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A NEW NUMERICAL TECHNIQUE FOR INVESTIGATING BOUNDARY VALUE PROBLEMS WITH Ψ-CAPUTO FRACTIONAL OPERATOR

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Abstract This article introduces a new numerical approach for solving linear and non-linear boundary value problems for Ψ-fractional differential equations (Ψ-FDEs). This approach relies on the Ψ-Haar wavelet operational integration matrices. The Ψ-operational matrices (Ψ-OMs) are used to convert the Ψ-FDE to an algebraic system of equations. The non-linear fractional boundary value problems are first linearized using the quasi-linearization technique, and then the Ψ-Haar wavelet technique is applied to the linearized problem. The solution is updated by the Ψ-Haar wavelet method in each iteration of the quasi-linearization technique. The proposed method is a good and simple mathematical technique for numerically solving non-linear Ψ-FDEs. The operational matrix (OM) method is computationally more efficient. Several linear and non-linear boundary value problems are discussed to demonstrate the applicability, efficiency, and simplicity of the method. Moreover, the error analysis is carried out resulting a rigorous error bound for the proposed method.

Keywords Ψ-Haar wavelet operational matrices, Ψ-Caputo fractional integration and derivative, quasilinearization, collocation points, convergence.

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1. Introduction

Fractional calculus is the generalization of classical calculus. Numerous scientific and engineering disciplines, including physics, biology, economics, biochemistry, and many others, use fractional calculus , see for example [14, 15, 18, 22]. In literature, a large number of definitions exist for fractional differential and integral operators including the Riemann-Liouville, the Caputo, the Caputo-Hadamard, the Hilfer, the Erdelyi-Kober, and many more [1, 11, 16, 21]. Usually it is difficult to choose the appropriate operator for simulating various physical phenomena. As a result, generalized fractional order operators may be introduced, including classical operators as special cases. Introducing the fractional derivatives of a function with respect to another function is an effective approach to cope with these difficulties. Fractional order Riemann-Liouville operators are modified by introducing fractional order differentiation and integration with respect to a general function Ψ [20, 21].

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In [2, 3] the Ψ-Caputo fractional differential and integral operators were defined and described by R.Almeida et al, this study is very significant in putting together a diversified set of fractional operators (FO). Furthermore, recent study on the Ψ-Caputo derivative suggests that mathematical models based on Ψ-Caputo fractional differential operators are more versatile and produce satisfactory results in a variety of scenarios [4,8,10,17,19]. In [2] R.Almeida used the Ψ-Caputo derivative to assess global population increase and demonstrated that the model's precision is controlled on the selection of fractional operator. Utilizing fixed-point theory, R.Almeida et al. investigated the uniqueness and existence results for solutions of non-linear Ψ-FDEs [6]. To solve Relaxation-oscillation equation with fractional order Ψ-Caputo derivative, R.Almeida et al. devised the Ψ-shifted legendre polynomials [5]. As a result, we believe that Ψ-FDE theory is a potential research field. Motivated by the above-mentioned work, in this study we devised a new approach for addressing linear and non-linear boundary value problems in Ψ-FDEs numerically.

The outline of this paper is as follows: In Section 2 we present the basics of Ψ-fractional calculus that is the Ψ-Caputo and Ψ-Riemann-Liouville fractional differential and integral operators and their properties. Haar wavelet and estimation of a function by Haar wavelet have also been discussed. In Section 3 we developed the Ψ-Haar wavelet OM of fractional integration that is used for solving boundary value problems in Ψ-FDEs. In Section 4 we presented an error estimate of our method in the form of Theorems 4.1 and 4.2. In Section 5 numerical solutions of some linear and non-linear boundary value problems are obtained by the proposed method, elaborating the efficacy and accuracy of the described approach. Finally, the paper is completed with a conclusion section.

2. Basics of Ψ-Fractional Calculus

This section reviews several concepts, definitions, and basic results from Ψ-fractional calculus that are essential for subsequent advancements in this paper.

Ψ-Fractional integral: [3, 7] Let $h : \mathbf{J} \to \mathbb{R}$ be an integrable function, where $\mathbf{J} = [\delta_1, \delta_2]$ and $\alpha \in \mathbb{R}, n \in \mathbb{N}$ and $\Psi(t) \in C^n(\mathbf{J})$ such that $\Psi'(t) > 0 \ \forall \ t \in \mathbf{J}$. The Ψ-Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$\mathcal{I}_{\delta_1}^{\alpha,\Psi} h(t) = \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^{t} \big(\Psi(t) - \Psi(s)\big)^{\alpha-1} \Psi'(s) h(s) ds.$$

Property 2.1.

$$\mathcal{I}_{\delta_1}^{\eta,\Psi} \mathcal{I}_{\delta_1}^{\zeta,\Psi} h(t) = \mathcal{I}_{\delta_1}^{\eta+\zeta,\Psi} h(t).$$

Ψ-Fractional Derivative : [3, 7, 16].
For $\alpha > 0$, $n - 1 < \alpha \le n$, the Ψ-Reimann-Liouville fractional derivative is given as

$$D_{\delta_1}^{\alpha,\Psi} h(t) = \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^n \mathcal{I}_{\delta_1}^{n-\alpha,\Psi} h(t).$$

Ψ-Caputo Fractional Derivative : [2]
For $n - 1 < \alpha \le n$, Ψ-Caputo fractional derivative of order α is defined as

$${}^{C}D_{\delta_1}^{\alpha,\Psi} h(t) = \mathcal{I}_{\delta_1}^{n-\alpha,\Psi} \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^n h(t)$$

or

$$^{C}D^{\alpha,\Psi}_{\delta_1}h(t)=\mathcal{I}^{n-\alpha,\Psi}_{\delta_1}h^{[n]}_{\Psi}(t),\quad \text{where}\quad h^{[n]}_{\Psi}(t)=\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{n}h(t).$$

Ψ-Caputo fractional derivative can also be defined as

$$^{C}D^{\alpha,\Psi}_{\delta_1}h(t)=D^{\alpha,\Psi}_{\delta_1}\big[h(t)-\sum_{\kappa=0}^{n-1}\frac{h^{[\kappa]}_{\Psi}(\delta_1)}{\kappa!}(\Psi(t)-\Psi(\delta_1))^{\kappa}\big]$$

where, $n=\lceil\alpha\rceil$ for $\alpha\notin\mathbb{N}$ and $n=\alpha$ for $\alpha\in\mathbb{N}$.

Property 2.2 ([2,3]). *If* $h(t)=\big(\Psi(t)-\Psi(\delta_1)\big)^{\zeta}$ *where* $\zeta>n$ *and* $\alpha>0$ *then*

$$^{c}D^{\alpha,\Psi}_{\delta_1}h(t)=\frac{\Gamma(\zeta+1)}{\Gamma(\zeta-\alpha+1)}\Big(\Psi(t)-\Psi(\delta_1)\Big)^{\zeta-\alpha}.$$

Property 2.3 ([2,3]).

$$^{c}D^{\alpha,\Psi}_{\delta_1}\mathcal{I}^{\alpha,\Psi}_{\delta_1}h(t)=h(t).$$

Proof. By definition

$$^{c}D^{\alpha,\Psi}_{\delta_1}\mathcal{I}^{\alpha,\Psi}_{\delta_1}h(t)=D^{\alpha,\Psi}_{\delta_1}\big[\mathcal{I}^{\alpha,\Psi}_{\delta_1}h(t)-\sum_{\kappa=0}^{n-1}\frac{[\mathcal{I}^{\alpha,\Psi}_{\delta_1}h(t)]^{[\kappa]}_{\Psi}(\delta_1)}{\kappa!}(\Psi(t)-\Psi(\delta_1))^{\kappa}\big].\quad (2.1)$$

Note that

$$\begin{aligned}
\mathcal{I}^{\alpha,\Psi}_{\delta_1}ht^{[\kappa]}_{\Psi}(t)&=\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{\kappa}\mathcal{I}^{\alpha,\Psi}_{\delta_1}h(t)\\
&=\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{\kappa-1}\frac{1}{\Psi'(t)}\frac{d}{dt}\int_{\delta_1}^{t}\frac{\big(\Psi(t)-\Psi(s)\big)^{\alpha-1}}{\Gamma(\alpha)}\Psi'(s)h(s)ds\\
&=\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{\kappa-1}\frac{1}{\Psi'(t)}\int_{\delta_1}^{t}\frac{(\alpha-1)\big(\Psi(t)-\Psi(s)\big)^{\alpha-2}}{\Gamma(\alpha)}\Psi'(t)\Psi'(s)h(s)ds\\
&=\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{\kappa-1}\int_{\delta_1}^{t}\frac{\big(\Psi(t)-\Psi(s)\big)^{\alpha-2}}{\Gamma(\alpha-1)}\Psi'(s)h(s)ds\\
&=\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{\kappa-1}\mathcal{I}^{\alpha-1,\Psi}_{\delta_1}h(t),\\
[\mathcal{I}^{\alpha,\Psi}_{\delta_1}h]^{[\kappa]}_{\Psi}(t)&=\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{\kappa-1}\mathcal{I}^{\alpha-1,\Psi}_{\delta_1}h(t).
\end{aligned}$$

Repeating the process κ-times we are at

$$[\mathcal{I}^{\alpha,\Psi}_{\delta_1}h]^{[\kappa]}_{\Psi}(t)=\mathcal{I}^{\alpha-\kappa,\Psi}_{\delta_1}h(t)\quad (2.2)$$

using equation (2.2) in equation (2.1), we have

$$^{c}D^{\alpha,\Psi}_{\delta_1}\mathcal{I}^{\alpha,\Psi}_{\delta_1}h(t)=D^{\alpha,\Psi}_{\delta_1}\big[\mathcal{I}^{\alpha,\Psi}_{\delta_1}h(t)-\sum_{\kappa=0}^{n-1}\frac{\mathcal{I}^{\alpha-\kappa,\Psi}_{\delta_1}h(\delta_1)}{\kappa!}(\Psi(t)-\Psi(\delta_1))^{\kappa}\big].\quad (2.3)$$

