Journal of Applied Analysis and Computation
Volume 13, Number 1, February 2023, 376–423
Website:http://www.jaac-online.com
DOI:10.11948/20220177

NUMERICAL METHODS FOR THE CAPUTO-TYPE FRACTIONAL DERIVATIVE WITH AN EXPONENTIAL KERNEL*

Enyu Fan[1], Changpin Li[1,†] and Zhiqiang Li[2]

Abstract In the present article, several typical numerical discrete formulas for the Caputo-type fractional derivative with an exponential kernel (call "exponential Caputo derivative" for brevity) with order $\alpha \in (0,1)$ and $\alpha \in (1,2)$ are constructed, which are L1, L1-2, L2-1_σ formulas for $\alpha \in (0,1)$, and H2N2 and $L2_1$ formulas for $\alpha \in (1,2)$, respectively. And the estimates of the truncation errors are determined. Meanwhile, the properties of the coefficients in these formulas are studied. Finally, some numerical examples are displayed which support the theoretical analysis.

Keywords Exponential Caputo derivative, numerical discrete formula, truncation error.

MSC(2010) 26A33, 65D25.

1. Introduction

On one hand, it is often said that the solution to the (integer-order) ordinary differential equation is exponentially asymptotic and that the solution to fractional ordinary differential equation has algebraic asymptotics. But in effect, the fractional derivative in the latter case is the usual Caputo derivative or Riemann-Liouville derivative [13, 16]. This naturally leads us to ask which kind of fractional ordinary differential equation has the solution with exponential asymptotics if it exists.

On the other hand, the Cauchy problem of partial differential equation with time integer-order derivative or time Caputo/Riemann-Liouville fractional derivative has algebraically asymptotical solution, e.g., [12]. It is also natural to ask which kind of Cauchy problem of time fractional partial differential equation has the solution with exponential asymptotics if there is any.

The answer to the above two questions is positive. There exists indeed fractional derivative, i.e., Caputo/Riemann-Liouville fractional derivative with an exponential kernel (call exponential Caputo/Riemann-Liouville fractional derivative for brevity) [9]. In the following, the corresponding definitions in this respect are introduced.

Definition 1.1 ([9]). For $\alpha > 0$ and $f(t) \in L^1(a,b)\,(-\infty \le a < b \le +\infty)$, the

[†]The corresponding author. Email: lcp@shu.edu.cn(C. Li)
[1]Department of Mathematics, Shanghai University, Shanghai 200444, China
[2]Department of Mathematics, Lvliang University, Lvliang 0033001, China
*The authors were supported by National Natural Science Foundation of China (No. 12271339).

α-th exponential Riemann-Liouville fractional integral of $f(t)$ is defined as follows

$$ {}_{e}\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} f(s) e^s \mathrm{d}s,\ t > a. \tag{1.1} $$

Remark 1.1. In the definition of Riemann-Liouville fractional integral, the condition $f(t) \in L^1(a,b)$ in most of papers is default albeit not clearly stated. Hereafter, we re-state the condition in this definition in order to attract the attention of beginners.

Definition 1.2 ([9]). For $n-1 < \alpha < n \in \mathbb{Z}^+$ and $f(t) \in AC^n_{\delta_e}[a,b] = \big\{f : [a,b] \to \mathbb{R} : \delta_e^{n-1} f(t) \in AC[a,b]\big\}(-\infty \le a < b \le +\infty)$, the α-th exponential Riemann-Liouville fractional derivative of $f(t)$ is defined as follows

$$ {}_{e}\mathrm{D}_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \delta_e^n \int_a^t \left(e^t - e^s\right)^{n-\alpha-1} f(s) e^s \mathrm{d}s,\ t > a, \tag{1.2} $$

where the operator δ_e^n is represented by

$$ \delta_e^n f(s) = \left(e^{-s}\frac{\mathrm{d}}{\mathrm{d}s}\right)^n f(s) = \delta_e\left(\delta_e^{n-1} f(s)\right),\quad \delta_e^0 f(s) = f(s). $$

Remark 1.2. In the definition of Riemann-Liouville fractional derivative, in most of papers the condition $f(t) \in AC^n[a,b]$ is admitted despite being not clearly pointed out. Hereafter, we emphasize this condition $f(t) \in AC^n_{\delta_e}[a,b]$ in the above definition. This explanation applies also to the next definition.

Definition 1.3 ([9]). For $n-1 < \alpha < n \in \mathbb{Z}^+$ and $f(t) \in AC^n_{\delta_e}[a,b] = \big\{f : [a,b] \to \mathbb{R} : \delta_e^{n-1} f(t) \in AC[a,b]\big\}(-\infty \le a < b \le +\infty)$, the α-th exponential Caputo fractional derivative of $f(t)$ is defined as below

$$ {}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(e^t - e^s\right)^{n-\alpha-1} \delta_e^n f(s) e^s \mathrm{d}s,\ t > a. \tag{1.3} $$

The exponential Riemann-Liouville fractional derivative (1.2) and the exponential Caputo fractional derivative (1.3) for order $n-1 < \alpha < n \in \mathbb{Z}^+$ are not equivalent with each other, but they have the following relationship [6]

$$ \begin{aligned} {}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t) &= {}_{e}\mathrm{D}_{a,t}^{\alpha}\left[f(t) - \sum_{k=0}^{n-1} \frac{\delta_e^k f(a)}{k!}\left(e^t - e^a\right)^k\right] \\ &= {}_{e}\mathrm{D}_{a,t}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{\delta_e^k f(a)}{\Gamma(k-\alpha+1)}\left(e^t - e^a\right)^{k-\alpha}. \end{aligned} \tag{1.4} $$

Due to this relation, this paper only focuses on constructing computational formulae for the exponential Caputo fractional derivative.

Because it is difficult and even impossible to exactly solve fractional differential equations, seeking numerical solutions to these equations becomes more and more important. So deriving various kinds of numerical discrete formulas for the corresponding fractional derivatives in these equations is one of the vital tasks. This paper aims to establish the typical approximate formulas for the exponential Caputo fractional derivatives.

The rest of the paper is outlined as follows. In Section 2, the necessary preliminaries are provided. In Section 3, some numerical formulas are obtained for approximating the exponential Caputo fractional derivatives defined in (1.3) under a uniform partition, namely, L1-A, L1-2-A, L2-1_σ-A formulas for $\alpha \in (0,1)$, and H2N2-A, L2_1-A formulas for $\alpha \in (1,2)$, where the corresponding truncation errors are determined. And properties of coefficients in the above formulas have been studied. In Section 4, L1-B, L1-2-B, L2-1_σ-B, H2N2-B and L2_1-B formulas under a special nonuniform partition are presented. Numerical experiments for the discrete formulas are displayed in Section 5, which support the theoretical analysis. Finally, the summary is given in Section 6.

2. Preliminaries

Before constructing the numerical formulas, divide the interval $[a,T]$ into subintervals $[t_{i-1},t_i]$ with $a = t_0 < t_1 < \cdots < t_N = T$ for $N \in \mathbb{Z}^+$. Hereafter, only two kinds of partitions are considered.

Partition A : Uniform partition

$$t_k = t_0 + k\tau, \quad \tau = t_k - t_{k-1} = \frac{T-a}{N} \; (1 \le k \le N)\,. \tag{2.1}$$

Partition B : Non-uniform partition (i.e., uniform partition in exponent's sense)

$$t_k = \log\left(e^{t_0} + k\widetilde{\tau}\right), \quad \widetilde{\tau} = e^{t_k} - e^{t_{k-1}} = \frac{e^T - e^a}{N} \; (1 \le k \le N)\,. \tag{2.2}$$

For convenience, call the numerical approximate formulas under **Partition A** and **Partition B** A-type formulas and B-type formulas, respectively. Denote $f^k = f(t_k)$ and introduce the following operator

$$\nabla_{\exp,t} f^{k-\frac{1}{2}} = \frac{f^k - f^{k-1}}{e^{t_k} - e^{t_{k-1}}}.$$

Next, we introduce the interpolation functions in the exponential sense in order to facilitate the construction of the approximate formulae.

Two points $(t_{j-1}, f(t_{j-1}))$, $(t_j, f(t_j))$ can determine the linear Lagrange interpolation function $L_{\exp,1,j}f(t)$ (in the sense of exponential function) for function $f(t)$ on the interval $[t_{j-1},t_j]$ $(1 \le j \le N)$, which reads as

$$L_{\exp,1,j}f(t) = \frac{e^{t_j} - e^t}{e^{t_j} - e^{t_{j-1}}} f^{j-1} + \frac{e^t - e^{t_{j-1}}}{e^{t_j} - e^{t_{j-1}}} f^j. \tag{2.3}$$

The corresponding truncation error on $[t_{j-1},t_j]$ is given below,

$$\begin{aligned} r_1^j(t) &= f(t) - L_{\exp,1,j}f(t) \\ &= \frac{1}{2}\delta_e^2 f(\eta_j)\left(e^t - e^{t_{j-1}}\right)\left(e^t - e^{t_j}\right), \quad \eta_j \in (t_{j-1},t_j). \end{aligned} \tag{2.4}$$

Similarly, three points $(t_{j-1}, f(t_{j-1}))$, $(t_j, f(t_j))$ and $(t_{j+1}, f(t_{j+1}))$, can determine quadratic polynomial interpolations (in the sense of exponential function) for function $f(t)$ on $[t_{j-1}, t_{j+1}]$ $(1 \le j \le N-1)$ as follows,

$$\begin{aligned} L_{\exp,2,j} f(t) = & \frac{(e^t - e^{t_j})(e^t - e^{t_{j+1}})}{(e^{t_{j-1}} - e^{t_j})(e^{t_{j-1}} - e^{t_{j+1}})} f^{j-1} + \frac{(e^t - e^{t_{j-1}})(e^t - e^{t_{j+1}})}{(e^{t_j} - e^{t_{j-1}})(e^{t_j} - e^{t_{j+1}})} f^j \\ & + \frac{(e^t - e^{t_{j-1}})(e^t - e^{t_j})}{(e^{t_{j+1}} - e^{t_{j-1}})(e^{t_{j+1}} - e^{t_j})} f^{j+1}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} N_{\exp,2,j} f(t) = & f(t_{j-1}) + \nabla_{\exp,t} f^{j-\frac{1}{2}} (e^t - e^{t_{j-1}}) \\ & + \frac{\nabla_{\exp,t} f^{j+\frac{1}{2}} - \nabla_{\exp,t} f^{j-\frac{1}{2}}}{e^{t_{j+1}} - e^{t_{j-1}}} (e^t - e^{t_{j-1}})(e^t - e^{t_j}), \end{aligned} \tag{2.6}$$

where the two interpolations are essentially the same, albeit with different expressions in form. They are called Lagrange interpolation and Newton interpolation (in the exponential sense) respectively, and their truncation errors are denoted as

$$r_2^j(t) = f(t) - L_{\exp,2,j} f(t), \quad R_N^j(t) = f(t) - N_{\exp,2,j} f(t),$$

which are the same

$$r_2^j(t) = R_N^j(t) = \frac{1}{6} \delta_e^3 f(\xi_j) \left(e^t - e^{t_{j-1}}\right) \left(e^t - e^{t_j}\right) \left(e^t - e^{t_{j+1}}\right), \ \xi_j \in (t_{j-1}, t_{j+1}). \tag{2.7}$$

On the other hand, $(t_0, f(t_0))$, $(t_1, f(t_1))$ and $(t_0, \delta_e f(t_0))$ can determine quadratic Hermite interpolation $H_{\exp,2,0} f(t)$ (in the exponential sense) for function $f(t)$ on $\left[t_0, t_1\right]$

$$H_{\exp,2,0} f(t) = f(t_0) + \delta_e f(t_0) \left(e^t - e^{t_0}\right) + \frac{\nabla_{\exp,t} f^{\frac{1}{2}} - \delta_e f(t_0)}{e^{t_1} - e^{t_0}} \left(e^t - e^{t_0}\right)^2, \tag{2.8}$$

with truncation error in the following form,

$$R_H(t) = f(t) - H_{\exp,2,0} f(t) = \frac{1}{6} \delta_e^3 f(\xi_0) \left(e^t - e^{t_0}\right)^2 \left(e^t - e^{t_1}\right), \ \xi_0 \in (t_0, t_1). \tag{2.9}$$

In addition, function $\varphi(x)$ at $x = x_0$ can be expanded in the the following form,

$$\begin{aligned} \varphi(x) = & \varphi(x_0) + \delta_e \varphi(x_0) \left(e^x - e^{x_0}\right) + \frac{1}{2!} \delta_e^2 \varphi(x_0) \left(e^x - e^{x_0}\right)^2 \\ & + \cdots + \frac{1}{n!} \delta_e^n \varphi(x_0) \left(e^x - e^{x_0}\right)^n + \frac{1}{(n+1)!} \delta_e^{n+1} \varphi(\xi) \left(e^x - e^{x_0}\right)^{n+1}, \end{aligned} \tag{2.10}$$

where ξ is between x and x_0. This is just the Taylor expansion in $\mathrm{span}\{1, e^x, e^{2x}, \cdots\}$.

3. A-type formulas

In present section, typical numerical approximate formulas for the exponential Caputo fractional derivative (1.3) with $\alpha \in (0,1)$ and $\alpha \in (1,2)$ under **Partition A** (2.1) are constructed. All these formulas are called A-type formulas.

3.1. L1-A formula

Now, the L1-A formula under **Partition A** for approximating fractional derivative (1.3) with $\alpha \in (0,1)$ is constructed as follows.

Using the interpolation function $L_{\exp,1,j}f(t)$ on $[t_{j-1}, t_j]$ to approximate the function $f(t)$ on the same interval gives

$$f(t) \approx L_{\exp,1,j}f(t),\ t \in [t_{j-1}, t_j]. \tag{3.1}$$

From the interpolation function (2.3) on $[t_{j-1}, t_j]$, one has

$$\delta_e\left(L_{\exp,1,j}f(t)\right) = \nabla_{\exp,t} f^{j-\frac{1}{2}}. \tag{3.2}$$

By means of formula (3.2), the exponential Caputo fractional derivative ${}_{Ce}\mathrm{D}^{\alpha}_{a,t}f(t)$ at $t = t_k$ $(1 \le k \le N)$ can be written as

$$\begin{aligned}
{}_{Ce}\mathrm{D}^{\alpha}_{a,t}f(t)\Big|_{t=t_k} &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left(e^{t_k}-e^{s}\right)^{-\alpha}\delta_e f(s)\,e^{s}\mathrm{d}s \\
&\approx \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left(e^{t_k}-e^{s}\right)^{-\alpha}\delta_e\left(L_{\exp,1,j}f(s)\right)e^{s}\mathrm{d}s \\
&= \frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{k}c^{(\alpha)}_{j,k}\left(f^{j}-f^{j-1}\right), \\
&:= {}_{Ce}\mathscr{D}^{\alpha}_{a,t}f^{k},
\end{aligned} \tag{3.3}$$

where

$$c^{(\alpha)}_{j,k} = \frac{1}{e^{t_j}-e^{t_{j-1}}}\left[\left(e^{t_k}-e^{t_{j-1}}\right)^{1-\alpha}-\left(e^{t_k}-e^{t_j}\right)^{1-\alpha}\right],\quad j=1,2,\ldots,k. \tag{3.4}$$

The numerical formula ${}_{Ce}\mathscr{D}^{\alpha}_{a,t}f^{k}$ in (3.3) is called L1-A formula for the exponential Caputo fractional derivative with $\alpha \in (0,1)$. The truncation error is analyzed in the following.

Theorem 3.1. *For $0 < \alpha < 1$ and $\delta_e^2 f(t) \in C[a,T]$, the following truncation errors R^k $(1 \le k \le N)$ with $\tau = t_k - t_{k-1} = \frac{T-a}{N}$,*

$$\begin{aligned}
R^{k} &= {}_{Ce}\mathrm{D}^{\alpha}_{a,t}f(t)\Big|_{t=t_k} - {}_{Ce}\mathscr{D}^{\alpha}_{a,t}f^{k} \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left(e^{t_k}-e^{s}\right)^{-\alpha}\delta_e\left[f(s)-L_{\exp,1,j}f(s)\right]e^{s}\mathrm{d}s,
\end{aligned} \tag{3.5}$$

has the estimate below

$$\begin{aligned}
\left|R^{k}\right| \le{}& \frac{1}{8\Gamma(1-\alpha)}\max_{t_0\le t\le t_{k-1}}\left|\delta_e^2 f(t)\right|\max_{1\le l\le k-1}\left(e^{t_l}-e^{t_{l-1}}\right)^{2}\left(e^{t_k}-e^{t_{k-1}}\right)^{-\alpha} \\
&+ \frac{\alpha}{2\Gamma(3-\alpha)}\max_{t_{k-1}\le t\le t_k}\left|\delta_e^2 f(t)\right|\left(e^{t_k}-e^{t_{k-1}}\right)^{2-\alpha}.
\end{aligned} \tag{3.6}$$

Proof. Using integration by parts in the right side of (3.5) and the truncation error given in (2.4) gives

$$\begin{aligned} R^k =& \frac{-\alpha}{\Gamma(1-\alpha)} \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} r_1^j(s) \left(e^{t_k} - e^s\right)^{-\alpha-1} e^s \mathrm{d}s \\ &+ \frac{-\alpha}{\Gamma(1-\alpha)} \int_{t_{k-1}}^{t_k} r_1^k(s) \left(e^{t_k} - e^s\right)^{-\alpha-1} e^s \mathrm{d}s \\ =& R_1^k + R_2^k. \end{aligned} \tag{3.7}$$

It follows from $r_1^j(s)\,(1 \le j \le k)$ (see (2.4)) that

$$\begin{aligned} \left|R_1^k\right| \le& \frac{\alpha}{8\Gamma(1-\alpha)} \max_{t_0 \le t \le t_{k-1}} \left|\delta_e^2 f(t)\right| \max_{1 \le l \le k-1} \left(e^{t_l} - e^{t_{l-1}}\right)^2 \\ &\times \int_{t_0}^{t_{k-1}} \left(e^{t_k} - e^s\right)^{-\alpha-1} e^s \mathrm{d}s \\ \le& \frac{1}{8\Gamma(1-\alpha)} \max_{t_0 \le t \le t_{k-1}} \left|\delta_e^2 f(t)\right| \max_{1 \le l \le k-1} \left(e^{t_l} - e^{t_{l-1}}\right)^2 \left(e^{t_k} - e^{t_{k-1}}\right)^{-\alpha}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \left|R_2^k\right| \le& \frac{\alpha}{2\Gamma(1-\alpha)} \max_{t_{k-1} \le t \le t_k} \left|\delta_e^2 f(t)\right| \int_{t_{k-1}}^{t_k} \left(e^s - e^{t_{k-1}}\right) \left(e^{t_k} - e^s\right)^{-\alpha} e^s \mathrm{d}s \\ =& \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_{k-1} \le t \le t_k} \left|\delta_e^2 f(t)\right| \left(e^{t_k} - e^{t_{k-1}}\right)^{2-\alpha}. \end{aligned} \tag{3.9}$$

Using the above two inequalities ends the proof. □

Remark 3.1. Due to $e^{t_k} - e^{t_{k-1}} = O(\tau)$ with $t_k \to t_{k-1}\,(1 \le k \le N)$, the estimate of truncation error R^k in (3.3) can be expressed as

$$\left|R^k\right| \le C\tau^{2-\alpha},$$

where C is a positive constant.

Next the properties of coefficients $c_{j,k}^{(\alpha)}$ in L1-A formula are shown below.

Theorem 3.2. *For order $\alpha \in (0,1)$, the coefficients $c_{j,k}^{(\alpha)}\,(1 \le j \le k, 1 \le k \le N)$ in (3.4) with $\tau = t_k - t_{k-1} = \frac{T-a}{N}$ satisfy*

$$c_{k,k}^{(\alpha)} > c_{k-1,k}^{(\alpha)} > \cdots > c_{1,k}^{(\alpha)} > 0. \tag{3.10}$$

Proof. It is obvious that the function $(e^{t_k} - e^s)^{1-\alpha}$ is a monotone decreasing function with respect to s, so

$$c_{j,k}^{(\alpha)} = \frac{1}{e^{t_j} - e^{t_{j-1}}} \left[\left(e^{t_k} - e^{t_{j-1}}\right)^{1-\alpha} - \left(e^{t_k} - e^{t_j}\right)^{1-\alpha}\right] > 0.$$

In addition, using the differential mean value theorem yields

$$c_{j+1,k}^{(\alpha)} - c_{j,k}^{(\alpha)} = (1-\alpha)(\xi_{j+1}^{-\alpha} - \xi_j^{-\alpha}) > 0, \tag{3.11}$$

where $e^{t_k} - e^{t_{j+1}} < \xi_{j+1} < e^{t_k} - e^{t_j} < \xi_j < e^{t_k} - e^{t_{j-1}}\,(1 \le j \le k-1)$, and hence (3.10). □

3.2. L1-2-A formula

In the subsection, L1-2-A formula under **Partition A** for numerical approach to fractional derivative (1.3) with $\alpha \in (0,1)$ is established as follows.

Using

$$\begin{cases} f(t) \approx L_{\exp,1,1} f(t), \quad t \in [t_0, t_1], \\ f(t) \approx L_{\exp,2,j-1} f(t), \quad t \in [t_{j-1}, t_j]\,(2 \leq j \leq N)\,, \end{cases} \tag{3.12}$$

and

$$\begin{aligned} &\delta_e\left(L_{\exp,2,j-1} f(t)\right) \\ =&\delta_e\left(L_{\exp,1,j} f(t)\right) + \frac{\nabla_{\exp,t} f^{j-\frac{1}{2}} - \nabla_{\exp,t} f^{j-\frac{3}{2}}}{e^{t_j} - e^{t_{j-2}}} \left(2e^t - e^{t_{j-1}} - e^{t_j}\right), \end{aligned} \tag{3.13}$$

one has

$$\begin{aligned} &{}_{Ce}\mathrm{D}^{\alpha}_{a,t} f(t)\Big|_{t=t_k} \\ =&\frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left(e^{t_k} - e^s\right)^{-\alpha} \delta_e f(s) e^s \mathrm{d}s \\ \approx&\frac{1}{\Gamma(1-\alpha)} \left\{ \int_{t_0}^{t_1} \left(e^{t_k} - e^s\right)^{-\alpha} \delta_e\left(L_{\exp,1,1} f(s)\right) e^s \mathrm{d}s \right. \\ &\left. + \sum_{j=2}^{k} \int_{t_{j-1}}^{t_j} \left(e^{t_k} - e^s\right)^{-\alpha} \delta_e\left(L_{\exp,2,j-1} f(s)\right) e^s \mathrm{d}s \right\} \\ =&{}_{Ce}\mathscr{D}^{\alpha}_{a,t} f^k - \frac{1}{\Gamma(2-\alpha)} \sum_{j=2}^{k} b^{(\alpha)}_{j,k} \left(\nabla_{\exp,t} f^{j-\frac{1}{2}} - \nabla_{\exp,t} f^{j-\frac{3}{2}}\right) \\ =&\frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^{k} c^{(\alpha)}_{j,k} \left(f^j - f^{j-1}\right) \\ :=&{}_{Ce}\mathbb{D}^{\alpha}_{a,t} f^k, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} c^{(\alpha)}_{j,k} &= \begin{cases} \frac{1}{e^{t_1} - e^{t_0}} \left(a^{(\alpha)}_{1,k} + b^{(\alpha)}_{2,k}\right), \ j = 1, \\ \frac{1}{e^{t_j} - e^{t_{j-1}}} \left(a^{(\alpha)}_{j,k} - b^{(\alpha)}_{j,k} + b^{(\alpha)}_{j+1,k}\right), \ 2 \leq j \leq k-1, \\ \frac{1}{e^{t_k} - e^{t_{k-1}}} \left(a^{(\alpha)}_{k,k} - b^{(\alpha)}_{k,k}\right), \ j = k, \end{cases} \\ a^{(\alpha)}_{j,k} &= \left(e^{t_k} - e^{t_{j-1}}\right)^{1-\alpha} - \left(e^{t_k} - e^{t_j}\right)^{1-\alpha}, \\ b^{(\alpha)}_{j,k} &= \left\{ \left(e^{t_j} - e^{t_{j-1}}\right) \left[\left(e^{t_k} - e^{t_j}\right)^{1-\alpha} + \left(e^{t_k} - e^{t_{j-1}}\right)^{1-\alpha}\right] \right. \\ &\quad \left. + \frac{2}{2-\alpha} \left[\left(e^{t_k} - e^{t_j}\right)^{2-\alpha} - \left(e^{t_k} - e^{t_{j-1}}\right)^{2-\alpha}\right] \right\} \frac{1}{e^{t_j} - e^{t_{j-2}}}. \end{aligned} \tag{3.15}$$

Formula ${}_{Ce}\mathbb{D}_{a,t}^{\alpha}f^k$ in (3.14) is called L1-2-A formula under **Partition A** (2.1) for approaching fractional derivative (1.3) with $\alpha \in (0,1)$.

