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MULTIDIMENSIONAL REVERSE HÖLDER INEQUALITY ON TIME SCALES

H. M. Rezk[1], Ghada ALNemer[2], Ahmed I. Saied[3], E. Awwad[3,4] and M. Zakarya[5,6,†]

Abstract This paper develops the study of Hölder's inequality with weighted functions where we can establish some new multidimensional reverse Hölder inequality on time scale measure spaces. Our results will be proved by using the definition and some properties of a Specht's ratio function. We will prove these inequalities in a time scale calculus to avoid proving them twice once in the continuous case and the second in the discrete case.

Keywords Hölder's inequality, multidimensional reverse Hölder inequality, time scale measure spaces, weight functions.

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1. Introduction

In [12], Hölder proved that

$$\sum_{k=1}^{n}\zeta_k y_k \le \left(\sum_{k=1}^{n}\zeta_k^{\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{k=1}^{n}y_k^{\beta}\right)^{\frac{1}{\beta}}, \tag{1.1}$$

where (ζ_k) and (y_k) are positive sequences and α, $\beta > 1$ such that $1/\alpha + 1/\beta = 1$. The integral form of (1.1) is

$$\int_d^l \psi(\tau)\varpi(\tau)d\tau \le \left(\int_d^l \psi^{\alpha}(\tau)d\tau\right)^{\frac{1}{\alpha}}\left(\int_d^l \varpi^{\beta}(\tau)d\tau\right)^{\frac{1}{\beta}}, \tag{1.2}$$

[†]The corresponding author.
Email: haythamrezk@azhar.edu.eg(H.M. Rezk), gnnemer@pnu.edu.sa(G. ALNemer), as0863289@gmail.com(A. I. Saied), mzibrahim@kku.edu.sa (M. Zakarya)
[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Egypt
[2]Department of Mathematical Science, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh, 11671, Saudi Arabia
[3]Department of Mathematics, Faculty of Science, Benha University, Benha, 13518, Egypt
[4]Department of Mathematics, College of Science and Arts,Jouf University, Gurayat, Saudi Arabia
[5]King Khalid University, College of Science, Department of Mathematics, P.O. Box 9004, 61413. Abha, Saudi Arabia
[6]Department of Mathematics, Faculty of Science, Al-Azhar University, 71524, Assiut, Egypt

where $\alpha,\ \beta>1$ such that $1/\alpha+1/\beta=1$ and $\psi,\ \varpi\in C\left((d,l),\mathbb{R}^{+}\right)$. In [18] Wang defined $L_\alpha=L_\alpha(S,\sum,\mu)$, $-\infty<\alpha<\infty$, as a space of all $\alpha-$th power nonnegative integrable functions over a given finite measure space $(S,\sum,\mu)$ (where S may be considered as a bounded subset of real numbers). For ψ in L_α, we write $\|\psi\|_\alpha=\left(\int_S\psi^\alpha d\mu\right)^{\frac{1}{\alpha}}$. He generalized (1.1) and (1.2) and proved that if ψ_1 is in L_α and ψ_2 is in L_β, then $\psi_1\psi_2$ is in L_r and

$$\|\psi_1\psi_2\|_r\leq\|\psi_1\|_\alpha\|\psi_2\|_\beta, \tag{1.3}$$

where $(1/\alpha)+(1/\beta)=(1/r)$, $\alpha,\ \beta,\ r>0$. Also, he established the inverse of (1.3) and proved that if ψ_1 is in L_α and ψ_2 is in L_β, such that

$$0<m_i\leq\psi_i(\zeta)\leq M_i<\infty,$$

on S where $m_i=\inf\psi_i(\zeta)$, $M_i=\sup\psi_i(\zeta)$, $i=1,2$ and $(1/\alpha)+(1/\beta)=(1/r)$, α, β, $r>0$, then

$$\|\psi_1\|_\alpha\|\psi_2\|_\beta\leq C^r_{\alpha\beta}\|\psi_1\psi_2\|_r,$$

where

$$C^r_{\alpha\beta}=\frac{\left[r\left(M_1^{\alpha/2}M_2^{\beta/2}+m_1^{\alpha/2}m_2^{\beta/2}\right)M\left(\alpha,\beta,r\right)\right]^{\frac{1}{r}}}{\left(\alpha\left(m_2M_2\right)^{\beta/2}\right)^{1/\alpha}\left(\beta\left(m_1M_1\right)^{\alpha/2}\right)^{1/\beta}},$$

with

$$M\left(\alpha,\beta,r\right)=\max\left\{s^{(\alpha/2)-r}\tau^{(\beta/2)-r}\mid s=M_1,m_1,\ \tau=M_2,m_2\right\}.$$

In [20], Zhao and Cheung proved that if $\psi(\zeta)$ and $\varpi(\zeta)$ are nonnegative continuous functions and $\psi^{1/\alpha}(\zeta)\varpi^{1/\beta}(\zeta)$ is integrable on $[d,l]$, then

$$\left(\int_d^l\psi^\alpha(\zeta)d\zeta\right)^{\frac{1}{\alpha}}\left(\int_d^l\varpi^\beta(\zeta)d\zeta\right)^{\frac{1}{\beta}}\leq\int_d^l S\left(\frac{Y\psi^\alpha(\zeta)}{X\varpi^\beta(\zeta)}\right)\psi(\zeta)\varpi(\zeta)d\zeta, \tag{1.4}$$

where

$$X=\int_d^l\psi^\alpha(\zeta)d\zeta,\ Y=\int_d^l\varpi^\beta(\zeta)d\zeta,\ \alpha>1\text{ and }\frac{1}{\alpha}+\frac{1}{\beta}=1,$$

and

$$S(h)=\frac{h^{1/(h-1)}}{e\log h^{1/(h-1)}},\ h\neq1.$$

Also, they proved the discrete case of (1.4) and established that if (d_i) and (l_i) are positive sequences, then

$$\left(\sum_{i=1}^n d_i^\alpha\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^n l_i^\beta\right)^{\frac{1}{\beta}}\leq\sum_{i=1}^n S\left(\frac{Zd_i^\alpha}{Yl_i^\beta}\right)d_il_i, \tag{1.5}$$

where $Y=\sum_{i=1}^n d_i^\alpha$ and $Z=\sum_{i=1}^n l_i^\beta$. They applied (1.4) to get Radon's reverse integral inequality and proved that if $\psi,\ \varpi\in C\left((d,l),\mathbb{R}^{+}\right)$ and $m>0$, then

$$\int_d^l\frac{\psi^{m+1}(\zeta)}{\varpi^m(\zeta)}d\zeta\leq\frac{\left(\int_d^l S\left(\frac{G\psi^{m+1}(\zeta)}{F\varpi^{m+1}(\zeta)}\right)\psi(\zeta)d\zeta\right)^{m+1}}{\left(\int_d^l\varpi(\zeta)d\zeta\right)^m}, \tag{1.6}$$

where

$$G=\int_d^l \varpi(\zeta)d\zeta \text{ and } F=\int_d^l \frac{\psi^{m+1}(\zeta)}{\varpi^m(\zeta)}d\zeta.$$

Also, they proved the discrete case of (1.6) and established that

$$\sum_{i=1}^n \frac{d_i^{m+1}}{l_i^m} \leq \frac{\sum_{i=1}^n S\left(\frac{Bd_i^{m+1}}{Al_i^{m+1}}\right) d_i}{\left(\sum_{i=1}^n l_i\right)^m}, \tag{1.7}$$

where $B=\sum_{i=1}^n l_i$ and $A=\sum_{i=1}^n d_i^{m+1}/l_i^m$. They applied (1.4) to get Jensen's reverse integral inequality and proved that if $\psi, p \in C\left((d,l),\mathbb{R}^+\right)$ and $\int_d^l p(\zeta)d\zeta=1$, then for $0<s<\tau$, we have

$$\left(\int_d^l S\left(\frac{\psi^\tau(\zeta)}{P}\right)\psi^s(\zeta)\alpha(\zeta)d\zeta\right)^{1/s} \geq \left(\int_d^l P(\zeta)\psi^\tau(\zeta)d\zeta\right)^{1/\tau}, \tag{1.8}$$

where $P=\int_d^l p(\zeta)\psi^\tau(\zeta)d\zeta$. Also, they proved the discrete case of (1.8) and established that if $\sum_{i=1}^n \lambda_i=1$ and $0<s<\tau$, then

$$\left(\sum_{i=1}^n S\left(\frac{d_i^\tau}{\Lambda}\right)d_i^s\lambda_i\right)^{1/s} \geq \left(\sum_{i=1}^n \lambda_i d_i^\tau\right)^{1/\tau}, \tag{1.9}$$

where $\Lambda=\sum_{i=1}^n \lambda_i d_i^\tau$.