Now we show that $\mathcal{I}_{\delta_1}^{\alpha-\kappa,\Psi}h(\delta_1)=0$. We will prove that $\lim_{t\to a}\mathcal{I}_{\delta_1}^{\alpha-\kappa,\Psi}h(t)=0$. Now

$$\begin{aligned}\|\,\mathcal{I}_{\delta_1}^{\alpha,\Psi}h(t)\| &= \|\frac{1}{\Gamma(\alpha)}\int_{\delta_1}^{t}\big(\Psi(t)-\Psi(s)\big)^{\alpha-1}\Psi'(s)h(s)ds\| \\ &\le \frac{1}{\Gamma(\alpha)}\int_{\delta_1}^{t}\|\big(\Psi(t)-\Psi(s)\big)^{\alpha-1}\Psi'(s)h(s)ds\| \\ &\le \frac{\|h\|}{\Gamma(\alpha)}\int_{\delta_1}^{t}\big(\Psi(t)-\Psi(s)\big)^{\alpha-1}\Psi'(s)h(s)ds \qquad \text{Assume} \quad \Psi^{'}(t)>0 \\ &\le \|h\|\frac{\big(\Psi(t)-\Psi(\delta_1)\big)^{\alpha}}{\Gamma(\alpha+1)} \qquad \text{integrating and using} \quad \Gamma(\alpha+1)=(\alpha)\Gamma(\alpha).\end{aligned}$$

Hence $\mathcal{I}_{\delta_1}^{\alpha,\Psi}h(t)\to 0$ as $t\to a$. Therefore, from equation (2.3), we have

$$\begin{aligned}{}^{c}D_{\delta_1}^{\alpha,\Psi}\mathcal{I}_{\delta_1}^{\alpha,\Psi}h(t) &= D_{\delta_1}^{\alpha,\Psi}\mathcal{I}_{\delta_1}^{\alpha,\Psi}h(t) \\ &= \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{n}\mathcal{I}_{\delta_1}^{n-\alpha,\Psi}\mathcal{I}_{\delta_1}^{\alpha,\Psi}h(t) \\ &= \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{n}\mathcal{I}_{\delta_1}^{n-\alpha+\alpha,\Psi}h(t) \\ &= \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{n}\mathcal{I}_{\delta_1}^{n,\Psi}h(t).\end{aligned}$$

Consequently, ${}^{c}D_{\delta_1}^{\alpha,\Psi}\mathcal{I}_{\delta_1}^{\alpha,\Psi}h(t)=h(t)$. Note

$$\begin{aligned}\frac{1}{\Psi'(t)}\frac{d}{dt}\mathcal{I}_{\delta_1}^{1,\Psi}h(t) &= \frac{1}{\Psi'(t)}\frac{d}{dt}\int_{\delta_1}^{t}\big(\Psi(t)-\Psi(s)\big)^{1-1}\Psi'(s)h(s)ds \\ &= \frac{1}{\Psi'(t)}\Psi^{'}(t)h(t) \qquad \text{by Leibniz rule} \\ &= h(t).\end{aligned}$$

Repeating above process n-times we have

$$\Big(\frac{1}{\Psi'(t)}\frac{d}{dt}\Big)^{n}\mathcal{I}_{\delta_1}^{n,\Psi}h(t)=h(t).$$

□

Lemma 2.1.

$$\mathcal{I}_{\delta_1}^{n,\Psi}h_{\Psi}^{[n]}(t)=h(t)-\sum_{\kappa=0}^{n-1}\frac{h_{\Psi}^{[n]}(\delta_1)}{\kappa!}\big(\Psi(t)-\Psi(\delta_1)\big)^{\kappa}.$$

Proof. For $n=1$

$$\begin{aligned}\mathcal{I}_{\delta_1}^{1,\Psi}h_{\Psi}^{[1]}(t) &= \int_{\delta_1}^{t}\frac{\big(\Psi(t)-\Psi(s)\big)^{1-1}}{\Gamma(1)}\Psi'(s)\frac{1}{\Psi'(s)}\frac{d}{ds}h(s)ds \\ &= \int_{\delta_1}^{t}\frac{d}{ds}h(s)ds \\ &= h(t)-h(\delta_1).\end{aligned} \tag{2.4}$$

For $n = 2$

$$\begin{aligned}\mathcal{I}_{\delta_1}^{2,\Psi} h_{\Psi}^{[2]}(t) &= \mathcal{I}_{\delta_1}^{1,\Psi}\mathcal{I}_{\delta_1}^{1,\Psi}\left[h_{\Psi}^{[1]}\right]_{\Psi}^{1}(t)\\ &= \mathcal{I}_{\delta_1}^{1,\Psi}\left[h_{\Psi}^{[1]} - h_{\Psi}^{[1]}\right] \quad \text{by} \quad (2.4)\\ &= \mathcal{I}_{\delta_1}^{1,\Psi} h_{\Psi}^{[1]}(t) - \mathcal{I}_{\delta_1}^{1,\Psi} h_{\Psi}^{[1]}(\delta_1)\\ &= h(t) - h(\delta_1) - h_{\Psi}^{[1]}(\delta_1)\int_{\delta_1}^{t}(\Psi(t)-\Psi(\delta_1))^{1-1}\Psi'(s)ds\\ &= h(t) - h(\delta_1) - h_{\Psi}^{[1]}(\delta_1)(\Psi(t)-\Psi(\delta_1)).\end{aligned} \tag{2.5}$$

Repeating above process n-times, we have

$$\mathcal{I}_{\delta_1}^{n,\Psi} h_{\Psi}^{[n]}(t) = h(t) - \sum_{\kappa=0}^{n-1}\frac{h_{\Psi}^{[\kappa]}(\delta_1)}{\kappa!}\big(\Psi(t)-\Psi(\delta_1)\big)^{\kappa}.$$

The proof of lemma 2.1 is completed. □

Lemma 2.2.

$$\mathcal{I}_{\delta_1}^{\alpha,\Psi}\ {}^{C}D_{\delta_1}^{\alpha,\Psi} h(t) = h(t) - \sum_{\kappa=0}^{n-1}\frac{h_{\Psi}^{[\kappa]}(\delta_1)}{\kappa!}(\Psi(t)-\Psi(\delta_1))^{\kappa}.$$

Proof. Since

$${}^{C}D_{\delta_1}^{\alpha,\Psi} h(t) = \mathcal{I}_{\delta_1}^{n-\alpha,\Psi} h_{\Psi}^{[n]}(t),$$

thus

$$\begin{aligned}\mathcal{I}_{\delta_1}^{\alpha,\Psi\,C} D_{\delta_1}^{\alpha,\Psi} h(t) &= \mathcal{I}_{\delta_1}^{\alpha,\Psi}\mathcal{I}_{\delta_1}^{n-\alpha,\Psi} h_{\Psi}^{[n]}(t)\\ &= \mathcal{I}_{\delta_1}^{\alpha+n-\alpha,\Psi} h_{\Psi}^{[n]}(t)\\ &= \mathcal{I}_{\delta_1}^{n,\Psi} h_{\Psi}^{[n]}(t)\\ \mathcal{I}_{\delta_1}^{\alpha,\Psi\,C} D_{\delta_1}^{\alpha,\Psi} h(t) &= h(t) - \sum_{\kappa=0}^{n-1}\frac{h_{\Psi}^{[\kappa]}(\delta_1)}{\kappa!}\big(\Psi(t)-\Psi(\delta_1)\big)^{\kappa}.\end{aligned}$$

□

Methodology for solution of Ψ-fractional boundary value problem:
Consider the boundary value problem

$$\begin{aligned}&{}^{C}D_{\delta_1}^{\alpha,\Psi}\varkappa(t) = g(t), \qquad 1 < \alpha \le 2,\\ &\varkappa(\delta_1) = \varkappa_{\delta_1}, \qquad \varkappa(\delta_2) = \varkappa_{\delta_2},\end{aligned} \tag{2.6}$$

apply $\mathcal{I}_{\delta_1}^{\alpha,\Psi}$ on equation (2.6)

$$\mathcal{I}_{\delta_1}^{\alpha,\Psi\,C} D_{\delta_1}^{\alpha,\Psi}\varkappa(t) = \mathcal{I}_{\delta_1}^{\alpha,\Psi} g(t),$$

apply lemma 2.2 on L.H.S, we get

$$\varkappa(t) - c_0 - c_1\big(\Psi(t)-\Psi(\delta_1)\big) = \frac{1}{\Gamma(\alpha)}\int_{\delta_1}^{t}\big(\Psi(t)-\Psi(s)\big)^{\alpha-1}\Psi'(s)g(s)ds. \tag{2.7}$$

Apply $\varkappa(\delta_1) = \varkappa_{\delta_1}$, equation (2.7) implies $\varkappa_{\delta_1} = c_0$.
Again apply $\varkappa(\delta_2) = \varkappa_{\delta_2}$ (put $t = \delta_2$ in equation (2.7)) we have,

$$\varkappa_{\delta_2} - \varkappa_{\delta_1} - c_1\big(\Psi(\delta_2) - \Psi(\delta_1)\big) = \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^{\delta_2} \big(\Psi(\delta_2) - \Psi(s)\big)^{\alpha-1} \Psi'(s) g(s) ds$$

$$\Rightarrow c_1 = \frac{\varkappa_{\delta_1} - \varkappa_{\delta_2}}{\Psi(\delta_2) - \Psi(\delta_1)} + \frac{1}{\Psi(s) - \Psi(\delta_1)} \int_{\delta_1}^{\delta_2} \frac{(\Psi(\delta_2) - \Psi(s))}{\Gamma(\alpha)} \Psi'(s) g(s) ds.$$

Substituting c_0 and c_1 in equation (2.7) we obtained

$$\begin{aligned} \varkappa(t) =& \varkappa_{\delta_1} + \left[\frac{\varkappa_{\delta_1} - \varkappa_{\delta_2}}{\Psi(\delta_2) - \Psi(\delta_1)}\right] [\Psi(t) - \Psi(\delta_1)] \\ &+ \frac{\Psi(t) - \Psi(\delta_1)}{\Psi(\delta_2) - \Psi(\delta_1)} \int_{\delta_1}^{\delta_2} \frac{[\Psi(\delta_2) - \Psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Psi'(s) g(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^{t} (\Psi(t) - \Psi(s))^{\alpha-1} \Psi'(s) g(s) ds. \end{aligned}$$