Remark 3.2. Particularly, when $k=1$ or $k=2$, then one has

$$\begin{aligned} &c_{1,1}^{(\alpha)} = \frac{1}{e^{t_1}-e^{t_0}} a_{1,1}^{(\alpha)}, \quad k=1; \\ &c_{1,2}^{(\alpha)} = \frac{1}{e^{t_1}-e^{t_0}} \left(a_{1,2}^{(\alpha)} + b_{2,2}^{(\alpha)}\right), \quad c_{2,2}^{(\alpha)} = \frac{1}{e^{t_2}-e^{t_1}} \left(a_{2,2}^{(\alpha)} - b_{2,2}^{(\alpha)}\right), \quad k=2. \end{aligned} \tag{3.16}$$

In the following, the truncation error of L1-2-A formula will be considered.

Theorem 3.3. *For $0<\alpha<1$ and $\delta_e^3 f(t) \in C\,[a,T]$, the truncation errors $R^k\big(1 \le k \le N\big)$ under* **Partition A**,

$$\begin{aligned} R^k =& {}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)\Big|_{t=t_k} - {}_{Ce}\mathbb{D}_{a,t}^{\alpha} f^k \\ =& \frac{1}{\Gamma(1-\alpha)} \Bigg\{ \int_{t_0}^{t_1} \left(e^{t_k}-e^s\right)^{-\alpha} \delta_e \left[f(s) - L_{\exp,1,1} f(s)\right] e^s \mathrm{d}s \\ &+ \sum_{j=2}^{k} \int_{t_{j-1}}^{t_j} \left(e^{t_k}-e^s\right)^{-\alpha} \delta_e \left[f(s) - L_{\exp,2,j-1} f(s)\right] e^s \mathrm{d}s \Bigg\}, \end{aligned} \tag{3.17}$$

have following estimates,

$$\begin{aligned} \left|R^1\right| \le& \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^2 f(t)\right| \left(e^{t_1}-e^{t_0}\right)^{2-\alpha}, \quad k=1, \\ \left|R^k\right| \le& \frac{\alpha}{8\Gamma(1-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^2 f(t)\right| \left(e^{t_k}-e^{t_1}\right)^{-1-\alpha} \left(e^{t_1}-e^{t_0}\right)^3 \\ &+ \frac{1}{12\Gamma(1-\alpha)} \max_{t_0 \le t \le t_{k-1}} \left|\delta_e^3 f(t)\right| \max_{1 \le l \le k-1} \left(e^{t_l}-e^{t_{l-1}}\right)^3 \left(e^{t_k}-e^{t_{k-1}}\right)^{-\alpha} \\ &+ \frac{\alpha}{3\Gamma(2-\alpha)} \max_{t_{k-2} \le t \le t_k} \left|\delta_e^3 f(t)\right| \max_{k-1 \le l \le k} \left(e^{t_l}-e^{t_{l-1}}\right)^{3-\alpha}, \quad k \ge 2. \end{aligned} \tag{3.18}$$

Proof. The case with $k=1$ is clear. For case with $k \ge 2$, using integration by parts in (3.17), and the truncation errors (2.4) and (2.7), one arrives at

$$\begin{aligned} R^k =& \frac{-\alpha}{\Gamma(1-\alpha)} \int_{t_0}^{t_1} \left(e^{t_k}-e^s\right)^{-\alpha-1} r_1^1(s) e^s \mathrm{d}s \\ &+ \frac{-\alpha}{\Gamma(1-\alpha)} \sum_{j=2}^{k-1} \int_{t_{j-1}}^{t_j} \left(e^{t_k}-e^s\right)^{-\alpha-1} r_2^{j-1}(s) e^s \mathrm{d}s \\ &+ \frac{-\alpha}{\Gamma(1-\alpha)} \int_{t_{k-1}}^{t_k} \left(e^{t_k}-e^s\right)^{-\alpha-1} r_2^{k-1}(s) e^s \mathrm{d}s \\ =& R_1^k + R_2^k + R_3^k. \end{aligned} \tag{3.19}$$

Simple calculations lead to

$$\begin{aligned} \left|R_1^k\right| \le& \frac{\alpha}{2\Gamma(1-\alpha)} \int_{t_0}^{t_1} \left|\delta_e^2 f(\eta_1)\right| \left(e^s-e^{t_0}\right) \left(e^{t_1}-e^s\right) \left(e^{t_k}-e^s\right)^{-\alpha-1} e^s \mathrm{d}s \\ \le& \frac{\alpha}{8\Gamma(1-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^2 f(t)\right| \left(e^{t_k}-e^{t_1}\right)^{-\alpha-1} \left(e^{t_1}-e^{t_0}\right)^3, \end{aligned} \tag{3.20}$$

$$\begin{aligned}
\left|R_2^k\right| \leq & \frac{\alpha}{24\Gamma(1-\alpha)}\sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j} \left|\delta_e^3 f(\xi_{j-1})\right| \left(e^s - e^{t_{j-2}}\right) \left(e^{t_j} - e^{t_{j-1}}\right)^2 \left(e^{t_k} - e^s\right)^{-\alpha-1} e^s \mathrm{d}s \\
\leq & \frac{\alpha}{12\Gamma(1-\alpha)} \max_{t_0\leq t\leq t_{k-1}} \left|\delta_e^3 f(t)\right| \max_{1\leq l\leq k-1}\left(e^{t_l} - e^{t_{l-1}}\right)^3 \int_{t_1}^{t_{k-1}}\left(e^{t_k} - e^s\right)^{-\alpha-1} e^s \mathrm{d}s \\
\leq & \frac{1}{12\Gamma(1-\alpha)} \max_{t_0\leq t\leq t_{k-1}} \left|\delta_e^3 f(t)\right| \max_{1\leq l\leq k-1} \left(e^{t_l} - e^{t_{l-1}}\right)^3 \left(e^{t_k} - e^{t_{k-1}}\right)^{-\alpha},
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\left|R_3^k\right| \leq & \frac{\alpha}{6\Gamma(1-\alpha)} \int_{t_{k-1}}^{t_k} \left|\delta_e^3 f(\xi_{k-1})\right| \left(e^s - e^{t_{k-2}}\right) \left(e^s - e^{t_{k-1}}\right) \left(e^{t_k} - e^s\right)^{-\alpha} e^s \mathrm{d}s \\
\leq & \frac{\alpha}{3\Gamma(2-\alpha)} \max_{t_{k-2}\leq t\leq t_k} \left|\delta_e^3 f(t)\right| \max_{k-1\leq l\leq k} \left(e^{t_l} - e^{t_{l-1}}\right)^{3-\alpha},
\end{aligned} \tag{3.22}$$

where $\eta_1 \in (t_0, t_1)$ and $\xi_{j-1} \in (t_{j-2}, t_j)\,(2 \leq j \leq k)$.

All this completes the proof. □

Now, the properties of coefficients defined in (3.15) are shown.

Theorem 3.4. *For order $\alpha \in (0,1)$ and step size $\tau = t_k - t_{k-1} = \frac{T-a}{N}$, the coefficients $c_{j,k}^{(\alpha)}\,(1 \leq j \leq k, 1 \leq k \leq N)$ in (3.15) satisfy*

$$c_{j,k}^{(\alpha)} > 0,\ \ j \neq k-1. \tag{3.23}$$

Proof. Firstly, the proof is divided into the following parts,

(i) $c_{1,1}^{(\alpha)} > 0$,

(ii) $c_{k,k}^{(\alpha)} > 0,\ k \geq 2$,

(iii) $c_{j,k}^{(\alpha)} > 0,\ 2 \leq j \leq k-2,\ k \geq 4$,

(iv) $c_{1,k}^{(\alpha)} > 0,\ k \geq 3$.

Part (i) is obvious. Part (ii) still holds due to

$$\begin{aligned}
c_{k,k}^{(\alpha)} &= \frac{1}{e^{t_k} - e^{t_{k-1}}} \left(a_{k,k}^{(\alpha)} - b_{k,k}^{(\alpha)}\right) \\
&= \left(e^{t_k} - e^{t_{k-1}}\right)^{-\alpha} \left(1 + \frac{\alpha}{2-\alpha} \frac{e^{t_k} - e^{t_{k-1}}}{e^{t_k} - e^{t_{k-2}}}\right) > 0.
\end{aligned} \tag{3.24}$$

For Part (iii) with $2 \leq j \leq k-2\,(k \geq 4)$, by the definition of $c_{j,k}^{(\alpha)}$ in (3.15), one gets

$$\begin{aligned}
& a_{j,k}^{(\alpha)} - b_{j,k}^{(\alpha)} + b_{j+1,k}^{(\alpha)} \\
= & \frac{1}{e^{t_j} - e^{t_{j-2}}} \left\{ \frac{2}{2-\alpha} \left[\left(e^{t_k} - e^{t_{j-1}}\right)^{2-\alpha} - \left(e^{t_k} - e^{t_j}\right)^{2-\alpha}\right] \right. \\
& \left. - \left(e^{t_j} - e^{t_{j-1}}\right) \left[\left(e^{t_k} - e^{t_j}\right)^{1-\alpha} + \left(e^{t_k} - e^{t_{j-1}}\right)^{1-\alpha}\right] \right\} + \left(e^{t_k} - e^{t_{j-1}}\right)^{1-\alpha}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{e^{t_{j+1}}-e^{t_{j-1}}}\left\{\frac{2}{2-\alpha}\left[\left(e^{t_k}-e^{t_j}\right)^{2-\alpha}-\left(e^{t_k}-e^{t_{j+1}}\right)^{2-\alpha}\right]\right.\\
&\left.-\left(e^{t_{j+1}}-e^{t_j}\right)\left[\left(e^{t_k}-e^{t_{j+1}}\right)^{1-\alpha}+\left(e^{t_k}-e^{t_j}\right)^{1-\alpha}\right]\right\}-\left(e^{t_k}-e^{t_j}\right)^{1-\alpha}\\
=&\,I_{j-1}-I_j.
\end{aligned}
\tag{3.25}
$$

Denote $x=e^s$ $(s\in[t_1,t_{k-2}])$ and $x_j=e^{t_j}$ $(j=1,2,\ldots,k-2)$. Then one has

$$
\begin{aligned}
I_j=&\ \frac{1-\alpha}{e^{t_{j+1}}-e^{t_{j-1}}}\int_{e^{t_j}}^{e^{t_{j+1}}}\left(e^{t_k}-s\right)^{-\alpha}\left(2s-e^{t_j}-e^{t_{j+1}}\right)\mathrm{d}s+\left(e^{t_k}-e^{t_j}\right)^{1-\alpha}\\
=&\ \frac{1-\alpha}{x_j\left(e^{\tau}-e^{-\tau}\right)}\int_{x_j}^{e^{\tau}x_j}\left(e^{t_k}-s\right)^{-\alpha}\left(2s-x_j-e^{\tau}x_j\right)\mathrm{d}s+\left(e^{t_k}-x_j\right)^{1-\alpha}\\
=&\ F(x_j),\quad 1\le j\le k-2,
\end{aligned}
\tag{3.26}
$$

where for $x\in[e^{t_1},e^{t_{k-2}}]$,

$$
F(x)=\frac{1-\alpha}{x\left(e^{\tau}-e^{-\tau}\right)}\int_{x}^{e^{\tau}x}\left(e^{t_k}-s\right)^{-\alpha}\left(2s-x-e^{\tau}x\right)\mathrm{d}s+\left(e^{t_k}-x\right)^{1-\alpha}. \tag{3.27}
$$

In order to find the monotonicity of I_j with respect to j, one needs to consider the monotonicity of $F(x)$ with x. Differentiating $F(x)$ gives

$$
\begin{aligned}
F'(x)=&-(1-\alpha)\left\{\frac{2}{x^2\left(e^{\tau}-e^{-\tau}\right)}\int_{x}^{e^{\tau}x}\left(e^{t_k}-s\right)^{-\alpha}s\,\mathrm{d}s\right.\\
&\left.-\frac{e^{\tau}\left(e^{\tau}-1\right)}{e^{\tau}-e^{-\tau}}\left(e^{t_k}-e^{\tau}x\right)^{-\alpha}+\frac{1-e^{-\tau}}{e^{\tau}-e^{-\tau}}\left(e^{t_k}-x\right)^{-\alpha}\right\}.
\end{aligned}
\tag{3.28}
$$

Let $\psi(s)=s\left(e^{t_k}-s\right)^{-\alpha}$. Applying the Hermite interpolation function to $\psi(s)$ on interval $[x,e^{\tau}x]$ yields

$$
\begin{aligned}
\psi(s)=&\,\psi(x)+\psi\left[x,e^{\tau}x\right](s-x)+\frac{\psi'(e^{\tau}x)-\psi\left[x,e^{\tau}x\right]}{e^{\tau}x-x}(s-x)\left(s-e^{\tau}x\right)\\
&+\frac{1}{6}\psi^{(3)}(\xi)(s-x)\left(s-e^{\tau}x\right)^2,\quad \xi\in\left(x,e^{\tau}x\right),
\end{aligned}
\tag{3.29}
$$

where

$$
\psi\left[x,e^{\tau}x\right]=\frac{\psi(e^{\tau}x)-\psi(x)}{e^{\tau}x-x}. \tag{3.30}
$$

Hence one gets

$$
\begin{aligned}
\frac{2}{x^2\left(e^{\tau}-e^{-\tau}\right)}\int_{x}^{e^{\tau}x}\psi(s)\mathrm{d}s=&\,\frac{e^{\tau}-1}{3(e^{\tau}-e^{-\tau})}\left\{2\left(e^{t_k}-x\right)^{-\alpha}+4e^{\tau}\left(e^{t_k}-e^{\tau}x\right)^{-\alpha}\right.\\
&\left.-\left(e^{\tau}-1\right)\left[\left(e^{t_k}-e^{\tau}x\right)^{-\alpha}+\alpha\left(e^{t_k}-e^{\tau}x\right)^{-\alpha-1}e^{\tau}x\right]\right\}\\
&+\frac{1}{3x^2\left(e^{\tau}-e^{-\tau}\right)}\int_{x}^{e^{\tau}x}\psi^{(3)}(\xi)(s-x)\left(s-e^{\tau}x\right)^2\mathrm{d}s.
\end{aligned}
\tag{3.31}
$$

Substituting (3.31) into (3.28) gives

$$\begin{aligned} F'(x) = \frac{-(1-\alpha)}{3\left(e^{\tau}-e^{-\tau}\right)} \Bigg\{ & \left(e^{t_k}-e^{\tau}x\right)^{-\alpha-1}\left(e^{\tau}-1\right)\left[\left(e^{t_k}-e^{\tau}x\right)-\alpha e^{\tau}x\left(e^{\tau}-1\right)\right] \\ & + \frac{\left(e^{\tau}-1\right)\left(2e^{\tau}+3\right)}{e^{\tau}}\left(e^{t_k}-x\right)^{-\alpha} \\ & + \frac{1}{x^2}\int_x^{e^{\tau}x}\psi^{(3)}(\xi)\left(s-x\right)\left(s-e^{\tau}x\right)^2\mathrm{d}s\Bigg\}. \end{aligned} \tag{3.32}$$

It is evident to know that

$$\psi^{(3)}(\xi) = 3\alpha(\alpha+1)\left(e^{t_k}-\xi\right)^{-\alpha-2} + \alpha(\alpha+1)(\alpha+2)\left(e^{t_k}-\xi\right)^{-\alpha-3}\xi > 0. \tag{3.33}$$

And for $x \in [e^{t_1}, e^{t_{k-2}}]$, one reaches

$$\begin{aligned} \left(e^{t_k}-e^{\tau}x\right)-\alpha e^{\tau}x\left(e^{\tau}-1\right) &= e^{t_k}-e^{\tau}x\left(1+\alpha e^{\tau}-\alpha\right) \\ &\geq e^{t_k}-e^{t_{k-1}}\left(1+\alpha e^{\tau}-\alpha\right) \\ &= e^{t_{k-1}}(1-\alpha)(e^{\tau}-1) \\ &> 0. \end{aligned} \tag{3.34}$$

Combining (3.33), (3.34) with (3.32), one has $F'(x) < 0$, which means that the sequence I_j is monotonically decreasing, i.e.,

$$I_1 > I_2 > \cdots > I_{k-2}.$$

So one has

$$\begin{aligned} c_{j,k}^{(\alpha)} &= \frac{1}{e^{t_j}-e^{t_{j-1}}}\left(a_{j,k}^{(\alpha)} - b_{j,k}^{(\alpha)} + b_{j+1,k}^{(\alpha)}\right) \\ &= \frac{1}{e^{t_j}-e^{t_{j-1}}}\left(I_{j-1}-I_j\right) > 0, \quad 2 \leq j \leq k-2\ (k \geq 4). \end{aligned} \tag{3.35}$$

For part (iv), when $j = 1$ ($k \geq 3$), one only needs to determine the sign of

$$a_{1,k}^{(\alpha)} + b_{2,k}^{(\alpha)} = (1-\alpha)\left\{\int_{e^{t_0}}^{e^{t_1}}\phi(s)\mathrm{d}s - \frac{1}{e^{t_2}-e^{t_0}}\int_{e^{t_1}}^{e^{t_2}}\varphi(s)\mathrm{d}s\right\}, \tag{3.36}$$

where

$$\phi(s) = \left(e^{t_k}-s\right)^{-\alpha}, \qquad \varphi(s) = \left(e^{t_k}-s\right)^{-\alpha}\left(2s-e^{t_1}-e^{t_2}\right). \tag{3.37}$$

Considering Hermite interpolation for $\phi(s)$ on $[e^{t_0}, e^{t_1}]$ and Lagrange interpolation for $\varphi(s)$ on $[e^{t_1}, e^{t_2}]$, respectively, one has the following formulas,

$$\begin{aligned} & \int_{e^{t_0}}^{e^{t_1}}\phi(s)\mathrm{d}s \\ = & \frac{1}{6}\Bigg\{\left(e^{t_1}-e^{t_0}\right)\left[2\left(e^{t_k}-e^{t_0}\right)^{-\alpha} + 4\left(e^{t_k}-e^{t_1}\right)^{-\alpha}\right] \\ & -\alpha\left(e^{t_1}-e^{t_0}\right)^2\left(e^{t_k}-e^{t_1}\right)^{-\alpha-1} + \int_{e^{t_0}}^{e^{t_1}}\phi^{(3)}(\xi)\left(s-e^{t_0}\right)\left(s-e^{t_1}\right)^2\mathrm{d}s\Bigg\}, \end{aligned} \tag{3.38}$$

where $\xi \in (e^{t_0}, e^{t_1})$, and

$$\int_{e^{t_1}}^{e^{t_2}} \varphi(s)\mathrm{d}s = \frac{1}{2}\Big\{ \left(e^{t_2} - e^{t_1}\right)^2 \left[\left(e^{t_k} - e^{t_2}\right)^{-\alpha} - \left(e^{t_k} - e^{t_1}\right)^{-\alpha}\right] + \int_{e^{t_1}}^{e^{t_2}} \varphi''(\eta) \left(s - e^{t_1}\right) \left(s - e^{t_2}\right) \mathrm{d}s \Big\}, \tag{3.39}$$

in which $\eta \in (e^{t_1}, e^{t_2})$. Combination of (3.36), (3.38) with (3.39) leads to

$$\begin{aligned} & a_{1,k}^{(\alpha)} + b_{2,k}^{(\alpha)} \\ = & \frac{1}{6}(1-\alpha)\left(e^{t_1} - e^{t_0}\right)\Bigg\{ \left(e^{t_k} - e^{t_1}\right)^{-\alpha-1} \left[\left(e^{t_k} - e^{t_1}\right) - \alpha\left(e^{t_1} - e^{t_0}\right)\right] \\ & + 3\left[\left(1 + \frac{e^{2\tau}}{e^{\tau}+1}\right)\left(e^{t_k} - e^{t_1}\right)^{-\alpha} - \frac{e^{2\tau}}{e^{\tau}+1}\left(e^{t_k} - e^{t_2}\right)^{-\alpha}\right] + 2\left(e^{t_k} - e^{t_0}\right)^{-\alpha} \Bigg\} \\ & + (1-\alpha)\Bigg\{\frac{1}{6}\int_{e^{t_0}}^{e^{t_1}} \phi'''(\xi)\left(s - e^{t_0}\right)\left(s - e^{t_1}\right)^2 \mathrm{d}s \\ & - \frac{1}{2\left(e^{t_2} - e^{t_0}\right)} \int_{e^{t_1}}^{e^{t_2}} \varphi''(\eta)\left(s - e^{t_1}\right)\left(s - e^{t_2}\right)\mathrm{d}s\Bigg\} \\ > & 0, \end{aligned} \tag{3.40}$$

where $\phi'''(\xi) > 0$, $\varphi''(\eta) > 0$, and

$$\begin{aligned} & \left(1 + \frac{e^{2\tau}}{e^{\tau}+1}\right)\left(e^{t_k} - e^{t_1}\right)^{-\alpha} - \frac{e^{2\tau}}{e^{\tau}+1}\left(e^{t_k} - e^{t_2}\right)^{-\alpha} \\ = & \frac{e^{2\tau}}{e^{\tau}+1}\left(e^{t_k} - e^{t_2}\right)^{-\alpha}\left\{\frac{e^{2\tau}+e^{\tau}+1}{e^{2\tau}}\left(1 - \frac{e^{t_2} - e^{t_1}}{e^{t_k} - e^{t_1}}\right)^{\alpha} - 1\right\} \\ \geq & \frac{e^{2\tau}}{e^{\tau}+1}\left(e^{t_k} - e^{t_2}\right)^{-\alpha}\left\{\frac{e^{2\tau}+e^{\tau}+1}{e^{2\tau}}\left(\frac{e^{\tau}}{e^{\tau}+1}\right)^{\alpha} - 1\right\} \\ > & \frac{e^{2\tau}}{e^{\tau}+1}\left(e^{t_k} - e^{t_2}\right)^{-\alpha}\left\{\frac{e^{2\tau}+e^{\tau}+1}{e^{2\tau}+e^{\tau}} - 1\right\} \\ > & 0. \end{aligned} \tag{3.41}$$

So it holds that

$$c_{1,k}^{(\alpha)} = \frac{1}{e^{t_1} - e^{t_0}}\left(a_{1,k}^{(\alpha)} + b_{2,k}^{(\alpha)}\right) > 0. \tag{3.42}$$

Therefore, the proof is over. □

Remark 3.3. For $j = k-1$, one can verify that the sign of $c_{k-1,k}^{(\alpha)}$ $(k \geq 2)$ concerning α is changeable, which is indicated in the framed part in Table 1. For case with

$k \geq 3$, by means of definition (3.15), one can attain that

$$\begin{aligned} & a_{k-1,k}^{(\alpha)} - b_{k-1,k}^{(\alpha)} + b_{k,k}^{(\alpha)} \\ = & \left(e^{t_k} - e^{t_{k-2}}\right)^{1-\alpha} - \left(e^{t_k} - e^{t_{k-1}}\right)^{1-\alpha} \\ & + \frac{1}{e^{t_k-1} - e^{t_{k-3}}} \left\{ \frac{2}{2-\alpha} \left[\left(e^{t_k} - e^{t_{k-2}}\right)^{2-\alpha} - \left(e^{t_k} - e^{t_{k-1}}\right)^{2-\alpha} \right] \right. \\ & \left. - \left(e^{t_{k-1}} - e^{t_{k-2}}\right) \left[\left(e^{t_k} - e^{t_{k-2}}\right)^{1-\alpha} + \left(e^{t_k} - e^{t_{k-1}}\right)^{1-\alpha} \right] \right\} \\ & - \frac{1}{e^{t_k} - e^{t_{k-2}}} \frac{\alpha}{2-\alpha} \left(e^{t_k} - e^{t_{k-1}}\right)^{2-\alpha}. \end{aligned} \tag{3.43}$$

Taking the limit of the above formula with respect to α, one gets

$$\begin{aligned} & \lim_{\alpha \to 0} \left(a_{k-1,k}^{(\alpha)} - b_{k-1,k}^{(\alpha)} + b_{k,k}^{(\alpha)} \right) = e^{t_{k-1}} - e^{t_{k-2}} > 0, \\ & \lim_{\alpha \to 1} \left(a_{k-1,k}^{(\alpha)} - b_{k-1,k}^{(\alpha)} + b_{k,k}^{(\alpha)} \right) = -\frac{e^{\tau}}{e^{\tau}+1} < 0. \end{aligned} \tag{3.44}$$

Considering (3.44) and the formula below together,

$$c_{k-1,k}^{(\alpha)} = \frac{1}{e^{t_{k-1}} - e^{t_{k-2}}} \left(a_{k-1,k}^{(\alpha)} - b_{k-1,k}^{(\alpha)} + b_{k,k}^{(\alpha)} \right), \tag{3.45}$$

one knows that the sign of $c_{k-1,k}^{(\alpha)}\,(k \geq 3)$ is indefinite. For $k = 2$, because of

$$\lim_{\alpha \to 0} \left(a_{1,2}^{(\alpha)} + b_{2,2}^{(\alpha)} \right) = e^{t_1} - e^{t_0} > 0, \tag{3.46}$$

and

$$\lim_{\alpha \to 1} \left(a_{1,2}^{(\alpha)} + b_{2,2}^{(\alpha)} \right) = -\frac{e^{\tau}}{e^{\tau}+1} < 0, \tag{3.47}$$

the sign of

$$c_{1,2}^{(\alpha)} = \frac{1}{e^{t_1} - e^{t_0}} \left(a_{1,2}^{(\alpha)} + b_{2,2}^{(\alpha)} \right), \tag{3.48}$$

can be variable.

