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THE DYNAMICAL BEHAVIOR AND PERIODIC SOLUTION IN DELAYED NONAUTONOMOUS CHEMOSTAT MODELS*

Ningning Ye[1], Long Zhang[1,†] and Zhidong Teng[2]

Abstract In this paper, the global dynamics and existence of positive periodic solutions in a general delayed nonautonomous chemostat model are investigated. The positivity and ultimate boundedness of solutions are firstly obtained. The sufficient conditions on the uniform persistence and strong persistence of solutions are established. Furthermore, the criterion on the global attractivity of trivial solution is also established. As the applications of main results, the periodic delayed chemostat model is discussed, and the necessary and sufficient criteria on the existence of positive periodic solutions, and uniform persistence and extinction of microorganism species are obtained. Lastly, the numerical examples are presented to illustrate the main conclusions.

Keywords Nonautonomous chemostat model, delay, persistence, global attractivity, positive periodic solution.

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1. Introduction

The chemostat is an experimental device invented in the 1950s, first used by microbiologists to study the growth of a given species of microorganisms, its usage greatly diversified with the time going on (see [26, 27, 33]). It is a standard tool for microbiologists to study relationships between microbial growth and environment parameters, and it is also the focus of great interest of theoretical ecology and mathematical ecology (see [10, 15, 38, 43] and the references cited therein). It is also used nowadays in analysis of antibiotic (see [18]) and to study recombinant problems related to genetically altered microorganisms (see [12, 20]).

It is precisely because of its importance that many scholars study the dynamical properties of chemostats by establishing dynamical models of differential equations, including the nonnegative boundedness of solutions, the extinction and persistence of microorganisms (see [7, 21, 29]), the stability of equilibrium (see [30, 31]), the existence of periodic solutions (see [1, 2]), the occurrence of bifurcations (see [32]) and the dynamical complexity (see [8, 9, 16, 41]), etc.

[†]The corresponding author. Email: longzhang_xj@sohu.com (L. Zhang)
[1]College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, Xinjiang, China
[2]College of Medical Engineering and Technology, Xinjiang Medical University, Urumqi 830017, Xinjiang, China
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We all know that in autonomous chemostat models, the parameters, input and output are all constants. However, the chemostat is a dynamic system with continuous material inputs and outputs, thus simulating the open system characteristics and temporal continuity of nature (see [33]). On the other hand, in the field of ecology the chemostat is often viewed as a model of a simple lake system, of the wastewater treatment process, or of biological waste decomposition (see [6,17]). In the field of environment, various factors such as climate and environment change dynamically, making it impossible for the parameters, input and output of the model to remain constant and change over time, which are more properly characterized by nonautonomous models (see [4,14,25]). If only seasonal periodic changes are considered, then the dynamics of chemostats are studied using periodic models (see [1, 2]). Moreover, when using chemostat models to study the growth of phytoplankton(see [29, 40]) and the cultivation of microorganisms in natural lakes (see [5,15,43]), the results obtained in a time-fluctuating environment can better explain the population dynamics. On this account, Rehim and Teng in [28] considered the following single-species nonautonomous chemostat model:

$$\begin{cases} \dfrac{\mathrm{d}s(t)}{\mathrm{d}t} =a(t) - b(t)s(t) - x(t)P(t, s(t)) \\ \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} =x(t)(-d(t) + Q(t, s(t))). \end{cases} \tag{1.1}$$

A series of interesting criteria on persistence, average persistence and extinction of solutions were established.

It is generally agreed that time delay can have a considerable impact on the nature of the ecosystem. In fact, in population models it may well become necessary to have lags, such as a generation lag coming after a lag of the kind considered in the Volterra model (see [22]). Without exception, the influence of time delay is inevitable in the chemostat model. Because microorganisms cannot immediately convert into their own biomass after absorbing nutrients, but there is a time delay. Therefore, the chemostat model with time delay can more clearly show that the absorption of nutrients by microorganisms is not instantaneous. Moreover, considering the time delay in the chemostat model can better explain some non-stationary situations, such as periodic fluctuations and instability. In order to better simulate these actual natural phenomena, many scholars incorporated time delay into the model to study its consequences (see [1, 3, 11, 13, 19, 23, 24, 37, 39, 42]). Especially, Amster et al. in [1] studied a single-species chemostat model with periodic nutrient supply and delay in the growth as follows:

$$\begin{cases} \dfrac{\mathrm{d}s(t)}{\mathrm{d}t} =Ds^0(t) - Ds(t) - \gamma^{-1}\mu(s(t))x(t) \\ \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} =x(t)(\mu(s(t-\tau)) - D). \end{cases} \tag{1.2}$$

The necessary and sufficient conditions for the existence of positive periodic solution were established by constructing Poincare type mapping and using Whyburn's lemma and Leray-Schauder's degree. Further, the criterion on the extinction of microorganism species x is also obtained.

Motivated by above work, in this article we propose a nonautonomous chemostat model with general delay in microorganism growth as follows:

$$\begin{cases} \dfrac{\mathrm{d}s(t)}{\mathrm{d}t} =a(t)-b(t)s(t)-x(t)P(t,s(t)) \\ \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} =x(t)(-d(t)+Q(t,s_t)), \end{cases} \tag{1.3}$$

where $s(t)$ and $x(t)$ are the concentrations of nutrient and microorganism when time is t, respectively, and $s_t = s(t+\theta)$ with $\theta \in [-\tau, 0]$. $P(t,s)$ is the per capita nutrient absorption rate of the microorganism at the concentration s and time t. $Q(t,s_t)$ is the growth rate of the microorganism at time t, which shows that the growth of microorganisms in biomass depends on the amounts of the nutrient consumed in whole interval $[t-\tau, t]$, where $\tau \geq 0$ is a constant. Functions $a(t)$, $b(t)$ and $d(t)$ denote, respectively, the input nutrient concentration, the dilution rate and the removal rate of microorganism.

We can easily see that the following delayed nonautonomous chemostat models are special cases of model (1.3):

$$\begin{cases} \dfrac{\mathrm{d}s(t)}{\mathrm{d}t} =a(t)-b(t)s(t)-x(t)P(t,s(t)) \\ \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} =x(t)(-d(t)+Q(t,s(t-\tau(t)))), \end{cases} \tag{1.4}$$

where $\tau(t)$ is nonnegatively bounded and continuously differentiable for all $t \geq 0$ and satisfies $\max_{t\geq 0} \tau'(t) < 1$, and

$$\begin{cases} \dfrac{\mathrm{d}s(t)}{\mathrm{d}t} =a(t)-b(t)s(t)-x(t)P(t,s(t)) \\ \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} =x(t)(-d(t)+Q(t,\displaystyle\int_{-\tau}^{0} c(t,\theta)s(t+\theta)\mathrm{d}\theta)), \end{cases} \tag{1.5}$$

where $c(t,\theta)$ is defined and nonnegative for all $(t,\theta) \in R_+ \times [-\tau, 0]$, and continuous for $t \in R_+$ and integrable for $\theta \in [-\tau, 0]$ with $\int_{-\tau}^{0} c(t,\theta)\mathrm{d}\theta \equiv 1$. It is clear that model (1.2) is a special case of model (1.4).

The purpose of this paper is to investigate the global dynamic behavior and existence of positive periodic solutions of model (1.3). We will establish a series criteria on the ultimate boundedness of solutions, the uniform persistence and strong persistence of nutrient and microorganism, and the global attractivity of trivial solution. Particularly, when model (1.3) degrades into the periodic case, we will further establish the necessary and sufficient conditions for the existence of positive periodic solutions and, the uniform persistence and extinction of microorganism, respectively. The main methods used in this research are the differential inequality principle, the inequalities analysis techniques and the reduction to absurdity.

This paper is organized as follows. In Section 2, as the preliminaries, we first introduce some basic assumptions for model (1.3). Then, some useful lemmas are given, and the positivity of solutions for model (1.3) with positive initial values is proved. Section 3 is devoted to demonstrate that the solutions of model (1.3) are ultimately bounded. In Section 4, the sufficient conditions on the uniform persistence and strong persistence of nutrient and microorganism are stated and proved. In Section 5, a criterion on the global attractivity of trivial solution of microorganisms vanishing is established. In allusion to time-periodic model (1.3), the necessary and sufficient conditions for the existence of positive periodic solutions, the uniform persistence and extinction of microorganism are stated in Section 6, respectively. In Section 7, the numerical experiments are presented to illustrate the main conclusions established in this paper. Finally, in Section 8, a brief conclusion is given, and some new interesting problems are proposed for the future research.

2. Preliminaries

We denote $R_+ = [0,\infty)$, $R_+^2 = R_+ \times R_+$ and by $C = C([-\tau,0],R)$ the Banach space of real valued continuous functions ϕ defined in $[-\tau,0]$ with the supreme norm $\|\phi\| = \sup_{\theta\in[-\tau,0]}|\phi(\theta)|$, and its positive cone as $C^+ = C([-\tau,0],R_+)$. For any $\phi_1,\phi_2 \in C^+$, we define $\phi_1 > \phi_2$ if $\phi_1(s) \geq \phi_2(s)$, and $\phi_1(s) \not\equiv \phi_2(s)$ for any $s \in [-\tau,0]$.

The initial condition for any solution of model (1.3) is as follows:

$$s(\theta) = \phi(\theta),\ x(0) = x_0 \quad \text{for all} \quad \theta \in [-\tau,0], \tag{2.1}$$

where $\phi \in C^+$, $\phi(0) > 0$ and $x_0 > 0$.

We introduce the following assumptions for model (1.3).

($\mathbf{A_1}$) $a(t)$, $b(t)$ and $d(t)$ are bounded continuous functions defined for all $t \geq 0$, and $a(t) \geq 0$ for $t \geq 0$. There are constants $\omega_i > 0$ $(i = 1,2)$ such that $\liminf_{t\to\infty}\int_t^{t+\omega_1} b(\xi)\mathrm{d}\xi > 0$ and $\liminf_{t\to\infty}\int_t^{t+\omega_2} d(\xi)\mathrm{d}\xi > 0$.

($\mathbf{A_2}$) $P(t,s)$ and $Q(t,\phi)$ are continuous functions for $(t,s) \in R_+^2$ and $(t,\phi) \in R_+ \times C^+$, respectively. $P(t,0) = Q(t,0) \equiv 0$ for any $t \geq 0$. For any positive constant H there exists a $K = K(H) > 0$ such that $|P(t,s_1) - P(t,s_2)| \leq K|s_1 - s_2|$ for any $(t,s_i) \in R_+^2$ with $0 \leq s_i \leq H$ $(i = 1,2)$ and $|Q(t,\phi_1) - Q(t,\phi_2)| \leq K|\phi_1 - \phi_2|$ for any $(t,\phi_i) \in R_+ \times C^+$ with $|\phi_i| \leq H$ $(i = 1,2)$.

($\mathbf{A_3}$) For any $s > 0$, $\liminf_{t\to\infty} P(t,s) > 0$. For any $t \geq 0$, $P(t,s)$ is nondecreasing for $s \in R_+$, and $Q(t,\phi_1) \leq Q(t,\phi_2)$ for any $\phi_1,\phi_2 \in C^+$ with $\phi_1 \leq \phi_2$.

($\mathbf{A_4}$) For any constants $H > \beta > 0$ there exists a continuous function $h(t)$ defined for $t \geq 0$ satisfying $\liminf_{t\to\infty}\int_t^{t+\alpha} h(\xi)\mathrm{d}\xi > 0$ for some constant $\alpha > 0$ such that for any $\phi_1,\phi_2 \in C^+$ with $\phi_i \leq H$ $(i = 1,2)$ and $\phi_1 - \phi_2 \geq \beta$, one has $Q(t,\phi_1) - Q(t,\phi_2) \geq h(t)$ for any $t \geq 0$.

Remark 2.1. Assumptions ($\mathbf{A_1}$) and ($\mathbf{A_2}$) are fundamental. The Lipschitz conditions of $P(t,s)$ and $Q(t,\phi)$ are given to assure the existence, uniqueness and continuability of solutions of model (1.3). For assumption ($\mathbf{A_3}$), $P(t,s)$ is increasing for $s \geq 0$ to show that the increase of nutrient s will result in that microorganism x acquires more many nutrient. $Q(t,\phi)$ is increasing for $\phi \geq 0$ to show that the increase of nutrient s will make that microorganism x acquires greater growth. The condition $\liminf_{t\to\infty} P(t,s) > 0$ for any $s > 0$ will be used in the proof of ultimate boundedness of solutions (See Theorem 3.1 below). Assumption ($\mathbf{A_4}$) will be used in the proof of extinction of microorganism x (See Theorem 5.1 below).

We see that for special models (1.4) and (1.5) assumptions ($\mathbf{A_2}$) – ($\mathbf{A_4}$) will degenerate into the following forms.

($\mathbf{A_2'}$) $P(t,s)$ and $Q(t,s)$ are continuous for any $(t,s) \in R_+^2$. $P(t,0) = Q(t,0) \equiv 0$ for any $t \geq 0$. For any positive constant H there exists a $K = K(H) > 0$ such that $|P(t,s_1) - P(t,s_2)| \leq K|s_1 - s_2|$ and $|Q(t,s_1) - Q(t,s_2)| \leq K|s_1 - s_2|$ for any $(t,s_i) \in R_+^2$ with $0 \leq s_i \leq H$ $(i = 1,2)$.