The general case:
Consider the boundary value problem

$$D_{\delta_1}^{\alpha,\Psi} \varkappa(t) = g(t), t \in [\delta_1, \delta_2], \qquad n-1 < \alpha \leq n, \tag{2.8}$$

with the initial and boundary conditions given by

$$\varkappa_{\Psi}^{[\kappa]}(\delta_1) = \varkappa_{\delta_1}^{\kappa}, \quad \varkappa_{\Psi}^{[n-1]}(\delta_2) = \varkappa_{\delta_2}, \quad \kappa = 0, 1, 2, \cdots n-2.$$

Apply $\mathcal{I}_{\delta_1}^{\alpha,\Psi}$ on (2.8)

$$\mathcal{I}_{\delta_1}^{\alpha,\Psi} \; {}^{C}D_{\delta_1}^{\alpha,\Psi} \varkappa(t) = \mathcal{I}_{\delta_1}^{\alpha,\Psi} g(t), \tag{2.9}$$

using Lemma 2.2, that is

$$\mathcal{I}_{\delta_1}^{\alpha,\Psi} \; {}^{C}D_{\delta_1}^{\alpha,\Psi} h(t) = h(t) - \textstyle\sum_{\kappa=0}^{n-1} C_\kappa (\Psi(t) - \Psi(\delta_1))^{\kappa}, \quad \text{where} \quad C_\kappa = \frac{h_{\Psi}^{[\kappa]}(\delta_1)}{\kappa!}$$

in equation (2.9), we have

$$\begin{aligned} \varkappa(t) - \sum_{\kappa=0}^{n-1} \frac{\varkappa_{\Psi}^{[\kappa]}(\delta_1)}{\kappa!} (\Psi(t) - \Psi(\delta_1))^{\kappa} &= I^{\alpha,\Psi} g(t), \\ \varkappa(t) = \sum_{\kappa=0}^{n-2} \frac{\varkappa_{\Psi}^{[\kappa]}(\delta_1)}{\kappa!} (\Psi(t) - \Psi(\delta_1))^{\kappa} &+ \frac{\varkappa_{\Psi}^{[n-1)}(\delta_1)}{(n-1)!} (\Psi(t) - \Psi(\delta_1))^{n-1} + \mathcal{I}_{\delta_1}^{\alpha,\Psi} g(t), \\ \varkappa_{(t)} = \sum_{\kappa=0}^{n-2} \frac{\varkappa_{\delta_1}^{\kappa}}{\kappa!} (\Psi(t) - \Psi(\delta_1))^{\kappa} &+ C_{n-1} (\Psi(t) - \Psi(\delta_1))^{n-1} + \mathcal{I}_{\delta_1}^{\alpha,\Psi} g(t), \end{aligned}$$

where

$$C_{n-1} := \frac{\varkappa_{\Psi}^{[n-1]}(\delta_1)}{(n-1)!}. \tag{2.10}$$

Now apply the boundary conditions $\varkappa_{\Psi}^{(n-1)}(\delta_2) = \varkappa_{\delta_2}$ we get

$$\left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^{n-1} \varkappa(t)$$

$$= \sum_{\kappa=0}^{n-2} \frac{\varkappa_{\delta_1}^{\kappa}}{\kappa!} \left[\frac{1}{\Psi'(t)}\frac{d}{dt}\right]^{n-1} (\Psi(t)-\Psi(\delta_1))^{\kappa} \quad (\kappa = 0, 1, \cdots n-2)$$
$$+ C_{n-1}\left[\frac{1}{\Psi'(t)}\frac{d}{dt}\right]^{n-1} (\Psi(t)-\Psi(\delta_1))^{n-1} + \left[\frac{1}{\Psi'(t)}\frac{d}{dt}\right]^{n-1} \mathcal{I}_{\delta_1}^{\alpha,\Psi} g(t). \tag{2.11}$$

Note (1)

$$\frac{1}{\Psi'(t)}\frac{d}{dt}(\Psi(t)-\Psi(\delta_1)) = \frac{1}{\Psi'(t)}\Psi'(t) = 1$$

and $\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^2 (\Psi(t)-\Psi(\delta_1))' = \frac{1}{\Psi'(t)}\frac{d}{dt}(1) = 0.$

In general $\left[\frac{1}{\Psi'(t)}\frac{d}{dt}\right]^n (\Psi(t)-\Psi(\delta_1))^m = 0$ if $m < n$.

Note (2)

$$\frac{1}{\Psi'(t)}\frac{d}{dt}(\Psi(t)-\Psi(\delta_1))^n = \frac{1}{\Psi'(t)} n(\Psi(t)-\Psi(\delta_1))^{n-1}\Psi'(t),$$
$$\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^2 (\Psi(t)-\Psi(\delta_1))^n = n(n-1)(\Psi(t)-\Psi(\delta_1))^{n-2},$$
$$\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^3 (\Psi(t)-\Psi(\delta_1))^n = n(n-1)(n-2)(\Psi(t)-\Psi(\delta_1))^{n-3},$$
$$\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m (\Psi(t)-\Psi(\delta_1))^n$$
$$= n(n-1)(n-2)\cdots(n-m+1)(\Psi(t)-\Psi(\delta_1))^{n-m}$$
$$= \frac{n(n-1)(n-2)\cdots(n-m+1)(n-m)!}{(n-m)!}(\Psi(t)-\Psi(\delta_1))^{n-m}$$
$$= \frac{n!}{(n-m)!}[\Psi(t)-\Psi(\delta_1)]^{n-m},$$

if $m = n$, $\left[\frac{1}{\Psi_1}\frac{d}{dt}\right]^n [\Psi(t)-\Psi(\delta_1)]^n = n!$.

Note (3)

$$\left[\mathcal{I}_{\delta_1}^{\alpha,\Psi} h(t)\right]_{\Psi}^{[\kappa]}(t) = \mathcal{I}_{\delta_1}^{\alpha-\kappa} h(t) \quad \text{or} \quad \left[\frac{1}{\Psi'(t)}\frac{d}{dt}\right]^{\kappa} \mathcal{I}_{\delta_1}^{\alpha,\Psi} h(t) = \mathcal{I}_{\delta_1}^{\alpha-\kappa} h(t),$$

using note (1), (2), and (3) in equation (2.11), we have

$$\varkappa_{\Psi}^{(n-1)}(t) = (n-1)! C_{n-1} + \mathcal{I}_{\delta_1}^{\alpha-n+1} g(t),$$

apply the boundary conditions

$$\varkappa_{\Psi}^{[n-1]}(\delta_2) = \varkappa_{\delta_2},$$
$$\varkappa^{(n-1)}(\delta_2) = (n-1)! C_{n-1} + \mathcal{I}_{\delta_1}^{\alpha-n+1,\Psi} g(\delta_2),$$
$$C_{n-1} = \frac{1}{(n-1)!}\left[\varkappa_{\delta_2} - \mathcal{I}_{\delta_1}^{\alpha-n+1,\Psi} g(\delta_2)\right].$$

Substituting C_{n-1} in equation (2.10), we have

$$\varkappa_{(t)} = \sum_{\kappa=0}^{n-2} \frac{\varkappa_{\delta_1}^{\kappa}}{\kappa!}(\Psi(t)-\Psi(\delta_1))^{\kappa} + \frac{[\Psi(t)-\Psi(\delta_1)]^{n-1}}{(n-1)!}\left[\varkappa_{\delta_2} - \mathcal{I}_{\delta_1}^{\alpha-n+1,\Psi} g(\delta_2)\right] + \mathcal{I}_{\delta_1}^{\alpha,\Psi} g(t),$$

$$\varkappa_{(t)} = \sum_{\kappa=0}^{n-2} \frac{\varkappa_{\delta_1}^{\kappa}[\Psi(t)-\Psi(\alpha))}{\kappa!} + \frac{\varkappa_{\delta_2}}{(n-1)!}[\Psi(t)-\Psi(1))^{n-1}$$
$$- \frac{[\Psi(t)-\Psi(\delta_1)]^{n-1}}{(n-1)!}\int_{\delta_1}^{\delta_2} \frac{(\Psi(\delta_2)-\Psi(s)\Psi'(s)}{\Gamma(\alpha-n+1)} g(s)ds$$
$$+ \int_{\delta_1}^{t} \frac{(\Psi(t)-\Psi(s))^{\alpha-1}}{\Gamma(\alpha)}\Psi'(s)g(s)ds.$$

2.1. The Haar wavelet

In 1990, the Hungarian mathematician Alfred Haar first introduced Haar wavelet. Haar wavelet is the first order wavelet from Daubechies family of wavelets. Haar wavelet have been shown to be impeccable for approximation of other functions.In recent decades this wavelet are widely used for numerical approximation of solutions of differential equations. The operational matrices of fractional order integration for Haar wavelet are first introduced by Chen and Hasiao [12]. The basic aspect of the Haar wavelet Operational matrix (OM) approach is to transform differential equations into a system of algebraic equations of finite order. First and foremost, the higher order derivative in the given problem is approximated by the Haar series. Then by using the Haar wavelet OM of integration, the lower order derivatives and the solutions can then be obtained very easily.

On the interval $[x_1, x_2]$, the ith Haar wavelet is defined as:

$$h_i(t) = \begin{cases} 1, & \text{if } \delta_1(i) \le t < \delta_2(i); \\ -1, & \text{if } \delta_2(i) \le t < \delta_3(i); \\ 0, & \text{elsewhere,} \end{cases} \tag{2.12}$$

where, $\delta_1(i) = x_1+(x_2-x_1)\frac{\kappa}{m}$, $\delta_2(i) = x_1+(x_2-x_1)\frac{\kappa+0.5}{m}$, $\delta_3(i) = x_1+(x_2-x_1)\frac{\kappa+1}{m}$. The wavelet number i is identified by the relation $i = m+\kappa+1$, where $m = 2^j$, $j = 0, 1, 2, 3, ..., J$ and $\kappa = 0, 1, 2, 3, ..., 2^j-1$. j and κ are the dilation and translation parameters, respectively, whereas J is the maximum resolution level.