Table 1. The coefficients $c_{j,k}^{(\alpha)}$ of L1-2-A formula

$k=1$	$c_{1,1}^{(\alpha)}$					
$k=2$	$\boxed{c_{1,2}^{(\alpha)}}$	$c_{2,2}^{(\alpha)}$				
$k=3$	$c_{1,3}^{(\alpha)}$	$\boxed{c_{2,3}^{(\alpha)}}$	$c_{3,3}^{(\alpha)}$			
$k=4$	$c_{1,4}^{(\alpha)}$	$c_{2,4}^{(\alpha)}$	$\boxed{c_{3,4}^{(\alpha)}}$	$c_{4,4}^{(\alpha)}$		
$k=5$	$c_{1,5}^{(\alpha)}$	$c_{2,5}^{(\alpha)}$	$c_{3,5}^{(\alpha)}$	$\boxed{c_{4,5}^{(\alpha)}}$	$c_{5,5}^{(\alpha)}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Lemma 3.1. *For $\alpha \in (0,1)$ and $t_j = t_0 + j\tau\,(0 \le j \le k)$, the coefficients $b_{j,k}^{(\alpha)}$ defined in (3.15) are negative, i.e.,*

$$b_{j,k}^{(\alpha)} < 0,\ 2 \le j \le k,\ k \ge 2. \tag{3.49}$$

Proof. Considering the monotonicity of $\varphi(s) = \left(e^{t_k} - s\right)^{-\alpha}$ and using the integral mean value theorem yield the proof. □

Theorem 3.5. *For $\alpha \in (0,1)$ and sufficiently small $\tau = t_k - t_{k-1} = \frac{T-a}{N}$, the coefficients $c_{j,k}^{(\alpha)}\,(1 \le j \le k, 1 \le k \le N)$ in (3.15) satisfy*

(1) $c_{j,k}^{(\alpha)} < c_{j+1,k}^{(\alpha)},\ 1 \le j \le k-3,\ k \ge 4,$

(2) $\left|c_{k-1,k}^{(\alpha)}\right| < c_{k,k}^{(\alpha)},\ k \ge 2,$

(3) $c_{k-2,k}^{(\alpha)} < c_{k,k}^{(\alpha)},\ k \ge 3.$

Proof. (1) Let

$$\begin{aligned} F(x) =& \frac{1-\alpha}{x\left(e^{\tau} - e^{-\tau}\right)} \int_{x}^{e^{\tau}x} \left[\left(e^{t_k} - s\right)^{-\alpha} - \left(e^{t_k} - e^{-\tau}s\right)^{-\alpha}\right] \left(2s - x - e^{\tau}x\right) \mathrm{d}s \\ &+ \left(e^{t_k} - x\right)^{1-\alpha} - e^{\tau}\left(e^{t_k} - e^{-\tau}x\right)^{1-\alpha}, \end{aligned} \tag{3.50}$$

where $x \in [e^{t_1}, e^{t_{k-2}}]$. And set $x_j = e^{t_j}$ $(1 \le j \le k-2)$. By differentiation and the integral mean value theorem, one gets

$$\begin{aligned} F'(x) =& -\frac{e^{\tau}(1-\alpha)}{e^{\tau}+1}\Big\{ \left(e^{\tau}+1\right)\varphi\left(\xi x\right) - \left[e^{\tau}\varphi\left(e^{\tau}x\right) - e^{-\tau}\varphi\left(x\right)\right] \Big\} \\ \le& -\frac{e^{2\tau}(1-\alpha)}{e^{\tau}+1}\varphi\left(e^{\tau}x\right)\left\{ \frac{e^{\tau}+1+e^{-\tau}}{e^{\tau}} \frac{\varphi\left(x\right)}{\varphi\left(e^{\tau}x\right)} - 1 \right\}, \end{aligned} \tag{3.51}$$

where $\xi \in [1, e^{\tau}]$,

$$\varphi(s) = \left(e^{t_k} - s\right)^{-\alpha} - \left(e^{t_k} - e^{-\tau}s\right)^{-\alpha},\ s \in \left[x, e^{\tau}x\right], \tag{3.52}$$

which is a positive monotone increasing function.

Consider the following limit

$$\lim_{\tau \to 0} \frac{e^{\tau}+1+e^{-\tau}}{e^{\tau}} \frac{\varphi\left(x\right)}{\varphi\left(e^{\tau}x\right)} = \lim_{e^{\tau} \to 1} \frac{3\left[\left(e^{t_k} - x\right)^{-\alpha} - \left(e^{t_k} - e^{-\tau}x\right)^{-\alpha}\right]}{\left(e^{t_k} - e^{\tau}x\right)^{-\alpha} - \left(e^{t_k} - x\right)^{-\alpha}} = 3. \tag{3.53}$$

So there exist $\tau^* > 0$ such that for $(0 <)\tau < \tau^*$ the following inequality

$$\frac{e^{\tau}+1+e^{-\tau}}{e^{\tau}} \frac{\varphi\left(x\right)}{\varphi\left(e^{\tau}x\right)} > 1 \tag{3.54}$$

holds, which implies $F'(x) < 0$ for sufficiently small τ.

For $2 \le j \le k-3\,(k \ge 5)$ and sufficiently small τ, from (3.15) one has

$$\begin{aligned}
& c_{j+1,k}^{(\alpha)} - c_{j,k}^{(\alpha)} \\
=& \frac{1}{e^{t_{j+1}} - e^{t_j}} \left(a_{j+1,k}^{(\alpha)} - b_{j+1,k}^{(\alpha)} + b_{j+2,k}^{(\alpha)}\right) - \frac{1}{e^{t_j} - e^{t_{j-1}}} \left(a_{j,k}^{(\alpha)} - b_{j,k}^{(\alpha)} + b_{j+1,k}^{(\alpha)}\right) \\
=& \frac{1}{e^{t_{j+1}} - e^{t_j}} \Big[\big(I_j - e^{\tau} I_{j-1}\big) - \big(I_{j+1} - e^{\tau} I_j\big)\Big] \\
=& \frac{1}{e^{t_{j+1}} - e^{t_j}} \Big(F(x_j) - F(x_{j+1})\Big) > 0,
\end{aligned} \tag{3.55}$$

where

$$\begin{aligned}
I_j =& \frac{1}{e^{t_{j+1}} - e^{t_{j-1}}} \left\{ \frac{2}{2-\alpha} \left[\left(e^{t_k} - e^{t_j}\right)^{2-\alpha} - \left(e^{t_k} - e^{t_{j+1}}\right)^{2-\alpha}\right] \right. \\
& \left. - \left(e^{t_{j+1}} - e^{t_j}\right) \left[\left(e^{t_k} - e^{t_j}\right)^{1-\alpha} + \left(e^{t_k} - e^{t_{j+1}}\right)^{1-\alpha}\right] \right\} + \left(e^{t_k} - e^{t_j}\right)^{1-\alpha}.
\end{aligned}$$

So $c_{j,k}^{(\alpha)} < c_{j+1,k}^{(\alpha)}$ holds for sufficiently small τ and $2 \le j \le k-3$ with $k \ge 5$.

For $k \ge 4$, set $t_{-1} = t_0 - \tau$, and let

$$\begin{aligned}
b_{1,k}^{(\alpha)} =& -\frac{1}{e^{t_1} - e^{t_{-1}}} \left\{ \frac{2}{2-\alpha} \left[\left(e^{t_k} - e^{t_0}\right)^{2-\alpha} - \left(e^{t_k} - e^{t_1}\right)^{2-\alpha}\right] \right. \\
& \left. - \left(e^{t_1} - e^{t_0}\right) \left[\left(e^{t_k} - e^{t_0}\right)^{1-\alpha} + \left(e^{t_k} - e^{t_1}\right)^{1-\alpha}\right] \right\}.
\end{aligned} \tag{3.56}$$

It is easy to verify that $b_{1,k}^{(\alpha)} < 0$ by using almost the same method as that used in the proof of Lemma 3.1. Then one can achieve

$$\begin{aligned}
& c_{2,k}^{(\alpha)} - c_{1,k}^{(\alpha)} \\
=& \frac{1}{e^{t_2} - e^{t_1}} \left(a_{2,k}^{(\alpha)} - b_{2,k}^{(\alpha)} + b_{3,k}^{(\alpha)}\right) - \frac{1}{e^{t_1} - e^{t_0}} \left(a_{1,k}^{(\alpha)} - b_{1,k}^{(\alpha)} + b_{2,k}^{(\alpha)}\right) - \frac{b_{1,k}^{(\alpha)}}{e^{t_1} - e^{t_0}} \\
=& \frac{1}{e^{t_2} - e^{t_1}} \Big(F(x_1) - F(x_2)\Big) - \frac{b_{1,k}^{(\alpha)}}{e^{t_1} - e^{t_0}} \\
>& \, 0,
\end{aligned} \tag{3.57}$$

which means $c_{1,k}^{(\alpha)} < c_{2,k}^{(\alpha)}$ for sufficiently small τ and $k \ge 4$.

(2) When $k \ge 3$, one gets

$$\begin{aligned}
& c_{k,k}^{(\alpha)} - \left|c_{k-1,k}^{(\alpha)}\right| \\
=& \frac{1}{e^{t_k} - e^{t_{k-1}}} \left\{ \left(a_{k,k}^{(\alpha)} - b_{k,k}^{(\alpha)}\right) - e^{\tau} \left|a_{k-1,k}^{(\alpha)} - b_{k-1,k}^{(\alpha)} + b_{k,k}^{(\alpha)}\right| \right\}.
\end{aligned} \tag{3.58}$$

For the case of $c_{k-1,k}^{(\alpha)} > 0$, one has

$$c_{k,k}^{(\alpha)} - \left|c_{k-1,k}^{(\alpha)}\right| = \frac{1}{e^{t_k} - e^{t_{k-1}}} \left\{ \left(a_{k,k}^{(\alpha)} - b_{k,k}^{(\alpha)}\right) - e^{\tau} \left(a_{k-1,k}^{(\alpha)} - b_{k-1,k}^{(\alpha)}\right) - e^{\tau} b_{k,k}^{(\alpha)} \right\}. \tag{3.59}$$

Using the definition of the coefficients leads to

$$\begin{aligned}&\left(a_{k,k}^{(\alpha)}-b_{k,k}^{(\alpha)}\right)-e^{\tau}\left(a_{k-1,k}^{(\alpha)}-b_{k-1,k}^{(\alpha)}\right)\\&=(1-\alpha)\int_{e^{t_{k-1}}}^{e^{t_k}}\left[\left(e^{t_k}-s\right)^{-\alpha}-\left(e^{t_k}-e^{-\tau}s\right)^{-\alpha}\right]\frac{2s-e^{t_{k-1}}-e^{t_{k-2}}}{e^{t_k}-e^{t_{k-2}}}\,\mathrm{d}s\\&\geq\frac{1-\alpha}{e^{\tau}+1}\int_{e^{t_{k-1}}}^{e^{t_k}}\left[\left(e^{t_k}-s\right)^{-\alpha}-\left(e^{t_k}-e^{-\tau}s\right)^{-\alpha}\right]\mathrm{d}s>0.\end{aligned}\tag{3.60}$$

Combining (3.59), (3.60) with Lemma 3.1, one gets $c_{k-1,k}^{(\alpha)}<c_{k,k}^{(\alpha)}\,(k\geq 3)$.

For the case of $c_{k-1,k}^{(\alpha)}<0$, one has

$$c_{k,k}^{(\alpha)}-\left|c_{k-1,k}^{(\alpha)}\right|=\frac{1}{e^{t_k}-e^{t_{k-1}}}\left\{a_{k,k}^{(\alpha)}+\left(e^{\tau}-1\right)b_{k,k}^{(\alpha)}+e^{\tau}a_{k-1,k}^{(\alpha)}-e^{\tau}b_{k-1,k}^{(\alpha)}\right\}.\tag{3.61}$$

It is evident that $e^{\tau}a_{k-1,k}^{(\alpha)}-e^{\tau}b_{k-1,k}^{(\alpha)}>0$. Then, one can attain that

$$a_{k,k}^{(\alpha)}+\left(e^{\tau}-1\right)b_{k,k}^{(\alpha)}=\left(e^{t_k}-e^{t_{k-1}}\right)^{1-\alpha}\left\{1-\frac{\alpha}{2-\alpha}\frac{e^{\tau}\left(e^{\tau}-1\right)}{e^{\tau}+1}\right\}.\tag{3.62}$$

Because

$$\lim_{\tau\to 0}\frac{e^{\tau}\left(e^{\tau}-1\right)}{e^{\tau}+1}=0,\tag{3.63}$$

there exist $\tau^*>0$, for $(0<)\tau<\tau^*$, such that $\frac{e^{\tau}(e^{\tau}-1)}{e^{\tau}+1}<1$, that is, $-c_{k-1,k}^{(\alpha)}<c_{k,k}^{(\alpha)}\,(k\geq 3)$ holds for sufficiently small τ.

For $k=2$, one similarly derives that

$$\begin{aligned}c_{2,2}^{(\alpha)}-\left|c_{1,2}^{(\alpha)}\right|&=\frac{1}{e^{t_2}-e^{t_1}}\left\{\left(a_{2,2}^{(\alpha)}-b_{2,2}^{(\alpha)}\right)-e^{\tau}\left(a_{1,2}^{(\alpha)}+b_{2,2}^{(\alpha)}\right)\right\}\\&=\frac{e^{\tau}\left(e^{t_2}-e^{t_0}\right)^{1-\alpha}}{e^{t_2}-e^{t_1}}\left\{\left(\frac{e^{\tau}}{e^{\tau}+1}\right)^{-\alpha}-1\right\}-\frac{e^{\tau}+1}{e^{t_2}-e^{t_1}}b_{2,2}^{(\alpha)}\\&>0,\end{aligned}\tag{3.64}$$

where $c_{1,2}^{(\alpha)}>0$ and τ is sufficiently small; and

$$\begin{aligned}c_{2,2}^{(\alpha)}-\left|c_{1,2}^{(\alpha)}\right|&=\frac{1}{e^{t_2}-e^{t_1}}\left\{\left(a_{2,2}^{(\alpha)}-b_{2,2}^{(\alpha)}\right)+e^{\tau}\left(a_{1,2}^{(\alpha)}+b_{2,2}^{(\alpha)}\right)\right\}\\&=\left(e^{t_2}-e^{t_1}\right)^{-\alpha}\left\{1-\frac{\alpha}{2-\alpha}\frac{e^{\tau}\left(e^{\tau}-1\right)}{e^{\tau}+1}\right\}+\frac{e^{\tau}a_{1,2}^{(\alpha)}}{e^{t_2}-e^{t_1}}\\&>0,\end{aligned}\tag{3.65}$$

where $c_{1,2}^{(\alpha)}<0$ and τ is sufficiently small.

(3) For $k\geq 4$, one gets

$$\begin{aligned}&c_{k,k}^{(\alpha)}-c_{k-2,k}^{(\alpha)}\\&=\frac{1}{e^{t_k}-e^{t_{k-1}}}\left(a_{k,k}^{(\alpha)}-b_{k,k}^{(\alpha)}\right)-\frac{1}{e^{t_{k-2}}-e^{t_{k-3}}}\left(a_{k-2,k}^{(\alpha)}-b_{k-2,k}^{(\alpha)}+b_{k-1,k}^{(\alpha)}\right)\\&=\frac{1}{e^{t_k}-e^{t_{k-1}}}\left\{\left(a_{k,k}^{(\alpha)}-b_{k,k}^{(\alpha)}\right)-e^{2\tau}\left(a_{k-2,k}^{(\alpha)}-b_{k-2,k}^{(\alpha)}\right)-e^{2\tau}b_{k-1,k}^{(\alpha)}\right\}\\&>0,\end{aligned}\tag{3.66}$$

where

$$\begin{aligned}&\left(a_{k,k}^{(\alpha)}-b_{k,k}^{(\alpha)}\right)-e^{2\tau}\left(a_{k-2,k}^{(\alpha)}-b_{k-2,k}^{(\alpha)}\right)\\&=(1-\alpha)\int_{e^{t_{k-1}}}^{e^{t_k}}\left[\left(e^{t_k}-s\right)^{-\alpha}-\left(e^{t_k}-e^{-2\tau}s\right)^{-\alpha}\right]\frac{2s-e^{t_{k-1}}-e^{t_{k-2}}}{e^{t_k}-e^{t_{k-2}}}\,\mathrm{d}s>0.\end{aligned}\tag{3.67}$$

Similarly, for $k=3$, one has

$$\begin{aligned}&c_{3,3}^{(\alpha)}-c_{1,3}^{(\alpha)}\\&=\frac{1}{e^{t_3}-e^{t_2}}\left(a_{3,3}^{(\alpha)}-b_{3,3}^{(\alpha)}\right)-\frac{1}{e^{t_1}-e^{t_0}}\left(a_{1,3}^{(\alpha)}+b_{2,3}^{(\alpha)}\right)\\&=\frac{1}{e^{t_3}-e^{t_2}}\left\{\left(a_{3,3}^{(\alpha)}-e^{2\tau}a_{1,3}^{(\alpha)}\right)-\left(b_{3,3}^{(\alpha)}+e^{2\tau}b_{2,3}^{(\alpha)}\right)\right\}\\&>0,\end{aligned}\tag{3.68}$$

where

$$a_{3,3}^{(\alpha)}-e^{2\tau}a_{1,3}^{(\alpha)}=(1-\alpha)\int_{e^{t_2}}^{e^{t_3}}\left[\left(e^{t_3}-s\right)^{-\alpha}-\left(e^{t_3}-e^{-2\tau}s\right)^{-\alpha}\right]\mathrm{d}s>0.\tag{3.69}$$

So all this ends the proof. □

3.3. L2-1_σ-A formula

Next, another higher-order numerical formula called L2-1_σ-A formula under **Partition A**, for numerical approaching to fractional derivative (1.3) with order $\alpha\in(0,1)$, is established below.