($\mathbf{A_3'}$) $\liminf_{t\to\infty} P(t,s) > 0$ for any $s > 0$. For any $t \geq 0$, $P(t,s)$ and $Q(t,s)$ are nondecreasing for $s \in R_+$.

($\mathbf{A_4'}$) For any constants $H > \beta > 0$ there exists a continuous function $h(t)$ defined for $t \geq 0$ satisfying $\liminf_{t\to\infty} \int_t^{t+\alpha} h(\xi)\mathrm{d}\xi > 0$ for some constant $\alpha > 0$ such that for any $t \in R_+$ and $s_1, s_2 \in R_+$ with $s_i \leq H$ $(i = 1,2)$ and $s_1 - s_2 \geq \beta$, one has $Q(t,s_1) - Q(t,s_2) \geq h(t)$.

Now, we consider the following nonautonomous linear equation:

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = c(t) - l(t)v(t), \tag{2.2}$$

where $c(t)$ and $l(t)$ are continuous bounded functions defined on $t \geq 0$ and $c(t) \geq 0$ for all $t \geq 0$. We have the following lemma.

Lemma 2.1. *Assume that there is a constant* $\alpha > 0$ *such that* $\liminf_{t\to\infty} \int_t^{t+\alpha} l(\eta)\mathrm{d}\eta > 0$. *Then, we have*

- *(i) Each fixed solution* $v^*(t)$ *of equation* (2.2) *is bounded on* $t \geq 0$ *and globally uniformly attractive;*
- *(ii) Let* $v(t)$ *be the solution of equation* (2.2) *and* $\bar{v}(t)$ *be the solution of equation* (2.2) *after replacing* $c(t)$ *with another continuous function* $\bar{c}(t)$. *If* $\bar{v}(0) = v(0)$, *then there exists a constant* $L > 0$ *that only depends on* $l(t)$, *such that* $\sup_{t\in R_+} |v(t) - \bar{v}(t)| \leq L \sup_{t\in R_+} |c(t) - \bar{c}(t)|$;
- *(iii) If there exists a constant* $k > 0$ *such that* $\liminf_{t\to\infty} \int_t^{t+k} c(\eta)\mathrm{d}\eta > 0$, *then* $m_2^{-1}e^{-r_2 k} \leq \liminf_{t\to\infty} v(t) \leq \limsup_{t\to\infty} v(t) \leq m_1^{-1}e^{r_1\alpha}$ *for any solution* $v(t)$ *of equation* (2.2), *where the positive constants* m_1 *and* m_2 *satisfy*

$$\liminf_{t\to\infty} \int_t^{t+\alpha} (l(\eta) - c(\eta)m_1)\mathrm{d}\eta > 0, \quad \limsup_{t\to\infty} \int_t^{t+k} (l(\eta) - c(\eta)m_2)\mathrm{d}\eta < 0,$$

and $r_1 = \sup_{t\geq 0} \{|l(t)| + c(t)m_1\}$ *and* $r_2 = \sup_{t\geq 0} \{|l(t)| + c(t)m_2\}$.

Lemma 2.1 can be proved by using the similar method given in [34], we hence omits it here.

For the convenience of narrations, we denote the functions $g_1(t,s,x) = a(t) - b(t)s - P(t,s)x$ and $g_2(t,s_t) = -d(t) + Q(t,s_t)$.

Lemma 2.2. *Assume that* ($\mathbf{A_1}$) *and* ($\mathbf{A_2}$) *hold. Then the solution* $(s(t),x(t))$ *of model* (1.3) *with initial condition* (2.1) *exists and is positive for all* $t \geq 0$.

Proof. Firstly, according to the fundamental theory of functional differential equations, model (1.3) has a unique solution $(s(t),x(t))$ satisfying initial condition (2.1) defined on some interval $[0,T)$ with $T \leq \infty$.

The proof of positivity of solution $(s(t),x(t))$ is simple. In fact, integrating the second equation of model (1.3) from 0 to any $t \in (0,T)$ we directly have

$$x(t) = x(0)e^{\int_0^t g_2(\xi,s_\xi)\mathrm{d}\xi} > 0. \tag{2.3}$$

Suppose that there is a $t_1 > 0$ such that $s(t_1) = 0$ and $s(t) > 0$ for any $t \in [0,t_1)$. Assumption ($\mathbf{A_2}$) implies that $P(t,s(t)) \leq Ks(t)$ for any $t \in [0,t_1]$. From the first equation of model (1.3) it follows that

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = g_1(t,s(t),x(t)) \geq -s(t)[b(t) + x(t)K]$$

for all $t \in [0, t_1]$. From this, we directly get $s(t_1) \geq s(0)e^{-\int_0^{t_1}[b(t)+x(t)K]\mathrm{d}t} > 0$, a contradiction. Therefore, solution $(s(t), x(t))$ of model (1.3) is positive on its existence interval.

Now, we prove that the solution of model (1.3) is defined for all $t \in [0, \infty)$. Suppose that solution $(s(t), x(t))$ is defined only on $[0, T)$ with $T < \infty$. Then, $(s(t), x(t))$ is unbounded when $t \to T$. By the first equation of model (1.3), one gets

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = g_1(t, s(t), x(t)) \leq a(t) - b(t)s(t).$$

From conclusion (i) of Lemma 2.1 and the comparison principle, we easily obtain the boundedness of $s(t)$ on $[-\tau, T)$. Then, by assumption $(\mathbf{A_2})$, $-d(t) + Q(t, s_t)$ is also bounded for $t \in [0, T)$. Accordingly, (2.3) indicates that $x(t)$ is bounded on $[0, T)$, a contradiction. Consequently, $(s(t), x(t))$ is defined for all $t \in [0, \infty)$. The proof is completed. □

3. Ultimate boundedness

In this section, we investigate the ultimate boundedness of solutions of model (1.3). If there exists a constant $C > 0$ such that any positive solution $(s(t), x(t))$ of model (1.3) satisfies $\limsup_{t\to\infty} s(t) < C$ and $\limsup_{t\to\infty} x(t) < C$, then we say that the solution of model (1.3) is ultimately bounded.

For model (1.3), if $x(t) \equiv 0$, i.e., there is no microorganism, then the subsystem of nutrient species is given as follows:

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = a(t) - b(t)s(t). \tag{3.1}$$

Let $s^*(t)$ be some fixed solution of equation (3.1) with initial value $s^*(0) = s_0^* > 0$. Obviously, model (1.3) has a trivial solution $(s^*(t), 0)$, which shows that microorganisms species x vanishes. From assumption $(\mathbf{A_1})$ and conclusion (i) of Lemma 2.1, it follows that $s^*(t)$ is defined for all $t \in R_+$ and is positive, bounded and globally attractive for equation (3.1).

Theorem 3.1. *Assume that* $(\mathbf{A_1})$–$(\mathbf{A_3})$ *hold. Then the solution* $(s(t), x(t))$ *of model* (1.3) *with initial condition* (2.1) *is ultimately bounded.*

Proof. Let $(s(t), x(t))$ be any solution of model (1.3) with initial condition (2.1). From the first equation of model (1.3), one has

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = g_1(t, s(t), x(t)) \leq a(t) - b(t)s(t).$$

By the comparison principle, for any constant $\varepsilon > 0$ there exists a $T = T(\varepsilon) > 0$ such that

$$s(t) < s^*(t) + \varepsilon \quad \text{for all} \quad t \geq T. \tag{3.2}$$

Hence, there exists a large $T_0 > 0$ such that $s(t) \leq H_1$ for all $t \geq T_0$, where the constant $H_1 > \sup_{t\geq 0} s^*(t)$. Consequently, $s(t)$ is ultimately bounded.

Now, we prove the ultimate boundedness of $x(t)$. Assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$ imply that there is a constant $\varepsilon_0 > 0$ small enough such that

$$\limsup_{t\to\infty} \int_t^{t+\omega_2} g_2(\xi, 2\varepsilon_0)\mathrm{d}\xi < 0. \tag{3.3}$$

From this, it follows that for any $t_0 \geq 0$

$$\lim_{t\to\infty}\int_{t_0}^{t} g_2(\xi, 2\varepsilon_0)\mathrm{d}\xi = -\infty. \tag{3.4}$$

Then, from assumption ($\mathbf{A_3}$) we can choose an enough large constant $H > 0$ such that

$$\frac{a_M}{b_m + \min_{t\geq 0}\{P(t,\varepsilon_0)\}\frac{H}{H_1}} < \frac{1}{2}\varepsilon_0,$$

where $a_M = \max_{t\geq 0} a(t)$ and $b_m = \min_{t\geq 0} b(t)$.

We first prove

$$\liminf_{t\to\infty} x(t) \leq H. \tag{3.5}$$

Suppose that $\liminf_{t\to\infty} x(t) > H$. Then, there exists a $T_1 \geq T_0$ such that $x(t) > H$ for all $t \geq T_1$. If $s(t) \geq \varepsilon_0$ for all $t \geq T_1$, then since $\varepsilon_0 \leq s(t) \leq H_1$ for all $t \geq T_1$, we can get that for all $t \geq T_1$,

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = g_1(t, s(t), x(t)) \leq a_M - (b_m + \min_{t\geq 0}\{P(t,\varepsilon_0)\}\frac{H}{H_1})s(t).$$

Therefore, the comparison principle implies

$$\limsup_{t\to\infty} s(t) \leq \frac{a_M}{b_m + \min_{t\geq 0}\{P(t,\varepsilon_0)\}\frac{H}{H_1}} < \frac{1}{2}\varepsilon_0,$$

a contradiction. Accordingly, there exists a $t_1 > T_1$ such that $s(t_1) < \varepsilon_0$. Furthermore, we prove $s(t) \leq 2\varepsilon_0$ for all $t \geq t_1$. Oppositely, there exists a $t_2 > t_1$ such that $s(t_2) > 2\varepsilon_0$. Due to the continuity of $s(t)$, there exists a $t_3 \in (t_1, t_2)$ such that $s(t_3) = \varepsilon_0$ and $s(t) > \varepsilon_0$ for all $t \in (t_3, t_2]$. Since $\varepsilon_0 \leq s(t) \leq H_1$ for all $t \in [t_3, t_2]$, one obtains that for all $t \in [t_3, t_2]$

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = g_1(t, s(t), x(t)) \leq a_M - (b_m + \min_{t\geq 0}\{P(t,\varepsilon_0)\}\frac{H}{H_1})s(t). \tag{3.6}$$

Let $B = b_m + \min_{t\geq 0}\{P(t,\varepsilon_0)\}\frac{H}{H_1}$. Solving equation (3.6) for $t \in [t_3, t_2]$ by using the method of variation of constant, we obtain

$$s(t_2) \leq e^{-B(t_2-t_3)}(s(t_3) + \int_{t_3}^{t_2} a_M e^{B(u-t_3)}\mathrm{d}u) \leq \varepsilon_0 + \frac{a_M}{B} < 2\varepsilon_0,$$

which is impossible. Hence, $s(t) \leq 2\varepsilon_0$ for all $t \geq t_1$. Consider the second equation of model (1.3), we have for any $t \geq t_1 + \tau$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)g_2(t, s_t) \leq x(t)g_2(t, 2\varepsilon_0). \tag{3.7}$$

For any $t > t_1 + \tau$ integrating (3.7) one obtains $x(t) \leq x(t_1+\tau)e^{\int_{t_1+\tau}^{t} g_2(\xi,2\varepsilon_0)\mathrm{d}\xi}$. Then, from (3.4) we further get that $x(t) \to 0$ as $t \to \infty$, a contradiction. This shows that (3.5) holds.

We further prove that there exists a constant $H_2 > 0$ such that

$$\limsup_{t\to\infty} x(t) \leq H_2. \tag{3.8}$$

Suppose that (3.8) does not hold, then there exists a sequence of initial values $\{r_n\} = \{(\phi_n, x_n)\} \subset C^+ \times (0,\infty)$ with $\phi_n(0) > 0$ such that the solution $(s(t, r_n),$ $x(t, r_n))$ of model (1.3) with initial value r_n satisfies $\limsup_{t\to\infty} x(t, r_n) > (2H+1)n$ for $n = 1, 2, \cdots$.