2.2. Function approximation by Haar wavelet

Any square integrable function $\varkappa(t)$ defined on the interval $[0, 1)$ can be written in terms of Haar wavelet as:

$$\varkappa(t) = \sum_{i=0}^{\infty} c_i h_i(t), \tag{2.13}$$

where the coefficients c_i of the Haar wavelet are defined by

$$c_i = \langle \varkappa(t), h_i(t) \rangle = \int_0^1 \varkappa(t) h_i(t) dt.$$

Only the first m terms in equation (2.13) are taken into consideration, that is

$$\varkappa(t) \cong \varkappa_m(t) = \sum_{i=0}^{m-1} c_i h_i(t)$$

with vector form as:

$$\varkappa(t) \cong \varkappa_m(t) = C_m^T H_m(t), \tag{2.14}$$

where $C_m^T = [c_0, c_1, c_2, \cdots, c_{m-1}]$ and $H_m(t) = [h_0(t), h_1(t), h_2(t), \cdots, h_{m-1}(t)]^T$.

Figure 1. shows the actual and approximated integral of the function $h(t) = \frac{t(e^t+2)}{e+2}$ at $J = 7$ and varied α values.

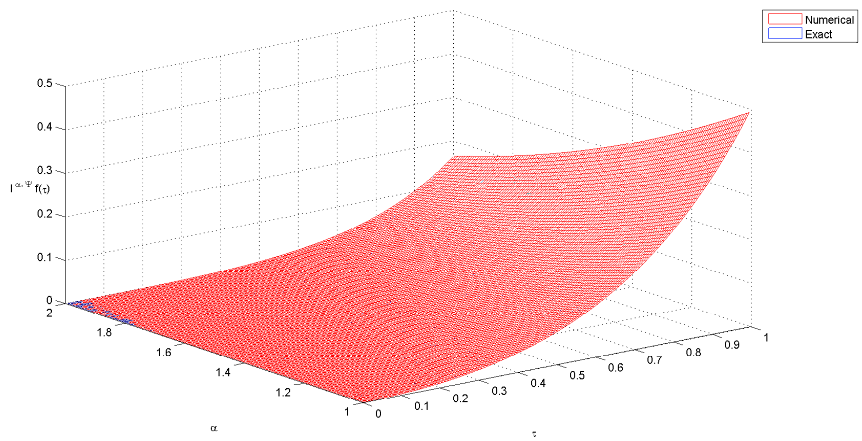


Figure 1. exact and numerical integration of $h(t) = \frac{t(e^t+2)}{e+2}$

3. The Ψ-operational matrix

In this section, we derived the Ψ-Haar wavelet OM of integration, $P^{\alpha,\Psi}$, of fractional order α, which is utilized in solving of Ψ-FDEs numerically. In general the Ψ-fractional integral of the Haar wavelet $H_m = [h_0, h_1, h_2, \cdots, h_{m-1}]$ is defined as:

$$P_i^{\alpha,\Psi}(t) = \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^{t} \Psi^{'}(t)\big(\Psi(t) - \Psi(s)\big)^{\alpha-1} h_i(s)ds. \tag{3.1}$$

The Ψ-fractional integrals in (3.1) can be approximated by

$$P_i^{\alpha,\Psi}(t) = \begin{cases} 0, & \text{if } t < \delta_1(i); \\ \Phi_1, & \text{if } t \in [\delta_1(i), \delta_2(i)); \\ \Phi_2, & \text{if } t \in (\delta_2(i), \delta_3(i)]; \\ \Phi_3, & \text{if } t > \delta_3(i). \end{cases} \tag{3.2}$$

where,

$$\Phi_1 = \frac{\big(\Psi(t) - \Psi(\delta_1(i))\big)^{\alpha}}{\Gamma(\alpha+1)},$$

$$\Phi_2 = \frac{\big(\Psi(t) - \Psi(\delta_1(i))\big)^{\alpha} - 2\big(\Psi(t) - \Psi(\delta_2(i))\big)^{\alpha}}{\Gamma(\alpha+1)},$$

$$\Phi_3 = \frac{\big(\Psi(t) - \Psi(\delta_1(i))\big)^{\alpha} - 2\big(\Psi(t) - \Psi(\delta_2(i))\big)^{\alpha} + \big(\Psi(t) - \Psi(\delta_3(i))\big)^{\alpha}}{\Gamma(\alpha+1)}.$$

Equation (3.2) is valid in the case of $i > 1$, for $i = 1$ we have:

$$P_1^{\alpha,\Psi}(t) = \frac{\big(\Psi(t) - \Psi(\delta_1)\big)^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.3}$$

For $\alpha = 1.8$, $J = 6$ and $\Psi(t) = \frac{t(e^t+2)}{e+2}$ the Ψ-Haar wavelet OM is given by

$$P_{8\times 8}^{\alpha,\Psi} = \begin{bmatrix} 0.1785 & -0.1403 & -0.0287 & -0.1224 & -0.0073 & -0.0220 & -0.0441 & -0.0806 \\ 0.0951 & -0.0568 & -0.0287 & 0.0044 & -0.0073 & -0.0220 & -0.0133 & 0.0191 \\ 0.0193 & 0.0055 & -0.0152 & 0.0092 & -0.0073 & -0.0013 & 0.0052 & 0.0043 \\ 0.0261 & -0.0261 & 0 & -0.0323 & 0 & 0 & -0.0154 & -0.0020 \\ 0.0042 & 0.0028 & 0 & 0.0016 & -0.0046 & 0.0013 & 0.0008 & 0.0008 \\ 0.0055 & -0.0007 & -0.0048 & 0.0032 & 0 & -0.0065 & 0.0019 & 0.0014 \\ 0.0068 & -0.0068 & 0 & 0.0006 & 0 & 0 & -0.0096 & 0.0031 \\ 0.0055 & -0.0055 & 0 & -0.0111 & 0 & 0 & 0 & -0.0149 \end{bmatrix}.$$

4. convergence analysis of Ψ-haar wavelet method

Recently,the error analysis of Caputo type FDEs and non-linear Fredholm integral equation are discussed in [9, 13]. This section deals with the detailed investigation of the error analysis of our method.

Theorem 4.1. *Let us Suppose that $\varkappa^n(t)$ is continuous on $[\delta_1, \delta_2]$, and that there exists $K > 0$ such that $|\varkappa_{\Psi}^{[n]}(t)| \le K$ for all $t \in [\delta_1, \delta_2]$, where $\delta_1, \delta_2 \in \mathbb{R}^+$, $\varkappa_{\Psi}^{[n]}(t) = \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \varkappa(t)$ and $\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m(t)$ is the approximation of $\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t)$, then*

$$\left\|\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t) - \mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m(t)\right\|_E \le \frac{(\delta_2-\delta_1)K\Big(\Psi'(\delta_2)\Big)^{m-\alpha}}{\Gamma(m-\alpha+1)} \frac{1}{\kappa^{(m-\alpha)}} \frac{1}{[1-2^{2(\alpha-m)}]^{\frac{1}{2}}}.$$

Proof. The function $\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t)$ defined on $[\delta_1, \delta_2]$ is approximated as follows:

$$\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t) = \sum_{i=a}^{\infty} c_i h_i(t),$$

where

$$c_i = \langle \mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t), h_i(t)\rangle = \int_{\delta_1}^{\delta_2} \Big(\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t)\Big) h_i(t)dt, \tag{4.1}$$

suppose

$$\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m(t) = \sum_{i=0}^{m-1} c_i h_i(t), \tag{4.2}$$

where $m = 2^{(1+\zeta)}$, and $\zeta = 1, 2, 3, \cdots$, then

$$\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t) - \mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m(t) = \sum_{i=m}^{\infty} c_i h_i(t) = \sum_{i=2^{\zeta+1}}^{\infty} c_i h_i(t), \tag{4.3}$$

which implies that

$$\begin{aligned}\left\|\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t) - \mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m(t)\right\|_E^2 &= \int_{\delta_1}^{t}\left(\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t) - \mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m(t)\right)^2 dt \\ &= \sum_{i=2^{\zeta+1}}^{\infty}\sum_{i'=2^{\zeta+1}}^{\infty} c_i c_{i'} \int_{\delta_1}^{t} h_i(t) h_{i'}(t) dt.\end{aligned}$$

As the sequence $\{h_m(t)\}$ is orthogonal, therefore, we have

$$\int_{\delta_1}^{\delta_2} h_m(t) h_m(t) dt = I_m,$$

where I_m is the order m identity matrix.