Used

$$\begin{cases}f(t)\approx L_{\exp,2,j}f(t), & t\in[t_{j-1},t_j]\,(2\le j\le k)\,,\\ f(t)\approx L_{\exp,1,k+1}f(t), & t\in[t_k,t_{k+\sigma}],\end{cases}\tag{3.70}$$

in which $t_{k+\sigma}=t_k+\sigma\tau$ (σ will be determined later), and considered

$$\begin{aligned}&\delta_e\left(L_{\exp,2,j}f(t)\right)\\=&\delta_e\left(L_{\exp,1,j}f(t)\right)+\frac{\nabla_{\exp,t}f^{j+\frac12}-\nabla_{\exp,t}f^{j-\frac12}}{e^{t_{j+1}}-e^{t_{j-1}}}\left(2e^t-e^{t_{j-1}}-e^{t_j}\right),\end{aligned}\tag{3.71}$$

the exponential Caputo fractional derivative ${}_{Ce}\mathrm{D}_{a,t}^{\alpha}f(t)$ with $\alpha\in(0,1)$ at $t=t_{k+\sigma}$ $(0\le k\le N-1)$ can be expressed below,

$$\begin{aligned}&{}_{Ce}\mathrm{D}_{a,t}^{\alpha}f(t)\Big|_{t=t_{k+\sigma}}\\&=\frac{1}{\Gamma(1-\alpha)}\left\{\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left(e^{t_{k+\sigma}}-e^s\right)^{-\alpha}\delta_e f(s)e^s\mathrm{d}s+\int_{t_k}^{t_{k+\sigma}}\left(e^{t_{k+\sigma}}-e^s\right)^{-\alpha}\delta_e f(s)e^s\mathrm{d}s\right\}\\&\approx\frac{1}{\Gamma(1-\alpha)}\left\{\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left(e^{t_{k+\sigma}}-e^s\right)^{-\alpha}\delta_e\left(L_{\exp,2,j}f(s)\right)e^s\mathrm{d}s\right.\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_k}^{t_{k+\sigma}}\left(e^{t_{k+\sigma}}-e^s\right)^{-\alpha}\delta_e\left(L_{\exp,1,k+1}f(s)\right)e^s\mathrm{d}s\Bigg\}\\
&=\frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{k+1}a_{j,k}^{(\alpha,\sigma)}\nabla_{\exp,t}f^{j-\frac{1}{2}}+\frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_{j,k}^{(\alpha,\sigma)}\left(\nabla_{\exp,t}f^{j+\frac{1}{2}}-\nabla_{\exp,t}f^{j-\frac{1}{2}}\right)\\
&=\frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{k+1}c_{j,k}^{(\alpha,\sigma)}\left(f^j-f^{j-1}\right)\\
&:={}_{Ce}\mathfrak{D}_{a,t}^{\alpha}f^{k+\sigma},
\end{aligned}
\tag{3.72}
$$

where

$$
\begin{aligned}
&c_{j,k}^{(\alpha,\sigma)}=\begin{cases}\frac{1}{e^{t_1}-e^{t_0}}\left(a_{1,k}^{(\alpha,\sigma)}-b_{1,k}^{(\alpha,\sigma)}\right),\ j=1,\\ \frac{1}{e^{t_j}-e^{t_{j-1}}}\left(a_{j,k}^{(\alpha,\sigma)}+b_{j-1,k}^{(\alpha,\sigma)}-b_{j,k}^{(\alpha,\sigma)}\right),\ 2\le j\le k,\\ \frac{1}{e^{t_{k+1}}-e^{t_k}}\left(a_{k+1,k}^{(\alpha,\sigma)}+b_{k,k}^{(\alpha,\sigma)}\right),\ j=k+1,\end{cases}\\
&a_{j,k}^{(\alpha,\sigma)}=\begin{cases}\left(e^{t_{k+\sigma}}-e^{t_{j-1}}\right)^{1-\alpha}-\left(e^{t_{k+\sigma}}-e^{t_j}\right)^{1-\alpha},\ 1\le j\le k,\\ \left(e^{t_{k+\sigma}}-e^{t_{j-1}}\right)^{1-\alpha},\ j=k+1,\end{cases}\\
&b_{j,k}^{(\alpha,\sigma)}=\Bigg\{-\left(e^{t_j}-e^{t_{j-1}}\right)\left[\left(e^{t_{k+\sigma}}-e^{t_{j-1}}\right)^{1-\alpha}+\left(e^{t_{k+\sigma}}-e^{t_j}\right)^{1-\alpha}\right]\\
&\qquad+\frac{2}{2-\alpha}\left[\left(e^{t_{k+\sigma}}-e^{t_{j-1}}\right)^{2-\alpha}-\left(e^{t_{k+\sigma}}-e^{t_j}\right)^{2-\alpha}\right]\Bigg\}\frac{1}{e^{t_{j+1}}-e^{t_{j-1}}}.
\end{aligned}
\tag{3.73}
$$

Call formula ${}_{Ce}\mathfrak{D}_{a,t}^{\alpha}f^{k+\sigma}$ given in (3.72) L2-1_σ-A formula.

Remark 3.4. In particular, when $k=0$ or $k=1$, one has

$$
\begin{aligned}
&c_{1,0}^{(\alpha,\sigma)}=\frac{1}{e^{t_1}-e^{t_0}}a_{1,0}^{(\alpha,\sigma)},\ k=0;\\
&c_{1,1}^{(\alpha,\sigma)}=\frac{1}{e^{t_1}-e^{t_0}}\left(a_{1,1}^{(\alpha,\sigma)}-b_{1,1}^{(\alpha,\sigma)}\right),\ c_{2,1}^{(\alpha,\sigma)}=\frac{1}{e^{t_2}-e^{t_1}}\left(a_{2,1}^{(\alpha,\sigma)}+b_{1,1}^{(\alpha,\sigma)}\right),\ k=1.
\end{aligned}
\tag{3.74}
$$

Now, the following theorem indicates the truncation error.

Theorem 3.6. *Assume $\delta_e^3 f(t)\in C[a,T]$ and $\alpha\in(0,1)$. For the fixed constant $\sigma=1-\frac{\alpha}{2}$ and sufficiently small τ, the following truncation errors $R^{k+\sigma}$ $(0\le k\le N-1)$ under* **Partition A**

$$
\begin{aligned}
R^{k+\sigma}&={}_{Ce}\mathrm{D}_{a,t}^{\alpha}f(t)\Big|_{t=t_{k+\sigma}}-{}_{Ce}\mathfrak{D}_{a,t}^{\alpha}f^{k+\sigma}\\
&=\frac{1}{\Gamma(1-\alpha)}\Bigg\{\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left(e^{t_{k+\sigma}}-e^s\right)^{-\alpha}\delta_e\left[f(s)-L_{\exp,2,j}f(s)\right]e^s\mathrm{d}s\\
&\qquad+\int_{t_k}^{t_{k+\sigma}}\left(e^{t_{k+\sigma}}-e^s\right)^{-\alpha}\delta_e\left[f(s)-L_{\exp,1,k+1}f(s)\right]e^s\mathrm{d}s\Bigg\},
\end{aligned}
\tag{3.75}
$$

have the estimate as follows

$$\begin{aligned}\left|R^{k+\sigma}\right| \le & \frac{1}{12\Gamma(1-\alpha)} \max_{t_0\le t\le t_{k+1}} \left|\delta_e^3 f(t)\right| \max_{1\le l\le k+1} \left(e^{t_l}-e^{t_{l-1}}\right)^3 \left(e^{t_{k+\sigma}}-e^{t_k}\right)^{-\alpha} \\ & + \left\{ \frac{1}{\Gamma(3-\alpha)} \frac{\sigma(1-\sigma)}{2e^a} \max_{t_k\le t\le t_{k+1}} \left|\delta_e^2 f(t)\right| \right. \\ & \left. + \frac{1}{6\Gamma(2-\alpha)} \max_{t_k\le t\le t_{k+1}} \left|\delta_e^3 f(t)\right| \right\} \left(e^{t_{k+1}}-e^{t_k}\right)^2 \left(e^{t_{k+\sigma}}-e^{t_k}\right)^{1-\alpha}.\end{aligned} \tag{3.76}$$

Proof. For $R^{k+\sigma}$ given in (3.75), let

$$R^{k+\sigma} = R_1^{k+\sigma} + R_2^{k+\sigma}. \tag{3.77}$$

Applying (2.7) and integration by parts to $R_1^{k+\sigma}$ gives

$$\begin{aligned}\left|R_1^{k+\sigma}\right| \le & \frac{\alpha}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left(e^{t_{k+\sigma}}-e^{s}\right)^{-\alpha-1} \left|r_2^j(s)\right| e^s \mathrm{d}s \\ \le & \frac{1}{12\Gamma(1-\alpha)} \max_{t_0\le t\le t_{k+1}} \left|\delta_e^3 f(t)\right| \max_{1\le l\le k+1} \left(e^{t_l}-e^{t_{l-1}}\right)^3 \left(e^{t_{k+\sigma}}-e^{t_k}\right)^{-\alpha}.\end{aligned} \tag{3.78}$$

Next, estimate $R_2^{k+\sigma}$. It is obvious that

$$R_2^{k+\sigma} = \frac{1}{\Gamma(1-\alpha)} \int_{t_k}^{t_{k+\sigma}} \left(e^{t_{k+\sigma}}-e^{s}\right)^{-\alpha} \left(\delta_e f(s) - \nabla_{\exp,t} f^{k+\frac{1}{2}}\right) e^s \mathrm{d}s. \tag{3.79}$$

Denote $\tilde{t}_{k+\frac{1}{2}} = t_k + \log \frac{e^\tau+1}{2} \in (t_k, t_{k+1})$, i.e., $e^{\tilde{t}_{k+\frac{1}{2}}} = \frac{1}{2}\left(e^{t_k}+e^{t_{k+1}}\right)$. Then using the Taylor expansion (2.10) to $\delta_e f(s)\,(s\in[t_k, t_{k+\sigma}])$ at $s=\tilde{t}_{k+\frac{1}{2}}$ gives

$$\delta_e f(s) = \delta_e f(\tilde{t}_{k+\frac{1}{2}}) + \delta_e^2 f(\tilde{t}_{k+\frac{1}{2}}) \left(e^s - e^{\tilde{t}_{k+\frac{1}{2}}}\right) + \frac{1}{2}\delta_e^3 f(\zeta) \left(e^s - e^{\tilde{t}_{k+\frac{1}{2}}}\right)^2, \tag{3.80}$$

where ζ is between s and $\tilde{t}_{k+\frac{1}{2}}$. Thus, from (3.79) and (3.80), one obtains

$$\begin{aligned}R_2^{k+\sigma} = & \frac{1}{\Gamma(1-\alpha)} \left(\delta_e f(\tilde{t}_{k+\frac{1}{2}}) - \nabla_{\exp,t} f^{k+\frac{1}{2}}\right) \int_{t_k}^{t_{k+\sigma}} \left(e^{t_{k+\sigma}}-e^{s}\right)^{-\alpha} e^s \mathrm{d}s \\ & + \frac{1}{\Gamma(1-\alpha)} \delta_e^2 f(\tilde{t}_{k+\frac{1}{2}}) \int_{t_k}^{t_{k+\sigma}} \left(e^{t_{k+\sigma}}-e^{s}\right)^{-\alpha} \left(e^s - e^{\tilde{t}_{k+\frac{1}{2}}}\right) e^s \mathrm{d}s \\ & + \frac{1}{2\Gamma(1-\alpha)} \int_{t_k}^{t_{k+\sigma}} \delta_e^3 f(\zeta) \left(e^{t_{k+\sigma}}-e^{s}\right)^{-\alpha} \left(e^s - e^{\tilde{t}_{k+\frac{1}{2}}}\right)^2 e^s \mathrm{d}s \\ = & I_1 + I_2 + I_3.\end{aligned} \tag{3.81}$$

Next duty is to estimate I_1, I_2 and I_3 . In order to investigate the item I_1, applying the Taylor formula (2.10) at $t=\tilde{t}_{k+\frac{1}{2}}$ to $f(t_k)$ and $f(t_{k+1})$ gives

$$\nabla_{\exp,t} f^{k+\frac{1}{2}} = \delta_e f(\tilde{t}_{k+\frac{1}{2}}) + \frac{1}{48}\left[\delta_e^3 f(\zeta_1) + \delta_e^3 f(\zeta_2)\right] \left(e^{t_{k+1}}-e^{t_k}\right)^2, \tag{3.82}$$

in which $\zeta_1 \in (t_k, \tilde{t}_{k+\frac{1}{2}})$, $\zeta_2 \in (\tilde{t}_{k+\frac{1}{2}}, t_{k+1})$. Therefore, one can arrive at

$$|I_1| \leq \frac{1}{24\Gamma(2-\alpha)} \max_{t_k \leq t \leq t_{k+1}} \left|\delta_e^3 f(t)\right| \left(e^{t_{k+1}} - e^{t_k}\right)^2 \left(e^{t_{k+\sigma}} - e^{t_k}\right)^{1-\alpha}. \tag{3.83}$$

For I_3, one has

$$\begin{aligned} |I_3| &\leq \frac{1}{8\Gamma(1-\alpha)} \max_{t_k \leq t \leq t_{k+1}} \left|\delta_e^3 f(t)\right| \left(e^{t_{k+1}} - e^{t_k}\right)^2 \int_{t_k}^{t_{k+\sigma}} \left(e^{t_{k+\sigma}} - e^s\right)^{-\alpha} e^s \mathrm{d}s \\ &= \frac{1}{8\Gamma(2-\alpha)} \max_{t_k \leq t \leq t_{k+1}} \left|\delta_e^3 f(t)\right| \left(e^{t_{k+1}} - e^{t_k}\right)^2 \left(e^{t_{k+\sigma}} - e^{t_k}\right)^{1-\alpha}. \end{aligned} \tag{3.84}$$

In order to estimate item I_2, first consider the following integral,

$$\begin{aligned} &\int_{t_k}^{t_{k+\sigma}} \left(e^{t_{k+\sigma}} - e^s\right)^{-\alpha} \left(e^s - e^{\tilde{t}_{k+\frac{1}{2}}}\right) e^s \mathrm{d}s \\ =& \frac{1}{(1-\alpha)(2-\alpha)} \left(e^{t_{k+\sigma}} - e^{t_k}\right)^{1-\alpha} \left[\left(e^{t_{k+\sigma}} - e^{t_k}\right) - \frac{2-\alpha}{2}\left(e^{t_{k+1}} - e^{t_k}\right)\right]. \end{aligned} \tag{3.85}$$

For guaranteeing high accuracy of L2-1_σ-A formula, the following condition must be met at least

$$\left(e^{t_{k+\sigma}} - e^{t_k}\right) - \frac{2-\alpha}{2}\left(e^{t_{k+1}} - e^{t_k}\right) = O\left(\left(e^{t_{k+1}} - e^{t_k}\right)^2\right). \tag{3.86}$$

Denote $x = e^\tau$. Then one has

$$\begin{aligned} \frac{\left(e^{t_{k+\sigma}} - e^{t_k}\right) - \frac{2-\alpha}{2}\left(e^{t_{k+1}} - e^{t_k}\right)}{\left(e^{t_{k+1}} - e^{t_k}\right)^2} &= \frac{(x^\sigma - 1) - \frac{2-\alpha}{2}(x-1)}{e^{t_k}(x-1)^2} \\ &= -\frac{1}{e^{t_k}} F(x), \end{aligned} \tag{3.87}$$

where $F(x) = \frac{\frac{2-\alpha}{2}(x-1) - (x^\sigma - 1)}{(x-1)^2}$. It is clear that

$$\lim_{x \to 1+} F(x) = \lim_{x \to 1+} \frac{\left(\frac{2-\alpha}{2} - \sigma x^{\sigma-1}\right)}{2(x-1)}. \tag{3.88}$$

So the limit (3.88) exists if and only if $\sigma = \frac{2-\alpha}{2}$. In this situation,

$$F(x) = \frac{\sigma(x-1) - (x^\sigma - 1)}{(x-1)^2}, \quad \lim_{x \to 1+} F(x) = \frac{\sigma(1-\sigma)}{2}.$$

Obviously, $F(x)$ is a monotonically decreasing positive function and satisfies $F(x) < \frac{\sigma(1-\sigma)}{2}$. Thus, one has

$$\frac{\left|\left(e^{t_{k+\sigma}} - e^{t_k}\right) - \frac{2-\alpha}{2}\left(e^{t_{k+1}} - e^{t_k}\right)\right|}{\left(e^{t_{k+1}} - e^{t_k}\right)^2} = \frac{1}{e^{t_k}} F(x) < \frac{1}{e^{t_k}} \frac{\sigma(1-\sigma)}{2} \leq \frac{1}{e^a} \frac{\sigma(1-\sigma)}{2}, \tag{3.89}$$

that is,

$$\left|\left(e^{t_{k+\sigma}} - e^{t_k}\right) - \frac{2-\alpha}{2}\left(e^{t_{k+1}} - e^{t_k}\right)\right| < \frac{1}{e^a} \frac{\sigma(1-\sigma)}{2} \left(e^{t_{k+1}} - e^{t_k}\right)^2.$$

Therefore, for $\sigma = \frac{2-\alpha}{2}$, I_2 has following estimate,

$$|I_2| \le \frac{1}{\Gamma(3-\alpha)} \frac{\sigma(1-\sigma)}{2e^a} \max_{t_k \le t \le t_{k+1}} \left|\delta_e^2 f(t)\right| \left(e^{t_{k+1}} - e^{t_k}\right)^2 \left(e^{t_{k+\sigma}} - e^{t_k}\right)^{1-\alpha}. \quad (3.90)$$

It immediately follows from (3.78), (3.81)-(3.84) and (3.90) that the proof is ended. □

Remark 3.5. Since $e^{t_k} - e^{t_{k-1}} = O(\tau)$ as $t_k \to t_{k-1}$ $(1 \le k \le N)$ and $e^{t_{k+\sigma}} - e^{t_k} = O(\tau)$ with $t_{k+\sigma} \to t_k$ $(0 \le k \le N-1)$, the estimation of truncation error $R^{k+\sigma}$ in (3.75) can be rewritten as

$$\left|R^{k+\sigma}\right| \le C\tau^{3-\alpha},$$

where C is a positive constant.

Next, the positivity and monotonicity of coefficients $c_{j,k}^{(\alpha,\sigma)}$ $(1 \le j \le k+1)$ will be shown.

Lemma 3.2. *For order $\alpha \in (0,1)$ and $t_j = t_0 + j\tau$ $(0 \le j \le k+1)$, the coefficients $b_{j,k}^{(\alpha,\sigma)}$ defined in (3.73) are positive, that is,*

$$b_{j,k}^{(\alpha,\sigma)} > 0,\ 1 \le j \le k,\ 1 \le k \le N-1. \quad (3.91)$$

Proof. The proof is almost the same as that of Lemma 3.1 and so is omitted here. □

Lemma 3.3. *For order $\alpha \in (0,1)$ and $t_j = t_0 + j\tau$ $(0 \le j \le k+1)$, the coefficients $a_{j,k}^{(\alpha,\sigma)}$ and $b_{j,k}^{(\alpha,\sigma)}$ defined in (3.73) satisfy*

$$a_{j,k}^{(\alpha,\sigma)} - b_{j,k}^{(\alpha,\sigma)} > 0,\ 1 \le j \le k,\ 1 \le k \le N-1. \quad (3.92)$$

Proof. Using the traditional Taylor formula for $\left(e^{t_{k+\sigma}} - e^{t_j}\right)^{2-\alpha}$ at $e^{t_{k+\sigma}} - e^{t_{j-1}}$ yields the conclusion. □

Theorem 3.7. *For order $\alpha \in (0,1)$ and $t_j = t_0 + j\tau$ $(0 \le j \le k+1)$, the coefficients $c_{j,k}^{(\alpha,\sigma)}$ defined in (3.73) are positive, i.e.,*

$$c_{j,k}^{(\alpha,\sigma)} > 0,\ 1 \le j \le k+1,\ 0 \le k \le N-1. \quad (3.93)$$

Proof. For $k \ge 1$, the following result can be attained by Lemma 3.3,

$$c_{1,k}^{(\alpha,\sigma)} = \frac{1}{e^{t_1} - e^{t_0}} \left(a_{1,k}^{(\alpha,\sigma)} - b_{1,k}^{(\alpha,\sigma)}\right) > 0.$$

For the case of $2 \le j \le k$, $k \ge 2$, using Lemmas 3.2 and 3.3, one knows that

$$c_{j,k}^{(\alpha,\sigma)} = \frac{1}{e^{t_j} - e^{t_{j-1}}} \left(\left(a_{j,k}^{(\alpha,\sigma)} - b_{j,k}^{(\alpha,\sigma)}\right) + b_{j-1,k}^{(\alpha,\sigma)}\right) > 0.$$

Obviously, $a_{j,k}^{(\alpha,\sigma)} > 0\,(1 \le j \le k+1)$. Then, when $k \ge 1$, one has

$$c_{k+1,k}^{(\alpha,\sigma)} = \frac{1}{e^{t_{k+1}} - e^{t_k}} \left(a_{k+1,k}^{(\alpha,\sigma)} + b_{k,k}^{(\alpha,\sigma)}\right) > 0,$$

and for $k = 0$, $c_{1,0}^{(\alpha,\sigma)} = \frac{1}{e^{t_1} - e^{t_0}} a_{1,0}^{(\alpha,\sigma)} > 0$. The proof is thus completed. □

Theorem 3.8. *For order $\alpha \in (0,1)$ and sufficiently small $\tau = t_{k+1} - t_k = \frac{T-a}{N}$, the coefficients $c_{j,k}^{(\alpha,\sigma)}$ $(1 \le j \le k+1,\ 0 \le k \le N-1)$ defined in (3.73) satisfy*

$$c_{1,k}^{(\alpha,\sigma)} < c_{2,k}^{(\alpha,\sigma)} < \cdots < c_{k+1,k}^{(\alpha,\sigma)}. \tag{3.94}$$

Proof. For $2 \le j \le k-1\,(k \ge 3)$, by the definition of $c_{j,k}^{(\alpha,\sigma)}$ in (3.73), one has

$$\begin{aligned}
& c_{j+1,k}^{(\alpha,\sigma)} - c_{j,k}^{(\alpha,\sigma)} \\
=& \frac{1}{e^{t_{j+1}} - e^{t_j}} \left(a_{j+1,k}^{(\alpha,\sigma)} + b_{j,k}^{(\alpha,\sigma)} - b_{j+1,k}^{(\alpha,\sigma)} \right) - \frac{1}{e^{t_j} - e^{t_{j-1}}} \left(a_{j,k}^{(\alpha,\sigma)} + b_{j-1,k}^{(\alpha,\sigma)} - b_{j,k}^{(\alpha,\sigma)} \right) \\
=& \frac{1}{e^{t_{j+1}} - e^{t_j}} \left[\left(I_j - I_{j+1}\right) - e^{\tau}\left(I_{j-1} - I_j\right) \right] \\
=& \frac{1}{e^{t_{j+1}} - e^{t_j}} \left[\left(I_j - e^{\tau} I_{j-1}\right) - \left(I_{j+1} - e^{\tau} I_j\right) \right],
\end{aligned} \tag{3.95}$$

where

$$\begin{aligned}
I_j =& \frac{1}{e^{t_{j+1}} - e^{t_{j-1}}} \left\{ \frac{2}{2-\alpha} \left[\left(e^{t_{k+\sigma}} - e^{t_{j-1}}\right)^{2-\alpha} - \left(e^{t_{k+\sigma}} - e^{t_j}\right)^{2-\alpha} \right] \right. \\
& \left. - \left(e^{t_j} - e^{t_{j-1}}\right) \left[\left(e^{t_{k+\sigma}} - e^{t_{j-1}}\right)^{1-\alpha} + \left(e^{t_{k+\sigma}} - e^{t_j}\right)^{1-\alpha} \right] \right\} + \left(e^{t_{k+\sigma}} - e^{t_j}\right)^{1-\alpha}.
\end{aligned}$$

Let

$$\begin{aligned}
F(x) =& \frac{1-\alpha}{x\left(e^{\tau} - e^{-\tau}\right)} \int_{e^{-\tau}x}^{x} \left[\left(e^{t_{k+\sigma}} - s\right)^{-\alpha} - \left(e^{t_{k+\sigma}} - e^{-\tau}s\right)^{-\alpha} \right] \left(2s - e^{-\tau}x - x\right) \mathrm{d}s \\
& + \left[\left(e^{t_{k+\sigma}} - x\right)^{1-\alpha} - e^{\tau} \left(e^{t_{k+\sigma}} - e^{-\tau}x\right)^{1-\alpha} \right],
\end{aligned} \tag{3.96}$$

where $x = e^t \in [e^{t_2}, e^{t_k}]$. Denote $x_j = e^{t_j}$ $(2 \le j \le k)$. And set

$$\varphi(s) = \left[\left(e^{t_{k+\sigma}} - s\right)^{-\alpha} - \left(e^{t_{k+\sigma}} - e^{-\tau}s\right)^{-\alpha} \right] s,\ s \in \left[e^{-\tau}x, x\right]. \tag{3.97}$$

Differentiation of the above function $F(x)$ gives

$$\begin{aligned}
F'(x) =& -\frac{2\left(1-\alpha\right)}{x^2\left(e^{\tau} - e^{-\tau}\right)} \int_{e^{-\tau}x}^{x} \left[\left(e^{t_{k+\sigma}} - s\right)^{-\alpha} - \left(e^{t_{k+\sigma}} - e^{-\tau}s\right)^{-\alpha} \right] s\,\mathrm{d}s \\
& + \frac{1-\alpha}{e^{\tau}+1} \left\{ \left[\left(e^{t_{k+\sigma}} - x\right)^{-\alpha} - \left(e^{t_{k+\sigma}} - e^{-\tau}x\right)^{-\alpha} \right] \right. \\
& \left. + e^{-\tau} \left[\left(e^{t_{k+\sigma}} - e^{-\tau}x\right)^{-\alpha} - \left(e^{t_{k+\sigma}} - e^{-2\tau}x\right)^{-\alpha} \right] \right\} \\
& - \left(1-\alpha\right) \left[\left(e^{t_{k+\sigma}} - x\right)^{-\alpha} - \left(e^{t_{k+\sigma}} - e^{-\tau}x\right)^{-\alpha} \right] \\
=& -\frac{2\left(1-\alpha\right)}{x^2\left(e^{\tau} - e^{-\tau}\right)} \int_{e^{-\tau}x}^{x} \varphi(s)\mathrm{d}s - \frac{1-\alpha}{x\left(e^{\tau}+1\right)} \left\{ e^{\tau}\varphi(x) - \varphi(e^{-\tau}x) \right\}.
\end{aligned} \tag{3.98}$$

It is evident that

$$\begin{aligned}\varphi'(s) = &\left[\left(e^{t_{k+\sigma}} - s\right)^{-\alpha} - \left(e^{t_{k+\sigma}} - e^{-\tau}s\right)^{-\alpha}\right]\\ &+ \alpha s\left[\left(e^{t_{k+\sigma}} - s\right)^{-\alpha-1} - \left(e^{t_{k+\sigma}} - e^{-\tau}s\right)^{-\alpha-1} e^{-\tau}\right] > 0.\end{aligned} \tag{3.99}$$

From $\varphi(s) > 0$, $\varphi'(s) > 0$ and (3.98), one knows that $F'(x) < 0$, that is, $F(x)$ is a monotonically decreasing function. Now, one can arrive at

$$\begin{aligned}c_{j+1,k}^{(\alpha,\sigma)} - c_{j,k}^{(\alpha,\sigma)} &= \frac{1}{e^{t_{j+1}} - e^{t_j}}\left[\left(I_j - e^{\tau} I_{j-1}\right) - \left(I_{j+1} - e^{\tau} I_j\right)\right]\\ &= \frac{1}{e^{t_{j+1}} - e^{t_j}}\left[F(x_j) - F(x_{j+1})\right] > 0,\end{aligned} \tag{3.100}$$

which means $c_{j+1,k}^{(\alpha,\sigma)} > c_{j,k}^{(\alpha,\sigma)}$ $(2 \le j \le k-1, k \ge 3)$.

