end{aligned}
$$

□

According to Theorem 4.2, we will formulate the Kirchhoff index and the number of spanning trees of the splitting $S$-vertex join for a connected regular graph $G_1$ and an arbitrary regular graph $G_2$.

**Corollary 4.1.** *For $i=1,2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices. Then*

$$
\begin{aligned}
Kf(G_1\bar{\wedge}G_2)=&(2n_1+n_2)\Big(\frac{r_1+n_1+n_2+2}{2n_1+2n_2+r_1n_1}+\sum_{j=2}^{n_2}\frac{1}{n_1+\mu_j(G_2)}\\
&+\sum_{i=2}^{n_1}\frac{2r_1+n_2+\mu_i(G_1)}{n_2r_1-\mu_i^2(G_1)+3r_1\mu_i(G_1)+n_2\mu_i(G_1)}\Big).
\end{aligned}
$$

**Proof.** Using (2.1), we have $Kf(G)=n\sum_{i=2}^{n}\frac{1}{\mu_i(G)}$, where $\mu_i(G)$ is the Laplacian eigenvalues of $G$. Now the Kirchhoff index $Kf(G_1\bar{\wedge}G_2)$ can be computed as follows. According to Theorem 4.2 $(i)$, we obtain

$$
\mu_i(G_1\bar{\wedge}G_2)=n_1+\mu_j(G_2), \tag{4.6}
$$

where $j=2,3,\ldots,n_2$. Thus

$$
\frac{1}{\mu_i(G_1\bar{\wedge}G_2)}=\frac{1}{n_1+\mu_j(G_2)}.
$$

Next, by Theorem 4.2 (*ii*), we know two roots $\alpha_1$ , $\alpha_2$ of the equation

$$x^2 - \big(2r_1 + n_2 + \mu_i(G_1)\big)x + n_2r_1 - \mu_i^2(G_1) + 3r_1\mu_i(G_1) + n_2\mu_i(G_1) = 0 \quad (4.7)$$

are the Laplacian eigenvalues of $G_1 \veebar G_2$, for each eigenvalue $\mu_i(G_1)$, $i = 2, 3, \ldots, n_1$. Hence, applying Vieta Theorem, we have

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2} = \frac{2r_1 + n_2 + \mu_i(G_1)}{n_2r_1 - \mu_i^2(G_1) + 3r_1\mu_i(G_1) + n_2\mu_i(G_1)}, \quad \text{for } i = 2, 3, \ldots, n_1.$$

Finally, on the basis of Theorem 4.2 (*iii*), we have $\mu_1(G_1 \overline{\wedge} G_2) = 0$. Then the other two roots of the equation

$$x^3 - (r_1 + n_1 + n_2 + 2)x^2 + (r_1n_1 + 2n_1 + 2n_2)x = 0 \quad (4.8)$$

are $y_1$,$y_2$. In the light of Vieta Theorem, we get

$$\frac{1}{y_1} + \frac{1}{y_2} = \frac{y_1 + y_2}{y_1y_2} = \frac{r_1 + n_1 + n_2 + 2}{r_1n_1 + 2n_1 + 2n_2}.$$

Note that $|V(G_1 \overline{\wedge} G_2)| = 2n_1 + n_2$. Therefore, the Kirchhoff index of $G_1 \overline{\wedge} G_2$ is below

$$\begin{aligned} Kf(G_1 \overline{\wedge} G_2) =& (2n_1 + n_2)\Big(\frac{r_1 + n_1 + n_2 + 2}{2n_1 + 2n_2 + r_1n_1} + \sum_{j=2}^{n_2} \frac{1}{n_1 + \mu_j(G_2)} \\ &+ \sum_{i=2}^{n_1} \frac{2r_1 + n_2 + \mu_i(G_1)}{n_2r_1 - \mu_i^2(G_1) + 3r_1\mu_i(G_1) + n_2\mu_i(G_1)}\Big). \end{aligned}$$

□

**Corollary 4.2.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices. Then*

$$\begin{aligned} \tau(G_1 \overline{\wedge} G_2) =& \frac{2n_1 + 2n_2 + r_1n_1}{2n_1 + n_2} \prod_{i=2}^{n_1} \big(n_2r_1 - \mu_i^2(G_1) + 3r_1\mu_i(G_1) \\ &+ n_2\mu_i(G_1)\big) \prod_{j=2}^{n_2} \big(n_1 + \mu_j(G_2)\big). \end{aligned}$$