Therefore,

$$\left\|\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t) - \mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m(t)\right\|_E^2 = \sum_{i'=2^{\zeta+1}}^{\infty} c_i^2. \tag{4.4}$$

Equation (4.1) implies:

$$\begin{aligned}c_i &= \int_{\delta_1}^{\delta_2}\left(\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t)\right) h_i(t) dt \\ &= 2^{\frac{j}{2}}\left\{\int_{\delta_1+(\delta_2-\delta_1)\kappa 2^{-j}}^{\delta_1+(\delta_2-\delta_1)(\kappa+\frac{1}{2})2^{-j}} \mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t) dt - \int_{\delta_1+(\delta_2-\delta_1)(\kappa+\frac{1}{2})2^{-j}}^{\delta_1+(\delta_2-\delta_1)(\kappa+1)2^{-j}} \mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t) dt\right\}.\end{aligned} \tag{4.5}$$

According to the mean value theorem (MVT) of integrals:

$$\exists \;\; t_1, t_2 \in (\delta_1, \delta_2)$$

where

$$\begin{aligned}&\delta_1 + \frac{(\delta_2-\delta_1)\kappa}{2^j} < t_1 < \delta_1 + \frac{(\delta_2-\delta_1)\big(\kappa+0.5\big)}{2^j}, \\ &\delta_1 + \frac{(\delta_2-\delta_1)\big(\kappa+0.5\big)}{2^j} < t_2 < \delta_1 + \frac{(\delta_2-\delta_1)(\kappa+1)}{2^j},\end{aligned}$$

such that

$$\begin{aligned}c_i =& 2^{\frac{j}{2}}(\delta_2-\delta_1)\left\{\left(\delta_1 + \frac{(\kappa+0.5)}{2^j} \;-\; (\delta_1 + \frac{\kappa}{2^j})\right)\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_1)\right. \\ &\left. - \left(\delta_1 + \frac{(\kappa+1)}{2^j} - (\delta_1 + \frac{(\kappa+0.5)}{2^j}\right)\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_2)\right\} \\ =& 2^{\frac{j}{2}}(\delta_2-\delta_1)\left\{2^{-j-1}\left(\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_1) - \mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_2)\right)\right\}.\end{aligned} \tag{4.6}$$

Therefore,

$$c_i^2 = 2^{-j-2}(\delta_2-\delta_1)^2(\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_1)-\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_2))^2. \tag{4.7}$$

Using the condition $|\varkappa_\Psi^{[n]}(t)| \leq K$, the $\Psi-$Caputo FO and the fact that $\Psi(t)$ is increasing we have:

$$\begin{aligned}
&|\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_1)-\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_2)|\\
=&\frac{1}{\Gamma(m-\alpha)}\Bigg|\int_{\delta_1}^{t_1}\Psi'(t)\Big(\Psi(t_1)-\Psi(t)\Big)^{m-\alpha-1}\varkappa_\Psi^{[n]}(t)dt\\
&-\int_{\delta_1}^{t_2}\Psi'(t)\Big(\Psi(t_2)-\Psi(t)\Big)^{m-\alpha-1}\varkappa_\Psi^{[n]}(t)dt\Bigg|\\
\leq&\frac{1}{\Gamma(m-\alpha)}\Bigg|\int_{\delta_1}^{t_1}\Psi'(t)\Big(\Psi(t_1)-\Psi(t)\Big)^{m-\alpha-1}\varkappa_\Psi^{[n]}(t)dt\\
&-\int_{\delta_1}^{t_1}\Psi'(t)\Big(\Psi(t_2)-\Psi(t)\Big)^{m-\alpha-1}\varkappa_\Psi^{[n]}(t)dt\Bigg|\\
&+\Bigg|\int_{t_1}^{t_2}\Psi'(t)\Big(\Psi(t_2)-\Psi(t)\Big)^{m-\alpha-1}\varkappa_\Psi^{[n]}(t)dt\Bigg|\\
\leq&\frac{1}{\Gamma(m-\alpha)}\Bigg(\int_{\delta_1}^{t_1}\Psi'(t)\Big[\Big(\Psi(t_1)-\Psi(t)\Big)^{m-\alpha-1}-\Big(\Psi(t_2)-\Psi(t)\Big)^{m-\alpha-1}\Big]\Big|\varkappa_\Psi^{[n]}(t)\Big|dt\\
&+\int_{t_1}^{t_2}\Psi'(t)\Big(\Psi(t_2)-\Psi(t)\Big)^{m-\alpha-1}\Big|\varkappa_\Psi^{[n]}(t)\Big|dt\Bigg),\ \text{where } m-\alpha-1>0\\
=&\frac{K}{\Gamma(m-\alpha+1)}\Bigg(\Big(\Psi(t_1)-\Psi(\delta_1)\Big)^{m-\alpha}-\Big(\Psi(t_2)-\Psi(\delta_1)\Big)^{m-\alpha}+2\Big(\Psi(t_2)-\Psi(t_1)\Big)^{m-\alpha}\Bigg).
\end{aligned}$$

Since $t_1>\delta_1,\ t_2>\delta_1$ and $t_2>t_1$ and $\Psi(t)$ is increasing, therefore,

$$\Big(\Psi(t_1)-\Psi(\delta_1)\Big)^{m-\alpha}-\Big(\Psi(t_2)-\Psi(\delta_1)\Big)^{m-\alpha}<0.$$

So,

$$|\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_1)-\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_2)|\leq\frac{2K}{\Gamma(m-\alpha+1)}\Big(\Psi(t_2)-\Psi(t_1)\Big)^{m-\alpha}.$$

According to the MVT, $\exists\ \zeta\in[t_1,t_2]\subseteq[\delta_1,\delta_2]$ such that $\Psi(t_2)-\Psi(t_1)\leq(t_2-t_1)\Psi'(\zeta)$.

Thus:

$$\begin{aligned}
|\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_1)-\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_2)| &\leq\frac{2K}{\Gamma(m-\alpha+1)}\Big((t_2-t_1)\Psi'(\zeta)\Big)^{m-\alpha}\\
&\leq\frac{2K}{\Gamma(m-\alpha+1)}\Big(\frac{\Psi'(\delta_2)}{2^j}\Big)^{m-\alpha},
\end{aligned}$$

implying,

$$\Big(\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_1)-\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t_2)\Big)^2\leq\frac{4K^2}{\Gamma^2(m-\alpha+1)}\Big(\frac{\Psi'(\delta_2)}{2^j}\Big)^{2(m-\alpha)}. \tag{4.8}$$

Putting (4.8) in (4.7), we get:

$$c_i^2 \leq 2^{-j-2}(\delta_2-\delta_1)^2 \frac{4K^2}{\Gamma^2(m-\alpha+1)}\Big(\frac{\Psi'(\delta_2)}{2^j}\Big)^{2(m-\alpha)}. \tag{4.9}$$

By incorporating equations (4.4) and (4.9), we get

$$\begin{aligned}
&\left\|\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t)-\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m(t)\right\|_E^2\\
&=\sum_{i=2^{\zeta+1}}^{\infty} c_i^2=\sum_{j=\zeta+1}^{\infty}\left(\sum_{i=2^j}^{2^{j+1}-1} c_i^2\right)\\
&\leq \sum_{j=\zeta+1}^{\infty}(\delta_2-\delta_1)^2\frac{K^2}{\Gamma^2(m-\alpha+1)2^{2j(m-\alpha)+j}}\Big(\Psi'(\delta_2)\Big)^{2(m-\alpha)}(2^{j+1}-1-2^j+1)\\
&=\frac{(\delta_2-\delta_1)^2K^2\Big(\Psi'(\delta_2)\Big)^{2(m-\alpha)}}{\Gamma^2(m-\alpha+1)}\sum_{j=\zeta+1}^{\infty}\frac{1}{2^{2j(m-\alpha)}}\\
&=\frac{(\delta_2-\delta_1)^2K^2\Big(\Psi'(\delta_2)\Big)^{2(m-\alpha)}}{\Gamma^2(m-\alpha+1)}\frac{1}{2^{2(\zeta+1)(m-\alpha)}}\frac{1}{1-2^{2(\alpha-m)}}.
\end{aligned} \tag{4.10}$$

This shows that:

$$\left\|\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t)-\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m(t)\right\|_E \leq \frac{(\delta_2-\delta_1)K\Big(\Psi'(\delta_2)\Big)^{m-\alpha}}{\Gamma(m-\alpha+1)}\frac{1}{2^{(\zeta+1)(m-\alpha)}}\frac{1}{[1-2^{2(\alpha-m)}]^{\frac{1}{2}}}. \tag{4.11}$$

Let $k=2^{\zeta+1}$, (4.11) can also be written as:

$$\left\|\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa(t)-\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m(t)\right\|_E \leq \frac{(\delta_2-\delta_1)K\Big(\Psi'(\delta_2)\Big)^{m-\alpha}}{\Gamma(m-\alpha+1)}\frac{1}{\kappa^{(m-\alpha)}}\frac{1}{[1-2^{2(\alpha-m)}]^{\frac{1}{2}}}. \tag{4.12}$$

We can compute the error bound as soon as we know the value of K.

First we estimate the value of K. As we know that $\varkappa^n(t)$ is continuous and abounded on the interval $[\delta_1,\delta_2]$ therefore, so is $\varkappa_\Psi^{[n]}(t)$ which can be estimated as:

$$\varkappa_\Psi^{[n]}(t) \cong \sum_{i=0}^{m-1} c_i h_i(t) = C_m^T H_m(t), \tag{4.13}$$

where $C_m=[c_0,c_1,c_2,\cdots,c_{m-1}]^T$ and $H_m(t)=[h_0(t),h_1(t),h_2(t),\cdots,h_{m-1}(t)]^T$.

Integrating (4.13), we have:

$$\varkappa_\Psi^{[n-1]}(t)=\int_{\delta_1}^t \varkappa_\Psi^{[n]}(t)dt+\varkappa_\Psi^{[n-1]}(\delta_1)=\int_{\delta_1}^t \varkappa_\Psi^{[n]}(t)dt \cong C_m^T P_{m\times m}^{1,\Psi} H_m(t). \tag{4.14}$$

Similarly,

$$\varkappa_\Psi^{[n-2]}(t)=\int_{\delta_1}^t \varkappa_\Psi^{[n-1]}(t)dt+\varkappa_\Psi^{[n-2]}(\delta_1)=\int_{\delta_1}^t \varkappa_\Psi^{[n-1]}(t)dt \cong C_m^T P_{m\times m}^{2,\Psi} H_m(t). \tag{4.15}$$

Proceeding in the same way we get:

$$\varkappa_\Psi(t) \cong C_m^T P_{m\times m}^{m,\Psi} H_m(t). \tag{4.16}$$

Consider $t_j = \frac{j-1/2}{m}$, $\quad j = 0, 1, 2, \cdots, m$. Substituting t_j in (4.16), we have:

$$\varkappa_\Psi(t_j) \cong C_m^T P_{m\times m}^{m,\Psi} H_m(t_j). \tag{4.17}$$

The matrix form of (4.17) is as:

$$\varkappa_\Psi \cong C_m^T P_{m\times m}^{m,\Psi} H_m(t_j) \text{ where } \varkappa_\Psi = [\varkappa_\Psi(t_1), \varkappa_\Psi(t_2), \varkappa_\Psi(t_3), \cdots, \varkappa_\Psi(t_m)]. \tag{4.18}$$

From equations (4.18) and (4.13) we can obtain $\mathcal{D}^{m,\Psi}(t)$ for every $t \in [\delta_1, \delta_2]$.