For every n, by the ultimate boundedness of $s(t)$ we first choose a $T^{(n)} > T_0$ such that $s(t, r_n) \le H_1$ for any $t \ge T^{(n)}$. Since $\liminf_{t\to\infty} x(t, r_n) \le H$, there exist two time sequences $\{u_q^{(n)}\}$ and $\{t_q^{(n)}\}$, satisfying $T^{(n)} + \tau < u_1^{(n)} < t_1^{(n)} < u_2^{(n)} < t_2^{(n)} < \cdots < u_q^{(n)} < t_q^{(n)} \cdots$ and $\lim_{q\to\infty} u_q^{(n)} = \infty$, such that

$$x(u_q^{(n)}, r_n) = 2H, \quad x(t_q^{(n)}, r_n) = (2H+1)n \tag{3.9}$$

and

$$2H < x(t, r_n) < (2H+1)n \quad \text{for all} \quad t \in [u_q^{(n)}, t_q^{(n)}]. \tag{3.10}$$

By assumption $(\mathbf{A_3})$ we deduce that for $t \in [u_q^{(n)}, t_q^{(n)}]$

$$\begin{aligned} x(t_q^{(n)}, r_n) =& x(u_q^{(n)}, r_n) e^{\int_{u_q^{(n)}}^{t_q^{(n)}} g_2(t, s_t)\mathrm{d}t} \\ \le& x(u_q^{(n)}, r_n) e^{\int_{u_q^{(n)}}^{t_q^{(n)}} g_2(t, H_1)dt} \le x(u_q^{(n)}, r_n) e^{L_0 (t_q^{(n)} - u_q^{(n)})} \end{aligned}$$

for $q = 1, 2, \cdots$, where constant $L_0 \ge \sup_{t\ge 0} g_2(t, H_1)$. Hence, $t_q^{(n)} - u_q^{(n)} > \frac{\ln n}{L_0}$ for $q = 1, 2, \cdots$. By (3.3), there exists a constant $p > 0$ such that

$$\int_t^{t+a} g_2(\xi, 2\varepsilon_0)\mathrm{d}\xi < 0, \; H_1 e^{-Bp} < \frac{1}{2}\varepsilon_0 \tag{3.11}$$

for all $t \in R_+$ and $a \ge p$. Then, we can choose an integer $N_0 > 0$ such that $t_q^{(n)} - u_q^{(n)} > 2p + \tau$ for all $q = 1, 2, \cdots, \; n \ge N_0$.

For all $n \ge N_0$ and positive integers q, if $s(t, r_n) \ge \varepsilon_0$ for any $t \in [u_q^{(n)}, u_q^{(n)} + p]$, then from assumption $(\mathbf{A_3})$, (3.6) and (3.11) we can obtain

$$\begin{aligned} s(u_q^{(n)} + p, r_n) \le& s(u_q^{(n)}, r_n) e^{-Bp} + \int_{u_q^{(n)}}^{u_q^{(n)}+p} a_M e^{B(u - u_q^{(n)} - p)}\mathrm{d}u \\ \le& H_1 e^{-Bp} + \frac{a_M}{B} < \varepsilon_0, \end{aligned}$$

which leads to a contradiction. Therefore, there is a $t_1 \in [u_q^{(n)}, u_q^{(n)} + p]$ such that $s(t_1, r_n) < \varepsilon_0$. A similar argument as in the above, we further obtain

$$s(t, r_n) \le 2\varepsilon_0 \quad \text{for all} \quad t \in [t_1, t_q^{(n)}]. \tag{3.12}$$

Finally, from assumption $(\mathbf{A_3})$, (3.10), (3.11) and (3.12), for $t \in [u_q^{(n)} + p + \tau, t_q^{(n)}]$ it follows that

$$\begin{aligned} x(t_q^{(n)}, r_n) =& x(u_q^{(n)} + p + \tau, r_n) e^{\int_{u_q^{(n)}+p+\tau}^{t_q^{(n)}} g_2(t, s_t)\mathrm{d}t} \\ \le& x(u_q^{(n)} + p + \tau, r_n) e^{\int_{u_q^{(n)}+p+\tau}^{t_q^{(n)}} g_2(t, 2\varepsilon_0)\mathrm{d}t} < (2H+1)n, \end{aligned}$$

which is contradictory with (3.9). Therefore, (3.8) holds. That is, $x(t)$ is also ultimately bounded. The proof is completed. □

As the consequences of Theorem 3.1, for special models (1.4) and (1.5) we have the following corollaries.

Corollary 3.1. *Assume that* $(\mathbf{A_1})$, $(\mathbf{A_2'})$ *and* $(\mathbf{A_3'})$ *hold. Then the solution* $(s(t), x(t))$ *of model* (1.4) *with initial condition* (2.1) *is ultimately bounded.*

Corollary 3.2. *Assume that* $(\mathbf{A_1})$, $(\mathbf{A_2'})$ *and* $(\mathbf{A_3'})$ *hold. Then the solution* $(s(t), x(t))$ *of model* (1.5) *with initial condition* (2.1) *is ultimately bounded.*

Remark 3.1. Notice that the condition $\liminf_{t\to\infty} P(t,s) > 0$ for any $s > 0$ in assumption $(\mathbf{A_3})$ is a stronger condition. Accordingly, a meaningful open problem is whether it is changed to that for $s > 0$ there exists a constant $\beta > 0$ such that $\liminf_{t\to\infty} \int_t^{t+\beta} P(\xi, s)\mathrm{d}\xi > 0$ to guarantee that the solution $(s(t), x(t))$ of model (1.3) with initial condition (2.1) is still ultimately bounded.

4. Persistence

In this section, the persistence of model (1.3) is investigated. Let $(s(t), x(t))$ be any positive solution of model (1.3), if $\liminf_{t\to\infty} x(t) > 0$, then one says that species x is strongly persistent. If there exist positive constants L and l with $L \geq l$ such that $l \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq L$ for any positive solution $(s(t), x(t))$ of model (1.3), then one says that species x is uniformly persistent. It is evident that if species x is uniformly persistent, then species x is also strongly persistent. The same concepts can be defined for nutrient s. The main results on the persistence for model (1.3) are established below.

Theorem 4.1. *Assume that* $(\mathbf{A_1}) - (\mathbf{A_3})$ *hold and* $\liminf_{t\to\infty} \int_t^{t+\beta} a(\eta)\mathrm{d}\eta > 0$ *for some constant* $\beta > 0$. *Then nutrient s in model (1.3) is uniformly persistent.*

Proof. Let $(s(t), x(t))$ be any positive solution of model (1.3) with initial point (ϕ, x_0). Since $(s(t), x(t))$ is ultimately bounded (See Theorem 3.1), there exists a constant $U > 0$ such that for any initial point (ϕ, x_0) there is a $T_0 = T_0(\phi, x_0) > 0$, one has $s(t) \leq U$ and $x(t) \leq U$ for all $t \geq T_0$. By assumption $(\mathbf{A_2})$ we deduce that

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = g_1(t, s(t), x(t)) \geq a(t) - (b(t) + UK)s(t).$$

Combining the comparison principle and conclusion (iii) of Lemma 2.1, it follows that there exists a constant $m > 0$ such that $\liminf_{t\to\infty} s(t) \geq m$. This shows that nutrient s is uniformly persistent. This completes the proof. □

Theorem 4.2. *Assume that* $(\mathbf{A_1}) - (\mathbf{A_3})$ *hold and* $\liminf_{t\to\infty} \int_t^{t+\beta} a(\eta)\mathrm{d}\eta > 0$ *for some positive constant* β. *If there exists a constant* $\lambda > 0$ *such that*

$$\liminf_{t\to\infty} \int_t^{t+\lambda} \left(-d(\xi) + Q\left(\xi, s_\xi^*\right)\right) \mathrm{d}\xi > 0 \tag{4.1}$$

then microorganism x in model (1.3) is uniformly persistent.

Proof. Let $(s(t), x(t))$ be any positive solution of model (1.3) with initial point (ϕ, x_0). From Theorem 3.1 and Theorem 4.1 it follows that there are constants

$0 < m < U$ such that for any initial point (ϕ, x_0) there is a $T_0 = T_0(\phi, x_0) > 0$, one has $m \leq s(t) \leq U$ and $x(t) \leq U$ for all $t \geq T_0$.

Firstly, by $\liminf_{t\to\infty} \int_t^{t+\lambda} g_2(\xi, s^*_\xi)\mathrm{d}\xi > 0$, there are constants $T_1 > T_0$, $\delta_1 > 0$ and $\varepsilon > 0$ small enough such that

$$\int_t^{t+\lambda} g_2(\xi, s^*_\xi - \varepsilon)\mathrm{d}\xi > \delta_1 \quad \text{for all} \quad t \geq T_1. \tag{4.2}$$

We consider the following linear equation:

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = a(t) - P(t, U)\alpha - b(t)s(t), \tag{4.3}$$

where $\alpha \in (0, \alpha_0]$, and constant $\alpha_0 > 0$ is chosen to satisfy $\liminf_{t\to\infty} \int_t^{t+\beta}(a(\xi) - P(\xi, U)\alpha_0)\mathrm{d}\xi > 0$ since $\liminf_{t\to\infty} \int_t^{t+\beta} a(\eta)\mathrm{d}\eta > 0$. Let $s_\alpha(t)$ be the solution of equation (4.3) with initial value $s_\alpha(0) = s^*(0)$. From conclusion (i) of Lemma 2.1 we know that $s_\alpha(t)$ is globally uniformly asymptotically stable. Then, conclusion (ii) of Lemma 2.1 indicates that $s_\alpha(t)$ converges to $s^*(t)$ uniformly for $t \in R_+$ as $\alpha \to 0$. Thus, there exists an enough small constant $\alpha > 0$ such that

$$s_\alpha(t) > s^*(t) - \frac{\varepsilon}{2} \quad \text{for all} \quad t \in R_+. \tag{4.4}$$

We first prove $\limsup_{t\to\infty} x(t) \geq \alpha$ for any positive solution $(s(t), x(t))$ of model (1.3). If the claim does not hold, then there is a positive solution $(s(t), x(t))$ of model (1.3) such that $\limsup_{t\to\infty} x(t) < \alpha$. Accordingly, there exists a $T_2 > T_1$ such that $x(t) \leq \alpha$ for all $t \geq T_2$. Then from the first equation of model (1.3) we get

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = g_1(t, s(t), x(t)) \geq a(t) - P(t, U)\alpha - b(t)s(t)$$

for all $t \geq T_2$. Combining the comparison principle and the global asymptotic stability of solution $s_\alpha(t)$, one deduces that there exists a $T_3 > T_2$ such that

$$s(t) > s_\alpha(t) - \frac{\varepsilon}{2} \quad \text{for all} \quad t \geq T_3. \tag{4.5}$$

Accordingly, (4.4) and (4.5) yield to

$$s(t) > s^*(t) - \varepsilon \quad \text{for all} \quad t \geq T_3. \tag{4.6}$$

For any $t \geq T_3 + \tau$, from assumption $(\mathbf{A_3})$ one deduces that

$$x(t) = x(T_3 + \tau)e^{\int_{T_3+\tau}^t g_2(\xi, s_\xi)\mathrm{d}\xi} \geq x(T_3 + \tau)e^{\int_{T_3+\tau}^t g_2(\xi, s^*_\xi - \varepsilon)\mathrm{d}\xi}.$$

Finally, from (4.2) it follows that $\lim_{t\to\infty} x(t) = \infty$, a contradiction. Therefore, $\limsup_{t\to\infty} x(t) > \alpha$ for any positive solution $(s(t), x(t))$ of model (1.3).

Next, we prove that there is a constant $\beta > 0$ such that $\liminf_{t\to\infty} x(t) > \beta$ for any positive solution $(s(t), x(t))$ of model (1.3). Suppose that the conclusion does not hold, then there exists a sequence of initial values $\{r_n\} = \{(\phi_n, x_n)\} \subset C^+ \times (0, \infty)$ with $\phi_n(0) > 0$ such that solution $(s(t, r_n), x(t, r_n))$ of model (1.3) with initial value r_n satisfies $\liminf_{t\to\infty} x(t, r_n) < \frac{\alpha}{n}$ for $n = 1, 2, \cdots$.