**Proof.** According to Lemma 2.5, it is easy to obtain that

$$\tau(G) = \frac{\prod_{i=2}^{n} \mu_i(G)}{n}. \quad (4.9)$$

Due to Theorem (4.2), we investigate the Laplacian eigenvalues of $G_1 \overline{\wedge} G_2$ in the following way:

From (4.7), we have

$$\alpha_1\alpha_2 = n_2r_1 - \mu_i^2(G_1) + 3r_1\mu_i(G_1) + n_2\mu_i(G_1), \quad (4.10)$$

where $i = 2, 3, \ldots, n_1$.

From (4.8), we get

$$y_1y_2 = 2n_1 + 2n_2 + r_1n_1. \quad (4.11)$$

By the above equation (4.9), combining the equations (4.6), (4.10) and (4.11), we can get

$$\begin{aligned}\tau(G_1 \barwedge G_2) =& \frac{2n_1+2n_2+r_1n_1}{2n_1+n_2}\prod_{i=2}^{n_1}\big(n_2r_1-\mu_i^2(G_1)+3r_1\mu_i(G_1)\\ &+n_2\mu_i(G_1)\big)\prod_{j=2}^{n_2}\big(n_1+\mu_j(G_2)\big).\end{aligned}$$

□

## 4.3. Q-spectra of splitting S-vertex join

By adding (4.1) to (4.2), the signless Laplacian matrix of $G_1 \barwedge G_2$ can be expressed as:

$$\begin{aligned}Q(G_1 \barwedge G_2) &= D(G_1 \barwedge G_2) + A(G_1 \barwedge G_2)\\ &= \begin{pmatrix} 2r_1I_{n_1}+A(G_1) & A(G_1) & O_{n_1\times n_2}\\ A(G_1) & (r_1+n_2)I_{n_1} & J_{n_1\times n_2}\\ O_{n_2\times n_1} & J_{n_2\times n_1} & (r_2+n_1)I_{n_2}+A(G_2)\end{pmatrix}\\ &= \begin{pmatrix} r_1I_{n_1}+Q(G_1) & A(G_1) & O_{n_1\times n_2}\\ A(G_1) & (r_1+n_2)I_{n_1} & J_{n_1\times n_2}\\ O_{n_2\times n_1} & J_{n_2\times n_1} & n_1I_{n_2}+Q(G_2)\end{pmatrix}. \end{aligned} \tag{4.12}$$

**Theorem 4.3.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices. Then the signless Laplacian spectrum of $G_1 \barwedge G_2$ consists of:*

*(i)* $n_1 + \nu_j(G_2)$ *for each* $j = 2, 3, \ldots, n_2$*;*

*(ii)* *two roots of the equation* $x^2-\big(2r_1+n_2+\nu_i(G_1)\big)x+n_2r_1-\nu_i^2(G_1)+3r_1\nu_i(G_1)+n_2\nu_i(G_1) = 0$ *for each* $i = 2, 3, \ldots, n_1$*;*

*(iii)* *three roots of the equation* $x^3-(4r_1+2r_2+n_1+n_2)x^2+(2r_1^2+4r_1n_1+3r_1n_2+2r_2n_2+8r_1r_2)x-4r_1^2r_2-2r_1^2n_1-6r_1r_2n_2 = 0$*.*

**Proof.** According to the (4.12), the signless Laplacian characteristic polynomial of $G_1 \barwedge G_2$ is

$$\begin{aligned}f_{Q(G_1\barwedge G_2)}(x) &= \det\big(xI_{2n_1+n_2} - Q(G_1 \barwedge G_2)\big)\\ &= \left(\begin{array}{cc|c} (x-r_1)I_{n_1}-Q(G_1) & -A(G_1) & O_{n_1\times n_2}\\ -A(G_1) & (x-r_1-n_2)I_{n_1} & -J_{n_1\times n_2}\\ \hline O_{n_2\times n_1} & -J_{n_2\times n_1} & (x-n_1)I_{n_2}-Q(G_2)\end{array}\right)\\ &= \det\big((x-n_1)I_{n_2}-Q(G_2)\big)\det(S) = \prod_{j=1}^{n_2}\big(x-n_1-\nu_j(G_2)\big)\det(S),\end{aligned}$$