Suppose that $t_i \in [\delta_1, \delta_2]$, for $i = 1, 2, 3, \cdots, l$, $t_i = \frac{i-1}{l}$ and compute $\varkappa_\Psi^{[n]}(t_i)$ for $i = 1, 2, 3, \cdots, l$.

The value of $\max |\varkappa_\Psi^n(t_i)| + \varepsilon$ can then be used as an estimate for K.

This estimation would become more accurate as l increases and ε is considered as δ_2. □

Theorem 4.2. *If the function $\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa_m$ derived by utilizing the Ψ-Haar wavelet is an accurate approximation of $\mathcal{D}_{\delta_1}^{\alpha,\Psi}\varkappa$, then we obtain the following exact upper error bound :*

$$\|\varkappa(t) - \varkappa_m(t)\|_E \le \frac{KN}{\Gamma(\alpha+1)\Gamma(m-\alpha+1)} \frac{1}{k^{(m-\alpha)}} \frac{1}{[1 - 2^{2(\alpha-m)}]^{\frac{1}{2}}}, \tag{4.19}$$

where $N = \max |(\delta_2 - \delta_1)(\Psi(\delta_2))^{m-\alpha}(\Psi(t) - \Psi(0))^\alpha|$.

Theorem 4.2. can be proved easily with the help of Theorem 4.1. Equation (4.19) implies that $\|\varkappa(t) - \varkappa_m(t)\|_E \longrightarrow 0$ as $m \longrightarrow \infty$. Thus our method is convergent.

5. numerical solutions of Ψ-FDEs

Here are some numerical examples of how to approximate the numerical solution of the linear and non-linear boundary value problems of Ψ-FDEs with our proposed method.

5.1. Linear Boundary Value Problems

Example 5.1. Consider the non-homogeneous fractional boundary value problem involving Ψ-Caputo fractional derivative

$$\mathcal{D}_0^{\alpha,\Psi}\varkappa(t) + a\varkappa(t) = g(t), \quad t \in [0,1], \quad \varkappa(0) = 0, \quad \varkappa(1) = \varkappa_1. \tag{5.1}$$

Where $1 < \alpha \le 2$.

For $g(t) = \Psi(t) + a\frac{\Psi(t)^{\alpha+1}}{\Gamma(\alpha+2)}$ and $\varkappa_1 = \frac{1}{\Gamma(\alpha+2)}$, the boundary value problem (5.1) has an exact solution $\varkappa(t) = \frac{(\Psi(t))^{\alpha+1}}{\Gamma(\alpha+2)}$.

The integral representation of (5.1) is given by

$$\varkappa(t) = -a\mathcal{I}_0^{\alpha,\Psi}\varkappa(t) + a\Psi(t)^{\alpha-1}\mathcal{I}_0^{\alpha,\Psi}\varkappa(1) + h(t) \tag{5.2}$$

where

$$h(t) = \mathcal{I}_0^{\alpha,\Psi} g(t) - \Psi(t)^{\alpha-1}\mathcal{I}_0^{\alpha,\Psi} g(1) + \frac{(\Psi(t))^{\alpha-1}}{\Gamma(\alpha+2)}.$$

For numerical solution, we approximate $\varkappa(t)$ as

$$\varkappa(t) = C_m^T H_m(t). \tag{5.3}$$

Then

$$\mathcal{I}_0^{\alpha,\Psi}\varkappa(t) = C_m^T \mathcal{I}_0^{\alpha,\Psi} H_m(t) = C_m^T P_{m\times m}^{\alpha,\Psi} H_m(t). \tag{5.4}$$

Let $\Phi(t) = (\Psi(t))^{\alpha-1}$, we have

$$\Phi(t)\mathcal{I}_0^{\alpha,\Psi}\varkappa(1) = C_m^T K_{m\times m}^{\alpha,\Psi} H_m(t), \tag{5.5}$$

using equations (5.3), (5.4) and (5.5) in equation (5.1), to have:

$$C_m^T H_m(t) = -aC_m^T P_{m\times m}^{\alpha,\Psi} H_m(t) + aC_m^T K_{m\times m}^{\alpha,\Psi} H_m(t) + F_m^T H_m(t), \tag{5.6}$$

where $F_m^T H_m(t)$ is the approximation of $h(t)$.

In Figure 2, numerical solutions, exact solutions, and the max absolute-error are plotted for various choices of the function $\Psi(t)$ and α. The max absolute-error is also shown in the Table 1 for various values of α and J. We discovered that as J increases, the max absolute-error also decreases.

Table 1. max absolute-error for different J and α values.

		$\Psi(t) = t$		
α	$J=5$	$J=6$	$J=7$	$J=8$
1.5	3.504635×10^{-5}	1.234185×10^{-5}	4.274255×10^{-6}	1.513357×10^{-6}
1.6	5.025384×10^{-5}	1.902764×10^{-5}	7.221228×10^{-5}	2.744912×10^{-5}
1.7	6.704913×10^{-5}	2.7187536×10^{-5}	1.103824×10^{-5}	4.482642×10^{-6}
1.8	7.817852×10^{-5}	3.393346×10^{-5}	1.474324×10^{-5}	6.404627×10^{-6}
1.9	6.675254×10^{-5}	3.148735×10^{-5}	1.458791×10^{-5}	6.812362×10^{-6}
		$\Psi(t) = \frac{t^2}{2} + \frac{t}{2}$		
1.5	2.717725×10^{-5}	9.021482×10^{-6}	3.073846×10^{-6}	1.068451×10^{-6}
1.6	4.135072×10^{-5}	1.572354×10^{-5}	6.053439×10^{-6}	2.346431×10^{-6}
1.7	5.948473×10^{-5}	2.457804×10^{-5}	1.017341×10^{-5}	4.143252×10^{-6}
1.8	7.227815×10^{-5}	3.163172×10^{-5}	1.386834×10^{-5}	6.154427×10^{-6}
1.9	6.335346×10^{-5}	2.952634×10^{-5}	1.381907×10^{-5}	6.456180×10^{-6}

Example 5.2. In this example we analyze the Ψ-fractional differential equation with variable coefficients by the proposed method.

$$\mathcal{D}_0^{\alpha,\Psi}\varkappa(t) + a(t)\varkappa(t) = h(t), \quad 2 \le \alpha < 3 \quad \text{and} \quad t \in [0,1], \tag{5.7}$$

with the boundary conditions

$$\varkappa(0) = 0, \quad \varkappa'(0) = 0, \quad \varkappa(1) = 0.$$

Approximate solutions of equation (5.1) for different choices of $\Psi(t)$ and α.

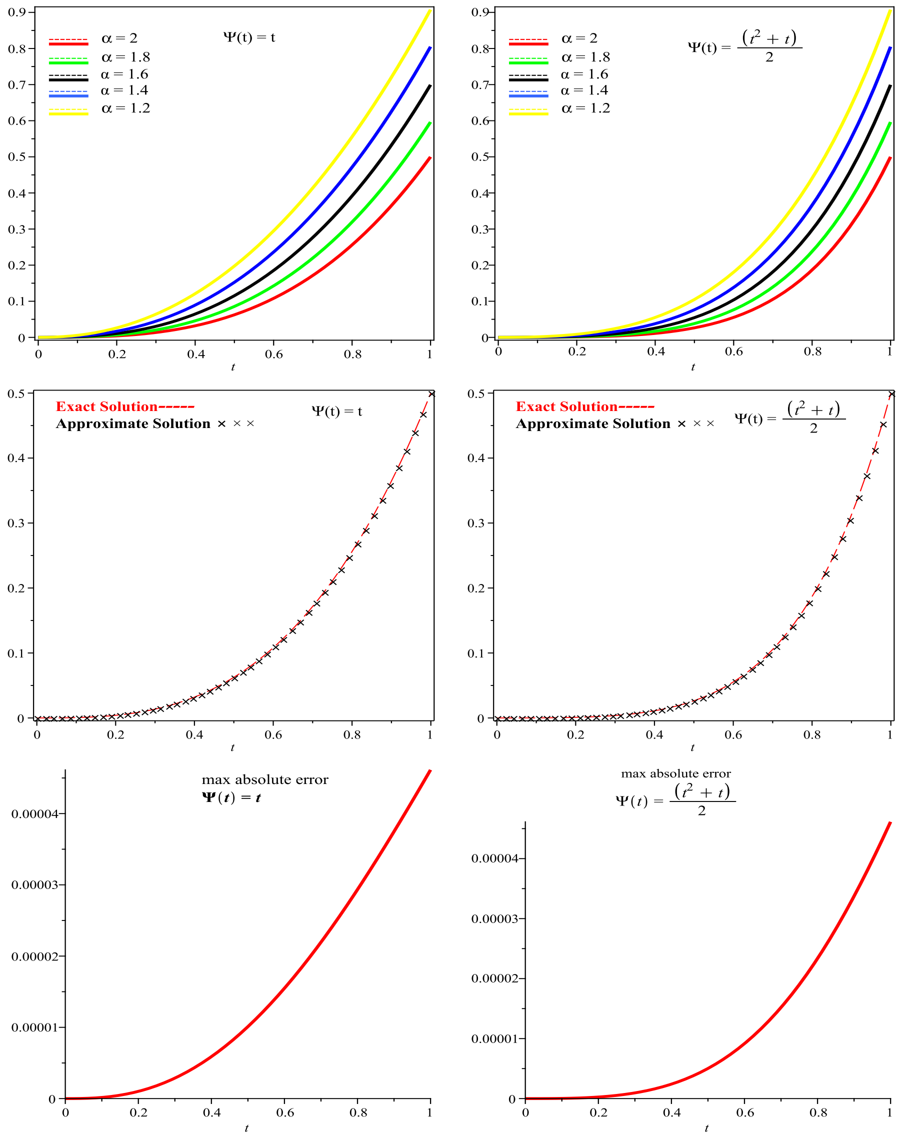


Figure 2. Exact and Approximate solutions and the corresponding max absolute error of equation (5.1) for $\Psi(t) = t$ and $\Psi(t) = (t^2 + t)/2$.