For every n, choose a $T^{(n)} > 0$ such that $m \leq s(t, r_n) \leq U$ for any $t \geq T^{(n)}$. Since $\limsup_{t\to\infty} x(t, r_n) \geq \alpha$, there exist two time sequences $\{u_q^{(n)}\}$ and $\{v_q^{(n)}\}$, satisfying $T^{(n)} + \tau < u_1^{(n)} < v_1^{(n)} < u_2^{(n)} < v_2^{(n)} < \cdots < u_q^{(n)} < v_q^{(n)} \cdots$ and $\lim_{q\to\infty} u_q^{(n)} = \infty$, such that

$$x(u_q^{(n)}, r_n) = \alpha, \quad x(v_q^{(n)}, r_n) = \frac{\alpha}{n+1} \tag{4.7}$$

and

$$\frac{\alpha}{n+1} < x(t, r_n) < \alpha \quad \text{for all} \quad t \in [u_q^{(n)}, v_q^{(n)}]. \tag{4.8}$$

By assumptions ($\mathbf{A_1}$) and ($\mathbf{A_3}$) we can choose a positive constant L such that $g_2(t, s_t) \geq -L$ for all $t \geq T^{(n)}$. Hence, from the second equation of model (1.3) one has

$$x(v_q^{(n)}, r_n) = x(u_q^{(n)}, r_n) e^{\int_{u_q^{(n)}}^{v_q^{(n)}} g_2(t,s_t)\mathrm{d}t} \geq x(u_q^{(n)}, r_n) e^{-L(v_q^{(n)} - u_q^{(n)})}$$

for all $q = 1, 2, \cdots$. Consequently, $v_q^{(n)} - u_q^{(n)} > \frac{\ln(n+1)}{L}$ for $q = 1, 2, \cdots$. By (4.2), there are positive constants p and l such that

$$\int_t^{t+a} g_2(\xi, s_\xi^* - \varepsilon)\mathrm{d}\xi > l \tag{4.9}$$

for all $t \in R_+$ and $a \geq p$. Let $\bar{s}_\alpha(t)$ be the solution of equation (4.3) with initial value $\bar{s}_\alpha(u_q^{(n)}) = s(u_q^{(n)}, r_n)$. From (4.8) we can obtain that for any n, q and $t \in [u_q^{(n)}, v_q^{(n)}]$

$$\frac{\mathrm{d}s(t, r_n)}{\mathrm{d}t} = g_1(t, s(t, r_n), x(t, r_n)) \geq a(t) - P(t, U)\alpha - b(t)s(t, r_n).$$

Accordingly, the comparison principle implies

$$s(t, r_n) \geq \bar{s}_\alpha(t) \quad \text{for all} \quad t \in [u_q^{(n)}, v_q^{(n)}]. \tag{4.10}$$

Since solution $s_\alpha(t)$ is globally uniformly asymptotically stable, there exists a constant $T \geq p$, and T does not depend on any n, such that

$$\bar{s}_\alpha(t) \geq s_\alpha(t) - \frac{\varepsilon}{2} \quad \text{for all} \quad t \geq u_q^{(n)} + T. \tag{4.11}$$

Choose an integer $N_0 > 0$ such that $v_q^{(n)} - u_q^{(n)} > 2T + \tau$ as $n \geq N_0$. Further, from (4.4) one has $s(t, r_n) \geq s^*(t) - \varepsilon$ for all $t \in [u_q^{(n)} + T, v_q^{(n)}]$. Thus, assumption ($\mathbf{A_3}$) and (4.8) yield that

$$\begin{aligned} x(v_q^{(n)}, r_n) =& x(u_q^{(n)} + T + \tau, r_n) e^{\int_{u_q^{(n)}+T+\tau}^{v_q^{(n)}} g_2(t,s_t)\mathrm{d}t} \\ \geq& x(u_q^{(n)} + T + \tau, r_n) e^{\int_{u_q^{(n)}+T+\tau}^{v_q^{(n)}} g_2(t,s_t^*-\varepsilon)\mathrm{d}t} \\ >& \frac{\alpha}{n+1} e^{\int_{u_q^{(n)}+T+\tau}^{v_q^{(n)}} g_2(t,s_t^*-\varepsilon)\mathrm{d}t} > \frac{\alpha}{n+1}, \end{aligned}$$

a contradiction. This completes the proof. □

Theorem 4.3. *Assume that* $(\mathbf{A_1}) - (\mathbf{A_3})$ *hold and let* $(s(t), x(t))$ *be any positive solution of model (1.3).*

(i) If $x(t)$ *is strongly persistent, then* $s(t)$ *is also strongly persistent;*

(ii) If $x(t)$ *is uniformly persistent, then* $s(t)$ *is also uniformly persistent.*

Proof. Firstly, we prove conclusion (i). Since $x(t)$ is strongly persistent and solution $(s(t), s(t))$ is ultimately bounded (See Theorem 3.1), there exist constants $M > m > 0$ and $T_0 > 0$ such that $m \leq x(t) \leq M$ and $s(t) \leq M$ for all $t \geq T_0$. From assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$ one can choose a constant $\eta > 0$ enough small such that

$$\liminf_{t\to\infty} \int_t^{t+\omega_2} g_2(\xi, \eta)\mathrm{d}\xi < 0. \tag{4.12}$$

We first prove $\limsup_{t\to\infty} s(t) > 0$. If the conclusion does not hold, then we have $\lim_{t\to\infty} s(t) = 0$. For above $\eta > 0$, there exists a $T_1 > T_0$ such that $s(t) < \eta$ for all $t \geq T_1$. By assumption $(\mathbf{A_3})$ one deduces that

$$x(t) = x(T_1 + \tau)e^{\int_{T_1+\tau}^t g_2(\xi, s_\xi)\mathrm{d}\xi} \leq x(T_1 + \tau)e^{\int_{T_1+\tau}^t g_2(\xi,\eta)\mathrm{d}\xi}$$

for all $t \geq T_1 + \tau$. Thus, (4.12) implies that $x(t) \to 0$ as $t \to \infty$, a contradiction. Therefore, $\limsup_{t\to\infty} s(t) > 0$.

Next, we prove $\liminf_{t\to\infty} s(t) > 0$. Let $\limsup_{t\to\infty} s(t) = \beta > 0$. If the claim does not hold, then there exist two time sequences $\{u_n\}$ and $\{v_n\}$, satisfying $T_0 < u_1 < v_1 < u_2 < v_2 < \cdots < u_n < v_n \cdots$ and $\lim_{n\to\infty} u_n = \infty$, such that

$$s(u_n) = \frac{\beta}{n}, \quad s(v_n) = \frac{\beta}{n^2} \tag{4.13}$$

and

$$\frac{\beta}{n^2} < s(t) < \frac{\beta}{n} \quad \text{for all} \quad t \in [u_n, v_n]. \tag{4.14}$$

Assumption $(\mathbf{A_2})$ implies that there exists a constant $c > 0$ such that

$$g_1(t, s, x) \geq -cs \tag{4.15}$$

for all $t \geq 0$, $0 \leq s \leq M$ and $m \leq x \leq M$. From the first equation of model (1.3) one has for any $t \geq T_0$

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = g_1(t, s(t), x(t)) \geq -cs(t).$$

Therefore, $s(v_n) \geq s(u_n)e^{-c(v_n-u_n)}$ for all $n = 1, 2, \cdots$. Consequently, $v_n - u_n > \frac{\ln n}{c}$ for $n = 1, 2, \cdots$. From (4.12) we can choose a constant $p > 0$ such that

$$Me^{\int_t^{t+a} g_2(\xi,\eta)\mathrm{d}\xi} < m \tag{4.16}$$

for all $t \geq 0$ and $a \geq p$. Accordingly, there is an integer $N_0 > 0$ such that $\frac{\beta}{N_0} < \eta$ and $v_n - u_n \geq 2p + \tau$ for all $n \geq N_0$. For any $n \geq N_0$, integrating the second equation of model (1.3), by assumption $(\mathbf{A_3})$ and (4.16) we obtain

$$x(v_n) = x(u_n + p + \tau)e^{\int_{u_n+p+\tau}^{v_n} g_2(t,s_t)\mathrm{d}t} \leq Me^{\int_{u_n+p+\tau}^{v_n} g_2(t,\eta)\mathrm{d}t} < m.$$

This leads to a contradiction. Consequently, $\liminf_{t\to\infty} s(t) > 0$.

Now, we prove conclusion (ii). Since $x(t)$ is uniformly persistent and solution $(s(t), x(t))$ is ultimately bounded, there are constants $M_1 > m_1 > 0$ and $T_0 > 0$ such that $m_1 \le x(t) \le M_1$ and $s(t) \le M_1$ for all $t \ge T_0$. We first show that $\limsup_{t\to\infty} s(t) > \eta$ for any positive solution $(s(t), x(t))$ of model (1.3), where constant $\eta > 0$ is given in (4.12). Indeed, if the claim does not hold, then there is a $T^* > T_0$ such that $s(t) \le \eta$ for all $t \ge T^*$. By assumption $(\mathbf{A_3})$ one gets

$$x(t) = x(T^* + \tau)e^{\int_{T^*+\tau}^{t} g_2(\xi, s_\xi)\mathrm{d}\xi} \le x(T^* + \tau)e^{\int_{T^*+\tau}^{t} g_2(\xi,\eta)\mathrm{d}\xi}.$$

From (4.12) we further get $\lim_{t\to\infty} x(t) = 0$, a contradiction. Hence, $\limsup_{t\to\infty} s(t) > \eta$ for any positive solution $(s(t), x(t))$ of model (1.3).

Next, we show that there exists a constant $m_2 > 0$ such that $\liminf_{t\to\infty} s(t) > m_2$ for any positive solution $(s(t), x(t))$ of model (1.3). Indeed, from (4.12) there exists a constant $p > 0$ such that

$$M_1 e^{\int_t^{t+a} g_2(\xi,\eta)\mathrm{d}\xi} < m_1 \tag{4.17}$$

for all $t \ge 0$ and $a \ge p$. If the above claim does not hold, then there exists a sequence of initial values $\{r_n\} = \{(\phi_n, x_n)\} \subset C^+ \times (0, \infty)$ with $\phi_n(0) > 0$ such that the solution $(s(t, r_n),\ x(t, r_n))$ of model (1.3) satisfies $\liminf_{t\to\infty} s(t, r_n) < \frac{\eta}{n}$ for $n = 1, 2, \cdots$.

For every n, choose a $T^{(n)} > T_0$ such that $s(t, r_n) \le M_1$ and $m_1 \le x(t, r_n) \le M_1$ for all $t \ge T^{(n)}$. Since $\limsup_{t\to\infty} s(t, r_n) \ge \eta$, there exist two time sequences $\{u_q^{(n)}\}$ and $\{v_q^{(n)}\}$, satisfying $T^{(n)} + \tau < u_1^{(n)} < v_1^{(n)} < u_2^{(n)} < v_2^{(n)} < \cdots < u_q^{(n)} < v_q^{(n)} \cdots$ and $\lim_{q\to\infty} u_q^{(n)} = \infty$, such that

$$s(u_q^{(n)}, r_n) = \eta, \quad s(v_q^{(n)}, r_n) = \frac{\eta}{n} \tag{4.18}$$

and

$$\frac{\eta}{n} < s(t, r_n) < \eta \ \text{ for all } \ t \in [u_q^{(n)}, v_q^{(n)}]. \tag{4.19}$$

Then, from (4.15) we have $s(v_q^{(n)}, r_n) \ge s(u_q^{(n)}, r_n)e^{-c(v_q^{(n)} - u_q^{(n)})}$ for all $q = 1, 2, \cdots$. Consequently, $v_q^{(n)} - u_q^{(n)} > \frac{\ln n}{c}$ for $q = 1, 2, \cdots$. Accordingly, there exists an integer $N_0 > 0$ such that $v_q^{(n)} - u_q^{(n)} \ge 2p + \tau$ for all $n \ge N_0$ and $q = 1, 2, \cdots$. For all $n \ge N_0$ and $q = 1, 2, \cdots$, integrating the second equation of model (1.3), by assumption $(\mathbf{A_3})$ and (4.17) we can obtain

$$x(v_q^{(n)}, r_n) = x(u_q^{(n)} + p + \tau, r_n)e^{\int_{u_q^{(n)}+p+\tau}^{v_q^{(n)}} g_2(t, s_t)\mathrm{d}t} \le M_1 e^{\int_{u_q^{(n)}+p+\tau}^{v_q^{(n)}} g_2(t,\eta)\mathrm{d}t} < m_1,$$

a contradiction. Combining Theorem 3.1, it yields that $s(t)$ is uniformly persistent. The proof is completed. □

Remark 4.1. From the proof of conclusion (i) in Theorem 4.3, we easily find that conclusion (i) can been extended to the following form. However, conclusion (ii) in Theorem 4.3 can not be changed.

Proposition 4.1. Assume that $(\mathbf{A_1}) - (\mathbf{A_3})$ hold. Let $(s(t), x(t))$ be some positive solution of model (1.3). If $x(t)$ is strongly persistent, then $s(t)$ is also strongly persistent.

Remark 4.2. Similarly to Remark 3.1, a meaningful opening question is whether the condition $\liminf_{t\to\infty} P(t,s) > 0$ for any $s > 0$ in assumption $(\mathbf{A_3})$ can be changed to that for any $s > 0$ there exists a constant $\beta > 0$ such that

$$\liminf_{t\to\infty} \int_t^{t+\beta} P(\xi, s)\mathrm{d}\xi > 0$$

to guarantee that all conclusions in Theorems 4.1-4.3 still hold.

As the consequences of Theorems 4.1-4.3, we have the following corollaries for special model (1.4).

Corollary 4.1. *Assume that* $(\mathbf{A_1})$ *and* $(\mathbf{A_2'}) - (\mathbf{A_3'})$ *hold and there is a constant* $\beta > 0$ *such that* $\liminf_{t\to\infty} \int_t^{t+\beta} a(\eta)\mathrm{d}\eta > 0$*. Then any positive solution* $(s(t), x(t))$ *of model (1.4) the conclusions given below hold.*

(i) $s(t)$ *is uniformly persistent;*

(ii) If there exists a constant $\lambda > 0$ *such that*

$$\liminf_{t\to\infty} \int_t^{t+\lambda} (-d(\xi) + Q(\xi, s^*(\xi - \tau(\xi)))\mathrm{d}\xi > 0,$$

then $x(t)$ *is uniformly persistent.*

Corollary 4.2. *Assume that* $(\mathbf{A_1})$ *and* $(\mathbf{A_2'}) - (\mathbf{A_3'})$ *hold. Let* $(s(t), x(t))$ *be any positive solution of model (1.4). Then the conclusions given below hold.*

(i) If $x(t)$ *is strongly persistent, then* $s(t)$ *is also strongly persistent;*

(ii) If $x(t)$ *is uniformly persistent, then* $s(t)$ *is also uniformly persistent.*

Remark 4.3. Similar conclusions as in Corollaries 4.1 and 4.2 also can be established for special model (1.5). We here omit them.