The theory of time scales, which has recently received a lot of attention, was initiated by Hilger in his PhD thesis in order to unify discrete and continuous analysis [13]. The general idea is to prove a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is a so called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$.

During the past decade a number of dynamic inequalities has been established by some authors which are motivated by practical problems.

Hölder's inequality is an important tool in different branches of modern mathematics such as classical real and complex analysis, numerical analysis, probability and differential equations. Since its discovery, it has been studied widely and has been generalized on many ways. Some reverse versions of Hölder's inequality on time scales $\mathbb{T}$ were established. For example, in [7], Agarwal et al. unified (1.1) and (1.2) on time scales and proved that if $d,\, l \in \mathbb{T}$ and $\psi,\, \varpi \in C_{rd}\left([d,l]_{\mathbb{T}},\mathbb{R}\right)$, then

$$\int_d^l |\psi(\tau)\varpi(\tau)|\,\Delta\tau \leq \left(\int_d^l |\psi(\tau)|^p\,\Delta\tau\right)^{\frac{1}{p}}\left(\int_d^l |\varpi(\tau)|^\beta\,\Delta\tau\right)^{\frac{1}{\beta}}, \tag{1.10}$$

where $p>1$, $\beta=p/(p-1)$. In [19], Wong et al. generalized (1.10) and proved that if $d,\, l\in\mathbb{T}$ and $\psi,\, \varpi,\, h \in C_{rd}\left([d,l]_{\mathbb{T}},\mathbb{R}\right)$, then

$$\int_d^l |h(\tau)|\,|\psi(\tau)\varpi(\tau)|\,\Delta\tau \leq \left(\int_d^l |h(\tau)|\,|\psi(\tau)|^\alpha\,\Delta\tau\right)^{\frac{1}{\alpha}}\left(\int_d^l |h(\tau)|\,|\varpi(\tau)|^\beta\,\Delta\tau\right)^{\frac{1}{\beta}}, \tag{1.11}$$

where $\alpha > 1$, $\beta = \alpha/(\alpha - 1)$. Also, they proved that (1.11) is reversed when $\alpha < 0$ or $\beta < 0$. They applied (1.11) to get Minkowski's inequality on time scales and established that if $d,\ l \in \mathbb{T}$, $\psi,\ \varpi,\ h \in C_{rd}([d, l]_{\mathbb{T}}, \mathbb{R})$ and $\alpha > 1$, then

$$\left(\int_d^l |h(\tau)|\,|\psi(\tau) + \varpi(\tau)|^{\alpha}\,\Delta\tau\right)^{\frac{1}{\alpha}} \le \left(\int_d^l |h(\tau)|\,|\psi(\tau)|^{\alpha}\,\Delta\tau\right)^{\frac{1}{\alpha}} + \left(\int_d^l |h(\tau)|\,|\varpi(\tau)|^{\alpha}\,\Delta\tau\right)^{\frac{1}{\alpha}}.$$

In [11], the authors proved the reverse Hölder inequality on time scales and established that if $\psi, \varpi \in C([d, l]_{\mathbb{T}}, \mathbb{R}^+)$ such that ψ^{α}, ϖ^{β} are $\diamondsuit_{\alpha}-$integrable on $[d, l]_{\mathbb{T}}$. Let $\alpha > 1$ and $1/\alpha + 1/\beta = 1$. Then

$$\int_d^l S\left(\frac{Y\psi^{\alpha}(\zeta)}{X\varpi^{\beta}(\zeta)}\right)\psi(\zeta)\varpi(\zeta)\diamondsuit_{\alpha}\zeta \ge \left(\int_d^l \psi^{\alpha}(\zeta)\diamondsuit_{\alpha}\zeta\right)^{\frac{1}{\alpha}}\left(\int_d^l \varpi^{\beta}(\zeta)\diamondsuit_{\alpha}\zeta\right)^{\frac{1}{\beta}}, \tag{1.12}$$

where $X = \int_d^l \psi^{\alpha}(\zeta)\diamondsuit_{\alpha}\zeta$, $Y = \int_d^l \varpi^{\beta}(\zeta)\diamondsuit_{\alpha}\zeta$ and $S(.)$ is the Specht's ratio (see [20]). Also, they proved (1.12) with weighted functions and established that if $\psi, \varpi, w \in C([d, l]_{\mathbb{T}}, \mathbb{R}^+)$ such that ψ^{α}, ϖ^{β} are $\diamondsuit_{\alpha}-$integrable on $[d, l]_{\mathbb{T}}$. If $\alpha > 1$ and $1/\alpha + 1/\beta = 1$, then

$$\int_d^l S\left(\frac{Y\psi^{\alpha}(\zeta)}{X\varpi^{\beta}(\zeta)}\right)w(\zeta)\psi(\zeta)\varpi(\zeta)\diamondsuit_{\alpha}\zeta \ge \left(\int_d^l w(\zeta)\psi^{\alpha}(\zeta)\diamondsuit_{\alpha}\zeta\right)^{\frac{1}{\alpha}}\left(\int_d^l w(\zeta)\varpi^{\beta}(\zeta)\diamondsuit_{\alpha}\zeta\right)^{\frac{1}{\beta}}, \tag{1.13}$$

where $X = \int_d^l w(\zeta)\psi^{\alpha}(\zeta)\diamondsuit_{\alpha}\zeta$ and $Y = \int_d^l w(\zeta)\varpi^{\beta}(\zeta)\diamondsuit_{\alpha}\zeta$.

The authors in [11], proved that if $\psi, \varpi \in C([d, l]_{\mathbb{T}}, \mathbb{R}^+)$ such that $0 < m \le \psi(t)/\varpi(t) \le M < \infty$ for all $t \in [d, l]_{\mathbb{T}}$. If $\alpha > 1$ and $1/\alpha + 1/\beta = 1$, then

$$\int_d^l S\left(\frac{Y\psi(\zeta)}{X\varpi(\zeta)}\right)\psi^{\frac{1}{\alpha}}(\zeta)\varpi^{\frac{1}{\beta}}(\zeta)\diamondsuit_{\alpha}\zeta \ge \frac{m^{\frac{1}{\alpha^2}}}{M^{\frac{1}{\beta^2}}}\int_d^l \psi^{\frac{1}{\beta}}(\zeta)\varpi^{\frac{1}{\alpha}}(\zeta)\diamondsuit_{\alpha}\zeta, \tag{1.14}$$

where $X = \int_d^l \psi(\zeta)\diamondsuit_{\alpha}\zeta$ and $Y = \int_d^l \varpi(\zeta)\diamondsuit_{\alpha}\zeta$. For more details about Hölder's inequality on time scales, we refer the reader to the books [8, 9] and the papers [1–3, 7, 16, 21, 22].

Following these trends and to develop the study of Hölder's inequality on time scales, we will prove some new multidimensional reverse Hölder inequalities (like (1.4), (1.5), (1.6), (1.7), (1.8), (1.9), (1.12) and (1.13)) on time scale measure spaces.