For numerical solution, we employ Ψ-Haar wavelet method. Suppose

$$\mathcal{D}^{\alpha,\Psi}\varkappa(t) = C_m^T H_m(t). \tag{5.8}$$

Using Ψ-Caputo integral operator and the boundary conditions, we have

$$\varkappa(t) = C_m^T P_{m\times m}^{\alpha,\Psi} H_m(t) - C_m^T K_{m\times m}^{\alpha,\Psi} H_m(t). \tag{5.9}$$

Substituting equations (5.8) and (5.9) in equation (5.7), we get

$$C_m^T(H_m(t) + C_m^T \hat{P}_{m\times m}^{\alpha,\Psi}(H_m(t) - C_m^T K_{m\times m}^{\alpha,\Psi} H_m(t) = F_m^T(t) H_m(t), \tag{5.10}$$

where the following approximations are used

$$a(t)\mathcal{I}_0^{\alpha,\Psi}H_m(t) = \hat{P}_{m\times m}^{\alpha,\Psi}H_m(t),$$
$$\Phi(t)\mathcal{I}_0^{\alpha,\Psi}H_m(1) = K_{m\times m}^{\alpha,\Psi}H_m(t),$$
$$h(t) = F_m^T(t)H_m(t),$$

where

$$\Phi(t) = a(t)(\Psi(t))^{\alpha-1}.$$

One may verify that for

$$a(t) = e^{-9\pi\Psi(t)},\ h(t) = e^{-9\pi\Psi(t)}\big((\Psi(t))^{\alpha-1} - (\Psi(t))^{\alpha}\big) - \Gamma(\alpha+1)$$

the boundary value problem (5.7) has the exact solution as $\varkappa(t) = \big(\frac{1}{\Psi(t)} - 1\big)(\Psi(t))^{\alpha}$.

For different choices of the function $\Psi(t)$ and α, numerical solutions, exact solution and the maximum absolute error are plotted in the Figure 3. Also the maximum absolute error is presented in the Table 2 for various J and α.

Table 2. max absolute-error for $\Psi(t) = t^3$ and different J and α values.

α	$J=5$	$J=6$	$J=7$	$J=8$
2.5	2.384237×10^{-5}	1.854308×10^{-5}	1.824315×10^{-5}	1.893752×10^{-5}
2.6	8.448743×10^{-6}	5.875834×10^{-6}	4.818904×10^{-6}	4.734526×10^{-6}
2.7	2.984073×10^{-6}	1.786233×10^{-6}	1.272453×10^{-6}	1.182537×10^{-6}
2.8	1.049082×10^{-6}	5.453306×10^{-7}	3.364195×10^{-7}	2.956827×10^{-7}

5.2. Non-linear Case

Example 5.3. Consider the non-linear boundary value problem of fractional order with Ψ-Caputo fractional derivative:

$$\mathcal{D}_0^{\alpha,\Psi}\varkappa(t) + a(t)\varkappa'^2(t) + b(t)\varkappa(t)\varkappa'(t) = h(t), \quad 1<\alpha\le 2, \quad t\in[0,1], \tag{5.11}$$

subject to boundary conditions $\varkappa(0)=0$, $\varkappa(1)=0$. The exact solution of equation (5.11) is given by $\varkappa(t) = (\Psi(t))^{\alpha} - (\Psi(t))^{70-\alpha}$. Where

$$\begin{aligned} h(t) =&\Gamma(\alpha+1) - \frac{71-\alpha}{71-2\alpha}(\Psi(t))^{70-2\alpha} + a(t)(\alpha(\Psi(t))^{\alpha-1} - (70-\alpha)(\Psi(t))^{69-\alpha})^2 \\ &+ b(t)(\alpha(\Psi(t))^{\alpha-1} - (70-\alpha)(\Psi(t))^{69-\alpha})((\Psi(t))^{\alpha} - (\Psi(t))^{70-\alpha}). \end{aligned}$$

We first linearize the non-linear terms in equation (5.11) by using quasi-linearization technique and then utilize Ψ-Haar wavelet method for numerical solution. Equation (5.11) in its linearized representation is given by:

$$\begin{aligned} &\mathcal{D}_0^{\alpha,\Psi}\varkappa_{r+1}(t) + b(t)\varkappa_r'(t)\varkappa_{r+1}(t) + (2a(t)\varkappa_r'(t) + b(t)\varkappa_r(t))\varkappa_{r+1}'(t) \\ =&h(t) + a(t)\varkappa_r'^2(t) + b(t)\varkappa_r(t)\varkappa_r'(t), \quad t>0 \quad \text{and} \quad 1<\alpha\le 2, \end{aligned} \tag{5.12}$$

having $\varkappa_{r+1}(0)=0$, $\varkappa_{r+1}(1)=0$ as the boundary conditions.

The Ψ-Haar wavelet approach is applied to equation (5.12).

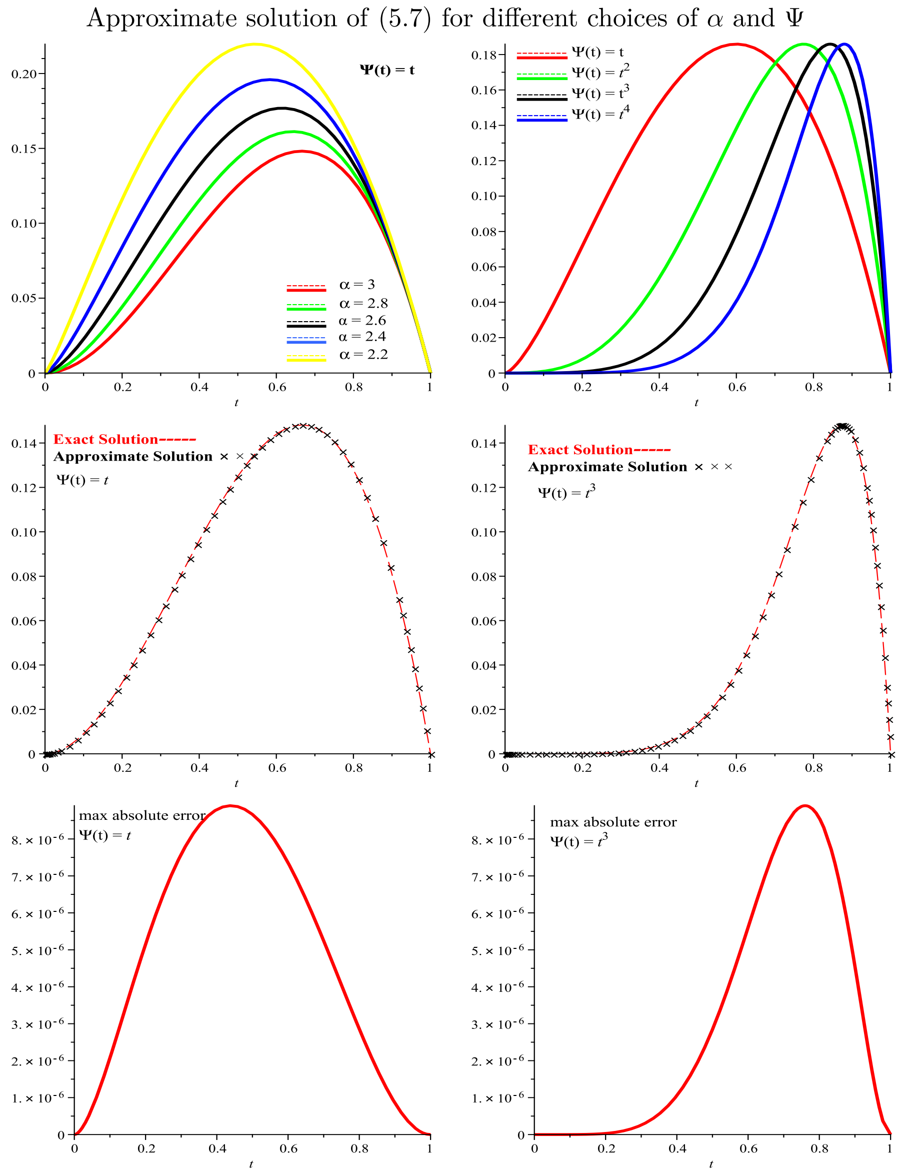


Figure 3. For equation (5.7). Exact and approximate solutions for $\Psi(t) = t$ and $\Psi(t) = t^3$ and the corresponding max absolute error.

Let

$$\mathcal{D}_0^{\alpha,\Psi} \varkappa_{r+1}(t) = C_m^T H_m(t). \tag{5.13}$$

Employing $\mathcal{I}^{\alpha,\Psi}$ and the boundary conditions on (5.13), we have

$$\varkappa_{r+1}(t) = \mathcal{I}^{\alpha,\Psi} C_m^T H_m(t) = C_m^T P_{m\times m}^{\alpha,\Psi} H_m(t). \tag{5.14}$$

$$\varkappa_{r+1}'(t) = C_m^T P_{m\times m}^{\alpha-1,\Psi} H_m(t). \tag{5.15}$$

Substituting (5.13), (5.14) and (5.15) in (5.12) we have,

$$C_m^T\big(H_m(t)+b(t)\varkappa_r'(t)P_{m\times m}^{\alpha,\Psi}H_m(t)+(2a(t)\varkappa_r'(t)+b(t)\varkappa_r(t))P_{m\times m}^{\alpha-1,\Psi}H_m(t)\big)$$
$$=h(t)+a(t)\varkappa_r'^2(t)+b(t)\varkappa_r(t)\varkappa_r'(t). \tag{5.16}$$

In matrix notation, equation (5.16) can be written as:

$$C_m^T\big(H_m(t)+b(t)\varkappa_r'(t)P_{m\times m}^{\alpha,\Psi}H_m(t)+(2a(t)\varkappa_r'(t)+b(t)\varkappa_r(t))P_{m\times m}^{\alpha-1,\Psi}H_m(t)\big)=F(t), \tag{5.17}$$

where $F(t)=h(t)+a(t)\varkappa_r'^2(t)+b(t)\varkappa_r(t)\varkappa_r'(t)$.