Remark 4.4. It is easy to see that when $\tau(t) \equiv 0$ then model (1.4) degrades into model (1.1). Clearly, in this case, Corollaries 4.1 and 4.2 become to Theorems 3.2 and 3.3 given in [28]. In addition, we also see that the corresponding results: Theorems 2 and 3 in [7] and Theorem 5.2.2 in [43] are as the special cases of Corollaries 4.1 and 4.2.

5. Global attractivity of trivial solution

In this section, we investigate the global attractivity of trivial solution of which microorganism species x vanishes in model (1.3). We have the following conclusion.

Theorem 5.1. *Assume that* $(\mathbf{A_1}) - (\mathbf{A_4})$ *hold. If there exists a positive constant* λ *such that*

$$\limsup_{t\to\infty} \int_t^{t+\lambda} (-d(\xi) + Q(\xi, s^*_\xi))\mathrm{d}\xi \le 0,$$

then trivial solution $(s^*(t), 0)$ *of model* (1.3) *is globally attractive. That is, for any positive solution* $(s(t), x(t))$ *of model* (1.3), $\lim_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty}(s(t) - s^*(t)) = 0$.

Proof. We first prove $\lim_{t\to\infty} x(t) = 0$. The proof process is divided into the following two cases.

Case 1. Assume $\limsup_{t\to\infty} \int_t^{t+\lambda}(-d(\xi) + Q(\xi, s_\xi^*))\mathrm{d}\xi < 0$. Then, it follows that $\limsup_{t\to\infty} t^{-1}\int_0^t(-d(\xi) + Q(\xi, s_\xi^*))\mathrm{d}\xi < 0$. From assumption ($\mathbf{A_2}$), we can choose small enough constants $\varepsilon > 0$, $\delta > 0$ and an enough large $T > 0$ such that

$$t^{-1}\int_0^t(-d(\xi) + Q(\xi, s_\xi^* + \varepsilon))\mathrm{d}\xi < -\delta \quad \text{for all} \quad t \ge T. \tag{5.1}$$

Let $(s(t), x(t))$ be any positive solution of model (1.3). Due to

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} \le a(t) - b(t)s(t) \quad \text{for all} \quad t \ge 0,$$

then combining conclusion (i) of Lemma 2.1 and the comparison principle, one deduces that there exists a $T_1 \ge T$ such that

$$s(t) \le s^*(t) + \varepsilon \quad \text{for all} \quad t \ge T_1. \tag{5.2}$$

Integrating the second equation of model (1.3), by assumption ($\mathbf{A_3}$) we obtain

$$x(t) = x(T_1+\tau)e^{\int_{T_1+\tau}^t(-d(\xi)+Q(\xi,s_\xi))\mathrm{d}\xi} \le x(T_1+\tau)e^{\int_{T_1+\tau}^t(-d(\xi)+Q(\xi,s_\xi^*+\varepsilon))\mathrm{d}\xi}$$

for any $t > T_1 + \tau$. Hence, (5.1) implies that $\lim_{t\to\infty} x(t) = 0$.

Case 2. Assume $\limsup_{t\to\infty} \int_t^{t+\lambda}(-d(\xi) + Q(\xi, s_\xi^*))\mathrm{d}\xi = 0$. Let $(s(t), x(t))$ be any positive solution of model (1.3) with initial condition (2.1). Firstly, Theorem 3.1 shows that there exists a constant $K > 0$ such that $s(t) \le K$ and $x(t) \le K$ for all $t \ge 0$.

For any given constant $\varepsilon > 0$, suppose that for any $t_0 \ge 0$ one has $x(t) \ge \varepsilon$ for all $t \ge t_0$, then $\liminf_{t\to\infty} x(t) > 0$, i.e., $x(t)$ is strongly persistent. By Proposition 4.1 in Remark 4.1, then $s(t)$ is also strongly persistent. Therefore, there exists a constant $\sigma > 0$, which is independent of any t_0, such that $s(t) \ge \sigma$ for all $t \ge 0$. Consequently, we obtain

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = g_1(t, s(t), x(t)) \le a(t) - b(t)s(t) - \varepsilon P(t, \sigma) \tag{5.3}$$

for all $t \ge t_0$. From (5.3) we further deduce that

$$s(t) \le s(t_0)e^{-\int_{t_0}^t b(\xi)\mathrm{d}\xi} + \int_{t_0}^t (a(\xi) - \varepsilon P(\xi, \sigma))e^{-\int_\xi^t b(u)\mathrm{d}u}\mathrm{d}\xi$$

for any $t \ge t_0$. Thanks to $s^*(t) = s^*(t_0)e^{-\int_{t_0}^t b(\xi)\mathrm{d}\xi} + \int_{t_0}^t a(\xi)e^{-\int_\xi^t b(u)\mathrm{d}u}\mathrm{d}\xi$. We conclude that

$$\begin{aligned}
s(t) - s^*(t) \le& (s(t_0) - s^*(t_0))e^{-\int_{t_0}^t b(\xi)\mathrm{d}\xi} - \int_{t_0}^t (\varepsilon P(\xi, \sigma))e^{-\int_\xi^t b(u)\mathrm{d}u}\mathrm{d}\xi \\
\le& 2Ke^{-\int_{t_0}^t b(\xi)\mathrm{d}\xi} - \varepsilon \inf_{t\ge0}\{P(t, \sigma)\}\int_{t_0}^t e^{-\int_\xi^t b(u)\mathrm{d}u}\mathrm{d}\xi \\
\le& 2Ke^{-\int_{t_0}^t b(\xi)\mathrm{d}\xi} - \varepsilon \inf_{t\ge0}\{P(t, \sigma)\}\frac{1}{\bar{b}}(1 - e^{-\bar{b}(t-t_0)}),
\end{aligned} \tag{5.4}$$

where $\bar{b} = \sup_{t\geq 0}\{b(t)\} > 0$. Accordingly, (5.4) indicates that there are constants $\beta > 0$ and $T_0 > 0$, which are independent of t_0, such that for any $t \geq T_0 + t_0$, $s(t) - s^*(t) \leq -\beta$. This together with assumption ($\mathbf{A_4}$) yields

$$Q(t, s_t) - Q(t, s_t^*) \leq -h(t) \quad \text{for all} \quad t \geq T_0 + t_0 + \tau. \tag{5.5}$$

By $\liminf_{t\to\infty} \int_t^{t+\alpha} h(\xi)\mathrm{d}\xi > 0$, we can choose a positive integer k such that $\liminf_{t\to\infty} \int_t^{t+k\lambda} h(\xi)\mathrm{d}\xi > \delta$, where $\delta > 0$ is a constant. Let constant $\varepsilon_0 > 0$ small enough such that $k\varepsilon_0 - \delta < 0$. Furthermore, there exists a $T_1 > T_0$, which does not depend on t_0, such that $\int_t^{t+\lambda}(-d(\xi) + Q(\xi, s_\xi^*))\mathrm{d}\xi < \varepsilon_0$ and $\int_t^{t+k\lambda} h(\xi)\mathrm{d}\xi > \delta$ for all $t \geq T_1$. For any $t > T_2 \triangleq T_1 + t_0 + \tau$, we choose an integer $p \geq 0$ such that $t \in [T_2 + pk\lambda, T_2 + (p+1)k\lambda)$, then we have

$$\begin{aligned}
&\int_{T_2}^{t} (-d(\xi) + Q(\xi, s_\xi))\mathrm{d}\xi \\
=&\int_{T_2}^{t} (-d(\xi) + Q(\xi, s_\xi^*))\mathrm{d}\xi + \int_{T_2}^{t} (Q(\xi, s_\xi) - Q(\xi, s_\xi^*))\mathrm{d}\xi \\
\leq&\int_{T_2}^{T_2+pk\lambda} (-d(\xi) + Q(\xi, s_\xi^*))\mathrm{d}\xi + \int_{T_2+pk\lambda}^{t} (-d(\xi) + Q(\xi, s_\xi^*))\mathrm{d}\xi \\
&- \int_{T_2}^{T_2+pk\lambda} h(\xi)\mathrm{d}\xi - \int_{T_2+pk\lambda}^{t} h(\xi)\mathrm{d}\xi \\
\leq& p(k\varepsilon_0 - \delta) + k\lambda(\max_{t\geq 0}\{d(t) + Q(t, s_t^*)\} + \max_{t\geq 0}|h(t)|).
\end{aligned} \tag{5.6}$$

This combines the second equation of model (1.3) for $t \in [T_2, t)$, one obtains

$$\begin{aligned}
x(t) =& x(T_2)e^{\int_{T_2}^{t}(-d(\xi)+Q(\xi,s_\xi))\mathrm{d}\xi} \\
\leq& Ke^{p(k\varepsilon_0-\delta)+k\lambda(\max_{t\geq 0}\{d(t)+Q(t,s_t^*)\}+\max_{t\geq 0}|h(t)|)}.
\end{aligned} \tag{5.7}$$

When $t \to \infty$ we have $p \to \infty$ and hence $p(k\varepsilon_0 - \delta) + k\lambda(\max_{t\geq 0}\{d(t) + Q(t, s_t^*)\} + \max_{t\geq 0}|h(t)|) \to -\infty$. Thus, from (5.7) we finally obtain $x(t) \to 0$ as $t \to \infty$, which leads to a contradiction with the fact that $x(t) \geq \varepsilon$ for all $t \geq t_0$. Therefore, there is a $t_1 > 0$ such that $x(t_1) < \varepsilon$.

Now, we prove $x(t) \leq \varepsilon e^M$ for all $t \geq t_1$, where

$$M = \max_{t\geq 0}\{d(t) + Q(t, K)\}(T_1 + \tau) + k\lambda(\max_{t\geq 0}\{d(t) + Q(t, s_t^*)\} + \max_{t\geq 0}|h(t)|).$$

In fact, if this conclusion does not hold, then there exists a $t_2 > t_1$ such that $x(t_2) > \varepsilon e^M$. Accordingly, there exists a $t_3 \in (t_1, t_2)$ such that $x(t_3) = \varepsilon$ and $x(t) > \varepsilon$ for all $t \in (t_3, t_2]$. If $t_2 - t_3 \leq T_1 + \tau$, then we have

$$x(t_2) = x(t_3)e^{\int_{t_3}^{t_2}(-d(\xi)+Q(\xi,s_\xi))\mathrm{d}\xi} \leq \varepsilon e^{\max_{t\geq 0}\{d(t)+Q(t,K)\}(T_1+\tau)}. \tag{5.8}$$

Hence, we have $x(t_2) \leq \varepsilon e^M$, a contradiction. If $t_2 - t_3 > T_1 + \tau$, then similar to the previous argument we can get $Q(t, s_t) - Q(t, s_t^*) \leq -h(t)$ for all $t \geq T_1 + t_3 + \tau$ and $\int_t^{t+k\lambda} h(\xi)\mathrm{d}\xi > \delta$ for all $t \geq T_1$. Let $T_2 = T_1 + \tau + t_3$. Using (5.6) and (5.7), we hence have

$$\begin{aligned}
x(t_2) =& x(t_3)e^{\int_{t_3}^{t_2}(-d(\xi)+Q(\xi,s_\xi))\mathrm{d}\xi} \leq \varepsilon e^{\int_{t_3}^{T_2}(-d(\xi)+Q(\xi,K))\mathrm{d}\xi + \int_{T_2}^{t_2}(-d(\xi)+Q(\xi,s_\xi))\mathrm{d}\xi} \\
\leq& \varepsilon e^{(\max_{t\geq 0}\{d(t)+Q(t,K)\}(T_1+\tau)+k\lambda(\max_{t\geq 0}\{d(t)+Q(t,s_t^*)\}+\max_{t\geq 0}|h(t)|))} = \varepsilon e^M,
\end{aligned}$$

which is impossible. Therefore, $x(t) \leq \varepsilon e^M$ for all $t \geq t_1$. By the arbitrariness of ε, we finally get $\lim_{t\to\infty} x(t) = 0$.

For any enough small constant $\beta > 0$, we consider the following equation:

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = a(t) - b(t)s(t) - \beta P(t, s^*(t) + \varepsilon). \tag{5.9}$$

Let $s^*_\beta(t)$ be the positive solution of (5.9) with initial value $s^*_\beta(0) = s^*(0)$. From conclusions (i) and (ii) of Lemma 2.1, it follows that $s^*_\beta(t)$ is globally uniformly attractive on $t \geq 0$, and for any $\varepsilon > 0$ there exists a $\beta > 0$ such that $|s^*_\beta(t) - s^*(t)| < \frac{\varepsilon}{2}$ for all $t \geq 0$. Since $\lim_{t\to\infty} x(t) = 0$, there exists a $T_2 > T_1$ such that $x(t) < \beta$ for all $t \geq T_2$. Thus,

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} \geq a(t) - b(t)s(t) - \beta P(t, s^*(t) + \varepsilon).$$

The global uniform attractivity of $s^*_\beta(t)$ and the comparison principle indicate that there exists a $T_3 > T_2$ such that $s(t) > s^*_\beta(t) - \frac{\varepsilon}{2}$ for all $t \geq T_3$. Thus, $s(t) > s^*(t) - \varepsilon$ for all $t \geq T_3$. Combining (5.2), it follows that $|s(t) - s^*(t)| < \varepsilon$ for all $t \geq T_3$. Therefore, we finally have $\lim_{t\to\infty}(s(t) - s^*(t)) = 0$. This completes the proof. □

As the consequences of Theorem 5.1, some corollaries on the global attractivity of trivial solution for special models (1.4) and (1.5) are given as follows.