The organization of paper as follows. In Section 2, we present some basics and some lemmas on time scales. In Section 3, we prove our main results. As special cases (in one dimension) the results give the inequalities proved by El-Deeb et al., [11]. Also, as special cases (in one dimension) when $\mathbb{T} = \mathbb{N}$ give the inequalities (1.5), (1.7) and (1.9) proved by Zhao and Cheung. Also, when $\mathbb{T} = \mathbb{R}$, our results give (1.4), (1.6) and (1.8) as special cases.

2. Preliminaries and basic lemmas

The forward jump operator is defined by: $\sigma(\tau) := \inf\{s \in \mathbb{T} : s > \tau\}$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(\tau) := \sigma(\tau) - \tau \geq 0$, and for any function $\psi : \mathbb{T} \to \mathbb{R}$ the notation $\psi^\sigma(\tau)$ denotes $\psi(\sigma(\tau))$. The derivative of product $\psi\varpi$ and quotient ψ/ϖ (where $\varpi\varpi^\sigma \neq 0$) are given by

$$(\psi\varpi)^\Delta = \psi^\Delta\varpi + \psi^\sigma\varpi^\Delta = \psi\varpi^\Delta + \psi^\Delta\varpi^\sigma, \quad \left(\frac{\psi}{\varpi}\right)^\Delta = \frac{\psi^\Delta\varpi - \psi\varpi^\Delta}{\varpi\varpi^\sigma}. \tag{2.1}$$

For more details of time scale analysis we refer the reader to the two books [9, 10]. In this paper, we will refer to the (delta) integral which we can define as follows. If $G^\Delta(\tau) = \varpi(\tau)$, then the Cauchy (delta) integral of ϖ is defined by $\int_d^\tau \varpi(\zeta)\Delta\zeta := G(\tau) - G(d)$. It can be shown (see [9]) that if $\varpi \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(\tau) := \int_{\tau_0}^\tau \varpi(\zeta)\Delta\zeta$ exists, $\tau_0 \in \mathbb{T}$ and satisfies $G^\Delta(\tau) = \varpi(\tau)$, $\tau \in \mathbb{T}$.

Lemma 2.1 (Specht's ratio [20])**.** *If d, l are positive numbers, $\epsilon > 1$ and $1/\epsilon + 1/\delta = 1$, then*

$$S\left(\frac{\alpha}{\beta}\right)\alpha^{1/\epsilon}\beta^{1/\delta} \geq \frac{\alpha}{\epsilon} + \frac{\beta}{\delta}, \tag{2.2}$$

where

$$S(h) = \frac{h^{1/(h-1)}}{e\log h^{1/(h-1)}}, h \neq 1.$$

Remark 2.1. In 2002, Tominaga [17] proved some properties of $S(h)$ in Lemma 2.1. He proved that

$$S(1) = 1, \; S(\tau) = S(\frac{1}{\tau}) \text{ for all } \tau > 0.$$

When $S(1) = 1$, we have that (2.2) holds with equality.

3. Main Results

In this section, we assume that the functions (without mentioning) are nonnegative, $\Delta-$integrable on $[d, l]_\mathbb{T}$ and the integrals considered are assumed to exist (finite i.e. convergent).

Let $(\Lambda_1, \mathcal{M}, \mu_\Delta), ..., (\Lambda_n, \mathcal{M}, \mu_\Delta)$ be finite dimensional time scale measure spaces. We define the product measure space $(\Lambda_1 \times ... \times \Lambda_n, \mathcal{M} \times ... \times \mathcal{M}, \mu_\Delta \times ... \times \mu_\Delta)$, where $\mathcal{M} \times ... \times \mathcal{M}$ is the product $\sigma-$algebra generated by $\{E \times ... \times F : E \in \mathcal{M}, ..., F \in \mathcal{M}\}$ and $(\mu_\Delta \times ... \times \mu_\Delta)(E \times ... \times F) = \mu_\Delta(E)...\mu_\Delta(F)$.

Also, in this section we can use the following

$$\psi(\zeta) := \psi(\zeta_1, ..., \zeta_n) \text{ and } d\mu_\Delta(\zeta) := d\mu_\Delta(\zeta_1)...d\mu_\Delta(\zeta_n).$$

Now, we can establish the first result.

3.1. The reversed Hölder inequality with two parameters

Theorem 3.1. *Let* $(\Lambda_1, \mathcal{M}, \mu_\Delta)$ *,...,* $(\Lambda_n, \mathcal{L}, \mu_\Delta)$ *be finite dimensional time scale measure spaces and* ψ, $\varpi : \mathbf{\Lambda} = \Lambda_1 \times ... \times \Lambda_n \to \mathbb{R}^+$. *If* $\epsilon > 1$ *and* $1/\epsilon + 1/\delta = 1$, *then*

$$\int ... \int_{\mathbf{\Lambda}} S\left(\frac{Y\psi^\epsilon(\zeta)}{X\varpi^\delta(\zeta)}\right).\psi(\zeta)\varpi(\zeta)d\mu_\Delta(\zeta)$$
$$\geq \left(\int ... \int_{\mathbf{\Lambda}} \psi^\epsilon(\zeta)d\mu_\Delta(\zeta)\right)^{\frac{1}{\epsilon}} \left(\int ... \int_{\mathbf{\Lambda}} \varpi^\delta(\zeta)d\mu_\Delta(\zeta)\right)^{\frac{1}{\delta}}, \tag{3.1}$$

where $X = \int ... \int_{\mathbf{\Lambda}} \psi^\epsilon(\zeta)d\mu_\Delta(\zeta)$, $Y = \int ... \int_{\mathbf{\Lambda}} \varpi^\delta(\zeta)d\mu_\Delta(\zeta)$ *and* $S(.)$ *is the Specht's ratio.*

Proof. Applying Lemma 2.1 with

$$\alpha = \frac{\psi^\epsilon(\zeta)}{X},\ \beta = \frac{\varpi^\delta(\zeta)}{Y},$$

where $X = \int ... \int_{\mathbf{\Lambda}} \psi^\epsilon(\zeta)d\mu_\Delta(\zeta)$, $Y = \int ... \int_{\mathbf{\Lambda}} \varpi^\delta(\zeta)d\mu_\Delta(\zeta)$, we see that

$$S\left(\frac{Y\psi^\epsilon(\zeta)}{X\varpi^\delta(\zeta)}\right)\frac{\psi(\zeta)}{X^{\frac{1}{\epsilon}}}\frac{\varpi(\zeta)}{Y^{\frac{1}{\delta}}} \geq \frac{1}{\epsilon}\frac{\psi^\epsilon(\zeta)}{X} + \frac{1}{\delta}\frac{\varpi^\delta(\zeta)}{Y}. \tag{3.2}$$

Integrating (3.2) on the region $\mathbf{\Lambda}$, we see (note $1/\epsilon + 1/\delta = 1$) that

$$\begin{aligned}
&\int ... \int_{\mathbf{\Lambda}} S\left(\frac{Y\psi^\epsilon(\zeta)}{X\varpi^\delta(\zeta)}\right).\psi(\zeta)\varpi(\zeta)d\mu_\Delta(\zeta)\\
&\geq X^{\frac{1}{\epsilon}}Y^{\frac{1}{\delta}} \int ... \int_{\mathbf{\Lambda}} \left[\frac{1}{\epsilon}\frac{\psi^\epsilon(\zeta)}{X} + \frac{1}{\delta}\frac{\varpi^\delta(\zeta)}{Y}\right] d\mu_\Delta(\zeta)\\
&= X^{\frac{1}{\epsilon}}Y^{\frac{1}{\delta}} \left[\frac{1}{\epsilon X}\int ... \int_{\mathbf{\Lambda}} \psi^\epsilon(\zeta)d\mu_\Delta(\zeta) + \frac{1}{\delta Y}\int ... \int_{\mathbf{\Lambda}} \varpi^\delta(\zeta)d\mu_\Delta(\zeta)\right]\\
&= X^{\frac{1}{\epsilon}}Y^{\frac{1}{\delta}} \left[\frac{1}{\epsilon} + \frac{1}{\delta}\right] = X^{\frac{1}{\epsilon}}Y^{\frac{1}{\delta}}\\
&= \left(\int ... \int_{\mathbf{\Lambda}} \psi^\epsilon(\zeta)d\mu_\Delta(\zeta)\right)^{\frac{1}{\epsilon}} \left(\int ... \int_{\mathbf{\Lambda}} \varpi^\delta(\zeta)d\mu_\Delta(\zeta)\right)^{\frac{1}{\delta}},
\end{aligned}$$

which is (3.1). □

Remark 3.1. As special cases (in one dimention), we get the inequality (1.12) proved by El-Deeb et al., [11].