The desired numerical solution is obtained by solving (5.17) for C_m^T and substituting the value of C_m^T in equation (5.15). Table 3 shows the max absolute-error for several J and α values. In Figure 4, the exact and approximate solutions for various selections of the function $\Psi(t)$ are shown.

Table 3. Absolute error for various J and α values

J	$\alpha=1.6$	$\alpha=1.7$	$\alpha=1.8$	$\alpha=1.9$	$\alpha=2.0$
5	1.981467×10^{-4}	1.637253×10^{-4}	1.253471×10^{-4}	9.497683×10^{-5}	7.13125×10^{-5}
6	6.389746×10^{-5}	4.824637×10^{-5}	3.526588×10^{-5}	2.516453×10^{-5}	1.753205×10^{-5}
7	2.0568703×10^{-5}	1.435892×10^{-5}	9.874581×10^{-6}	6.624682×10^{-6}	4.413058×10^{-6}
8	6.6420531×10^{-6}	4.346458×10^{-6}	2.785478×10^{-6}	1.753052×10^{-6}	1.105386×10^{-6}

Example 5.4. Consider the fractional order non-linear Lane Emden boundary value problem with Ψ-Caputo fractional derivative

$$\mathcal{D}_0^{\alpha,\Psi}\varkappa(t)+\frac{2}{\Psi(t)}\varkappa'(t)-6\varkappa^2(t)=h(t), \tag{5.18}$$

where, $1<\alpha\le 2$, and $t\in[0,1]$.

Subject to the boundary conditions

$$\varkappa(0)=0, \quad \varkappa(1)=2.$$

For $\alpha=2$ and $h(t)=6+\frac{2}{\Psi(t)}-6((\Psi(t))^2+\Psi(t))^2$, the exact solution of the problem (5.18) is given by $\varkappa(t)=(\Psi(t))^2+\Psi(t)$.

We first linearize the non-linear terms in equation (5.18) using the Quasilinearization technique, and then use the Ψ-Haar wavelet approach to determine the numerical solution of the linearized FDE using the same procedure as in example 5.3.

The max absolute-error for various α and J values is shown in Table 4. In Figure 5, the exact and approximate solutions for various selections of α and their max absolute-error are plotted.

6. Conclusion

Operational matrices approach has been applied for the fist time to Ψ-FDEs with boundary conditions. One of the major advantages of the technique is that it is a convenient and effective numerical scheme for solution of non-linear Ψ-FDEs. The operational matrices are sparse matrices, which help to reduce the computational

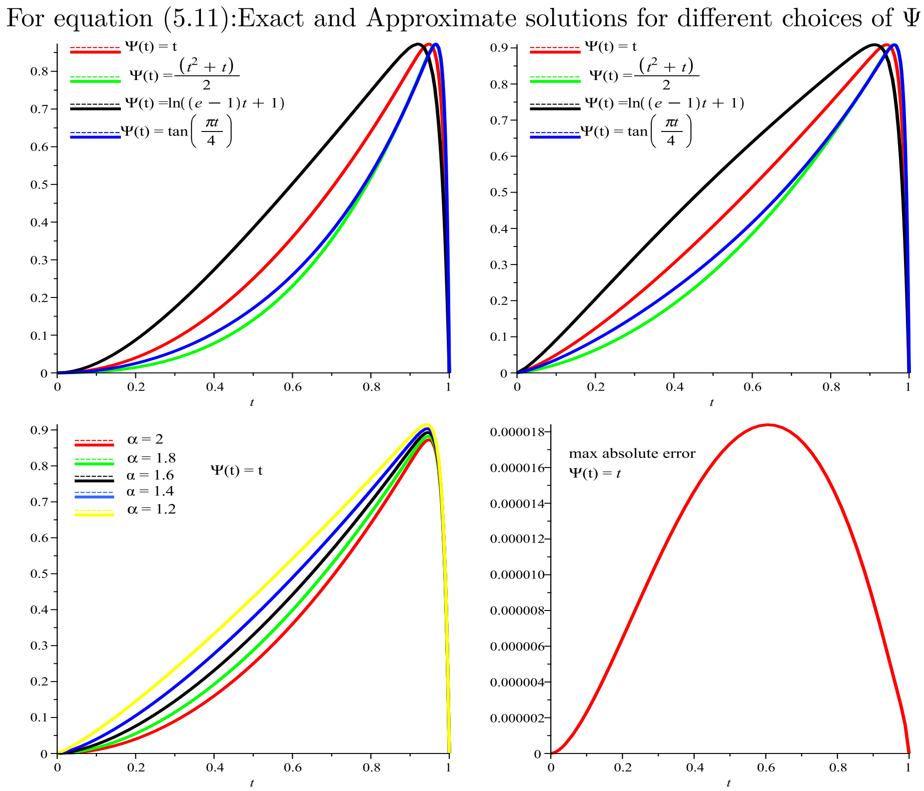


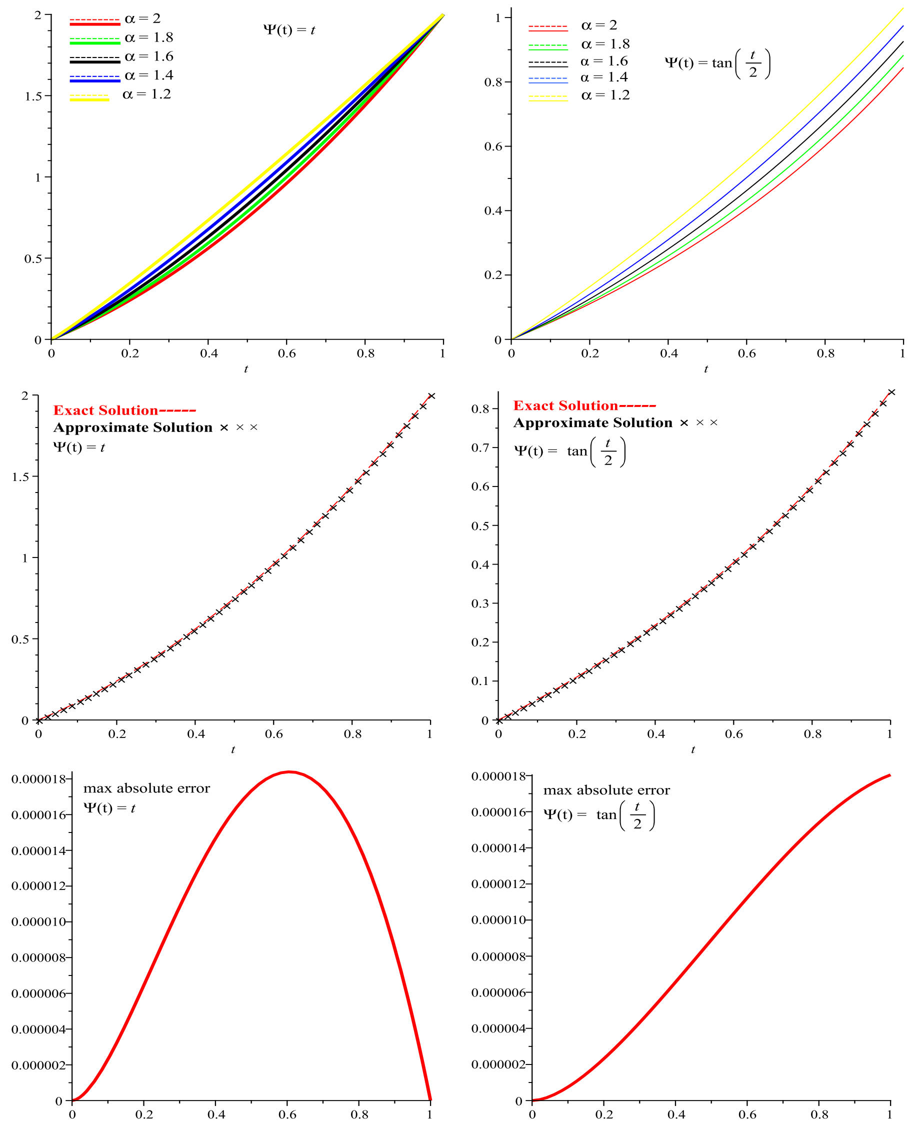
Figure 4. For equation (5.11):Approximate solutions for different choices of α and the max absolute error.

Table 4. max absolute error for $\Psi(t) = \tan(\frac{t}{2})$ and different choices of α, J.

$\zeta = 0.5$ and $\alpha = 0.75$				
α	$J = 5$	$J = 6$	$J = 7$	$J = 8$
1.5	6.405136×10^{-6}	2.372232×10^{-6}	8.34561×10^{-7}	2.450634×10^{-7}
1.6	3.866363×10^{-6}	1.233445×10^{-6}	3.756732×10^{-7}	1.234257×10^{-7}
1.7	3.213574×10^{-6}	8.766463×10^{-7}	2.391531×10^{-7}	5.643642×10^{-8}
1.8	2.827468×10^{-6}	7.563442×10^{-7}	1.746525×10^{-7}	4.686534×10^{-8}
1.9	2.457272×10^{-6}	5.846783×10^{-7}	1.584563×10^{-7}	3.532453×10^{-8}

cost of the method. The numerical scheme given in this study is based on Ψ-Haar wavelet OMs of integration. For linear and non-linear Ψ-FDEs, these OMs are generated and successfully used to solve two and multi-point boundary value problems. We have applied the scheme to linear and non-linear boundary value problems. Furthermore, an error analysis is performed, yielding a strict error bound for the suggested approach. Other wavelet bases, such as Gegenbauer, Chebyshev, and Legendre wavelet, can be used with the proposed technique. The proposed method can be used to solve Ψ-fractional partial differential equations as well.

Numerical solutions of (5.18) for $\Psi(t) = t$ and $\Psi(t) = \tan(\frac{t}{2})$ for different values of



α

Figure 5. exact and approximate solutions of (5.18) for $\Psi(t) = t$ and $\Psi(t) = \tan(\frac{t}{2})$ and the corresponding max absolute error.

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