For $k \ge 2$, one gets

$$\begin{aligned}&c_{k+1,k}^{(\alpha,\sigma)} - c_{k,k}^{(\alpha,\sigma)}\\ =&\frac{1}{e^{t_{k+1}} - e^{t_k}}\left(a_{k+1,k}^{(\alpha,\sigma)} + b_{k,k}^{(\alpha,\sigma)}\right) - \frac{1}{e^{t_k} - e^{t_{k-1}}}\left(a_{k,k}^{(\alpha,\sigma)} + b_{k-1,k}^{(\alpha,\sigma)} - b_{k,k}^{(\alpha,\sigma)}\right)\\ =&\frac{1}{e^{t_{k+1}} - e^{t_k}}\left\{\left(a_{k+1,k}^{(\alpha,\sigma)} - e^{\tau} a_{k,k}^{(\alpha,\sigma)}\right) + \left(\left(1 + e^{\tau}\right) b_{k,k}^{(\alpha,\sigma)} - e^{\tau} b_{k-1,k}^{(\alpha,\sigma)}\right)\right\}\\ =&\frac{1}{e^{t_{k+1}} - e^{t_k}}\left(J_1 + J_2\right).\end{aligned} \tag{3.101}$$

Firstly, define

$$\psi(s) = \left(1 + e^{\tau}\right)\left(e^{t_{k+\sigma}} - s\right)^{-\alpha} - \left(e^{t_{k+\sigma}} - e^{-\tau}s\right)^{-\alpha},\ s \in \left[e^{t_{k-1}}, e^{t_k}\right]. \tag{3.102}$$

One can get

$$\psi'(s) = \alpha\left[\left(1 + e^{\tau}\right)\left(e^{t_{k+\sigma}} - s\right)^{-\alpha-1} - e^{-\tau}\left(e^{t_{k+\sigma}} - e^{-\tau}s\right)^{-\alpha-1}\right] > 0. \tag{3.103}$$

Now one can start to show $J_2 > 0$. From (3.101) and (3.73), one has

$$\begin{aligned}J_2 =& \frac{1 + e^{\tau}}{e^{t_{k+1}} - e^{t_{k-1}}}\left\{\frac{2}{2-\alpha}\left[\left(e^{t_{k+\sigma}} - e^{t_{k-1}}\right)^{2-\alpha} - \left(e^{t_{k+\sigma}} - e^{t_k}\right)^{2-\alpha}\right]\right.\\ &\left.- \left(e^{t_k} - e^{t_{k-1}}\right)\left[\left(e^{t_{k+\sigma}} - e^{t_{k-1}}\right)^{1-\alpha} + \left(e^{t_{k+\sigma}} - e^{t_k}\right)^{1-\alpha}\right]\right\}\\ &- \frac{e^{\tau}}{e^{t_k} - e^{t_{k-2}}}\left\{\frac{2}{2-\alpha}\left[\left(e^{t_{k+\sigma}} - e^{t_{k-2}}\right)^{2-\alpha} - \left(e^{t_{k+\sigma}} - e^{t_{k-1}}\right)^{2-\alpha}\right]\right.\\ &\left.- \left(e^{t_{k-1}} - e^{t_{k-2}}\right)\left[\left(e^{t_{k+\sigma}} - e^{t_{k-2}}\right)^{1-\alpha} + \left(e^{t_{k+\sigma}} - e^{t_{k-1}}\right)^{1-\alpha}\right]\right\}\\ =& \frac{1-\alpha}{e^{t_{k+1}} - e^{t_{k-1}}}\int_{e^{t_{k-1}}}^{e^{t_k}} \psi(s)\left(2s - e^{t_{k-1}} - e^{t_k}\right) \mathrm{d}s.\end{aligned} \tag{3.104}$$

Furthermore, by means of the integral mean value theorem, one arrives at

$$\begin{aligned}J_2 &= \frac{1-\alpha}{e^{t_{k+1}}-e^{t_{k-1}}}\Bigg\{\int_{e^{t_{k-1}}}^{\frac{e^{t_{k-1}}+e^{t_k}}{2}}\psi(s)\left(2s-e^{t_{k-1}}-e^{t_k}\right)\mathrm{d}s \\ &\qquad + \int_{\frac{e^{t_{k-1}}+e^{t_k}}{2}}^{e^{t_k}}\psi(s)\left(2s-e^{t_{k-1}}-e^{t_k}\right)\mathrm{d}s\Bigg\} \\ &= \frac{\left(e^{t_k}-e^{t_{k-1}}\right)^2}{4\left(e^{t_{k+1}}-e^{t_{k-1}}\right)}(1-\alpha)\Big\{\psi(\xi_2)-\psi(\xi_1)\Big\}\geq 0,\end{aligned}\tag{3.105}$$

where $e^{t_{k-1}}\leq\xi_1\leq\frac{e^{t_{k-1}}+e^{t_k}}{2}\leq\xi_2\leq e^{t_k}$.

The next duty is to show $J_1>0$. Introduce

$$\phi(\sigma)=\left(\frac{\sigma}{\sigma+1}\right)^{2\sigma-1},\ \sigma\in\left(\frac{1}{2},1\right).\tag{3.106}$$

By elementary operation, one gets $\phi'(\sigma)<0$ and $\phi(\sigma)\in\left(\phi(1),\phi(\frac{1}{2})\right)=\left(\frac{1}{2},1\right)$. From (3.101) and $\sigma=1-\frac{\alpha}{2}$, one has

$$\begin{aligned}J_1 &= a_{k+1,k}^{(\alpha,\sigma)}-e^{\tau}a_{k,k}^{(\alpha,\sigma)} \\ &= (1+e^{\tau})\left(e^{t_{k+\sigma}}-e^{t_k}\right)^{1-\alpha}-e^{\tau}\left(e^{t_{k+\sigma}}-e^{t_{k-1}}\right)^{1-\alpha} \\ &= e^{\tau}\left(e^{t_{k+\sigma}}-e^{t_{k-1}}\right)^{1-\alpha}\left\{\frac{1+e^{\tau}}{e^{\tau}}\left(\frac{e^{\tau}\left(e^{\sigma\tau}-1\right)}{e^{(\sigma+1)\tau}-1}\right)^{2\sigma-1}-1\right\}.\end{aligned}\tag{3.107}$$

Let $\Phi(\tau,\sigma)=\frac{1+e^{\tau}}{e^{\tau}}\left(\frac{e^{\tau}(e^{\sigma\tau}-1)}{e^{(\sigma+1)\tau}-1}\right)^{2\sigma-1}$. And then the following limit holds

$$\lim_{\tau\to0}\Phi(\tau,\sigma)=2\phi(\sigma)>1,\tag{3.108}$$

which indicates that for $\forall\,\sigma\in(\frac{1}{2},1)$, $\exists\,\tau^*>0$, such that for $\tau<\tau^*$, $\Phi(\tau,\sigma)>1$. So $J_1>0$ for sufficiently small τ.

Combining with (3.101) and (3.105), one gets $c_{k+1,k}^{(\alpha,\sigma)}>c_{k,k}^{(\alpha,\sigma)}$ $(k\geq2)$ for sufficiently small τ.

Besides, for $k=1$, one has

$$\begin{aligned}&c_{2,1}^{(\alpha,\sigma)}-c_{1,1}^{(\alpha,\sigma)} \\ &= \frac{1}{e^{t_2}-e^{t_1}}\left(a_{2,1}^{(\alpha,\sigma)}+b_{1,1}^{(\alpha,\sigma)}\right)-\frac{1}{e^{t_1}-e^{t_0}}\left(a_{1,1}^{(\alpha,\sigma)}-b_{1,1}^{(\alpha,\sigma)}\right) \\ &= \frac{1}{e^{t_2}-e^{t_1}}\left\{\left(a_{2,1}^{(\alpha,\sigma)}-e^{\tau}a_{1,1}^{(\alpha,\sigma)}\right)+(1+e^{\tau})\,b_{1,1}^{(\alpha,\sigma)}\right\} \\ &= \frac{1}{e^{t_2}-e^{t_1}}\left[J+(1+e^{\tau})\,b_{1,1}^{(\alpha,\sigma)}\right].\end{aligned}\tag{3.109}$$

For the term J, one gets

$$J=e^{\tau}\left(e^{t_{1+\sigma}}-e^{t_0}\right)^{1-\alpha}\Big(\Phi(\tau,\sigma)-1\Big).\tag{3.110}$$

By (3.108) and Lemma 3.2, one can prove that $c_{2,1}^{(\alpha,\sigma)}>c_{1,1}^{(\alpha,\sigma)}$ for sufficiently small τ.

Finally, for $k \geq 2$, the following formula holds by using (3.73),

$$\begin{aligned}
&c_{2,k}^{(\alpha,\sigma)} - c_{1,k}^{(\alpha,\sigma)} \\
=& \frac{1}{e^{t_2} - e^{t_1}} \left(a_{2,k}^{(\alpha,\sigma)} + b_{1,k}^{(\alpha,\sigma)} - b_{2,k}^{(\alpha,\sigma)} \right) - \frac{1}{e^{t_1} - e^{t_0}} \left(a_{1,k}^{(\alpha,\sigma)} - b_{1,k}^{(\alpha,\sigma)} \right) \\
=& \frac{1}{e^{t_2} - e^{t_1}} \left\{ \left(a_{2,k}^{(\alpha,\sigma)} - e^{\tau} a_{1,k}^{(\alpha,\sigma)} - b_{2,k}^{(\alpha,\sigma)} + e^{\tau} b_{1,k}^{(\alpha,\sigma)} \right) + b_{1,k}^{(\alpha,\sigma)} \right\}.
\end{aligned} \tag{3.111}$$

Furthermore, one can attain that

$$\begin{aligned}
&a_{2,k}^{(\alpha,\sigma)} - e^{\tau} a_{1,k}^{(\alpha,\sigma)} - b_{2,k}^{(\alpha,\sigma)} + e^{\tau} b_{1,k}^{(\alpha,\sigma)} \\
=& (1-\alpha) \int_{e^{t_1}}^{e^{t_2}} \left(e^{t_{k+\sigma}} - s \right)^{-\alpha} \mathrm{d}s - (1-\alpha)\, e^{\tau} \int_{e^{t_0}}^{e^{t_1}} \left(e^{t_{k+\sigma}} - s \right)^{-\alpha} \mathrm{d}s \\
&- \frac{1-\alpha}{e^{t_3} - e^{t_1}} \left\{ \int_{e^{t_1}}^{e^{t_2}} \left(e^{t_{k+\sigma}} - s \right)^{-\alpha} \left(2s - e^{t_1} - e^{t_2} \right) \mathrm{d}s \right. \\
&\left. - e^{2\tau} \int_{e^{t_0}}^{e^{t_1}} \left(e^{t_{k+\sigma}} - s \right)^{-\alpha} \left(2s - e^{t_0} - e^{t_1} \right) \mathrm{d}s \right\} \\
=& (1-\alpha) \int_{e^{t_1}}^{e^{t_2}} \frac{e^{t_3} + e^{t_2} - 2s}{e^{t_3} - e^{t_1}} \left[\left(e^{t_{k+\sigma}} - s \right)^{-\alpha} - \left(e^{t_{k+\sigma}} - e^{-\tau} s \right)^{-\alpha} \right] \mathrm{d}s > 0.
\end{aligned} \tag{3.112}$$

Combining (3.111) and (3.112) with Lemma 3.2 leads to $c_{2,k}^{(\alpha,\sigma)} > c_{1,k}^{(\alpha,\sigma)}$ $(k \geq 2)$.

Till now, the proof is finally shown. □

The above three subsections are for the fractional derivative with order $\alpha \in (0,1)$. The following two subsections are for order $\alpha \in (1,2)$.

3.4. H2N2-A formula

In this subsection, we propose an H2N2-A formula to approximate the fractional derivative in (1.3) with $\alpha \in (1,2)$.

At first, we use the Hermite interpolation $H_{\exp,2,0} f(t)$ on $\left[t_0, t_1\right]$ to approximate the function $f(t)$ on the interval $\left[t_0, t_{\frac{1}{2}}\right]$ and the quadratic Newton interpolation $N_{\exp,2,j} f(t)$ on $\left[t_{j-1}, t_{j+1}\right]$ $(1 \leq j \leq k-1, 1 \leq k \leq N)$ to approximate the function $f(t)$ on the interval $\left[t_{j-\frac{1}{2}}, t_{j+\frac{1}{2}}\right]$ $(1 \leq j \leq k-1, 1 \leq k \leq N)$, that is,

$$\begin{cases} f(t) \approx H_{\exp,2,0} f(t), & t \in [t_0, t_{\frac{1}{2}}], \\ f(t) \approx N_{\exp,2,j} f(t), & t \in \left[t_{j-\frac{1}{2}}, t_{j+\frac{1}{2}}\right] (1 \leq j \leq k-1), \end{cases} \tag{3.113}$$

where $t_{k-\frac{1}{2}} = \frac{t_{k-1}+t_k}{2}$ $(1 \leq k \leq N)$. Then one can derive

$$\begin{aligned}
\delta_e^2 \left(H_{\exp,2,0} f(t) \right) =& \frac{2\left(\nabla_{\exp,t} f^{\frac{1}{2}} - \delta_e f(t_0) \right)}{e^{t_1} - e^{t_0}}, \\
\delta_e^2 \left(N_{\exp,2,j} f(t) \right) =& \frac{2\left(\nabla_{\exp,t} f^{j+\frac{1}{2}} - \nabla_{\exp,t} f^{j-\frac{1}{2}} \right)}{e^{t_{j+1}} - e^{t_{j-1}}}.
\end{aligned} \tag{3.114}$$

Used (3.114), for the order $\alpha \in (1,2)$, the fractional derivative ${}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)$ defined in (1.3) at $t = t_{k-\frac{1}{2}}$ $(1 \le k \le N)$ can be obtained as follows,

$$
\begin{aligned}
&{}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)\Big|_{t=t_{k-\frac{1}{2}}} \\
&= \frac{1}{\Gamma(2-\alpha)}\left\{\int_{t_0}^{t_{\frac{1}{2}}}\left(e^{t_{k-\frac{1}{2}}}-e^{s}\right)^{1-\alpha}\delta_e^2 f(s)e^s\mathrm{d}s+\sum_{j=1}^{k-1}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\left(e^{t_{k-\frac{1}{2}}}-e^{s}\right)^{1-\alpha}\delta_e^2 f(s)e^s\mathrm{d}s\right\} \\
&\approx \frac{1}{\Gamma(2-\alpha)}\left\{\int_{t_0}^{t_{\frac{1}{2}}}\left(e^{t_{k-\frac{1}{2}}}-e^{s}\right)^{1-\alpha}\delta_e^2\left(H_{\exp,2,0}f(s)\right)e^s\mathrm{d}s\right. \\
&\quad\left.+\sum_{j=1}^{k-1}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\left(e^{t_{k-\frac{1}{2}}}-e^{s}\right)^{1-\alpha}\delta_e^2\left(N_{\exp,2,j}f(s)\right)e^s\mathrm{d}s\right\} \\
&= \frac{2}{\Gamma(3-\alpha)}\left(\nabla_{\exp,t}f^{\frac{1}{2}}-\delta_e f(t_0)\right)a_{0,k}^{(\alpha)}+\frac{2}{\Gamma(3-\alpha)}\sum_{j=1}^{k-1}\left(\nabla_{\exp,t}f^{j+\frac{1}{2}}-\nabla_{\exp,t}f^{j-\frac{1}{2}}\right)a_{j,k}^{(\alpha)} \\
&= \frac{2}{\Gamma(3-\alpha)}\sum_{j=1}^{k}c_{j,k}^{(\alpha)}\left(f^j-f^{j-1}\right)-\frac{2}{\Gamma(3-\alpha)}a_{0,k}^{(\alpha)}\,\delta_e f(t_0) \\
&:= {}_{Ce}\mathbb{D}_{a,t}^{\alpha} f^{k-\frac{1}{2}},
\end{aligned}
\tag{3.115}
$$

where

$$
c_{j,k}^{(\alpha)} = \begin{cases}
\frac{1}{e^{t_1}-e^{t_0}}\left(a_{0,k}^{(\alpha)}-a_{1,k}^{(\alpha)}\right), & j=1, \\
\frac{1}{e^{t_j}-e^{t_{j-1}}}\left(a_{j-1,k}^{(\alpha)}-a_{j,k}^{(\alpha)}\right), & 2 \le j \le k-1, \\
\frac{1}{e^{t_k}-e^{t_{k-1}}}a_{k-1,k}^{(\alpha)}, & j=k,
\end{cases}
\tag{3.116}
$$

$$
a_{j,k}^{(\alpha)} = \begin{cases}
\frac{1}{e^{t_1}-e^{t_0}}\left[\left(e^{t_{k-\frac{1}{2}}}-e^{t_0}\right)^{2-\alpha}-\left(e^{t_{k-\frac{1}{2}}}-e^{t_{\frac{1}{2}}}\right)^{2-\alpha}\right], & j=0, \\
\frac{1}{e^{t_{j+1}}-e^{t_{j-1}}}\left[\left(e^{t_{k-\frac{1}{2}}}-e^{t_{j-\frac{1}{2}}}\right)^{2-\alpha}-\left(e^{t_{k-\frac{1}{2}}}-e^{t_{j+\frac{1}{2}}}\right)^{2-\alpha}\right], & 1 \le j \le k-1.
\end{cases}
$$

Call ${}_{Ce}\mathbb{D}_{a,t}^{\alpha} f^{k-\frac{1}{2}}$ in (3.115) H2N2-A formula under **Partition A** which is useful for approximating the exponential Caputo fractional derivative (1.3) with $\alpha \in (1,2)$. By the way, the Hermite interpolation has ever been used to approximate the Caputo derivative [5, 19]. Here the interpolation method is used by utilizing different basis functions.

Remark 3.6. When $k = 1$ or $k = 2$, one has

$$
\begin{aligned}
&c_{1,1}^{(\alpha)} = \frac{1}{e^{t_1}-e^{t_0}}a_{0,1}^{(\alpha)}, \quad k=1; \\
&c_{1,2}^{(\alpha)} = \frac{1}{e^{t_1}-e^{t_0}}\left(a_{0,2}^{(\alpha)}-a_{1,2}^{(\alpha)}\right), \quad c_{2,2}^{(\alpha)} = \frac{1}{e^{t_2}-e^{t_1}}a_{1,2}^{(\alpha)}, \quad k=2.
\end{aligned}
\tag{3.117}
$$

Now, the truncation error is estimated below.