Theorem 3.2. *Let* $(\Lambda_1, \mathcal{M}, \mu_\Delta)$ *,...,* $(\Lambda_n, \mathcal{L}, \mu_\Delta)$ *be finite dimensional time scale measure spaces and* w, ψ, $\varpi : \mathbf{\Lambda} = \Lambda_1 \times ... \times \Lambda_n \to \mathbb{R}^+$. *If* $\epsilon > 1$ *and* $1/\epsilon + 1/\delta = 1$, *then*

$$\int ... \int_{\mathbf{\Lambda}} S\left(\frac{Y\psi^\epsilon(\zeta)}{X\varpi^\delta(\zeta)}\right).w(\zeta)\psi(\zeta)\varpi(\zeta)d\mu_\Delta(\zeta) \tag{3.3}$$
$$\geq \left(\int ... \int_{\mathbf{\Lambda}} w(\zeta)\psi^\epsilon(\zeta)d\mu_\Delta(\zeta)\right)^{\frac{1}{\epsilon}} \left(\int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^\delta(\zeta)d\mu_\Delta(\zeta)\right)^{\frac{1}{\delta}},$$

where $X = \int ... \int_{\mathbf{\Lambda}} w(\zeta)\psi^\epsilon(\zeta)d\mu_\Delta(\zeta)$, $Y = \int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^\delta(\zeta)d\mu_\Delta(\zeta)$ *and* $S(.)$ *is the Specht's ratio.*

Proof. Applying Lemma 2.1 with

$$\alpha = \frac{w(\zeta)\psi^{\epsilon}(\zeta)}{X},\ \beta = \frac{w(\zeta)\varpi^{\delta}(\zeta)}{Y},$$

where $X = \int ... \int_{\mathbf{\Lambda}} w(\zeta)\psi^{\epsilon}(\zeta)d\mu_{\Delta}(\zeta)$, $Y = \int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\delta}(\zeta)d\mu_{\Delta}(\zeta)$, we have

$$S\left(\frac{Y\psi^{\epsilon}(\zeta)}{X\varpi^{\delta}(\zeta)}\right)\frac{w^{\frac{1}{\epsilon}}(\zeta)\psi(\zeta)}{X^{\frac{1}{\epsilon}}}\frac{w^{\frac{1}{\delta}}(\zeta)\varpi(\zeta)}{Y^{\frac{1}{\delta}}} \geq \frac{1}{\epsilon}\frac{w(\zeta)\psi^{\epsilon}(\zeta)}{X} + \frac{1}{\delta}\frac{w(\zeta)\varpi^{\delta}(\zeta)}{Y},$$

and then (since $1/\epsilon + 1/\delta = 1$), we obtain

$$S\left(\frac{Y\psi^{\epsilon}(\zeta)}{X\varpi^{\delta}(\zeta)}\right)w(\zeta)\psi(\zeta)\varpi(\zeta) \geq X^{\frac{1}{\epsilon}}Y^{\frac{1}{\delta}}\left[\frac{1}{\epsilon}\frac{w(\zeta)\psi^{\epsilon}(\zeta)}{X} + \frac{1}{\delta}\frac{w(\zeta)\varpi^{\delta}(\zeta)}{Y}\right]. \tag{3.4}$$

Integrating (3.4) on the region $\mathbf{\Lambda}$ where $1/\epsilon + 1/\delta = 1$, we get

$$\begin{aligned}
&\int ... \int_{\mathbf{\Lambda}} S\left(\frac{Y\psi^{\epsilon}(\zeta)}{X\varpi^{\delta}(\zeta)}\right).w(\zeta)\psi(\zeta)\varpi(\zeta)d\mu_{\Delta}(\zeta)\\
&\geq X^{\frac{1}{\epsilon}}Y^{\frac{1}{\delta}}\int ... \int_{\mathbf{\Lambda}}\left[\frac{1}{\epsilon}\frac{w(\zeta)\psi^{\epsilon}(\zeta)}{X} + \frac{1}{\delta}\frac{w(\zeta)\varpi^{\delta}(\zeta)}{Y}\right]d\mu_{\Delta}(\zeta)\\
&= X^{\frac{1}{\epsilon}}Y^{\frac{1}{\delta}}\left[\frac{1}{\epsilon X}\int ... \int_{\mathbf{\Lambda}} w(\zeta)\psi^{\epsilon}(\zeta)d\mu_{\Delta}(\zeta) + \frac{1}{\delta Y}\int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\delta}(\zeta)d\mu_{\Delta}(\zeta)\right]\\
&= X^{\frac{1}{\epsilon}}Y^{\frac{1}{\delta}}\left[\frac{1}{\epsilon} + \frac{1}{\delta}\right] = X^{\frac{1}{\epsilon}}Y^{\frac{1}{\delta}}\\
&= \left(\int ... \int_{\mathbf{\Lambda}} w(\zeta)\psi^{\epsilon}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\epsilon}}\left(\int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\delta}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\delta}},
\end{aligned}$$

which is (3.3). □

Remark 3.2. As a special case (in one dimention), we get the inequality (1.13) proved by El-Deeb et al., [11].

Theorem 3.3. *Let* $(\Lambda_1, \mathcal{M}, \mu_{\Delta})$ *, ..., * $(\Lambda_n, \mathcal{L}, \mu_{\Delta})$ *be finite dimensional time scale measure spaces and* u*,* $v : \mathbf{\Lambda} = \Lambda_1 \times ... \times \Lambda_n \to \mathbb{R}^+$*. If* $m > 0$*, then*

$$\begin{aligned}
&\left[\int ... \int_{\mathbf{\Lambda}} S\left(\frac{Yu^{m+1}(\zeta)}{Xv^{m+1}(\zeta)}\right)u(\zeta)d\mu_{\Delta}(\zeta)\right]^{m+1}\\
&\geq \left(\int ... \int_{\mathbf{\Lambda}}\frac{u^{m+1}(\zeta)}{v^{m}(\zeta)}d\mu_{\Delta}(\zeta)\right)\left(\int ... \int_{\mathbf{\Lambda}} v(\zeta)d\mu_{\Delta}(\zeta)\right)^{m},
\end{aligned} \tag{3.5}$$

where $X = \int ... \int_{\mathbf{\Lambda}} \frac{u^{m+1}(\zeta)}{v^{m}(\zeta)}d\mu_{\Delta}(\zeta)$*,* $Y = \int ... \int_{\mathbf{\Lambda}} v(\zeta)d\mu_{\Delta}(\zeta)$ *and* $S(.)$ *is the Specht's ratio.*