where

$$
\begin{aligned}
S=&\begin{pmatrix} (x-r_1)I_{n_1}-Q(G_1) & -A(G_1) \\ -A(G_1) & (x-r_1-n_2)I_{n_1} \end{pmatrix}\\
&-\begin{pmatrix} O_{n_1\times n_2} \\ -J_{n_1\times n_2} \end{pmatrix}\big((x-n_1)I_{n_2}-Q(G_2)\big)^{-1}\begin{pmatrix} O_{n_2\times n_1} & -J_{n_2\times n_1} \end{pmatrix}\\
=&\begin{pmatrix} (x-r_1)I_{n_1}-Q(G_1) & A(G_1) \\ A(G_1) & (x-r_1-n_2)I_{n_1}-\Gamma_{Q(G_2)}(x-n_1)J_{n_1\times n_1} \end{pmatrix}
\end{aligned}
$$

is the Schur complement of $(x-n_1)I_{n_2}-Q(G_2)$. Therefore, by Lemma 2.1, we get

$$
\begin{aligned}
&\det(S)\\
=&\det\big((x-r_1-n_2)I_{n_1}-\Gamma_{Q(G_2)}(x-n_1)J_{n_1\times n_1}\big)\\
&\det\Big((x-r_1)I_{n_1}-Q(G_1)-A(G_1)\big((x-r_1-n_2)I_{n_1}-\Gamma_{Q(G_2)}(x-n_1)J_{n_1\times n_1}\big)^{-1}A(G_1)\Big)\\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{Q(G_2)}(x-n_1)}{x-r_1-n_2}\big)\\
&\det\Bigg((x-r_1)I_{n_1}-Q(G_1)-A(G_1)\Big(\frac{1}{x-r_1-n_2}I(G_1)\\
&+\frac{\Gamma_{Q(G_2)}(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{Q(G_2)}(x-n_1)\big)}J_{n_1\times n_1}\Big)A(G_1)\Bigg)\\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{Q(G_2)}(x-n_1)}{x-r_1-n_2}\big)\det\Big((x-r_1)I_{n_1}-Q(G_1)\\
&-\frac{1}{x-r_1-n_2}A^2(G_1)-\frac{\Gamma_{Q(G_2)}(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{Q(G_2)}(x-n_1)\big)}\\
&\times A(G_1)J_{n_1\times n_1}A(G_1)\Big)\\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{Q(G_2)}(x-n_1)}{x-r_1-n_2}\big)\det\Big((x-r_1)I_{n_1}-Q(G_1)\\
&-\frac{1}{x-r_1-n_2}A^2(G_1)-\frac{r_1^2\Gamma_{Q(G_2)}(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{Q(G_2)}(x-n_1)\big)}J_{n_1\times n_1}\Big)\\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{Q(G_2)}(x-n_1)}{x-r_1-n_2}\big)\Big(\det\big((x-r_1)I_{n_1}-Q(G_1)-\frac{1}{x-r_1-n_2}A^2(G_1)\big)\\
&-\frac{r_1^2\Gamma_{Q(G_2)}(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{Q(G_2)}(x-n_1)\big)}\\
&\mathbf{1}_{n_1}^T adj\big((x-r_1)I_{n_1}-Q(G_1)-\frac{1}{x-r_1-n_2}A^2(G_1)\big)\mathbf{1}_{n_1}\Big)\\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{Q(G_2)}(x-n_1)}{x-r_1-n_2}\big)\det\big((x-r_1)I_{n_1}-Q(G_1)-\frac{1}{x-r_1-n_2}A^2(G_1)\big)\\
&\times\Big[1-\frac{r_1^2\Gamma_{Q(G_2)}(x-n_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{Q(G_2)}(x-n_1)\big)}\\
&\mathbf{1}_{n_1}^T\big((x-r_1)I_{n_1}-Q(G_1)-\frac{1}{x-r_1-n_2}A^2(G_1)\big)^{-1}\mathbf{1}_{n_1}\Big]\\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1\Gamma_{Q(G_2)}(x-n_1)}{x-r_1-n_2}\big)\det\big((x-r_1)I_{n_1}-Q(G_1)-\frac{1}{x-r_1-n_2}A^2(G_1)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big[1-\frac{r_1^2\Gamma_{Q(G_2)}(x-n_1)\Gamma_{Q(G_1)+\frac{1}{x-r_1-n_2}A^2(G_1)}(x-r_1)}{(x-r_1-n_2)\big(x-r_1-n_2-n_1\Gamma_{Q(G_2)}(x-n_1)\big)}\Big]\\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1n_2}{(x-r_1-n_2)(x-n_1-2r_2)}\big)\\
&\prod_{i=1}^{n_1}\Big(x-r_1-\nu_i(G_1)-\frac{1}{x-r_1-n_2}\big(\nu_i(G_1)-r_1\big)^2\Big)\\
&\times\Big[1-\frac{n_1n_2r_1^2}{(x-r_1-n_2)(x-n_1-2r_2)(x-r_1-n_2-\frac{n_1n_2}{x-n_1-2r_2})(x-3r_1-\frac{r_1^2}{x-r_1-n_2})}\Big].
\end{aligned}
$$