Corollary 5.1. *Assume that* $(\mathbf{A_1})$, $(\mathbf{A_2'}) - (\mathbf{A_4'})$ *hold. Provided that there exists a positive constant* λ *such that*

$$\limsup_{t\to\infty} \int_t^{t+\lambda} (-d(\xi) + Q(\xi, s^*(\xi - \tau(\xi))))\mathrm{d}\xi \leq 0,$$

then trivial solution $(s^*(t), 0)$ *of model* (1.4) *is globally attractive.*

Remark 5.1. Similar conclusions to Corollary 5.1 can also be established for special model (1.5). We omit it here.

Remark 5.2. It is easy to see that when $\tau(t) \equiv 0$, Corollary 5.1 extended and improved the corresponding result given in [28], that is Theorem 4.3 in [28].

Remark 5.3. On the basis of Case 1 in the proof of Theorem 5.1, where we used the fact that from $\limsup_{t\to\infty} \int_t^{t+\lambda}(-d(\xi) + Q(\xi, s^*_\xi))\mathrm{d}\xi < 0$ it follows that $\limsup_{t\to\infty} t^{-1} \int_0^t(-d(\xi) + Q(\xi, s^*_\xi))\mathrm{d}\xi < 0$, we can propose the following open question:

Whether the condition $\limsup_{t\to\infty} \int_t^{t+\lambda}(-d(\xi) + Q(\xi, s^*_\xi))\mathrm{d}\xi \leq 0$ in Theorem 5.1 can be weaken to $\limsup_{t\to\infty} t^{-1} \int_0^t(-d(\xi) + Q(\xi, s^*_\xi))\mathrm{d}\xi \leq 0$.

Remark 5.4. Similarly to Remark 4.2, a meaningful open question is whether the condition $\liminf_{t\to\infty} P(t, s) > 0$ for any $s > 0$ in assumption $(\mathbf{A_3})$ can be changed to that for any $s > 0$ there exists a positive constant β such that $\liminf_{t\to\infty} \int_t^{t+\beta} P(\xi, s)$ $\mathrm{d}\xi > 0$ to guarantee that the conclusions in Theorem 5.1 still hold.

Remark 5.5. We see that assumption $(\mathbf{A_4})$ is added in Theorem 5.1. However, in Theorems 4.2 and 4.3, only assumptions $(\mathbf{A_1})$–$(\mathbf{A_3})$ are required. Therefore, an interesting open problem is whether assumption $(\mathbf{A_4})$ can be removed in Theorem 5.1. That is, whether we also can prove $x(t) \to 0$ as $t \to \infty$ in Case 2 in the proof of Theorem 5.1 for any positive solution $(s(t), x(t))$ of model (1.3) only when assumptions $(\mathbf{A_1})$–$(\mathbf{A_3})$ are satisfied.

6. Periodic system

When $a(t)$, $b(t)$, $d(t)$, $P(t,s)$ and $Q(t,\phi)$ are ω-periodic functions with respect to time t, that is, $a(t+\omega)=a(t)$, $b(t+\omega)=b(t)$, $d(t+\omega)=d(t)$, $P(t+\omega,s)=P(t,s)$ and $Q(t+\omega,\phi)=Q(t,\phi)$ for any $t\in R_+$, then model (1.3) is said to be ω-periodic model. Particularly, in models (1.4) and (1.5), we further assume that $\tau(t)$ and $c(t,\theta)$ are ω-periodic functions with respect to t, then models (1.4) and (1.5) are also said to be ω-periodic models.

We see that for ω-periodic model (1.3) assumptions $(\mathbf{A_2})$–$(\mathbf{A_4})$ degenerate to the following form:

$(\mathbf{B_1})$ $a(t)$, $b(t)$ and $d(t)$ denote ω-periodic continuous functions defined on $t\geq 0$ and $a(t)\geq 0$ for $t\in[0,\omega]$, $\int_0^\omega b(\xi)\mathrm{d}\xi>0$ and $\int_0^\omega d(\xi)\mathrm{d}\xi>0$.

$(\mathbf{B_2})$ $P(t,s)$ and $Q(t,\phi)$ are continuous functions for $(t,s)\in R_+^2$ and $(t,\phi)\in R_+\times C^+$ and ω-periodic with respect to t, respectively. $P(t,0)=Q(t,0)\equiv 0$ for any $t\in[0,\omega]$. For any positive constant H, there exists a $K=K(H)>0$ such that $|P(t,s_1)-P(t,s_2)|\leq K|s_1-s_2|$ for any $(t,s_i)\in R_+^2$ with $0\leq s_i\leq H$ $(i=1,2)$ and $|Q(t,\phi_1)-Q(t,\phi_2)|\leq K|\phi_1-\phi_2|$ for any $(t,\phi_i)\in R_+\times C^+$ with $|\phi_i|\leq H$ $(i=1,2)$.

$(\mathbf{B_3})$ For any $s>0$, $P(t,s)>0$ for any $t\in[0,\omega]$. For any $t\in[0,\omega]$, $P(t,s)$ is nondecreasing for $s\in R_+$, and $Q(t,\phi_1)\leq Q(t,\phi_2)$ for any $\phi_1,\phi_2\in C^+$ with $\phi_1\leq\phi_2$.

$(\mathbf{B_4})$ For any constants $H>\beta>0$, there exists an ω-periodic continuous function $h(t)$ defined for $t\geq 0$ satisfying $\int_0^\omega h(\xi)\mathrm{d}\xi>0$ such that for any $t\in[0,\omega]$, $\phi_1,\phi_2\in C^+$ with $\phi_i\leq H$ $(i=1,2)$ and $\phi_1-\phi_2\geq\beta$, one has $Q(t,\phi_1)-Q(t,\phi_2)\geq h(t)$.

Particularly, for ω-periodic models (1.4) and (1.5) the above assumptions $(\mathbf{B_2})-(\mathbf{B_4})$ will degenerate into the following forms.

$(\mathbf{B_2'})$ $P(t,s)$ and $Q(t,s)$ are continuous functions for $(t,s)\in R_+^2$ and ω-periodic with respect to t, respectively. $P(t,0)=Q(t,0)\equiv 0$ for any $t\in[0,\omega]$. For any positive constant H, there exists a $K=K(H)>0$ such that $|P(t,s_1)-P(t,s_2)|\leq K|s_1-s_2|$ and $|Q(t,s_1)-Q(t,s_2)|\leq K|s_1-s_2|$ for any $(t,s_i)\in[0,\omega]\times R_+$ with $0\leq s_i\leq H$ $(i=1,2)$.

$(\mathbf{B_3'})$ For any $s>0$, $P(t,s)>0$ for any $t\in[0,\omega]$. For any $t\in[0,\omega]$, $P(t,s)$ and $Q(t,s)$ are nondecreasing for any $s\in R_+$.

$(\mathbf{B_4'})$ For any constants $H>\beta>0$, there exists an ω-periodic continuous function $h(t)$ defined for $t\geq 0$ satisfying $\int_0^\omega h(\xi)\mathrm{d}\xi>0$ such that for any $t\in[0,\omega]$ and $s_1,s_2\in R_+$ with $s_i\leq H$ $(i=1,2)$ and $s_1-s_2\geq\beta$ one has $Q(t,s_1)-Q(t,s_2)\geq h(t)$.

In addition, for ω-periodic model (1.2) we assume that function $\mu(s)$ satisfies the following assumption $(\mathbf{P})$. Then we can prove that all assumptions $(\mathbf{B_1})$ and $(\mathbf{B_2'})-(\mathbf{B_4'})$ are satisfied for ω-periodic model (1.2).

$(\mathbf{P})$ $\mu(s)$ is a continuous differentiable function and $\mu'(s)>0$ for any $s\geq 0$, and $\mu(0)=0$.

Consider the above nonautonomous linear equation (2.2), one further assumes that $c(t)$ and $l(t)$ denote ω-periodic continuous functions defined on $t\geq 0$ and $c(t)\geq 0$ for all $t\in[0,\omega]$. Based on Lemma 2.1 the result is given as follows.

Lemma 6.1. *For ω-periodic equation (2.2), suppose that $\int_0^\omega l(t)\mathrm{d}t>0$. Then we have the following conclusions.*

(i) Equation (2.2) has a unique ω-periodic solution $v^*(t)$ which is globally uni-

formly attractive. Particularly, if $\int_0^\omega c(t)\mathrm{d}t > 0$*, then* $v^*(t)$ *is also positive;*

(ii) Let $v(t)$ *be the solution of equation* (2.2) *and* $\bar{v}(t)$ *be the solution of equation* (2.2) *after replacing* $c(t)$ *with another* ω*-periodic continuous function* $\bar{c}(t)$*. If* $\bar{v}(0) = v(0)$*, then there exists a constant* $L > 0$ *that only depends on* $l(t)$*, such that* $\sup_{t\in R_+} |v(t) - \bar{v}(t)| \le L \sup_{t\in[0,\omega]} |c(t) - \bar{c}(t)|$*;*

(iii) If $\int_0^\omega c(t)\mathrm{d}t > 0$*, then* $m_2^{-1}e^{-r_2\omega} \le \liminf_{t\to\infty} v(t) \le \limsup_{t\to\infty} v(t) \le m_1^{-1}e^{r_1\omega}$ *for any solution* $v(t)$ *of equation* (2.2)*, where the positive constants* m_1 *and* m_2 *are chosen satisfying*

$$\int_0^\omega (l(t) - c(t)m_1)\mathrm{d}t > 0, \quad \int_0^\omega (l(t) - c(t)m_2)\mathrm{d}t < 0$$

and $r_1 = \sup_{t\in[0,\omega]} \{|l(t)| + c(t)m_1\}$ *and* $r_2 = \sup_{t\in[0,\omega]} \{|l(t)| + c(t)m_2\}$*.*

Furthermore, we consider subsystem (3.1) of nutrient species, when assumption $(\mathbf{B_1})$ holds, then by Lemma 6.1 the fixed solution $s^*(t)$ can be chosen as the ω-periodic solution of subsystem (3.1).

In addition, the periodicity of $-d(t) + Q(t, s_t^*)$, $-d(t) + Q(t, s^*(t - \tau(t))$ and $-d(t) + Q(t, \int_{-\tau}^0 c(t,\theta)s^*(t+\theta)d\theta)$ yields that

$$\begin{aligned}
&\liminf_{t\to\infty} \int_t^{t+\omega} (-d(\xi) + Q(\xi, s_\xi^*))\mathrm{d}\xi \\
=&\limsup_{t\to\infty} \int_t^{t+\omega} (-d(\xi) + Q(\xi, s_\xi^*))\mathrm{d}\xi = \int_0^\omega (-d(\xi) + Q(\xi, s_\xi^*))\mathrm{d}\xi, \\
&\liminf_{t\to\infty} \int_t^{t+\omega} (-d(\xi) + Q(\xi, s^*(\xi - \tau(\xi))))\mathrm{d}\xi \\
=&\limsup_{t\to\infty} \int_t^{t+\omega} (-d(\xi) + Q(\xi, s^*(\xi - \tau(\xi))))\mathrm{d}\xi \\
=&\int_0^\omega (-d(\xi) + Q(\xi, s^*(\xi - \tau(\xi))))\mathrm{d}\xi
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{t\to\infty} \int_t^{t+\omega} (-d(\xi) + Q(\xi, \int_{-\tau}^0 c(\xi,\theta)s^*(\xi+\theta)\mathrm{d}\theta))\mathrm{d}\xi \\
=&\limsup_{t\to\infty} \int_t^{t+\omega} (-d(\xi) + Q(\xi, \int_{-\tau}^0 c(\xi,\theta)s^*(\xi+\theta)\mathrm{d}\theta))\mathrm{d}\xi \\
=&\int_0^\omega (-d(\xi) + Q(\xi, \int_{-\tau}^0 c(\xi,\theta)s^*(\xi+\theta)\mathrm{d}\theta))\mathrm{d}\xi.
\end{aligned}$$

Therefore, based on the main results given in [35, 36] on the existence of positive periodic solution for general delayed periodic population dynamical models, as the applications of Theorems 4.1-4.3 and Corollary 4.1, one can obtain the conclusions for the existence of positive periodic solution for periodic models (1.3)–(1.5) as follows.

Corollary 6.1. *For* ω*-periodic model (1.3), assume that* $(\mathbf{B_1})$–$(\mathbf{B_4})$ *hold and* $\int_0^\omega a(\eta)\mathrm{d}\eta > 0$*. Then the conclusions given below are equivalent,*

(i) model (1.3) has a positive ω*-periodic solution;*

(ii) species x in model (1.3) is uniformly persistent;

(iii) $\int_0^\omega(-d(\xi)+Q(\xi,s^*_\xi))\mathrm{d}\xi>0$.

Corollary 6.2. *For ω-periodic model (1.4), assume that* $(\mathbf{B_1})$ *and* $(\mathbf{B_2'})$–$(\mathbf{B_4'})$ *hold and* $\int_0^\omega a(\eta)\mathrm{d}\eta>0$*. Then the conclusions given below are equivalent,*

(i) model (1.4) has a positive ω-periodic solution;

(ii) species x in model (1.4) is uniformly persistent;

(iii) $\int_0^\omega(-d(\xi)+Q(\xi,s^*(\xi-\tau(\xi))))\mathrm{d}\xi>0$.