Proof. Applying Theorem 3.1 with $\epsilon = m + 1$ and $\delta = (m + 1)/m$, we see

$$\begin{aligned}
&\int ... \int_{\mathbf{\Lambda}} S\left(\frac{Y\psi^{m+1}(\zeta)}{X\varpi^{\frac{m+1}{m}}(\zeta)}\right).\psi(\zeta)\varpi(\zeta)d\mu_{\Delta}(\zeta)\\
&\geq \left(\int ... \int_{\mathbf{\Lambda}}\psi^{m+1}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{m+1}}\left(\int ... \int_{\mathbf{\Lambda}}\varpi^{\frac{m+1}{m}}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{m}{m+1}},
\end{aligned} \tag{3.6}$$

where $X = \int ... \int_{\mathbf{\Lambda}} \psi^{m+1}(\zeta) d\mu_\Delta(\zeta)$ and $Y = \int ... \int_{\mathbf{\Lambda}} \varpi^{\frac{m+1}{m}}(\zeta) d\mu_\Delta(\zeta)$. Taking

$$\psi(\zeta) = u(\zeta)/v^{m/(m+1)}(\zeta) \text{ and } \varpi(\zeta) = v^{m/(m+1)}(\zeta),\ u,\ v > 0,$$

in (3.6), we have for

$$X = \int ... \int_{\mathbf{\Lambda}} \frac{u^{m+1}(\zeta)}{v^m(\zeta)} d\mu_\Delta(\zeta) \text{ and } Y = \int ... \int_{\mathbf{\Lambda}} v(\zeta) d\mu_\Delta(\zeta),$$

that

$$\int ... \int_{\mathbf{\Lambda}} S\left(\frac{Yu^{m+1}(\zeta)}{Xv^{m+1}(\zeta)}\right) u(\zeta) d\mu_\Delta(\zeta) \geq \left(\int ... \int_{\mathbf{\Lambda}} \frac{u^{m+1}(\zeta)}{v^m(\zeta)} d\mu_\Delta(\zeta)\right)^{\frac{1}{m+1}} \left(\int ... \int_{\mathbf{\Lambda}} v(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{m}{m+1}},$$

and then

$$\left[\int ... \int_{\mathbf{\Lambda}} S\left(\frac{Yu^{m+1}(\zeta)}{Xv^{m+1}(\zeta)}\right) u(\zeta) d\mu_\Delta(\zeta)\right]^{m+1} \geq \left(\int ... \int_{\mathbf{\Lambda}} \frac{u^{m+1}(\zeta)}{v^m(\zeta)} d\mu_\Delta(\zeta)\right) \left(\int ... \int_{\mathbf{\Lambda}} v(\zeta) d\mu_\Delta(\zeta)\right)^m,$$

which satisfies (3.5). □

Remark 3.3. As a special case (in one dimention when $\mathbb{T} = \mathbb{R}$), we get the inequality (1.6) proved by Zhao and Cheung [20].

In the following, we generalize Theorem 3.3 with weighted function.

Theorem 3.4. *Let $(\Lambda_1, \mathcal{M}, \mu_\Delta)$,..., $(\Lambda_n, \mathcal{L}, \mu_\Delta)$ be finite dimensional time scale measure spaces and u, v, $w : \mathbf{\Lambda} = \Lambda_1 \times ... \times \Lambda_n \to \mathbb{R}^+$. If $m > 0$, then*

$$\left[\int ... \int_{\mathbf{\Lambda}} S\left(\frac{Yu^{m+1}(\zeta)}{Xv^{m+1}(\zeta)}\right) w(\zeta) u(\zeta) d\mu_\Delta(\zeta)\right]^{m+1} \geq \left(\int ... \int_{\mathbf{\Lambda}} w(\zeta) \frac{u^{m+1}(\zeta)}{v^m(\zeta)} d\mu_\Delta(\zeta)\right) \left(\int ... \int_{\mathbf{\Lambda}} w(\zeta) v(\zeta) d\mu_\Delta(\zeta)\right)^m, \tag{3.7}$$

where $X = \int ... \int_{\mathbf{\Lambda}} w(\zeta) \frac{u^{m+1}(\zeta)}{v^m(\zeta)} d\mu_\Delta(\zeta)$, $Y = \int ... \int_{\mathbf{\Lambda}} w(\zeta) v(\zeta) d\mu_\Delta(\zeta)$ and $S(.)$ is the Specht's ratio.

Proof. Applying Theorem 3.2 with $\epsilon = m + 1$ and $\delta = (m + 1)/m$, we see

$$\int ... \int_{\mathbf{\Lambda}} S\left(\frac{Y\psi^{m+1}(\zeta)}{X\varpi^{\frac{m+1}{m}}(\zeta)}\right) .w(\zeta)\psi(\zeta)\varpi(\zeta) d\mu_\Delta(\zeta) \geq \left(\int ... \int_{\mathbf{\Lambda}} w(\zeta)\psi^{m+1}(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{m+1}} \left(\int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\frac{m+1}{m}}(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{m}{m+1}}, \tag{3.8}$$

where $X = \int ... \int_{\mathbf{\Lambda}} w(\zeta)\psi^{m+1}(\zeta) d\mu_\Delta(\zeta)$ and $Y = \int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\frac{m+1}{m}}(\zeta) d\mu_\Delta(\zeta)$. Taking

$$\psi(\zeta) = u(\zeta)/v^{m/(m+1)}(\zeta) \text{ and } \varpi(\zeta) = v^{m/(m+1)}(\zeta),\ u,\ v > 0 \text{ in (3.8)},$$

we observe that

$$X=\int\ldots\int_{\mathbf{\Lambda}}w(\zeta)\frac{u^{m+1}(\zeta)}{v^{m}(\zeta)}d\mu_{\Delta}(\zeta)\ \text{ and } Y=\int\ldots\int_{\mathbf{\Lambda}}w(\zeta)v(\zeta)d\mu_{\Delta}(\zeta),$$

and then

$$\begin{aligned}&\int\ldots\int_{\mathbf{\Lambda}}S\left(\frac{Yu^{m+1}(\zeta)}{Xv^{m+1}(\zeta)}\right)w(\zeta)u(\zeta)d\mu_{\Delta}(\zeta)\\ &\geq\left(\int\ldots\int_{\mathbf{\Lambda}}w(\zeta)\frac{u^{m+1}(\zeta)}{v^{m}(\zeta)}d\mu_{\Delta}(\zeta)\right)^{\frac{1}{m+1}}\left(\int\ldots\int_{\mathbf{\Lambda}}w(\zeta)v(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{m}{m+1}},\end{aligned}$$

thus

$$\begin{aligned}&\left[\int\ldots\int_{\mathbf{\Lambda}}S\left(\frac{Yu^{m+1}(\zeta)}{Xv^{m+1}(\zeta)}\right)w(\zeta)u(\zeta)d\mu_{\Delta}(\zeta)\right]^{m+1}\\ &\geq\left(\int\ldots\int_{\mathbf{\Lambda}}w(\zeta)\frac{u^{m+1}(\zeta)}{v^{m}(\zeta)}d\mu_{\Delta}(\zeta)\right)\left(\int\ldots\int_{\mathbf{\Lambda}}w(\zeta)v(\zeta)d\mu_{\Delta}(\zeta)\right)^{m},\end{aligned}$$

which satisfies (3.7). □

Theorem 3.5. *Let* $(\Lambda_1,\mathcal{M},\mu_{\Delta})$ *, ..., * $(\Lambda_n,\mathcal{L},\mu_{\Delta})$ *be finite dimensional time scale measure spaces and* h, $\psi:\mathbf{\Lambda}=\Lambda_1\times\ldots\times\Lambda_n\to\mathbb{R}^+$ *and* $\int\ldots\int_{\mathbf{\Lambda}}h(\zeta)d\mu_{\Delta}(\zeta)=1$. *If* $0<s<\tau$, *then*

$$\left(\int\ldots\int_{\mathbf{\Lambda}}S\left(\frac{\psi^{\tau}(\zeta)}{X}\right)\psi^{s}(\zeta)h(\zeta)d\mu_{\Delta}(\zeta)\right)^{1/s}\geq\left(\int\ldots\int_{\mathbf{\Lambda}}h(\zeta)\psi^{\tau}(\zeta)d\mu_{\Delta}(\zeta)\right)^{1/\tau}, \tag{3.9}$$

where $X=\int\ldots\int_{\mathbf{\Lambda}}h(\zeta)\psi^{\tau}(\zeta)d\mu_{\Delta}(\zeta)$ *and* $S(.)$ *is the Specht's ratio.*