Theorem 3.9. *Let $\delta_e^3 f(t) \in C[a,T]$ and $1<\alpha<2$. For the sufficiently small τ, the truncation errors $R^{k-\frac{1}{2}}$ $(1 \le k \le N)$,*

$$\begin{aligned} R^{k-\frac{1}{2}} =& {}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)\Big|_{t=t_{k-\frac{1}{2}}} - {}_{Ce}\mathbb{D}_{a,t}^{\alpha} f^{k-\frac{1}{2}} \\ =& \frac{1}{\Gamma(2-\alpha)} \left\{ \int_{t_0}^{t_{\frac{1}{2}}} \left(e^{t_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} \delta_e^2 \left(f(s) - H_{\exp,2,0} f(s)\right) e^s \mathrm{d}s \right. \\ & \left. + \sum_{j=1}^{k-1} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \left(e^{t_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} \delta_e^2 \left(f(s) - N_{\exp,2,j} f(s)\right) e^s \mathrm{d}s \right\}, \end{aligned} \tag{3.118}$$

with $t_j = t_0 + j\tau$ and $t_{j-\frac{1}{2}} = t_{j-1} + \frac{1}{2}\tau$ $(1 \le j \le k)$ have following estimates,

$$\begin{aligned} \left|R^{k-\frac{1}{2}}\right| \le& \frac{1}{\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^3 f(t)\right| \left(e^{t_1} - e^{t_0}\right) \left(e^{t_{\frac{1}{2}}} - e^{t_0}\right)^{2-\alpha}, \quad k=1, \\ \left|R^{k-\frac{1}{2}}\right| \le& \frac{1}{\Gamma(2-\alpha)} \left(2 + \frac{5\left(e^{T-a}+3\right)}{24}\right) \max_{t_0 \le t \le t_{k-1}} \left|\delta_e^3 f(t)\right| \\ & \times \max_{1 \le l \le k-1} \left(e^{t_l} - e^{t_{l-1}}\right)^2 \left(e^{t_{k-\frac{1}{2}}} - e^{t_{k-\frac{3}{2}}}\right)^{1-\alpha} \\ & + \frac{1}{\Gamma(3-\alpha)} \max_{t_{k-2} \le t \le t_k} \left|\delta_e^3 f(t)\right| \left(e^{t_k} - e^{t_{k-1}}\right) \left(e^{t_{k-\frac{1}{2}}} - e^{t_{k-\frac{3}{2}}}\right)^{2-\alpha}, \ k \ge 2. \end{aligned} \tag{3.119}$$

Proof. For $k=1$, one can get the following estimation

$$\begin{aligned} \left|R^{\frac{1}{2}}\right| \le& \frac{1}{\Gamma(2-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^3 f(t)\right| \left(e^{t_1} - e^{t_0}\right) \int_{t_0}^{t_{\frac{1}{2}}} \left(e^{t_{\frac{1}{2}}} - e^s\right)^{1-\alpha} e^s \mathrm{d}s \\ =& \frac{1}{\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^3 f(t)\right| \left(e^{t_1} - e^{t_0}\right) \left(e^{t_{\frac{1}{2}}} - e^{t_0}\right)^{2-\alpha}. \end{aligned} \tag{3.120}$$

For $k \ge 2$, $R^{k-\frac{1}{2}}$ can be divided into three subitems,

$$\begin{aligned} R^{k-\frac{1}{2}} =& \frac{1}{\Gamma(2-\alpha)} \int_{t_0}^{t_{\frac{1}{2}}} \left(e^{t_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} \delta_e^2 \left(R_H(s)\right) e^s \mathrm{d}s \\ & + \frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^{k-2} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \left(e^{t_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} \delta_e^2 \left(R_N^j(s)\right) e^s \mathrm{d}s \\ & + \frac{1}{\Gamma(2-\alpha)} \int_{t_{k-\frac{3}{2}}}^{t_{k-\frac{1}{2}}} \left(e^{t_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} \delta_e^2 \left(R_N^{k-1}(s)\right) e^s \mathrm{d}s \\ =& R_1^{k-\frac{1}{2}} + R_2^{k-\frac{1}{2}} + R_3^{k-\frac{1}{2}}. \end{aligned} \tag{3.121}$$

From the interpolation remainder (2.9), one has

$$\begin{aligned} & \left|R_1^{k-\frac{1}{2}}\right| \\ \le& \frac{1}{3\Gamma(2-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^3 f(t)\right| \int_{t_0}^{t_{\frac{1}{2}}} \left(e^{t_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} \left[2\left|e^s - e^{t_0}\right| + \left|e^s - e^{t_1}\right|\right] e^s \mathrm{d}s \\ \le& \frac{1}{\Gamma(2-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^3 f(t)\right| \left(e^{t_1} - e^{t_0}\right)^2 \left(e^{t_{k-\frac{1}{2}}} - e^{t_{k-\frac{3}{2}}}\right)^{1-\alpha}. \end{aligned} \tag{3.122}$$

By the same way, using the interpolation remainder (2.7) gives

$$\begin{aligned}
&\left|R_3^{k-\frac{1}{2}}\right| \\
\le&\frac{1}{3\Gamma(2-\alpha)}\int_{t_{k-\frac{3}{2}}}^{t_{k-\frac{1}{2}}}\left|\delta_e^3 f(\xi_{k-1})\right|\left(e^{t_{k-\frac{1}{2}}}-e^s\right)^{1-\alpha}\left[\left(e^{t_k}-e^{t_{k-2}}\right)+\left|e^s-e^{t_{k-1}}\right|\right]e^s\mathrm{d}s \\
\le&\frac{1}{\Gamma(3-\alpha)}\max_{t_{k-2}\le t\le t_k}\left|\delta_e^3 f(t)\right|\left(e^{t_k}-e^{t_{k-1}}\right)\left(e^{t_{k-\frac{1}{2}}}-e^{t_{k-\frac{3}{2}}}\right)^{2-\alpha},
\end{aligned} \tag{3.123}$$

where $\xi_{k-1}\in(t_{k-2},t_k)$.

After integration by parts, the item $R_2^{k-\frac{1}{2}}$ can be divided into two subitems,

$$\begin{aligned}
R_2^{k-\frac{1}{2}}=&\frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{k-2}\left(e^{t_{k-\frac{1}{2}}}-e^s\right)^{1-\alpha}\delta_e\left(R_N^j(s)\right)\Bigg|_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \\
&+\frac{1-\alpha}{\Gamma(2-\alpha)}\sum_{j=1}^{k-2}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\left(e^{t_{k-\frac{1}{2}}}-e^s\right)^{-\alpha}\delta_e\left(R_N^j(s)\right)e^s\mathrm{d}s \\
=&J_1+J_2.
\end{aligned} \tag{3.124}$$

By means of the interpolation remainder (2.7) again, one gets

$$\begin{aligned}
\delta_e\left(R_N^j(s)\right)=&\frac{1}{6}\delta_e^3 f(\xi_j)\Big\{\left(e^s-e^{t_j}\right)\left(e^s-e^{t_{j+1}}\right)+\left(e^s-e^{t_{j-1}}\right)\left(e^s-e^{t_{j+1}}\right) \\
&+\left(e^s-e^{t_{j-1}}\right)\left(e^s-e^{t_j}\right)\Big\},\quad \xi_j\in(t_{j-1},t_{j+1}).
\end{aligned} \tag{3.125}$$

Therefore, for $s\in\left[t_{j-\frac{1}{2}},t_{j+\frac{1}{2}}\right]$, one gets

$$\left|\delta_e\left(R_N^j(s)\right)\right|\le\max_{t_{j-1}\le t\le t_{j+1}}\left|\delta_e^3 f(t)\right|\left(e^{t_{j+1}}-e^{t_j}\right)^2. \tag{3.126}$$

Substituting (3.126) into J_2 leads to

$$\begin{aligned}
|J_2|\le&\frac{\alpha-1}{\Gamma(2-\alpha)}\max_{t_0\le t\le t_{k-1}}\left|\delta_e^3 f(t)\right|\max_{2\le l\le k-1}\left(e^{t_l}-e^{t_{l-1}}\right)^2\int_{t_{\frac{1}{2}}}^{t_{k-\frac{3}{2}}}\left(e^{t_{k-\frac{1}{2}}}-e^s\right)^{-\alpha}e^s\mathrm{d}s \\
\le&\frac{1}{\Gamma(2-\alpha)}\max_{t_0\le t\le t_{k-1}}\left|\delta_e^3 f(t)\right|\max_{2\le l\le k-1}\left(e^{t_l}-e^{t_{l-1}}\right)^2\left(e^{t_{k-\frac{1}{2}}}-e^{t_{k-\frac{3}{2}}}\right)^{1-\alpha}.
\end{aligned} \tag{3.127}$$

Before estimating the subitem J_1, the following equations are needed.

$$\begin{aligned}
&\delta_e\left(R_N^j(t_{j+\frac{1}{2}})\right) \\
=&\frac{1}{6}\,\delta_e^3 f(\xi_j)\,e^{2t_j}\left\{\left(e^{\frac{1}{2}\tau}-1\right)\left(e^{\frac{1}{2}\tau}-e^{\tau}\right)+\left(e^{\frac{1}{2}\tau}-e^{-\tau}\right)\left(e^{\frac{1}{2}\tau}-e^{\tau}\right)\right. \\
&\left.+\left(e^{\frac{1}{2}\tau}-e^{-\tau}\right)\left(e^{\frac{1}{2}\tau}-1\right)\right\},
\end{aligned} \tag{3.128}$$

$$
\begin{aligned}
&\delta_e\Big(R_N^j(t_{j-\frac{1}{2}})\Big)\\
=&\frac{1}{6}\,\delta_e^3 f(\xi_j)\,e^{2t_j}\Big\{\Big(e^{-\frac{1}{2}\tau}-1\Big)\Big(e^{-\frac{1}{2}\tau}-e^{\tau}\Big)+\Big(e^{-\frac{1}{2}\tau}-e^{-\tau}\Big)\Big(e^{-\frac{1}{2}\tau}-e^{\tau}\Big)\\
&+\Big(e^{-\frac{1}{2}\tau}-e^{-\tau}\Big)\Big(e^{-\frac{1}{2}\tau}-1\Big)\Big\},
\end{aligned}
$$

where $\xi_j \in (t_{j-1}, t_{j+1})$. Thus, let $x = e^{\tau}(x > 1)$. Then one has

$$
\begin{aligned}
&\Big|\delta_e\Big(R_N^j(t_{j-\frac{1}{2}})\Big)\Big| \le \frac{5}{6}\left|\delta_e^3 f(\xi_j)\right|\left(e^{t_{j+1}}-e^{t_j}\right)^2,\\
&\Big|\delta_e\Big(R_N^j(t_{j+\frac{1}{2}})\Big)-\delta_e\Big(R_N^j(t_{j-\frac{1}{2}})\Big)\Big| = \frac{1}{6}\left|\delta_e^3 f(\xi_j)\right| B_j,
\end{aligned} \tag{3.129}
$$

in which

$$
B_j = e^{2t_j}\left\{2\left(x^{\frac{3}{2}}-x^{-\frac{3}{2}}\right)-3\left(x-x^{-1}\right)\right\} > 0, \quad x > 1.
$$

In addition, let

$$
F(x) = \frac{2\left(x^{\frac{3}{2}}-x^{-\frac{3}{2}}\right)-3\left(x-x^{-1}\right)}{(x-1)^3}, \quad x > 1.
$$

Then it is easy to verify that $F(x)$ is a monotonically decreasing positive function and

$$
\lim_{x\to 1+} F(x) = \frac{5}{4}.
$$

So, $F(x) < \frac{5}{4}\,(x > 1)$. Then one can obtain

$$
\frac{B_j}{\left(e^{t_{j+1}}-e^{t_j}\right)^3} = \frac{1}{e^{t_j}}F(x) < \frac{5}{4e^{t_j}} < \frac{5}{4e^{a}}. \tag{3.130}
$$

Substituting the above relation into the second equation of (3.129) leads to

$$
\Big|\delta_e\Big(R_N^j(t_{j+\frac{1}{2}})\Big)-\delta_e\Big(R_N^j(t_{j-\frac{1}{2}})\Big)\Big| < \frac{5}{24e^{a}}\left|\delta_e^3 f(\xi_j)\right|\left(e^{t_{j+1}}-e^{t_j}\right)^3. \tag{3.131}
$$

It follows from the first inequality of (3.129) and (3.131) that

$$
\begin{aligned}
&|J_1|\\
\le&\frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{k-2}\Big\{\Big|\delta_e\left(R_N^j\left(t_{j+\frac{1}{2}}\right)\right)-\delta_e\left(R_N^j\left(t_{j-\frac{1}{2}}\right)\right)\Big|\left(e^{t_{k-\frac{1}{2}}}-e^{t_{j+\frac{1}{2}}}\right)^{1-\alpha}\\
&+\Big|\delta_e\left(R_N^j\left(t_{j-\frac{1}{2}}\right)\right)\Big|\Big[\left(e^{t_{k-\frac{1}{2}}}-e^{t_{j+\frac{1}{2}}}\right)^{1-\alpha}-\left(e^{t_{k-\frac{1}{2}}}-e^{t_{j-\frac{1}{2}}}\right)^{1-\alpha}\Big]\Big\}\\
\le&\frac{5\left(e^{T-a}+3\right)}{24\Gamma(2-\alpha)}\max_{t_0\le t\le t_{k-1}}\left|\delta_e^3 f(t)\right|\max_{2\le l\le k-1}\left(e^{t_l}-e^{t_{l-1}}\right)^2\left(e^{t_{k-\frac{1}{2}}}-e^{t_{k-\frac{3}{2}}}\right)^{1-\alpha}.
\end{aligned} \tag{3.132}
$$

It follows from that (3.122), (3.123), and (3.124) together with (3.127) and (3.132) the second inequality of (3.119) holds.

Till now, all this ends the proof. □

Remark 3.7. Because $e^{t_k} - e^{t_{k-1}} = O(\tau)$ with $t_k \to t_{k-1}$ and $e^{t_{k-\frac{1}{2}}} - e^{t_{k-\frac{3}{2}}} = O(\tau)$ with $t_{k-\frac{1}{2}} \to t_{k-\frac{3}{2}}$, the truncation errors $R^{k-\frac{1}{2}}$ in (3.118) can be estimated as follows

$$\left|R^{k-\frac{1}{2}}\right| \le C\tau^{3-\alpha},$$

where C is a positive constant.

The properties of the coefficients in the above H2N2 formula (3.115) are studied below.

Lemma 3.4. *For $\alpha \in (1,2)$, the coefficients $a_{j,k}^{(\alpha)}\,(0 \le j \le k-1, 1 \le k \le N)$ in (3.116) with $\tau = t_k - t_{k-1} = \frac{T-a}{N}$ and $t_{k-\frac{1}{2}} = t_k - \frac{1}{2}\tau$ satisfy*

$$a_{k-1,k}^{(\alpha)} > a_{k-2,k}^{(\alpha)} > \cdots > a_{0,k}^{(\alpha)} > 0. \tag{3.133}$$

Proof. It is obvious that $a_{j,k}^{(\alpha)} > 0\ (0 \le j \le k-1)$ from the definition (3.116). In addition, for $1 \le j \le k-2\,(k \ge 3)$, using the differential mean value theorem gives

$$\begin{aligned} a_{j+1,k}^{(\alpha)} - a_{j,k}^{(\alpha)} =& \frac{1}{e^{t_{j+2}} - e^{t_j}}\left[\left(e^{t_{k-\frac{1}{2}}} - e^{t_{j+\frac{1}{2}}}\right)^{2-\alpha} - \left(e^{t_{k-\frac{1}{2}}} - e^{t_{j+\frac{3}{2}}}\right)^{2-\alpha}\right] \\ &- \frac{1}{e^{t_{j+1}} - e^{t_{j-1}}}\left[\left(e^{t_{k-\frac{1}{2}}} - e^{t_{j-\frac{1}{2}}}\right)^{2-\alpha} - \left(e^{t_{k-\frac{1}{2}}} - e^{t_{j+\frac{1}{2}}}\right)^{2-\alpha}\right] \\ =& (2-\alpha)\frac{e^{\frac{1}{2}\tau} - e^{-\frac{1}{2}\tau}}{e^{\tau} - e^{-\tau}}\left(\xi_{j+1}^{1-\alpha} - \xi_j^{1-\alpha}\right) \\ >& 0, \end{aligned} \tag{3.134}$$

where $e^{t_{k-\frac{1}{2}}} - e^{t_{j+\frac{3}{2}}} < \xi_{j+1} < e^{t_{k-\frac{1}{2}}} - e^{t_{j+\frac{1}{2}}} < \xi_j < e^{t_{k-\frac{1}{2}}} - e^{t_{j-\frac{1}{2}}}$, which means that $a_{j,k}^{(\alpha)} < a_{j+1,k}^{(\alpha)}\,(1 \le j \le k-2,\ k \ge 3)$.

For $j = 0\,(k \ge 2)$, one has

$$\begin{aligned} a_{1,k}^{(\alpha)} - a_{0,k}^{(\alpha)} =& \frac{1}{e^{t_2} - e^{t_0}}\left[\left(e^{t_{k-\frac{1}{2}}} - e^{t_{\frac{1}{2}}}\right)^{2-\alpha} - \left(e^{t_{k-\frac{1}{2}}} - e^{t_{\frac{3}{2}}}\right)^{2-\alpha}\right] \\ &- \frac{1}{e^{t_1} - e^{t_0}}\left[\left(e^{t_{k-\frac{1}{2}}} - e^{t_0}\right)^{2-\alpha} - \left(e^{t_{k-\frac{1}{2}}} - e^{t_{\frac{1}{2}}}\right)^{2-\alpha}\right] \\ =& (2-\alpha)\left\{\frac{e^{\frac{1}{2}\tau} - e^{-\frac{1}{2}\tau}}{e^{\tau} - e^{-\tau}}\xi_1^{1-\alpha} - \frac{e^{\frac{1}{2}\tau} - 1}{e^{\tau} - 1}\xi_0^{1-\alpha}\right\} \\ >& 0, \end{aligned} \tag{3.135}$$

where $e^{t_{k-\frac{1}{2}}} - e^{t_{\frac{3}{2}}} < \xi_1 < e^{t_{k-\frac{1}{2}}} - e^{t_{\frac{1}{2}}} < \xi_0 < e^{t_{k-\frac{1}{2}}} - e^{t_0}$, that is, $a_{0,k}^{(\alpha)} < a_{1,k}^{(\alpha)}\,(k \ge 2)$.

The proof is thus finished. □

Theorem 3.10. *For $\alpha \in (1,2)$, the coefficients $c_{j,k}^{(\alpha)}\,(1 \le j \le k, 1 \le k \le N)$ in (3.116) with $\tau = t_k - t_{k-1} = \frac{T-a}{N}$ and $t_{k-\frac{1}{2}} = t_k - \frac{1}{2}\tau$ satisfy*

$$c_{j,k}^{(\alpha)} < 0\,(1 \le j \le k-1)\,,\ c_{k,k}^{(\alpha)} > 0. \tag{3.136}$$

Proof. It is easy to be proved by using Lemma 3.4. □

Theorem 3.11. *For $\alpha \in (1,2)$ and sufficiently small step τ, the coefficients $c_{j,k}^{(\alpha)}\big(1 \le j \le k, 1 \le k \le N\big)$ in (3.116) with $\tau = t_k - t_{k-1} = \frac{T-a}{N}$ and $t_{k-\frac{1}{2}} = t_k - \frac{1}{2}\tau$ satisfy*

(1) $c_{j+1,k}^{(\alpha)} < c_{j,k}^{(\alpha)},\ 2 \le j \le k-2,\ k \ge 4,$

(2) $\left|c_{k-1,k}^{(\alpha)}\right| < c_{k,k}^{(\alpha)},\ k \ge 2.$

Proof. (1) For $2 \le j \le k-2\,(k \ge 4)$, using (3.116) yields

$$\begin{aligned} c_{j+1,k}^{(\alpha)} - c_{j,k}^{(\alpha)} &= \frac{1}{e^{t_{j+1}} - e^{t_j}}\left(a_{j,k}^{(\alpha)} - a_{j+1,k}^{(\alpha)}\right) - \frac{1}{e^{t_j} - e^{t_{j-1}}}\left(a_{j-1,k}^{(\alpha)} - a_{j,k}^{(\alpha)}\right) \\ &= \frac{1}{e^{t_{j+1}} - e^{t_j}}\left\{\left(a_{j,k}^{(\alpha)} - e^{\tau}a_{j-1,k}^{(\alpha)}\right) - \left(a_{j+1,k}^{(\alpha)} - e^{\tau}a_{j,k}^{(\alpha)}\right)\right\}. \end{aligned} \tag{3.137}$$

Consider the following function $F(x)$ where $x \in [e^{t_1}, e^{t_{k-2}}]$ and $x_j = e^{t_j}(1 \le j \le k-2)$,

$$\begin{aligned} F(x) &= \frac{2-\alpha}{(e^{2\tau}-1)\,x}\left\{\int_{e^{\frac{1}{2}\tau}x}^{e^{\frac{3}{2}\tau}x}\left(e^{t_{k-\frac{1}{2}}} - s\right)^{1-\alpha}\mathrm{d}s - e^{2\tau}\int_{e^{-\frac{1}{2}\tau}x}^{e^{\frac{1}{2}\tau}x}\left(e^{t_{k-\frac{1}{2}}} - s\right)^{1-\alpha}\mathrm{d}s\right\} \\ &= \frac{2-\alpha}{(e^{2\tau}-1)\,x}\int_{e^{\frac{1}{2}\tau}x}^{e^{\frac{3}{2}\tau}x}\psi(s)\mathrm{d}s, \end{aligned} \tag{3.138}$$

in which

$$\psi(s) = \left(e^{t_{k-\frac{1}{2}}} - s\right)^{1-\alpha} - e^{\tau}\left(e^{t_{k-\frac{1}{2}}} - e^{-\tau}s\right)^{1-\alpha}. \tag{3.139}$$

Applying the Lagrange linear interpolation on interval $\left[e^{\frac{1}{2}\tau}x, e^{\frac{3}{2}\tau}x\right]$ to $\psi(s)$ leads to

$$\begin{aligned} \psi(s) = &\frac{e^{\frac{3}{2}\tau}x - s}{e^{\frac{3}{2}\tau}x - e^{\frac{1}{2}\tau}x}\psi(e^{\frac{1}{2}\tau}x) + \frac{s - e^{\frac{1}{2}\tau}x}{e^{\frac{3}{2}\tau}x - e^{\frac{1}{2}\tau}x}\psi(e^{\frac{3}{2}\tau}x) \\ &+ \frac{1}{2}\psi''(\xi)\left(s - e^{\frac{1}{2}\tau}x\right)\left(s - e^{\frac{3}{2}\tau}x\right),\quad \xi \in \left(e^{\frac{1}{2}\tau}x, e^{\frac{3}{2}\tau}x\right). \end{aligned} \tag{3.140}$$

So

$$\begin{aligned} \frac{1}{x}\int_{e^{\frac{1}{2}\tau}x}^{e^{\frac{3}{2}\tau}x}\psi(s)\mathrm{d}s = &\frac{1}{2}\left(e^{\frac{3}{2}\tau} - e^{\frac{1}{2}\tau}\right)\left[\psi\left(e^{\frac{3}{2}\tau}x\right) + \psi\left(e^{\frac{1}{2}\tau}x\right)\right] \\ &+ \frac{1}{2x}\int_{e^{\frac{1}{2}\tau}x}^{e^{\frac{3}{2}\tau}x}\psi''(\xi)\left(s - e^{\frac{1}{2}\tau}x\right)\left(s - e^{\frac{3}{2}\tau}x\right)\mathrm{d}s. \end{aligned} \tag{3.141}$$

Furthermore, one can obtain the following result using differentiation on $x \in$

$[e^{t_1}, e^{t_{k-2}})$ and formula (3.141)

$$\begin{aligned} F'(x) &= \frac{2-\alpha}{(e^{2\tau}-1)\,x}\left\{\psi\left(e^{\frac{3}{2}\tau}x\right)e^{\frac{3}{2}\tau} - \psi\left(e^{\frac{1}{2}\tau}x\right)e^{\frac{1}{2}\tau} - \frac{1}{x}\int_{e^{\frac{1}{2}\tau}x}^{e^{\frac{3}{2}\tau}x}\psi(s)\mathrm{d}s\right\} \\ &= \frac{2-\alpha}{(e^{2\tau}-1)\,x}\left\{\frac{1}{2}\left(e^{\frac{3}{2}\tau}+e^{\frac{1}{2}\tau}\right)\left[\psi\left(e^{\frac{3}{2}\tau}x\right)-\psi\left(e^{\frac{1}{2}\tau}x\right)\right]\right. \\ &\quad \left. + \frac{1}{2x}\int_{e^{\frac{1}{2}\tau}x}^{e^{\frac{3}{2}\tau}x}\psi''(\xi)\left(s-e^{\frac{1}{2}\tau}x\right)\left(e^{\frac{3}{2}\tau}x-s\right)\mathrm{d}s\right\}. \end{aligned} \tag{3.142}$$

Since

$$\begin{aligned} \psi'(s) &= (\alpha-1)\left[\left(e^{t_{k-\frac{1}{2}}}-s\right)^{-\alpha} - \left(e^{t_{k-\frac{1}{2}}}-e^{-\tau}s\right)^{-\alpha}\right] > 0, \\ \psi''(s) &= \alpha(\alpha-1)\left[\left(e^{t_{k-\frac{1}{2}}}-s\right)^{-\alpha-1} - e^{-\tau}\left(e^{t_{k-\frac{1}{2}}}-e^{-\tau}s\right)^{-\alpha-1}\right] > 0, \end{aligned} \tag{3.143}$$

it immediately follows that $F'(x) > 0$ $(x \in [e^{t_1}, e^{t_{k-2}}))$, which implies that $F(x)$ is an increasing function with respect to $x \in [e^{t_1}, e^{t_{k-2}}]$. Therefore,

$$c_{j+1,k}^{(\alpha)} - c_{j,k}^{(\alpha)} = \frac{1}{e^{t_{j+1}}-e^{t_j}}\left[F(x_{j-1}) - F(x_j)\right] < 0. \tag{3.144}$$

(2) For $k \geq 3$, one has

$$\begin{aligned} & c_{k,k}^{(\alpha)} - \left|c_{k-1,k}^{(\alpha)}\right| \\ =& \frac{1}{e^{t_k}-e^{t_{k-1}}}a_{k-1,k}^{(\alpha)} - \frac{1}{e^{t_{k-1}}-e^{t_{k-2}}}\left(a_{k-1,k}^{(\alpha)} - a_{k-2,k}^{(\alpha)}\right) \\ =& \frac{\left(e^{t_{k-\frac{1}{2}}}-e^{t_{k-\frac{3}{2}}}\right)^{2-\alpha}}{\left(e^{t_k}-e^{t_{k-1}}\right)\left(e^{t_k}-e^{t_{k-2}}\right)}\left\{e^{2\tau}\left(\frac{e^{\tau}+1}{e^{\tau}}\right)^{2-\alpha} - \left(e^{2\tau}+e^{\tau}-1\right)\right\}. \end{aligned} \tag{3.145}$$

Let

$$\varphi(\alpha) = e^{2\tau}\left(\frac{e^{\tau}+1}{e^{\tau}}\right)^{2-\alpha} - \left(e^{2\tau}+e^{\tau}-1\right).$$

Simple calculations imply

$$\varphi'(\alpha) = -e^{2\tau}\left(\frac{e^{\tau}+1}{e^{\tau}}\right)^{2-\alpha}\log\frac{e^{\tau}+1}{e^{\tau}} < 0,$$

$\varphi(1) > 0$, and $\varphi(2) < 0$. Set $\varphi(\alpha^*) = 0$. Then one gets

$$\alpha^* = 2 - \log\frac{e^{2\tau}+e^{\tau}-1}{e^{2\tau}}\bigg/\log\frac{e^{\tau}+1}{e^{\tau}}.$$

Therefore, according to the above results, one has

$$\begin{cases} \left|c_{k-1,k}^{(\alpha)}\right| > c_{k,k}^{(\alpha)}, & \alpha > \alpha^*, \\ \left|c_{k-1,k}^{(\alpha)}\right| < c_{k,k}^{(\alpha)}, & \alpha < \alpha^*. \end{cases} \tag{3.146}$$

Because of the following limit

$$\lim_{\tau \to 0} \alpha^* = 2,$$

one can get that for $\forall\, \alpha \in (1,2)$, $\exists\, \tau^* > 0$ such that when $\tau < \tau^*$, $\alpha < \alpha^*$ holds, which means $\left|c_{k-1,k}^{(\alpha)}\right| < c_{k,k}^{(\alpha)}$ for sufficiently small step τ.