Let $u=\xi-\tau(\xi)$. Since $\max_{\xi\in[0,\omega]}\tau'(\xi)<1$, then $u=\xi-\tau(\xi)$ is increasing for all $\xi\geq 0$. Hence, $u=\xi-\tau(\xi)$ has the inverse function $\xi=\psi(u)$. When $\tau(\xi)$ is ω-periodic, then $\tau'(\xi)$ is also ω-periodic and $\psi(u+\omega)=\psi(u)+\omega$. We can prove that the condition *(iii)* in Corollary 6.2 is equivalent to

$$\int_0^\omega(-d(u)+\frac{Q(\psi(u),s^*(u))}{1-\tau'(\psi(u))})\mathrm{d}u>0. \tag{6.1}$$

In fact, we have

$$\int_0^\omega(-d(\xi)+Q(\xi,s^*(\xi-\tau(\xi))))\mathrm{d}\xi=\int_0^\omega(-d(\xi))\mathrm{d}\xi+\int_0^\omega Q(\xi,s^*(\xi-\tau(\xi)))\mathrm{d}\xi.$$

Since the function $\frac{Q(\psi(u),s^*(u))}{1-\tau'(\psi(u))}$ is ω-periodic, we further have

$$\int_0^\omega Q(\xi,s^*(\xi-\tau(\xi)))\mathrm{d}\xi=\int_{-\tau(0)}^{\omega-\tau(\omega)}\frac{Q(\psi(u),s^*(u))}{1-\tau'(\psi(u))}\mathrm{d}u=\int_0^\omega\frac{Q(\psi(u),s^*(u))}{1-\tau'(\psi(u))}\mathrm{d}u.$$

Therefore, we finally obtain

$$\int_0^\omega(-d(\xi)+Q(\xi,s^*(\xi-\tau(\xi))))\mathrm{d}\xi=\int_0^\omega(-d(u)+\frac{Q(\psi(u),s^*(u))}{1-\tau'(\psi(u))})\mathrm{d}u.$$

Particularly, when $\tau(t)\equiv\tau$, we have $\psi(u)=u+\tau$ and hence condition (6.1) becomes into

$$\int_0^\omega(-d(u)+Q(u+\tau,s^*(u)))\mathrm{d}u>0.$$

Therefore, for periodic model (1.2), from Corollary 6.2 and the above discussions we can obtain the following result.

Corollary 6.3. *For ω-periodic model (1.2), assume that* $(\mathbf{P})$ *holds and* $\int_0^\omega s^0(t)\mathrm{d}t>0$*. Then the conclusions given below are equivalent,*

(i) model (1.2) has a positive ω-periodic solution;

(ii) species x in model (1.2) is uniformly persistent;

(iii) $\omega^{-1}\int_0^\omega\mu(v^*(u))\mathrm{d}u>D$*, where $v^*(t)$ is the unique positive ω-periodic solution of equation* $\frac{\mathrm{d}s(t)}{\mathrm{d}t}=Ds^0(t)-Ds(t)$.

Corollary 6.4. *For ω-periodic model (1.5), assume that* $(\mathbf{B_1})$ *and* $(\mathbf{B_2'})$–$(\mathbf{B_4'})$ *hold and* $\int_0^\omega a(\eta)\mathrm{d}\eta>0$*, then the conclusions given below are equivalent,* *(i)* *model (1.5) has a positive ω-periodic solution;*

(ii) *species x in model (1.5) is uniformly persistent;*

(iii) $\int_0^\omega(-d(\xi)+Q(\xi,\int_{-\tau}^0 c(\xi,\theta)s^*(\xi+\theta)\mathrm{d}\theta))\mathrm{d}\xi>0.$

Furthermore, as the consequences of Theorem 5.1 and Corollary 5.1, we can get the conclusions for the extinction of solutions for periodic models (1.3)–(1.5) as follows.

Corollary 6.5. *For ω-periodic model (1.3), assume that* $(\mathbf{B_1})$–$(\mathbf{B_4})$ *hold. Then the following statements are equivalent,*

(i) $\int_0^\omega(-d(\xi)+Q(\xi,s^*_\xi))\mathrm{d}\xi\le 0$;

(ii) *Trivial periodic solution* $(s^*(t),0)$ *of model* (1.3) *is globally attractive. That is, for any positive solution* $(s(t),x(t))$ *of model* (1.3), $\lim_{t\to\infty}x(t)=0$ *and* $\lim_{t\to\infty}(s(t)-s^*(t))=0$.

Corollary 6.6. *For ω-periodic model (1.4), assume that* $(\mathbf{B_1})$, $(\mathbf{B_2'})$–$(\mathbf{B_4'})$ *hold. Then the following statements are equivalent,*

(i) $\int_0^\omega(-d(\xi)+Q(\xi,s^*(\xi-\tau(\xi))))\mathrm{d}\xi\le 0$;

(ii) *Trivial periodic solution* $(s^*(t),0)$ *of model* (1.4) *is globally attractive.*

Remark 6.1. Obviously, the condition (i) in Corollary 6.5 is equivalent to

$$\int_0^\omega(-d(u)+\frac{Q(\psi(u),s^*(u))}{1-\tau'(\psi(u))})\mathrm{d}u\le 0. \tag{6.2}$$

Particularly, when $\tau(t)\equiv\tau$, condition (6.2) becomes to $\int_0^\omega(-d(u)+Q(u+\tau,s^*(u)))\mathrm{d}u\le 0$. Therefore, for periodic model (1.2), from Corollary 6.6 and Remark 6.1 we can obtain the following conclusions.

Corollary 6.7. *For ω-periodic model (1.2), assume that* $(\mathbf{P})$ *holds. Then the following statements are equivalent,*

(i) $\omega^{-1}\int_0^\omega\mu(v^*(u))\mathrm{d}u\le D$, *where $v^*(t)$ is the unique positive ω-periodic solution of equation* $\frac{\mathrm{d}s(t)}{\mathrm{d}t}=Ds^0(t)-Ds(t)$;

(ii) *Trivial periodic solution* $(v^*(t),0)$ *of model* (1.2) *is globally attractive.*

Corollary 6.8. *For ω-periodic model (1.5), assume that* $(\mathbf{B_1})$, $(\mathbf{B_2'})$–$(\mathbf{B_4'})$ *hold. Then the following statements are equivalent,*

(i) $\int_0^\omega(-d(\xi)+Q(\xi,\int_{-\tau}^0 c(\xi,\theta)s^*(\xi+\theta)\mathrm{d}\theta))\mathrm{d}\xi\le 0$;

(ii) *Trivial periodic solution* $(s^*(t),0)$ *of model* (1.5) *is globally attractive.*

Remark 6.2. From Corollaries 6.3 and 6.7, we easily see that the main results Theorem 1 and Theorem 2 established in [1] are extended and improved to general periodic chemostat model with delayed microorganism growth.

Remark 6.3. Based on Remarks 3.1, 4.2 and 5.4, a meaningful open problem is whether the condition $P(t,s)>0$ for any $s>0$ and $t\in[0,\omega]$ can be changed to that $\int_0^\omega P(\xi,s)\mathrm{d}\xi>0$ for any $s>0$ to guarantee that all conclusions in Corollaries 6.1, 6.2, 6.4-6.6 and 6.8 still hold.

Remark 6.4. For periodic models (1.3)–(1.5), as a special case of open problem proposed in Remark 5.5 we have that whether assumption $(\mathbf{B_4})$ or $(\mathbf{B_4'})$ can be removed in Corollaries 6.1, 6.2, 6.4-6.6 and 6.8, respectively, and to guarantee that the same conclusions still hold.

7. Numerical examples

In this section, several numerical examples for models (1.4) and (1.5) are given to illustrate the main conclusions established in the above sections. The numerical simulations are presented by Matlab.

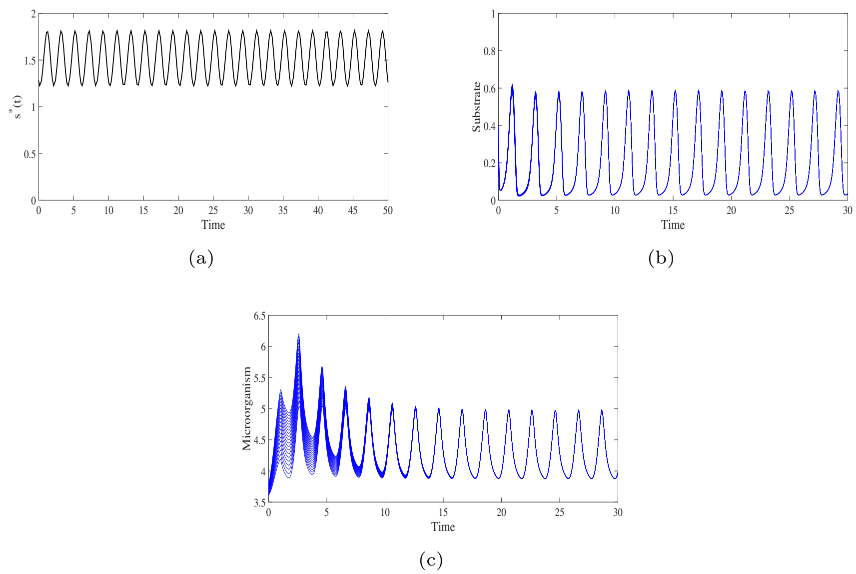

Figure 1. (a): the numerical simulation of $s^*(t)$; (b)-(c): the solutions for model (1.4) with the initial functions $(s(\eta), x(\eta)) = (0.2+0.01i, 3.6+0.01i)$ for all $\eta \in [-1.1, 0]$, $i = 1, 2, ..., 20$ converge to a positive periodic solution as $t \to \infty$.

Example 7.1. In model (1.4), we take $a(t) = 1.3+0.6\sin\pi t$, $b(t) = 0.9+0.5\cos\pi t$, $d(t) = 0.6 + 0.4\sin\pi t$, $P(t,s) = (1 + \cos\pi t)\frac{1.5s}{0.3+s}$, $Q(t,s) = (0.9 + 0.7\sin\pi t)\frac{1.1s}{0.1+s}$ and $\tau(t) = 1 + \frac{1}{10}\sin\pi t$. Obviously, model (1.4) is 2-periodic for time t. We have $\frac{\partial P(t,s)}{\partial s} = (1 + \cos\pi t)\frac{0.45}{(0.3+s)^2}$ and $\frac{\partial Q(t,s)}{\partial s} = (0.9 + 0.7\sin\pi t)\frac{0.11}{(0.1+s)^2}$. It is evident that all conditions in assumptions $(\mathbf{B_1})$ and $(\mathbf{B_2'})$–$(\mathbf{B_4'})$ hold, except for the condition "For any $s > 0$, $P(t,s) > 0$ for any $t \in [0,\omega]$" in $(\mathbf{B_3'})$. However, we have $\int_0^2 P(t,s)dt = 2\frac{1.5s}{0.3+s} > 0$ for any $s > 0$.

From the equation $\frac{\mathrm{d}s^*(t)}{\mathrm{d}t} = 1.3 + 0.6\sin\pi t - (0.9 + 0.5\cos\pi t)s^*(t)$, we get the numerical simulation of $s^*(t)$, see Figure 1(a). By numerical calculation we can obtain

$$\int_0^2 Q(t, s^*(t-\tau(t)))\mathrm{d}t = \int_0^2 (0.9+0.7\sin\pi t)\frac{1.1s^*(t-(1+\frac{1}{10}\sin\pi t))}{0.1+s^*(t-(1+\frac{1}{10}\sin\pi t))}\mathrm{d}t$$
$$\approx 1.2661 > \int_0^2 d(t)\mathrm{d}t = 1.2.$$

The numerical simulations in Figure 1 show that the open problem given in Remark 6.3 corresponding to Corollary 6.2 may be true.

Example 7.2. In model (1.5), we take $a(t) = 2+1.9\sin 2\pi t$, $b(t) = 1+0.7\cos 2\pi t$, $d(t) = 1+0.9\sin 2\pi t$, $P(t,s) = (5+4.5\cos 2\pi t)\frac{s}{0.2+s}$, $Q(t,s) = (14+9\sin 2\pi t)\frac{0.1s}{0.1+s}$ and $c(t,\theta) = 1+0.45\cos 2\pi\theta$ for $\theta \in [-1,0]$. Obviously, model (1.5) is 1-periodic for time t. We have $\frac{\partial P(t,s)}{\partial s} = (5+4.5\cos 2\pi t)\frac{0.2}{(0.2+s)^2}$, $\frac{\partial Q(t,s)}{\partial s} = (14+9\sin 2\pi t)\frac{0.01}{(0.1+s)^2}$ and $\int_{-\tau}^0 c(t,\theta)\mathrm{d}\theta = \int_{-1}^0 (1+0.45\cos 2\pi\theta)\mathrm{d}\theta$=1. It is evident that $(\mathbf{B_1})$ and $(\mathbf{B_2'})$ – $(\mathbf{B_4'})$ hold.