Proof. From assumptions, we have for

$$X=\int\ldots\int_{\mathbf{\Lambda}}h(\zeta)\psi^{\tau}(\zeta)d\mu_{\Delta}(\zeta)\ \text{ and } Y=\int\ldots\int_{\mathbf{\Lambda}}h(\zeta)d\mu_{\Delta}(\zeta)=1,$$

that

$$\begin{aligned}&\int\ldots\int_{\mathbf{\Lambda}}S\left(\frac{\psi^{\tau}(\zeta)}{X}\right)\psi^{s}(\zeta)h(\zeta)d\mu_{\Delta}(\zeta)\\ &=\int\ldots\int_{\mathbf{\Lambda}}S\left(\frac{Y\left[h^{s/\tau}(\zeta)\psi^{s}(\zeta)\right]^{\tau/s}}{X\left[h^{1-s/\tau}(\zeta)\right]^{\tau/(\tau-s)}}\right)h^{s/\tau}(\zeta)\psi^{s}(\zeta)h^{1-s/\tau}(\zeta)d\mu_{\Delta}(\zeta).\end{aligned} \tag{3.10}$$

Applying Theorem (3.1) with $\epsilon=\tau/s$ and $\delta=\tau/(\tau-s)$ to the right hand side of (3.10), we get

$$\begin{aligned}&\int\ldots\int_{\mathbf{\Lambda}}S\left(\frac{Y\left[h^{s/\tau}(\zeta)\psi^{s}(\zeta)\right]^{\tau/s}}{X\left[h^{1-s/\tau}(\zeta)\right]^{\tau/(\tau-s)}}\right)h^{s/\tau}(\zeta)\psi^{s}(\zeta)h^{1-s/\tau}(\zeta)d\mu_{\Delta}(\zeta)\\ &\geq\left(\int\ldots\int_{\mathbf{\Lambda}}h(\zeta)\psi^{\tau}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{s}{\tau}}\left(\int\ldots\int_{\mathbf{\Lambda}}h(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{\tau-s}{\tau}}\\ &=\left(\int\ldots\int_{\mathbf{\Lambda}}h(\zeta)\psi^{\tau}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{s}{\tau}}.\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), we have

$$\int \dots \int_{\mathbf{\Lambda}} S\left(\frac{\psi^{\tau}(\zeta)}{X}\right)\psi^{s}(\zeta)h(\zeta)d\mu_{\Delta}(\zeta) \geq \left(\int \dots \int_{\mathbf{\Lambda}} h(\zeta)\psi^{\tau}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{s}{\tau}},$$

and then

$$\left(\int \dots \int_{\mathbf{\Lambda}} S\left(\frac{\psi^{\tau}(\zeta)}{X}\right)\psi^{s}(\zeta)h(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{s}} \geq \left(\int \dots \int_{\mathbf{\Lambda}} h(\zeta)\psi^{\tau}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\tau}},$$

which satisfies (3.9). □

Remark 3.4. As a special case (in one dimention) when $\mathbb{T}=\mathbb{R}$, we get the inequality (1.8) proved by Zhao and Cheung [20].

Theorem 3.6. *Let* $(\Lambda_1, \mathcal{M}, \mu_{\Delta})$ *, ..., * $(\Lambda_n, \mathcal{L}, \mu_{\Delta})$ *be finite dimensional time scale measure spaces and* w, ψ, $\varpi : \mathbf{\Lambda} = \Lambda_1 \times \dots \times \Lambda_n \to \mathbb{R}^{+}$ *such that* $0 < m \leq \psi/\varpi \leq M < \infty$. *If* $\epsilon > 1$, $1/\epsilon + 1/\delta = 1$, *then*

$$\int \dots \int_{\mathbf{\Lambda}} S\left(\frac{Y\psi(\zeta)}{X\varpi(\zeta)}\right) w(\zeta)\psi^{\frac{1}{\epsilon}}(\zeta)\varpi^{\frac{1}{\delta}}(\zeta)d\mu_{\Delta}(\zeta) \geq \frac{m^{\frac{1}{\epsilon^2}}}{M^{\frac{1}{\delta^2}}}\int \dots \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\frac{1}{\epsilon}}(\zeta)\psi^{\frac{1}{\delta}}(\zeta)d\mu_{\Delta}(\zeta), \tag{3.12}$$

where $X = \int \dots \int_{\mathbf{\Lambda}} w(\zeta)\psi(\zeta)d\mu_{\Delta}(\zeta)$, $Y = \int \dots \int_{\mathbf{\Lambda}} w(\zeta)\varpi(\zeta)d\mu_{\Delta}(\zeta)$ *and* $S(.)$ *is the Specht's ratio.*

Proof. Applying Theorem 3.2 with replacing ψ, ϖ by $\psi^{\frac{1}{\epsilon}}$, $\varpi^{\frac{1}{\delta}}$ respectively, we have for

$$X = \int \dots \int_{\mathbf{\Lambda}} w(\zeta)\psi(\zeta)d\mu_{\Delta}(\zeta) \text{ and } Y = \int \dots \int_{\mathbf{\Lambda}} w(\zeta)\varpi(\zeta)d\mu_{\Delta}(\zeta),$$

that

$$\begin{aligned} &\int \dots \int_{\mathbf{\Lambda}} S\left(\frac{Y\psi(\zeta)}{X\varpi(\zeta)}\right) w(\zeta)\psi^{\frac{1}{\epsilon}}(\zeta)\varpi^{\frac{1}{\delta}}(\zeta)d\mu_{\Delta}(\zeta) \\ &\geq \left(\int \dots \int_{\mathbf{\Lambda}} w(\zeta)\psi(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\epsilon}} \left(\int \dots \int_{\mathbf{\Lambda}} w(\zeta)\varpi(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\delta}} \\ &= \left(\int \dots \int_{\mathbf{\Lambda}} w(\zeta)\psi^{\frac{1}{\epsilon}}(\zeta)\psi^{\frac{1}{\delta}}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\epsilon}} \left(\int \dots \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\frac{1}{\epsilon}}(\zeta)\varpi^{\frac{1}{\delta}}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\delta}}. \end{aligned} \tag{3.13}$$

Since $0 < m \leq \psi/\varpi \leq M < \infty$, we have

$$\psi^{\frac{1}{\epsilon}}(\zeta) \geq m^{\frac{1}{\epsilon}}\varpi^{\frac{1}{\epsilon}}(\zeta) \text{ and } \varpi^{\frac{1}{\delta}}(\zeta) \geq M^{\frac{-1}{\delta}}\psi^{\frac{1}{\delta}}(\zeta). \tag{3.14}$$

Substituting (3.14) into the right hand side of (3.13), we get (note $1/\epsilon + 1/\delta = 1$) that

$$\int \dots \int_{\mathbf{\Lambda}} S\left(\frac{Y\psi(\zeta)}{X\varpi(\zeta)}\right) w(\zeta)\psi^{\frac{1}{\epsilon}}(\zeta)\varpi^{\frac{1}{\delta}}(\zeta)d\mu_{\Delta}(\zeta)$$

$$\geq \frac{m^{\frac{1}{\epsilon^2}}}{M^{\frac{1}{\delta^2}}}\left(\int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\frac{1}{\epsilon}}(\zeta)\psi^{\frac{1}{\delta}}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\epsilon}}$$
$$\times\left(\int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\frac{1}{\epsilon}}(\zeta)\psi^{\frac{1}{\delta}}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\delta}}$$
$$= \frac{m^{\frac{1}{\epsilon^2}}}{M^{\frac{1}{\delta^2}}}\left(\int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\frac{1}{\epsilon}}(\zeta)\psi^{\frac{1}{\delta}}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\epsilon}+\frac{1}{\delta}}$$
$$= \frac{m^{\frac{1}{\epsilon^2}}}{M^{\frac{1}{\delta^2}}}\left(\int ... \int_{\mathbf{\Lambda}} w(\zeta)\varpi^{\frac{1}{\epsilon}}(\zeta)\psi^{\frac{1}{\delta}}(\zeta)d\mu_{\Delta}(\zeta)\right),$$

which is (3.12). □

Remark 3.5. As a special case (in one dimention) if $w(\zeta) = 1$, we get the inequality (1.14) proved by El-Deeb, Elsennary and Wing-Sum Cheung [11].