Thus, we know

$$
\begin{aligned}
&f_{Q(G_1\bar{\curlywedge}G_2)}(x)\\
=&(x-r_1-n_2)^{n_1}\big(1-\frac{n_1n_2}{(x-r_1-n_2)(x-n_1-2r_2)}\big)\\
&\prod_{j=1}^{n_2}\big(x-n_1-\nu_j(G_2)\big)\prod_{i=1}^{n_1}\Big(x-r_1-\nu_i(G_1)-\frac{1}{x-r_1-n_2}\big(\nu_i(G_1)-r_1\big)^2\Big)\\
&\times\Big[1-\frac{n_1n_2r_1^2}{(x-r_1-n_2)(x-n_1-2r_2)(x-r_1-n_2-\frac{n_1n_2}{x-n_1-2r_2})(x-3r_1-\frac{r_1^2}{x-r_1-n_2})}\Big]\\
=&\prod_{i=2}^{n_1}\Big(x^2-\big(2r_1+n_2+\nu_i(G_1)\big)x+n_2r_1-\nu_i^2(G_1)+3r_1\nu_i(G_1)\\
&+n_2\nu_i(G_1)\Big)\prod_{j=2}^{n_2}\big(x-n_1-\nu_j(G_2)\big)\times\big(x^3-(4r_1+2r_2+n_1+n_2)x^2\\
&+(2r_1^2+4r_1n_1+3r_1n_2+2r_2n_2+8r_1r_2)x-(4r_1^2r_2+2r_1^2n_1+6r_1r_2n_2)\big).
\end{aligned}
$$

□

## 5. Non-regular simultaneous cospectral graphs

In 2010, Butler [4] constructed simultaneous cospectral graphs for the adjacency and normalized Laplacian matrices, and asked the same for all three matrices, namely, adjacency, Laplacian and normalized Laplacian. In this section, we also consider simultaneous cospectral graphs but for the adjacency, Laplacian and signless Laplacian matrices. We use the spectra of graph operations based on splitting graph and construct several classes of non-regular A-cospectral, L-cospectral and Q-cospectral graphs which promote the problem asked by Butler [4].

Applying the definition of Laplacian and signless Laplacian matrices, we can easyly obtain the following lemma which is crucial for the proof of the simultaneous cospectral graphs.

**Lemma 5.1.** *(i) If $G$ is an $r$-regular graph then $L(G)=rI_n-A(G)$ and $Q(G)=rI_n+A(G)$;*

*(ii) If $G_1$ and $G_2$ are A-cospectral regular graphs then they are also cospectral with respect to the Laplacian and signless Laplacian matrices.*

Combining with Lemma 5.1 and all the theorems given in the previous section, we have the following Theorem which explain the existence of non-regular simultaneous cospectral graphs for the adjacency, Laplacian and signless Laplacian.

**Theorem 5.1.** *Let $G_i$, $H_i$ be $r_i$-regular graphs, $i = 1, 2$, where $G_1$ need not be different from $H_1$. If $G_1$ and $H_1$ are A-cospectral, and $G_2$ and $H_2$ are A-cospectral then $G_1 \veebar G_2$ (respectively $G_1 \barwedge G_2$) and $H_1 \veebar H_2$ (respectively $H_1 \barwedge H_2$) are simultaneously A-cospectral, L-cospectral and Q-cospectral.*

As a matter of fact, although $G_i$ and $H_i$ are regular graphs, $G_1 \veebar G_2$ ($G_1 \barwedge G_2$) and $H_1 \veebar H_2$ ($H_1 \barwedge H_2$) are non-regular graphs.

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