By almost the same way as above, for $k = 2$, one has

$$\begin{aligned} & c_{2,2}^{(\alpha)} - \left|c_{1,2}^{(\alpha)}\right| \\ = & \frac{1}{e^{t_2} - e^{t_1}} a_{1,2}^{(\alpha)} - \frac{1}{e^{t_1} - e^{t_0}} \left(a_{1,2}^{(\alpha)} - a_{0,2}^{(\alpha)}\right) \\ = & \frac{\left(e^{t_{\frac{3}{2}}} - e^{t_{\frac{1}{2}}}\right)^{2-\alpha}}{\left(e^{t_2} - e^{t_1}\right)\left(e^{t_1} - e^{t_0}\right)} \left\{ e^{\tau} \left(\frac{e^{\frac{3}{2}\tau} - 1}{e^{\frac{3}{2}\tau} - e^{\frac{1}{2}\tau}} \right)^{2-\alpha} - \left(e^{\tau} + \frac{e^{\tau} - 1}{e^{\tau} + 1} \right) \right\}. \end{aligned} \tag{3.147}$$

Set

$$\begin{aligned} \phi(\alpha) = & e^{\tau} \left(\frac{e^{\frac{3}{2}\tau} - 1}{e^{\frac{3}{2}\tau} - e^{\frac{1}{2}\tau}} \right)^{2-\alpha} - \left(e^{\tau} + \frac{e^{\tau} - 1}{e^{\tau} + 1} \right) \\ = & e^{\tau} \left(\frac{e^{\tau} + e^{\frac{1}{2}\tau} + 1}{e^{\tau} + e^{\frac{1}{2}\tau}} \right)^{2-\alpha} - \frac{e^{2\tau} + 2e^{\tau} - 1}{e^{\tau} + 1}. \end{aligned} \tag{3.148}$$

Then it is convenient to get that

$$\phi'(\alpha) = -\, e^{\tau} \left(\frac{e^{\tau} + e^{\frac{1}{2}\tau} + 1}{e^{\tau} + e^{\frac{1}{2}\tau}} \right)^{2-\alpha} \log \frac{e^{\tau} + e^{\frac{1}{2}\tau} + 1}{e^{\tau} + e^{\frac{1}{2}\tau}} < 0, \tag{3.149}$$

$\phi(1) > 0$, and $\phi(2) < 0$. Setting $\phi(\alpha^*) = 0$ yields

$$\alpha^* = 2 - \log \frac{e^{2\tau} + 2e^{\tau} - 1}{e^{2\tau} + e^{\tau}} \Big/ \log \frac{e^{\tau} + e^{\frac{1}{2}\tau} + 1}{e^{\tau} + e^{\frac{1}{2}\tau}}. \tag{3.150}$$

From (3.149) and (3.150), one knows that

$$\begin{cases} c_{2,2}^{(\alpha)} < \left|c_{1,2}^{(\alpha)}\right|, & \alpha > \alpha^*, \\ c_{2,2}^{(\alpha)} > \left|c_{1,2}^{(\alpha)}\right|, & \alpha < \alpha^*. \end{cases} \tag{3.151}$$

Considering the following limit

$$\lim_{\tau \to 0} \alpha^* = 2,$$

one can get that for $\forall\, \alpha \in (1,2)$, $\exists\, \tau^* > 0$ such that when $\tau < \tau^*$, the inequality $\alpha < \alpha^*$ holds, that is, $c_{2,2}^{(\alpha)} > \left|c_{1,2}^{(\alpha)}\right|$ for sufficiently small τ.

Finally, the theorem is proved. □

Remark 3.8. Choose $[a, T] = [0, 2]$, $\tau = 0.02$ and $\alpha = 1.5$ to get the values of $c_{2,k}^{(\alpha)} - c_{1,k}^{(\alpha)}\,(3 \le k \le 100)$ with different $t_{k-\frac{1}{2}}$ in Figure 1. By observation, one can find that the sign of $c_{2,k}^{(\alpha)} - c_{1,k}^{(\alpha)}\,(k \ge 3)$ is uncertain.

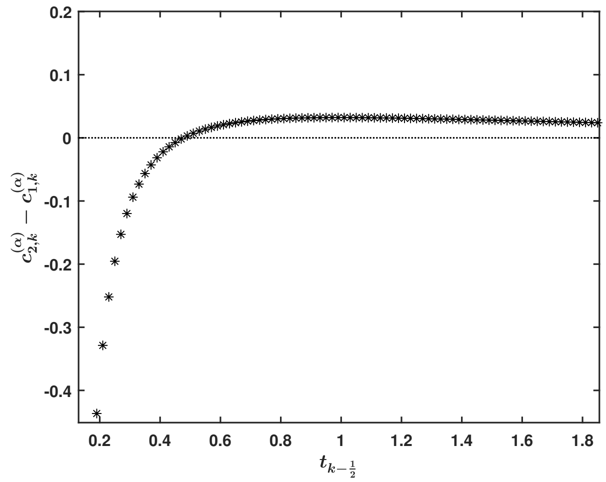

Figure 1. The values of $c_{2,k}^{(\alpha)} - c_{1,k}^{(\alpha)}$ for H2N2-A formula.

3.5. $L2_1$-A formula

In the following, adopting the order reduction method produces a new formula, called $L2_1$-A formula, which is somewhat similar to the H2N2-A formula.

For brevity, denote $g(t) = \delta_e f(t)$, $\beta = \alpha - 1$, and $\tilde{t}_{k-\frac{1}{2}} = \log\left(\frac{e^{t_k}+e^{t_{k-1}}}{2}\right)$ $(1 \le k \le N)$, that is, $e^{\tilde{t}_{k-\frac{1}{2}}} = \frac{1}{2}\left(e^{t_k} + e^{t_{k-1}}\right)$. Then one has the following approximate relations

$$\begin{cases} \delta_e\left(g(t)\right) \approx \dfrac{g(\tilde{t}_{\frac{1}{2}}) - g(t_0)}{e^{\tilde{t}_{\frac{1}{2}}} - e^{t_0}}, \quad t \in [t_0, \tilde{t}_{\frac{1}{2}}], \\ \delta_e\left(g(t)\right) \approx \dfrac{g(\tilde{t}_{j+\frac{1}{2}}) - g(\tilde{t}_{j-\frac{1}{2}})}{e^{\tilde{t}_{j+\frac{1}{2}}} - e^{\tilde{t}_{j-\frac{1}{2}}}}, \quad t \in [\tilde{t}_{j-\frac{1}{2}}, \tilde{t}_{j+\frac{1}{2}}]\,(1 \le j \le k-1)\,, \end{cases} \tag{3.152}$$

and

$$g(\tilde{t}_{j-\frac{1}{2}}) = \delta_e\, f(\tilde{t}_{j-\frac{1}{2}}) \approx \nabla_{\exp,t} f^{j-\frac{1}{2}}, \quad 1 \le j \le k. \tag{3.153}$$

According to the above relationships (3.152) and (3.153), the fractional derivative ${}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)$ defined in (1.3) at $t = \tilde{t}_{k-\frac{1}{2}}$ $(1 \le k \le N)$ with order $\alpha \in (1,2)$ have the following expression

$$\begin{aligned} {}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)\Big|_{t=\tilde{t}_{k-\frac{1}{2}}} &= {}_{Ce}\mathrm{D}_{a,t}^{\beta} g(t)\Big|_{t=\tilde{t}_{k-\frac{1}{2}}} \\ = \frac{1}{\Gamma(1-\beta)} \Bigg\{ &\int_{t_0}^{\tilde{t}_{\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}} - e^s\right)^{-\beta} \delta_e\left(g(s)\right) e^s \mathrm{d}s \\ &+ \sum_{j=1}^{k-1} \int_{\tilde{t}_{j-\frac{1}{2}}}^{\tilde{t}_{j+\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}} - e^s\right)^{-\beta} \delta_e\left(g(s)\right) e^s \mathrm{d}s \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\beta)} \Bigg\{ \frac{g(\tilde{t}_{\frac{1}{2}})-g(t_0)}{e^{\tilde{t}_{\frac{1}{2}}}-e^{t_0}} \int_{t_0}^{\tilde{t}_{\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}}-e^{s}\right)^{-\beta} e^{s}\mathrm{d}s \\
&\quad + \sum_{j=1}^{k-1} \frac{g(\tilde{t}_{j+\frac{1}{2}})-g(\tilde{t}_{j-\frac{1}{2}})}{e^{\tilde{t}_{j+\frac{1}{2}}}-e^{\tilde{t}_{j-\frac{1}{2}}}} \int_{\tilde{t}_{j-\frac{1}{2}}}^{\tilde{t}_{j+\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}}-e^{s}\right)^{-\beta} e^{s}\mathrm{d}s \Bigg\} + \Upsilon^{k-\frac{1}{2}} \\
&= \frac{1}{\Gamma(3-\alpha)} \Bigg\{ a_{0,k}^{(\alpha)} \left(g(\tilde{t}_{\frac{1}{2}})-g(t_0)\right) + \sum_{j=1}^{k-1} a_{j,k}^{(\alpha)} \left(g(\tilde{t}_{j+\frac{1}{2}})-g(\tilde{t}_{j-\frac{1}{2}})\right) \Bigg\} + \Upsilon^{k-\frac{1}{2}} \\
&= \frac{1}{\Gamma(3-\alpha)} \Bigg\{ a_{k-1,k}^{(\alpha)} \nabla_{\exp,t} f^{k-\frac{1}{2}} + \sum_{j=1}^{k-1} \left(a_{j-1,k}^{(\alpha)} - a_{j,k}^{(\alpha)}\right) \nabla_{\exp,t} f^{j-\frac{1}{2}} \\
&\quad - a_{0,k}^{(\alpha)} \delta_e\left(f(t_0)\right) \Bigg\} + \Upsilon^{k-\frac{1}{2}} + r^{k-\frac{1}{2}} \\
&= {}_{Ce}\mathcal{D}_{a,t}^{\alpha} f^{k-\frac{1}{2}} + R^{k-\frac{1}{2}},
\end{aligned} \tag{3.154}
$$

where $R^{k-\frac{1}{2}} = \Upsilon^{k-\frac{1}{2}} + r^{k-\frac{1}{2}}$, and

$$
\begin{aligned}
\Upsilon^{k-\frac{1}{2}} &= \frac{1}{\Gamma(1-\beta)} \Bigg\{ \int_{t_0}^{\tilde{t}_{\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}}-e^{s}\right)^{-\beta} \left(\delta_e\left(g(s)\right) - \frac{g(\tilde{t}_{\frac{1}{2}})-g(t_0)}{e^{\tilde{t}_{\frac{1}{2}}}-e^{t_0}}\right) e^{s}\mathrm{d}s \\
&\quad + \sum_{j=1}^{k-1} \int_{\tilde{t}_{j-\frac{1}{2}}}^{\tilde{t}_{j+\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}}-e^{s}\right)^{-\beta} \left(\delta_e\left(g(s)\right) - \frac{g(\tilde{t}_{j+\frac{1}{2}})-g(\tilde{t}_{j-\frac{1}{2}})}{e^{\tilde{t}_{j+\frac{1}{2}}}-e^{\tilde{t}_{j-\frac{1}{2}}}}\right) e^{s}\mathrm{d}s \Bigg\}, \\
r^{k-\frac{1}{2}} &= \frac{1}{\Gamma(3-\alpha)} \Bigg\{ a_{k-1,k}^{(\alpha)} \left(\delta_e f(\tilde{t}_{k-\frac{1}{2}}) - \nabla_{\exp,t} f^{k-\frac{1}{2}}\right) \\
&\quad + \sum_{j=1}^{k-1} \left(a_{j-1,k}^{(\alpha)} - a_{j,k}^{(\alpha)}\right) \left(\delta_e f(\tilde{t}_{j-\frac{1}{2}}) - \nabla_{\exp,t} f^{j-\frac{1}{2}}\right) \Bigg\},
\end{aligned} \tag{3.155}
$$

in which

$$
a_{j,k}^{(\alpha)} = \begin{cases} \frac{1}{e^{\tilde{t}_{\frac{1}{2}}}-e^{t_0}} \left[\left(e^{\tilde{t}_{k-\frac{1}{2}}}-e^{t_0}\right)^{2-\alpha} - \left(e^{\tilde{t}_{k-\frac{1}{2}}}-e^{\tilde{t}_{\frac{1}{2}}}\right)^{2-\alpha}\right], & j=0, \\ \frac{1}{e^{\tilde{t}_{j+\frac{1}{2}}}-e^{\tilde{t}_{j-\frac{1}{2}}}} \left[\left(e^{\tilde{t}_{k-\frac{1}{2}}}-e^{\tilde{t}_{j-\frac{1}{2}}}\right)^{2-\alpha} - \left(e^{\tilde{t}_{k-\frac{1}{2}}}-e^{\tilde{t}_{j+\frac{1}{2}}}\right)^{2-\alpha}\right], & 1 \le j \le k-1. \end{cases} \tag{3.156}
$$

Call formula ${}_{Ce}\mathcal{D}_{a,t}^{\alpha} f^{k-\frac{1}{2}}$ in (3.154) L2$_1$-A formula under **Partition A** for approximating the fractional derivative (1.3) with $\alpha \in (1,2)$.

Remark 3.9. Particularly, when $k=1$, one has

$$
\begin{aligned}
\Upsilon^{\frac{1}{2}} &= \frac{1}{\Gamma(1-\beta)} \int_{t_0}^{\tilde{t}_{\frac{1}{2}}} \left(e^{\tilde{t}_{\frac{1}{2}}}-e^{s}\right)^{-\beta} \left(\delta_e\left(g(s)\right) - \frac{g(\tilde{t}_{\frac{1}{2}})-g(t_0)}{e^{\tilde{t}_{\frac{1}{2}}}-e^{t_0}}\right) e^{s}\mathrm{d}s, \\
r^{\frac{1}{2}} &= \frac{1}{\Gamma(3-\alpha)} a_{0,1}^{(\alpha)} \left(\delta_e f(\tilde{t}_{\frac{1}{2}}) - \nabla_{\exp,t} f^{\frac{1}{2}}\right).
\end{aligned} \tag{3.157}
$$

The coefficients in (3.156) satisfy the following properties.

Theorem 3.12. *For $\alpha \in (1,2)$, the coefficients $a_{j,k}^{(\alpha)}$ $(0 \le j \le k-1, 1 \le k \le N)$ in (3.156) with $\tau = t_k - t_{k-1} = \frac{T-a}{N}$ satisfy*

$$a_{k-1,k}^{(\alpha)} > a_{k-2,k}^{(\alpha)} > \cdots > a_{0,k}^{(\alpha)} > 0. \tag{3.158}$$

Proof. It is easy to prove them using the mean value theorem and so the proof is omitted here. □

In the following, the truncation error of L2$_1$ formula will be analyzed.

Theorem 3.13. *Let $\delta_e^3 f(t) \in C[a,T]$ and $1 < \alpha < 2$. The truncation errors $R^{k-\frac{1}{2}}$ $(1 \le k \le N)$ in (3.154) with $t_k = t_0 + k\tau$ and $\tilde{t}_{k-\frac{1}{2}} = \log\left(\frac{e^{t_k}+e^{t_{k-1}}}{2}\right)$ have following estimations*

$$\begin{aligned} \left|R^{k-\frac{1}{2}}\right| &\le \frac{2^{\alpha-1}}{6\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^3 f(t)\right| \left(e^{t_1} - e^{t_0}\right)^{3-\alpha}, \ k = 1, \\ \left|R^{k-\frac{1}{2}}\right| &\le \left\{\frac{2^{\alpha-1}}{4\Gamma(2-\alpha)} + \frac{2^{\alpha-1}+6}{12\Gamma(3-\alpha)}\right\} \max_{t_0 \le t \le t_k} \left|\delta_e^3 f(t)\right| \max_{1 \le l \le k} \left(e^{t_l} - e^{t_{l-1}}\right)^{3-\alpha}, \ k \ge 2. \end{aligned} \tag{3.159}$$

Proof. The first step is to estimate the item $r^{k-\frac{1}{2}}$. For $k \ge 2$, using (3.82) gives

$$\begin{aligned} \left|r^{k-\frac{1}{2}}\right| &\le \frac{1}{24\Gamma(3-\alpha)} \max_{t_0 \le t \le t_k} \left|\delta_e^3 f(t)\right| \max_{1 \le l \le k} \left(e^{t_l} - e^{t_{l-1}}\right)^2 \left(2a_{k-1,k}^{(\alpha)} - a_{0,k}^{(\alpha)}\right) \\ &\le \frac{2^{\alpha-1}}{12\Gamma(3-\alpha)} \max_{t_0 \le t \le t_k} \left|\delta_e^3 f(t)\right| \max_{1 \le l \le k} \left(e^{t_l} - e^{t_{l-1}}\right)^{3-\alpha}. \end{aligned} \tag{3.160}$$

When $k = 1$, one has

$$\begin{aligned} \left|r^{\frac{1}{2}}\right| &\le \frac{1}{24\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^3 f(t)\right| \left(e^{t_1} - e^{t_0}\right)^2 a_{0,1}^{(\alpha)} \\ &= \frac{2^{\alpha-1}}{24\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^3 f(t)\right| \left(e^{t_1} - e^{t_0}\right)^{3-\alpha}. \end{aligned} \tag{3.161}$$

The next step is to consider the item $\Upsilon^{k-\frac{1}{2}}$ for $k \ge 2$, which can be broken down into three subitems,

$$\begin{aligned} &\Upsilon^{k-\frac{1}{2}} \\ =&\frac{1}{\Gamma(2-\alpha)} \int_{t_0}^{\tilde{t}_{\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} \left(\delta_e\left(g(s)\right) - \frac{g(\tilde{t}_{\frac{1}{2}}) - g(t_0)}{e^{\tilde{t}_{\frac{1}{2}}} - e^{t_0}}\right) e^s \mathrm{d}s \\ &+ \frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^{k-2} \int_{\tilde{t}_{j-\frac{1}{2}}}^{\tilde{t}_{j+\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} \left(\delta_e\left(g(s)\right) - \frac{g(\tilde{t}_{j+\frac{1}{2}}) - g(\tilde{t}_{j-\frac{1}{2}})}{e^{\tilde{t}_{j+\frac{1}{2}}} - e^{\tilde{t}_{j-\frac{1}{2}}}}\right) e^s \mathrm{d}s \\ &+ \frac{1}{\Gamma(2-\alpha)} \int_{\tilde{t}_{k-\frac{3}{2}}}^{\tilde{t}_{k-\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} \left(\delta_e\left(g(s)\right) - \frac{g(\tilde{t}_{k-\frac{1}{2}}) - g(\tilde{t}_{k-\frac{3}{2}})}{e^{\tilde{t}_{k-\frac{1}{2}}} - e^{\tilde{t}_{k-\frac{3}{2}}}}\right) e^s \mathrm{d}s \\ =&I_1 + I_2 + I_3. \end{aligned} \tag{3.162}$$

For the subitem I_1, using the linear interpolation on $\left[t_0, \tilde{t}_{\frac{1}{2}}\right]$ in the sense of exponential function gives

$$\begin{aligned}|I_1| &\leq \frac{1}{2\Gamma(2-\alpha)} \max_{t_0 \leq t \leq \tilde{t}_{\frac{1}{2}}} \left|\delta_e^2 g(t)\right| \left(e^{\tilde{t}_{\frac{1}{2}}} - e^{t_0}\right) \int_{t_0}^{\tilde{t}_{\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} e^s \mathrm{d}s \\ &\leq \frac{2^{\alpha-1}}{8\Gamma(2-\alpha)} \max_{t_0 \leq t \leq \tilde{t}_{\frac{1}{2}}} \left|\delta_e^2 g(t)\right| \max_{1 \leq l \leq k} \left(e^{t_l} - e^{t_{l-1}}\right)^{3-\alpha}.\end{aligned} \tag{3.163}$$

By the interpolation on $\left[\tilde{t}_{j-\frac{1}{2}}, \tilde{t}_{j+\frac{1}{2}}\right]$ and integration by parts, one can derive

$$\begin{aligned}|I_2| \leq &\frac{\alpha-1}{8\Gamma(2-\alpha)} \max_{\tilde{t}_{\frac{1}{2}} \leq t \leq \tilde{t}_{k-\frac{3}{2}}} \left|\delta_e^2 g(t)\right| \max_{1 \leq l \leq k-1} \left(e^{t_l} - e^{t_{l-1}}\right)^2 \int_{\tilde{t}_{\frac{1}{2}}}^{\tilde{t}_{k-\frac{3}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}} - e^s\right)^{-\alpha} e^s \mathrm{d}s \\ \leq &\frac{2^{\alpha-1}}{8\Gamma(2-\alpha)} \max_{\tilde{t}_{\frac{1}{2}} \leq t \leq \tilde{t}_{k-\frac{3}{2}}} \left|\delta_e^2 g(t)\right| \max_{1 \leq l \leq k} \left(e^{t_l} - e^{t_{l-1}}\right)^{3-\alpha}.\end{aligned} \tag{3.164}$$

By the similar technique, one knows that

$$\begin{aligned}|I_3| &\leq \frac{1}{2\Gamma(2-\alpha)} \max_{\tilde{t}_{k-\frac{3}{2}} \leq t \leq \tilde{t}_{k-\frac{1}{2}}} \left|\delta_e^2 g(t)\right| \max_{k-1 \leq l \leq k} \left(e^{t_l} - e^{t_{l-1}}\right) \int_{\tilde{t}_{k-\frac{3}{2}}}^{\tilde{t}_{k-\frac{1}{2}}} \left(e^{\tilde{t}_{k-\frac{1}{2}}} - e^s\right)^{1-\alpha} e^s \mathrm{d}s \\ &\leq \frac{1}{2\Gamma(3-\alpha)} \max_{\tilde{t}_{k-\frac{3}{2}} \leq t \leq \tilde{t}_{k-\frac{1}{2}}} \left|\delta_e^2 g(t)\right| \max_{k-1 \leq l \leq k} \left(e^{t_l} - e^{t_{l-1}}\right)^{3-\alpha}.\end{aligned} \tag{3.165}$$

If $k = 1$, then one gets

$$\begin{aligned}\left|\Upsilon^{\frac{1}{2}}\right| &\leq \frac{1}{4\Gamma(2-\alpha)} \max_{t_0 \leq t \leq \tilde{t}_{\frac{1}{2}}} \left|\delta_e^2 g(t)\right| \left(e^{t_1} - e^{t_0}\right) \int_{t_0}^{\tilde{t}_{\frac{1}{2}}} \left(e^{\tilde{t}_{\frac{1}{2}}} - e^s\right)^{1-\alpha} e^s \mathrm{d}s \\ &= \frac{2^{\alpha-1}}{8\Gamma(3-\alpha)} \max_{t_0 \leq t \leq \tilde{t}_{\frac{1}{2}}} \left|\delta_e^2 g(t)\right| \left(e^{t_1} - e^{t_0}\right)^{3-\alpha}.\end{aligned} \tag{3.166}$$

It follows from (3.160), (3.161) and (3.163)-(3.166) that the estimates hold. □

4. B-type formulas

If one selects **Partition B** (2.2), then B-type formulas can be derived. Borrowing the technique in [7], the exponential Caputo derivative ${}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)$ on $[a, T]$ can be changed into ${}_{C}\mathrm{D}_{0,t'}^{\alpha} \tilde{f}_a(t')$ on $[0, e^T - e^a]$. So the corresponding discrete formulas for ${}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)$ may be conveniently got.