3.2. The reversed Hölder inequality with three parameters

Theorem 3.7. *Let* $(\Lambda_1, \mathcal{M}, \mu_{\Delta})$ *, ...,* $(\Lambda_n, \mathcal{L}, \mu_{\Delta})$ *be finite dimensional time scale measure spaces and* ψ, ϖ, $h : \mathbf{\Lambda} = \Lambda_1 \times ... \times \Lambda_n \to \mathbb{R}^+$*. If* ϵ, δ, $r > 1$ *and* $1/\epsilon + 1/\delta + 1/r = 1$*, then*

$$\int ... \int_{\mathbf{\Lambda}} \Omega(\zeta)\psi(\zeta)\varpi(\zeta)h(\zeta)d\mu_{\Delta}(\zeta)$$
$$\geq \left(\int ... \int_{\mathbf{\Lambda}} \psi^{\epsilon}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\epsilon}}\left(\int ... \int_{\mathbf{\Lambda}} \varpi^{r}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{r}}\left(\int ... \int_{\mathbf{\Lambda}} h^{\delta}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\delta}},$$

with

$$\Omega(\zeta) = \frac{1}{\left(\int ... \int_{\mathbf{\Lambda}} [\varpi(\zeta)h(\zeta)]^{\frac{\epsilon}{\epsilon-1}} d\mu_{\Delta}(\zeta)\right)^{\frac{\epsilon-1}{\epsilon}}} S\left(\frac{Y\psi^{\epsilon}(\zeta)}{X[\varpi(\zeta)h(\zeta)]^{\frac{\epsilon}{\epsilon-1}}}\right)$$
$$\times\left(\int ... \int_{\mathbf{\Lambda}} S\left(\frac{W\varpi^{r}(\zeta)}{Zh^{\delta}(\zeta)}\right)\varpi^{r}(\zeta)h^{\delta}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{r}+\frac{1}{\delta}},$$

where $S(.)$ *is the Specht's ratio and*

$$X = \int ... \int_{\mathbf{\Lambda}} \psi^{\epsilon}(\zeta)d\mu_{\Delta}(\zeta), Y = \int ... \int_{\mathbf{\Lambda}} [\varpi(\zeta)h(\zeta)]^{\frac{\epsilon}{\epsilon-1}} d\mu_{\Delta}(\zeta),$$

and

$$Z = \int ... \int_{\mathbf{\Lambda}} \varpi^{r}(\zeta)d\mu_{\Delta}(\zeta), W = \int ... \int_{\mathbf{\Lambda}} h^{\delta}(\zeta)d\mu_{\Delta}(\zeta).$$

Proof. Denote $1/s = 1/\delta + 1/r$, then we have from the assumption $1/\epsilon + 1/\delta + 1/r = 1$ that $1/\epsilon + 1/s = 1$ and $s > 1$). Applying Lemma 2.1 with ϵ, $s > 1$ and

$$\alpha = \frac{\psi^{\epsilon}(\zeta)}{X}, \ \beta = \frac{(\varpi h)^{s}(\zeta)}{Y},$$

where $X = \int ... \int_{\mathbf{\Lambda}} \psi^{\epsilon}(\zeta)d\mu_{\Delta}(\zeta)$, $Y = \int ... \int_{\mathbf{\Lambda}} (\varpi h)^{s}(\zeta)d\mu_{\Delta}(\zeta)$, we see that

$$S\left(\frac{Y\psi^{\epsilon}(\zeta)}{X(\varpi h)^{s}(\zeta)}\right)\frac{\psi(\zeta)}{X^{\frac{1}{\epsilon}}}\frac{(\varpi h)(\zeta)}{Y^{\frac{1}{s}}} \geq \frac{1}{\epsilon}\frac{\psi^{\epsilon}(\zeta)}{X} + \frac{1}{s}\frac{(\varpi h)^{s}(\zeta)}{Y},$$

i.e.

$$S\left(\frac{Y\psi^{\epsilon}(\zeta)}{X\varpi^{s}(\zeta)h^{s}(\zeta)}\right)\frac{\psi(\zeta)}{X^{\frac{1}{\epsilon}}}\frac{\varpi(\zeta)h(\zeta)}{Y^{\frac{1}{s}}}\geq\frac{1}{\epsilon}\frac{\psi^{\epsilon}(\zeta)}{X}+\frac{1}{s}\frac{\varpi^{s}(\zeta)h^{s}(\zeta)}{Y}. \tag{3.15}$$

Integrating (3.15) on $\mathbf{\Lambda}$, we have (where $1/\epsilon+1/s=1$) that

$$\begin{aligned}
&\int\dots\int_{\mathbf{\Lambda}}S\left(\frac{Y\psi^{\epsilon}(\zeta)}{X\varpi^{s}(\zeta)h^{s}(\zeta)}\right)\psi(\zeta)\varpi(\zeta)h(\zeta)d\mu_{\Delta}(\zeta)\\
&\geq X^{\frac{1}{\epsilon}}Y^{\frac{1}{s}}\int\dots\int_{\mathbf{\Lambda}}\left[\frac{1}{\epsilon}\frac{\psi^{\epsilon}(\zeta)}{X}+\frac{1}{s}\frac{\varpi^{s}(\zeta)h^{s}(\zeta)}{Y}\right]d\mu_{\Delta}(\zeta)\\
&=X^{\frac{1}{\epsilon}}Y^{\frac{1}{s}}\left[\frac{1}{\epsilon X}\int\dots\int_{\mathbf{\Lambda}}\psi^{\epsilon}(\zeta)d\mu_{\Delta}(\zeta)+\frac{1}{sY}\int\dots\int_{\mathbf{\Lambda}}\varpi^{s}(\zeta)h^{s}(\zeta)d\mu_{\Delta}(\zeta)\right]\\
&=X^{\frac{1}{\epsilon}}Y^{\frac{1}{s}}\left[\frac{1}{\epsilon}+\frac{1}{s}\right]=X^{\frac{1}{\epsilon}}Y^{\frac{1}{s}}\\
&=\left(\int\dots\int_{\mathbf{\Lambda}}\psi^{\epsilon}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\epsilon}}\left(\int\dots\int_{\mathbf{\Lambda}}\varpi^{s}(\zeta)h^{s}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{s}},
\end{aligned}$$

i.e.

$$\begin{aligned}
&\int\dots\int_{\mathbf{\Lambda}}S\left(\frac{Y\psi^{\epsilon}(\zeta)}{X\left[\varpi(\zeta)h(\zeta)\right]^{\frac{\epsilon}{\epsilon-1}}}\right)\psi(\zeta)\varpi(\zeta)h(\zeta)d\mu_{\Delta}(\zeta)\\
&\geq\left(\int\dots\int_{\mathbf{\Lambda}}\psi^{\epsilon}(\zeta)d\mu_{\Delta}(\zeta)\right)^{\frac{1}{\epsilon}}\left(\int\dots\int_{\mathbf{\Lambda}}\left[\varpi(\zeta)h(\zeta)\right]^{\frac{\epsilon}{\epsilon-1}}d\mu_{\Delta}(\zeta)\right)^{\frac{\epsilon-1}{\epsilon}}.
\end{aligned} \tag{3.16}$$