From the equation $\frac{\mathrm{d}s^*(t)}{\mathrm{d}t} = 2+1.9\sin 2\pi t-(1+0.7\cos 2\pi t)s^*(t)$, we get the numerical simulation of $s^*(t)$, see Figure 2(a). By numerical calculation we further obtain

$$\begin{aligned}
&\int_0^\omega Q(t, \int_{-\tau}^0 c(t,\theta)s^*(t+\theta)\mathrm{d}\theta)\mathrm{d}t \\
=&\int_0^1 Q(t, \int_{-1}^0 (1+0.45\cos 2\pi\theta)s^*(t+\theta)\mathrm{d}\theta)\mathrm{d}t \\
=&\int_0^1 (14+9\sin 2\pi t)\frac{0.1\int_{-1}^0(1+0.45\cos 2\pi\theta)s^*(t+\theta)\mathrm{d}\theta}{0.1+\int_{-1}^0(1+0.45\cos 2\pi\theta)s^*(t+\theta)\mathrm{d}\theta}\mathrm{d}t \approx 1.3800 \\
>&\int_0^1 d(t)\mathrm{d}t = 1.
\end{aligned}$$

The numerical simulations in Figure 2 indicate that the conclusions in Corollary 6.4 are right.

Example 7.3. In model (1.4), we take $a(t) = 7.5+6\sin\pi t$, $b(t) = 1+0.5\cos\pi t$, $d(t) = 1.1+0.8\sin\pi t$, $P(t,s) = (1.2+1.2\cos\pi t)\frac{1.3s}{0.4+s}$, $Q(t,s) = (0.9+0.5\sin\pi t)\frac{1.2s}{0.6+s}$ and $\tau(t) = 1+\frac{1}{10}\sin\pi t$. Obviously, model (1.4) is 2-periodic for time t. We have $\frac{\partial P(t,s)}{\partial s} = (1.2+1.2\cos\pi t)\frac{0.52}{(0.4+s)^2}$ and $\frac{\partial Q(t,s)}{\partial s} = (0.9+0.5\sin\pi t)\frac{0.72}{(0.6+s)^2}$. It is easy to see that all conditions in assumptions $(\mathbf{B_1})$ and $(\mathbf{B_2'})-(\mathbf{B_4'})$ hold, except for the condition "For any $s>0$, $P(t,s)>0$ for any $t\in[0,\omega]$" in $(\mathbf{B_3'})$. However, we have $\int_0^2 P(t,s)dt = 2.4\frac{1.3s}{0.4+s} > 0$ for any $s>0$.

From $\frac{\mathrm{d}s^*(t)}{\mathrm{d}t} = 7.5+6\sin\pi t-(1+0.5\cos\pi t)s^*(t)$, we get the numerical simulation of $s^*(t)$, see Figure 3(a). By numerical calculation we further get

$$\int_0^2 Q(t, s^*(t-\tau(t)))\mathrm{d}t = \int_0^2 (0.9+0.5\sin\pi t)\frac{1.2s^*(t-(1+\frac{1}{10}\sin\pi t))}{0.6+s^*(t-(1+\frac{1}{10}\sin\pi t))}\mathrm{d}t$$
$$\approx 1.1035 < \int_0^2 d(t)\mathrm{d}t = 2.2.$$

The numerical simulations in Figure 3 imply that the open problem given in Remark 6.3 corresponding to Corollary 6.6 may be true.

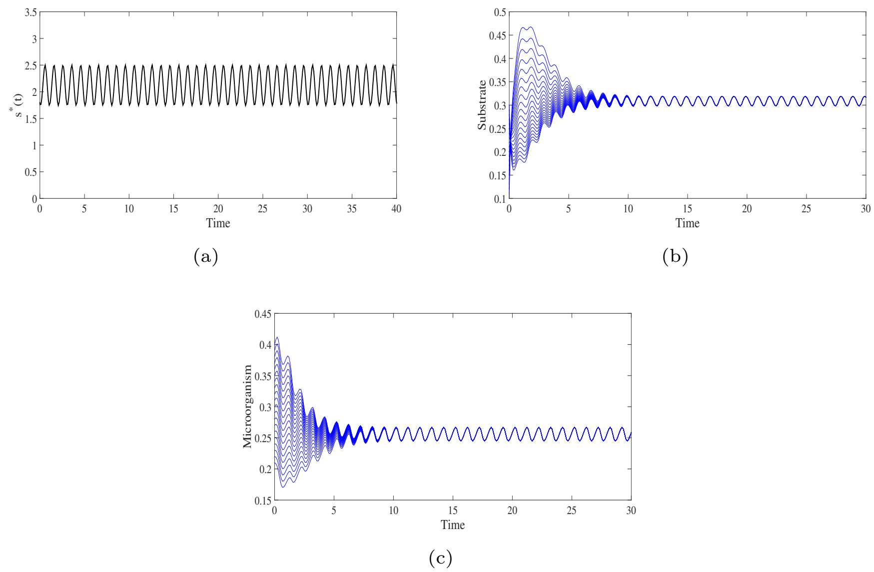

Figure 2. (a): the numerical simulation of $s^*(t)$; (b)-(c): the solutions for model (1.5) with the initial functions $(s(\eta), x(\eta)) = (0.1 + 0.01i, 0.2 + 0.01i)$ for all $\eta \in [-1, 0]$, $i = 1, 2, ..., 20$ converge to a positive periodic solution as $t \to \infty$.

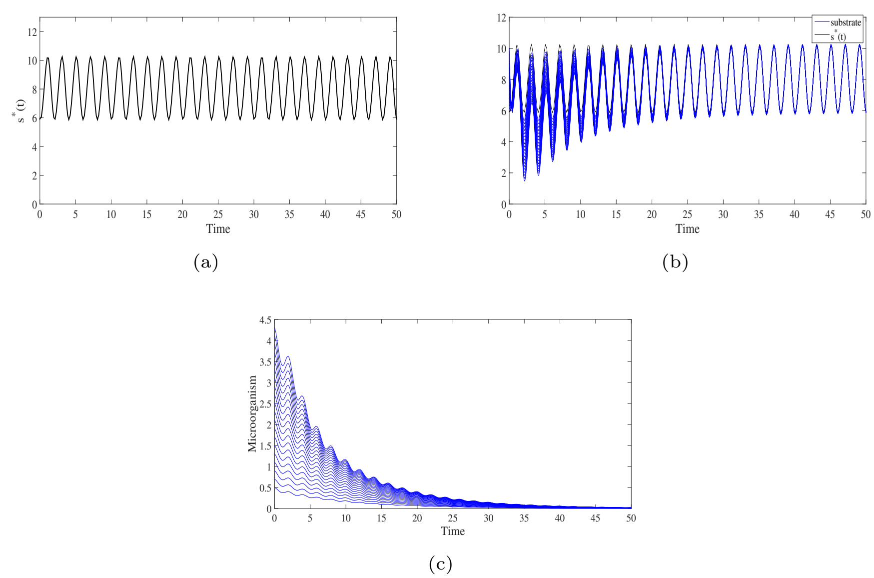

Figure 3. (a): the numerical simulation of $s^*(t)$; (b)-(c): the solutions for model (1.4) with the initial functions $(s(\eta), x(\eta)) = (6.2 + 0.2i, 0.5 + 0.2i)$ for all $\eta \in [-1.1, 0]$, $i = 1, 2, ..., 20$ converge to $(s^*(t), 0)$ as $t \to \infty$.

Example 7.4. In model (1.5),we take $a(t) = 0.5+0.4\sin\frac{1}{2}\pi t$, $b(t) = 0.4+0.3\cos\frac{1}{2}\pi t$, $d(t) = 0.8+0.7\sin\frac{1}{2}\pi t$, $P(t,s) = (1.2+0.8\cos\frac{1}{2}\pi t)\frac{1.2s}{0.2+s}$, $Q(t,s) = (0.8+0.7\sin\frac{1}{2}\pi t)$ $\frac{s}{0.2+s}$, $c(t,\theta) = 1+0.45\cos 2\pi\theta$, $\theta \in [-1, 0]$. Obviously, model (1.5) is 4-periodic for time t. We have $\frac{\partial P(t,s)}{\partial s}=(1.2+0.8\cos\frac{1}{2}\pi t)\frac{0.24}{(0.2+s)^2}$, $\frac{\partial Q(t,s)}{\partial s}=(0.8+0.7\sin\frac{1}{2}\pi t)\frac{0.2}{(0.2+s)^2}$ and $\int_{-1}^{0}(1+0.45\cos 2\pi\theta)\mathrm{d}\theta=1$. It is evident that $(\mathbf{B_1})$, $(\mathbf{B_2'})-(\mathbf{B_4'})$ hold.

From $\frac{\mathrm{d}s^*(t)}{\mathrm{d}t}=0.5+0.4\sin\frac{1}{2}\pi t-(0.4+0.3\cos\frac{1}{2}\pi t)s^*(t)$, we get the numerical simulation of $s^*(t)$, see Figure 4(a). By numerical calculation we obtain

$$\begin{aligned}
&\int_0^4 Q(t,\int_{-1}^0 c(t,\theta)s^*(t+\theta)\mathrm{d}\theta)\mathrm{d}t\\
=&\int_0^4 Q(t,\int_{-1}^0 (1+0.45\cos 2\pi\theta)s^*(t+\theta)\mathrm{d}\theta)\mathrm{d}t\\
=&\int_0^4 (0.8+0.7\sin t)\frac{\int_{-1}^0(1+0.45\cos 2\pi\theta)s^*(t+\theta)\mathrm{d}\theta}{0.2+\int_{-1}^0(1+0.45\cos 2\pi\theta)s^*(t+\theta)\mathrm{d}\theta}\mathrm{d}t\approx 1.1086\\
<&\int_0^4 d(t)\mathrm{d}t=3.2.
\end{aligned}$$

From the numerical simulations in Figure 4 we see that the conclusions in Corollary 6.8 are right.

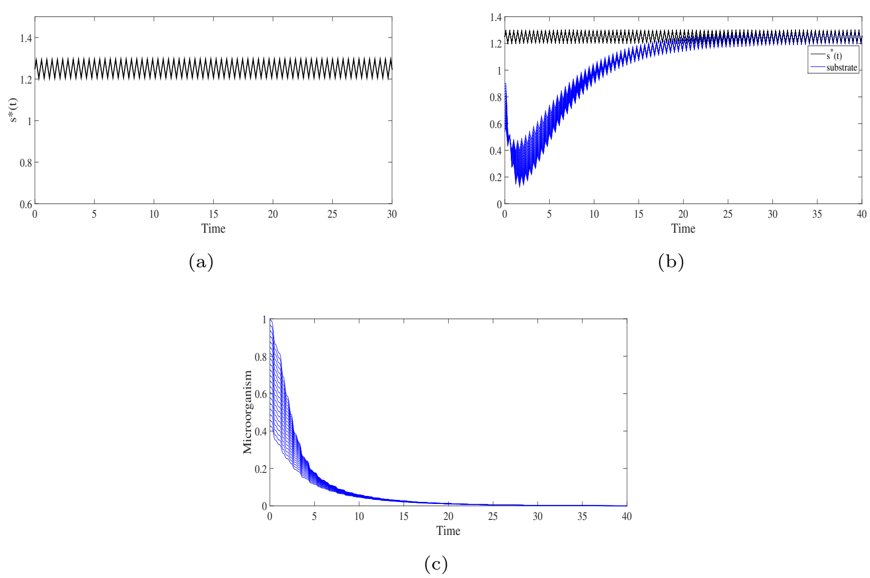

Figure 4. (a): the numerical simulation of $s^*(t)$; (b)-(c): the solutions for model (1.5) with the initial functions $(s(\eta),x(\eta))=(0.5+0.02i,0.4+0.03i)$ for all $\eta\in[-1,0]$, $i=1,2,\ldots,20$ converge to $(s^*(t),0)$ as $t\to\infty$, respectively.

8. Conclusion

In this article, we investigate a nonautonomous chemostat model with general delay in microorganism growth. A series of criteria on the positivity and ultimate boundedness of solutions, uniform persistence and strong persistence of system, global attractivity of trivial solution in which microorganisms species x vanishes are established by the approaches of reductionism, comparison principle and differential inequality techniques etc. The corresponding results of uniform persistence and extinction obtained in [7, 28, 43] are extended to the general delayed nonautonomous case. For the two special cases of model (1.3), i.e., model (1.4) and model (1.5), the sufficient criteria for the uniform persistence and strong persistence of microorganism species x are also obtained, respectively. Furthermore, for periodic model (1.3), we obtain the necessary and sufficient criteria for the existence of positive periodic solutions, the uniform persistence of microorganism species and the global attractivity of trivial periodc solution. Additionally, similar results are obtained for ω-periodic models (1.4) and (1.5). We also see that the main result on the existence of positive periodic solution established in [1,28] is improved and extended. Finally, our main theoretical results are illustrated by some special numerical examples.

We see that only one species and one nutrient is investigated in this paper, biologically, it is more proper to extend the model (1.3) to more general chemostat models, such as delayed nonautonomous chemostat models with multiple species or multiple nutrients. In addition, some more general and complicated results, e.g., bifurcation, chaos, and the average persistence would be valuable and interesting research subjects in future.

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