Since $1/s=1/\delta+1/r$, then $s/r+s/\delta=1$. Since $r>1$, then $1/r>0$, and then $1/s>1/\delta$, thus we see (note $\delta>1$) that $\delta/s>1$. Also, we observe (since $\delta/s>1$, $\delta/s-1>0$) that

$$\frac{r}{s}=\frac{\delta/s}{\delta/s-1}=1+\frac{1}{\delta/s-1}>1.$$

Again by applying Lemma 2.1 with r/s, $\delta/s>1$ such that $s/r+s/\delta=1$ and

$$\alpha=\frac{(\varpi^{s}(\zeta))^{\frac{r}{s}}}{Z}=\frac{\varpi^{r}(\zeta)}{Z},\ \beta=\frac{(h^{s}(\zeta))^{\frac{\delta}{s}}}{W}=\frac{h^{\delta}(\zeta)}{W},$$

where $Z=\int\dots\int_{\mathbf{\Lambda}}\varpi^{r}(\zeta)d\mu_{\Delta}(\zeta)$, $W=\int\dots\int_{\mathbf{\Lambda}}h^{\delta}(\zeta)d\mu_{\Delta}(\zeta)$, we see that

$$S\left(\frac{W\varpi^{r}(\zeta)}{Zh^{\delta}(\zeta)}\right)\frac{\varpi^{r}(\zeta)}{Z^{\frac{s}{r}}}\frac{h^{\delta}(\zeta)}{W^{\frac{s}{\delta}}}\geq\frac{s}{r}\frac{\varpi^{r}(\zeta)}{Z}+\frac{s}{\delta}\frac{h^{\delta}(\zeta)}{W}. \tag{3.17}$$

Integrating (3.17) on the region $\mathbf{\Lambda}$ with $s/r+s/\delta=1$, we get

$$\begin{aligned}
&\int\dots\int_{\mathbf{\Lambda}}S\left(\frac{W\varpi^{r}(\zeta)}{Zh^{\delta}(\zeta)}\right)\varpi^{r}(\zeta)h^{\delta}(\zeta)d\mu_{\Delta}(\zeta)\\
&\geq Z^{\frac{s}{r}}W^{\frac{s}{\delta}}\int\dots\int_{\mathbf{\Lambda}}\left[\frac{s}{r}\frac{\varpi^{r}(\zeta)}{Z}+\frac{s}{\delta}\frac{h^{\delta}(\zeta)}{W}\right]d\mu_{\Delta}(\zeta)\\
&=Z^{\frac{s}{r}}W^{\frac{s}{\delta}}\left[\frac{s}{rZ}\int\dots\int_{\mathbf{\Lambda}}\varpi^{r}(\zeta)d\mu_{\Delta}(\zeta)+\frac{s}{\delta W}\int\dots\int_{\mathbf{\Lambda}}h^{\delta}(\zeta)d\mu_{\Delta}(\zeta)\right]\\
&=Z^{\frac{s}{r}}W^{\frac{s}{\delta}}\left[\frac{s}{r}+\frac{s}{\delta}\right]=Z^{\frac{s}{r}}W^{\frac{s}{\delta}}
\end{aligned}$$

$$= \left(\int \ldots \int_{\mathbf{\Lambda}} \varpi^r(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{s}{r}} \left(\int \ldots \int_{\mathbf{\Lambda}} h^\delta(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{s}{\delta}},$$

and then

$$\left(\int \ldots \int_{\mathbf{\Lambda}} S\left(\frac{W\varpi^r(\zeta)}{Zh^\delta(\zeta)}\right).\varpi^r(\zeta) h^\delta(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{s}}$$
$$\geq \left(\int \ldots \int_{\mathbf{\Lambda}} \varpi^r(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{r}} \left(\int \ldots \int_{\mathbf{\Lambda}} h^\delta(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{\delta}},$$

thus we have (note $1/s = 1/\delta + 1/r$) that

$$\left(\int \ldots \int_{\mathbf{\Lambda}} S\left(\frac{W\varpi^r(\zeta)}{Zh^\delta(\zeta)}\right)\varpi^r(\zeta) h^\delta(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{r}+\frac{1}{\delta}}$$
$$\geq \left(\int \ldots \int_{\mathbf{\Lambda}} \varpi^r(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{r}} \left(\int \ldots \int_{\mathbf{\Lambda}} h^\delta(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{\delta}}. \tag{3.18}$$

From (3.16) and (3.18), we have

$$\frac{1}{\left(\int \ldots \int_{\mathbf{\Lambda}} [\varpi(\zeta)h(\zeta)]^{\frac{\epsilon}{\epsilon-1}} d\mu_\Delta(\zeta)\right)^{\frac{\epsilon-1}{\epsilon}}}$$
$$\times \int \ldots \int_{\mathbf{\Lambda}} S\left(\frac{Y\psi^\epsilon(\zeta)}{X[\varpi(\zeta)h(\zeta)]^{\frac{\epsilon}{\epsilon-1}}}\right)\psi(\zeta)\varpi(\zeta)h(\zeta) d\mu_\Delta(\zeta)$$
$$\times \left(\int \ldots \int_{\mathbf{\Lambda}} S\left(\frac{W\varpi^r(\zeta)}{Zh^\delta(\zeta)}\right)\varpi^r(\zeta) h^\delta(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{r}+\frac{1}{\delta}}$$
$$\geq \left(\int \ldots \int_{\mathbf{\Lambda}} \psi^\epsilon(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{\epsilon}} \left(\int \ldots \int_{\mathbf{\Lambda}} \varpi^r(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{r}} \left(\int \ldots \int_{\mathbf{\Lambda}} h^\delta(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{\delta}}.$$

The last inequality can be written as following

$$\int \ldots \int_{\mathbf{\Lambda}} \Omega(\zeta)\psi(\zeta)\varpi(\zeta)h(\zeta) d\mu_\Delta(\zeta)$$
$$\geq \left(\int \ldots \int_{\mathbf{\Lambda}} \psi^\epsilon(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{\epsilon}} \left(\int \ldots \int_{\mathbf{\Lambda}} \varpi^r(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{r}}$$
$$\times \left(\int \ldots \int_{\mathbf{\Lambda}} h^\delta(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{\delta}},$$

where

$$\Omega(\zeta) = \frac{1}{\left(\int \ldots \int_{\mathbf{\Lambda}} [\varpi(\zeta)h(\zeta)]^{\frac{\epsilon}{\epsilon-1}} d\mu_\Delta(\zeta)\right)^{\frac{\epsilon-1}{\epsilon}}} S\left(\frac{Y\psi^\epsilon(\zeta)}{X[\varpi(\zeta)h(\zeta)]^{\frac{\epsilon}{\epsilon-1}}}\right)$$
$$\times \left(\int \ldots \int_{\mathbf{\Lambda}} S\left(\frac{W\varpi^r(\zeta)}{Zh^\delta(\zeta)}\right).\varpi^r(\zeta) h^\delta(\zeta) d\mu_\Delta(\zeta)\right)^{\frac{1}{r}+\frac{1}{\delta}}.$$

The proof is complete. □

4. Conclusion and Future work

In this manuscript, we present some new reverse generalizations and refinements of multidimensional Hölder-type inequalities with Specht's ratio on an arbitrary time scale measure spaces. We generalize a number of those inequalities to a general time scale measure space. These inequalities extend some known dynamic inequalities on time scales, unify and extend some continuous inequalities and their corresponding discrete analogues. In the future, we will continue to generalize more fractional dynamic inequalities by using Specht's ratio, Kantorovich's ratio and n-tuple fractional integral. In particular, such inequalities can be introduced by using fractional integrals and fractional derivatives of the Riemann-Liouville type on time scales. In addition to this, we may generalize these results to be with multidimensional Hölder-type inequalities via supermultiplicative functions on time scales. It will also be very enjoyable to introduce such inequalities in quantum calculations.

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