Several definitions are introduced firstly.

Definition 4.1 ([18]). For $\alpha > 0$ and $f(t) \in L^1(a, b)$, the α-th Riemann-Liouville fractional integral of $f(t)$ is defined by

$${}_{RL}\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \mathrm{d}s. \tag{4.1}$$

Definition 4.2 ([18]). For $n-1<\alpha<n\in\mathbb{Z}^+$ and $f(t)\in AC^n[a,b]$, the α-th Riemann-Liouville fractional derivative of $f(t)$ is defined by

$$ {}_{RL}\mathrm{D}_{a,t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t (t-s)^{n-\alpha-1}f(s)\mathrm{d}s. \tag{4.2}$$

Definition 4.3 ([15]). For $n-1<\alpha<n\in\mathbb{Z}^+$ and $f(t)\in AC^n[a,b]$, the α-th Caputo fractional derivative of $f(t)$ is defined by

$$ {}_{C}\mathrm{D}_{a,t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)\mathrm{d}s. \tag{4.3}$$

The following lemmas reveal the relationship between Caputo/Riemann-Liouville derivatives and exponential Caputo/Riemann-Liouville derivatives.

Lemma 4.1. *For $n-1<\alpha<n\in\mathbb{Z}^+$ and $t>a$, the following nonlinear differential system with the exponential Caputo derivative*

$$\begin{cases} {}_{Ce}\mathrm{D}_{a,t}^{\alpha}x(t)=g\left(t,x\left(t\right)\right), \\ \delta_e^k x(a)=x_{ak},\ k=0,1,\dots,n-1, \end{cases} \tag{4.4}$$

where the continuous function $g\left(t,x\left(t\right)\right)$ satisfies the Lipschitz condition with the second variable in the given domain, is equivalent to the following nonlinear fractional differential system with Caputo derivative

$$\begin{cases} {}_{C}\mathrm{D}_{0,t'}^{\alpha}\tilde{x}_a(t')=g\left(\log(t'+e^a),\tilde{x}_a\left(t'\right)\right),\ t'>0, \\ \tilde{x}_a^{(k)}(0)=x_{ak},\ k=0,1,\dots,n-1. \end{cases} \tag{4.5}$$

Proof. From the transformations $\tilde{s}=e^s-e^a$ and $t'=e^t-e^a$, one finds that $\delta_e^n x(s)=\frac{\mathrm{d}^n}{\mathrm{d}\tilde{s}^n}x\left(\log(\tilde{s}+e^a)\right)$ and $\delta_e^n x(t)=\frac{\mathrm{d}^n}{\mathrm{d}(t')^n}x\left(\log(t'+e^a)\right)$. Then, using Definitions (4.3) and (1.3), one gets

$$\begin{aligned} {}_{Ce}\mathrm{D}_{a,t}^{\alpha}x(t) &=\frac{1}{\Gamma(n-\alpha)}\int_a^t \left(e^t-e^s\right)^{n-\alpha-1}\delta_e^n x(s)e^s\mathrm{d}s \\ &=\frac{1}{\Gamma(n-\alpha)}\int_0^{e^t-e^a} \left(e^t-e^a-\tilde{s}\right)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\tilde{s}^n}x\left(\log(\tilde{s}+e^a)\right)\mathrm{d}\tilde{s} \\ &=\frac{1}{\Gamma(n-\alpha)}\int_0^{t'} \left(t'-\tilde{s}\right)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\tilde{s}^n}\tilde{x}_a(\tilde{s})\mathrm{d}\tilde{s} \\ &={}_{C}\mathrm{D}_{0,t'}^{\alpha}\tilde{x}_a(t'). \end{aligned} \tag{4.6}$$

Under the above transformation, one can also get

$$g\left(t,x(t)\right)=g\big(\log(t'+e^a),x(\log(t'+e^a))\big)=g\big(\log(t'+e^a),\tilde{x}_a(t')\big).$$

For the initial value conditions, one can arrive at

$$x_{ak}=\delta_e^k x(a)=\delta_e^k x(t)|_{t=a}=\frac{\mathrm{d}^k}{\mathrm{d}(t')^k}\tilde{x}_a(t')|_{t'=0}=\tilde{x}_a^{(k)}(0),\ 0\le k\le n-1.$$

The proof is thus finished. □

From the above lemma, one can see that for $t\in[a,T]$, $x(t)=x(\log(t'+e^a))=\tilde{x}_a(t')$, where $t'\in[0,e^T-e^a]$.

Similarly, one can also get the lemma below.

Lemma 4.2. *For $n-1<\alpha<n\in\mathbb{Z}^+$, the following nonlinear differential system with the exponential Riemann-Liouville derivative*

$$\begin{cases} {}_e\mathrm{D}_{a,t}^{\alpha}x(t)=g\left(t,x\left(t\right)\right),\ t>a, \\ {}_e\mathrm{D}_{a,t}^{\alpha+k-n}x(a)=x_{ak},\ k=0,1,\ldots,n-1, \end{cases} \tag{4.7}$$

where the continuous function $g\left(t,x\left(t\right)\right)$ satisfies the Lipschitz condition with the second variable in the given domain, is equivalent to the following nonlinear fractional differential system with Riemann-Liouville derivative

$$\begin{cases} {}_{RL}\mathrm{D}_{0,t'}^{\alpha}\tilde{x}_a(t')=g\left(\log(t'+e^a),\tilde{x}_a\left(t'\right)\right),\ t'>0, \\ {}_{RL}\mathrm{D}_{0,t'}^{\alpha+k-n}\tilde{x}_a(0)=x_{ak},\ k=0,1,\ldots,n-1. \end{cases} \tag{4.8}$$

Proof. For $n-1<\alpha<n\in\mathbb{Z}^+$, using the transformations $\tilde{s}=e^s-e^a$ and $t'=e^t-e^a$ yields

$$\begin{aligned} {}_e\mathrm{D}_{a,t}^{\alpha}x(t)&=\frac{1}{\Gamma(n-\alpha)}\delta_e^n\int_a^t\left(e^t-e^s\right)^{n-\alpha-1}x(s)e^s\mathrm{d}s \\ &=\frac{1}{\Gamma(n-\alpha)}\delta_e^n\int_0^{e^t-e^a}\left(e^t-e^a-\tilde{s}\right)^{n-\alpha-1}x(\log(\tilde{s}+e^a))\mathrm{d}\tilde{s} \\ &=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}(t')^n}\int_0^{t'}\left(t'-\tilde{s}\right)^{n-\alpha-1}\tilde{x}_a(\tilde{s})\mathrm{d}\tilde{s} \\ &={}_{RL}\mathrm{D}_{0,t'}^{\alpha}\tilde{x}_a(t'). \end{aligned} \tag{4.9}$$

Similarly, one gets

$$g(t,x(t))=g(\log(t'+e^a),\tilde{x}_a(t')).$$

For $\alpha>0$, according to Definitions (1.1) and (4.1), the following relation holds

$$\begin{aligned} {}_e\mathrm{D}_{a,t}^{-\alpha}x(t)&=\frac{1}{\Gamma(\alpha)}\int_a^t\left(e^t-e^s\right)^{\alpha-1}x(s)e^s\mathrm{d}s \\ &=\frac{1}{\Gamma(\alpha)}\int_0^{e^t-e^a}\left(e^t-e^a-\tilde{s}\right)^{\alpha-1}x(\log(\tilde{s}+e^a))\mathrm{d}\tilde{s} \\ &=\frac{1}{\Gamma(\alpha)}\int_0^{t'}\left(t'-\tilde{s}\right)^{\alpha-1}\tilde{x}_a(\tilde{s})\mathrm{d}\tilde{s} \\ &={}_{RL}\mathrm{D}_{0,t'}^{-\alpha}\tilde{x}_a(t'). \end{aligned} \tag{4.10}$$

The initial value conditions are obtained as follows

$$\begin{aligned} x_{ak}&={}_e\mathrm{D}_{a,t}^{\alpha+k-n}x(t)\big|_{t=a} \\ &=\delta_e^k\,{}_e\mathrm{D}_{a,t}^{-(n-\alpha)}x(t)\big|_{t=a} \\ &=\frac{\mathrm{d}^k}{\mathrm{d}(t')^k}\,{}_{RL}\mathrm{D}_{0,t'}^{-(n-\alpha)}\tilde{x}_a(t')|_{t'=0} \\ &={}_{RL}\mathrm{D}_{0,t'}^{\alpha+k-n}\tilde{x}_a(t')|_{t'=0} \\ &={}_{RL}\mathrm{D}_{0,t'}^{\alpha+k-n}\tilde{x}_a(0),\quad 0\le k\le n-1. \end{aligned}$$

Hence, the proof is completed. □

From Lemma 4.2, one can see also that for $t \in (a, T]$, $x(t) = x(\log(t' + e^a)) = \tilde{x}_a(t')$, where $t' \in (0, e^T - e^a]$.

Through the above Lemma 4.1, one can find that the numerical approximation formulas under **Partition B** for the exponential Caputo derivatives, i.e., L1-B, L1-2-B, L2-1_σ-B, H2N2-B and L2$_1$-B formulas, are almost the same as the approximation formulas of the Caputo derivatives on the uniform partition [1,3,10,11,19,21]. The discretisations under **Partitions B** and **A** are those at different nodes. Hence **Partition B** discretization can be regarded as a supplement to **Partition A** discretization. Here we omit the mathematical details but directly list them. Recall in **Partition B** (2.2), $t_k = \log(e^{t_0} + k\widetilde{\tau})$, $\widetilde{\tau} = e^{t_k} - e^{t_{k-1}} = \frac{e^T - e^a}{N}$ $(1 \le k \le N)$.

4.1. L1-B formula

The L1-B formula for the fractional derivative (1.3) with $\alpha \in (0,1)$ at $t = t_k$ $(1 \le k \le N)$ is given by

$$ {}_{Ce}\mathscr{D}^{\alpha}_{a,t} f^k = \frac{(\widetilde{\tau})^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} \tilde{c}^{(\alpha)}_{j,k} \left(f^j - f^{j-1} \right), $$

where $\tilde{c}^{(\alpha)}_{j,k} = (k-j+1)^{1-\alpha} - (k-j)^{1-\alpha}$, $1 \le j \le k$.

Theorem 4.1. *For $0 < \alpha < 1$, $\delta_e^2 f(t) \in C[a, T]$ and $t_k = \log(e^{t_0} + k\widetilde{\tau})$, the truncation errors have following estimation,*

$$ \left| {}_{Ce}\mathrm{D}^{\alpha}_{a,t} f(t)\big|_{t=t_k} - {}_{Ce}\,\mathscr{D}^{\alpha}_{a,t} f^k \right| \le C\widetilde{\tau}^{2-\alpha}, \ 1 \le k \le N. $$

Theorem 4.2. *The coefficients $\tilde{c}^{(\alpha)}_{j,k}$ $(1 \le j \le k,\ 1 \le k \le N)$ of L1-B formula satisfy*

$$ 1 = \tilde{c}^{(\alpha)}_{k,k} > \tilde{c}^{(\alpha)}_{k-1,k} > \cdots > \tilde{c}^{(\alpha)}_{1,k} > 0. $$

4.2. L1-2-B formula

The L1-2-B formula for the fractional derivative (1.3) with $\alpha \in (0,1)$ at $t = t_k$ $(1 \le k \le N)$ is given by

$$ {}_{Ce}\mathbb{D}^{\alpha}_{a,t} f^k = \frac{(\widetilde{\tau})^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} \tilde{c}^{(\alpha)}_{j,k} \left(f^j - f^{j-1} \right), $$

where

$$ \begin{aligned} \tilde{c}^{(\alpha)}_{j,k} &= \begin{cases} \tilde{a}^{(\alpha)}_{1,k} + \tilde{b}^{(\alpha)}_{2,k}, \ j = 1, \\ \tilde{a}^{(\alpha)}_{j,k} - \tilde{b}^{(\alpha)}_{j,k} + \tilde{b}^{(\alpha)}_{j+1,k}, \ 2 \le j \le k-1, \\ \tilde{a}^{(\alpha)}_{k,k} - \tilde{b}^{(\alpha)}_{k,k}, \ j = k, \end{cases} \\ \tilde{a}^{(\alpha)}_{j,k} &= (k-j+1)^{1-\alpha} - (k-j)^{1-\alpha}, \\ \tilde{b}^{(\alpha)}_{j,k} &= \frac{1}{2}\left[(k-j)^{1-\alpha} + (k-j+1)^{1-\alpha} \right] \\ &\quad + \frac{1}{2-\alpha}\left[(k-j)^{2-\alpha} - (k-j+1)^{2-\alpha} \right]. \end{aligned} \tag{4.11} $$

Remark 4.1. When $k=1$ or $k=2$, one has

$$\tilde{c}_{1,1}^{(\alpha)}=\tilde{a}_{1,1}^{(\alpha)},\quad k=1;$$
$$\tilde{c}_{1,2}^{(\alpha)}=\tilde{a}_{1,2}^{(\alpha)}+\tilde{b}_{2,2}^{(\alpha)},\quad \tilde{c}_{2,2}^{(\alpha)}=\tilde{a}_{2,2}^{(\alpha)}-\tilde{b}_{2,2}^{(\alpha)},\quad k=2.$$

Theorem 4.3. *Let* $\delta_e^3 f(t)\in C[a,T]$, $0<\alpha<1$ *and* $t_k=\log\left(e^{t_0}+k\widetilde{\tau}\right)$. *Denote*

$$R^k={}_{Ce}\mathrm{D}_{a,t}^{\alpha}f(t)\big|_{t=t_k}-{}_{Ce}\mathbb{D}_{a,t}^{\alpha}f^k,\ 1\le k\le N.$$

Then the truncation errors satisfy

$$\begin{aligned}\left|R^1\right|&\le\frac{\alpha}{2\Gamma(3-\alpha)}\max_{t_0\le t\le t_1}\left|\delta_e^2f(t)\right|\widetilde{\tau}^{2-\alpha},\quad k=1,\\ \left|R^k\right|&\le\frac{\alpha}{8\Gamma(1-\alpha)}\max_{t_0\le t\le t_1}\left|\delta_e^2f(t)\right|\left(e^{t_k}-e^{t_1}\right)^{-1-\alpha}\widetilde{\tau}^3\\ &\quad+\max_{t_0\le t\le t_k}\left|\delta_e^3f(t)\right|\left\{\frac{1}{12\Gamma(1-\alpha)}+\frac{\alpha}{3\Gamma(2-\alpha)}\right\}\widetilde{\tau}^{3-\alpha},\quad k\ge2.\end{aligned}$$

Theorem 4.4. *For* $\alpha\in(0,1)$, *coefficients* $\tilde{c}_{j,k}^{(\alpha)}$ $(1\le j\le k,\ 3\le k\le N)$ *defined by (4.11) have following properties,*

(1) $\tilde{c}_{k,k}^{(\alpha)}>\left|\tilde{c}_{k-1,k}^{(\alpha)}\right|$,

(2) $\tilde{c}_{k,k}^{(\alpha)}>\tilde{c}_{k-2,k}^{(\alpha)}$,

(3) $\tilde{c}_{j,k}^{(\alpha)}>0,\ j\ne k-1$,

(4) $\tilde{c}_{k-2,k}^{(\alpha)}>\tilde{c}_{k-3,k}^{(\alpha)}>\cdots>\tilde{c}_{1,k}^{(\alpha)}$,

(5) $\sum_{j=1}^{k}\tilde{c}_{j,k}^{(\alpha)}=k^{1-\alpha}$.

In particular, when $k=1$ *one has* $\tilde{c}_{1,1}^{(\alpha)}=1$; *and when* $k=2$, *one can arrive at*

(1) $\tilde{c}_{1,2}^{(\alpha)}=2^{1-\alpha}-\left(\frac{1}{2}+\frac{1}{2-\alpha}\right)\in\left(-\frac{1}{2},1\right)$, $\quad\tilde{c}_{2,2}^{(\alpha)}=\frac{1}{2}+\frac{1}{2-\alpha}\in\left(1,\frac{3}{2}\right)$,

(2) $\left|\tilde{c}_{1,2}^{(\alpha)}\right|<\tilde{c}_{2,2}^{(\alpha)}$.

4.3. L2-1_σ-B formula

For $t_k=\log\left(e^{t_0}+k\widetilde{\tau}\right)$, $t_{k+\sigma}=\log\left(e^{t_0}+(k+\sigma)\widetilde{\tau}\right)$ and $\sigma=1-\frac{\alpha}{2}$, the L2-$1_\sigma$-B formula for the fractional derivative (1.3) with $\alpha\in(0,1)$ at $t=t_{k+\sigma}$ $(0\le k\le N-1)$ is given by

$${}_{Ce}\mathfrak{D}_{a,t}^{\alpha}f^{k+\sigma}=\frac{(\widetilde{\tau})^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k+1}\tilde{c}_{j,k}^{(\alpha,\sigma)}\left(f^j-f^{j-1}\right),$$

where

$$\tilde{c}_{j,k}^{(\alpha,\sigma)} = \begin{cases} \tilde{a}_{1,k}^{(\alpha,\sigma)} - \tilde{b}_{1,k}^{(\alpha,\sigma)}, \quad j=1, \\ \tilde{a}_{j,k}^{(\alpha,\sigma)} + \tilde{b}_{j-1,k}^{(\alpha,\sigma)} - \tilde{b}_{j,k}^{(\alpha,\sigma)}, \quad 2 \le j \le k, \\ a_{k+1,k}^{(\alpha,\sigma)} + b_{k,k}^{(\alpha,\sigma)}, \quad j = k+1, \end{cases}$$

$$\tilde{a}_{j,k}^{(\alpha,\sigma)} = \begin{cases} (k+\sigma-j+1)^{1-\alpha} - (k+\sigma-j)^{1-\alpha}, \ 1 \le j \le k, \\ \sigma^{1-\alpha}, \ j = k+1, \end{cases}$$

$$\tilde{b}_{j,k}^{(\alpha,\sigma)} = \frac{1}{2-\alpha}\left[(k+\sigma-j+1)^{2-\alpha} - (k+\sigma-j)^{2-\alpha}\right] - \frac{1}{2}\left[(k+\sigma-j+1)^{1-\alpha} + (k+\sigma-j)^{1-\alpha}\right].$$

Remark 4.2. In particular, when $k = 0$ or $k = 1$, one has

$$\begin{aligned} &\tilde{c}_{1,0}^{(\alpha,\sigma)} = \tilde{a}_{1,0}^{(\alpha,\sigma)}, \quad k=0; \\ &\tilde{c}_{1,1}^{(\alpha,\sigma)} = \tilde{a}_{1,1}^{(\alpha,\sigma)} - \tilde{b}_{1,1}^{(\alpha,\sigma)}, \quad \tilde{c}_{2,1}^{(\alpha,\sigma)} = \tilde{a}_{2,1}^{(\alpha,\sigma)} + \tilde{b}_{1,1}^{(\alpha,\sigma)}, \quad k=1. \end{aligned} \tag{4.12}$$

Theorem 4.5. *Let $\delta_e^3 f(t) \in C[a,T]$, $\alpha \in (0,1)$ and the fixed $\sigma = 1 - \frac{\alpha}{2}$. Denote*

$$R^{k+\sigma} = {}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)\big|_{t=t_{k+\sigma}} - {}_{Ce}\mathfrak{D}_{a,t}^{\alpha} f^{k+\sigma}, \ 0 \le k \le N-1.$$

For $t_k = \log\left(e^{t_0} + k\widetilde{\tau}\right)$ and $t_{k+\sigma} = \log\left(e^{t_0} + (k+\sigma)\widetilde{\tau}\right)$, the truncation errors are bounded as follows,

$$\left|R^{k+\sigma}\right| \le \left\{\frac{\sigma^{-\alpha}}{12\Gamma(1-\alpha)} \max_{t_0 \le t \le t_{k+1}} \left|\delta_e^3 f(t)\right| + \frac{\sigma^{1-\alpha}}{6\Gamma(2-\alpha)} \max_{t_k \le t \le t_{k+1}} \left|\delta_e^3 f(t)\right|\right\} \widetilde{\tau}^{3-\alpha}.$$

Theorem 4.6. *For $\alpha \in (0,1)$ and $\sigma = 1 - \frac{\alpha}{2}$, coefficients $\tilde{c}_{j,k}^{(\alpha,\sigma)}$ $(1 \le j \le k+1$, $0 \le k \le N-1)$ defined by (4.3) satisfy*

(1) $\tilde{c}_{j,k}^{(\alpha,\sigma)} > \frac{1-\alpha}{2}\left(k-j+1+\sigma\right)^{-\alpha}$,

(2) $\tilde{c}_{k+1,k}^{(\alpha,\sigma)} > \tilde{c}_{k,k}^{(\alpha,\sigma)} > \tilde{c}_{k-1,k}^{(\alpha,\sigma)} > \cdots > \tilde{c}_{2,k}^{(\alpha,\sigma)} > \tilde{c}_{1,k}^{(\alpha,\sigma)}$,

(3) $(2\sigma-1)\,\tilde{c}_{k+1,k}^{(\alpha,\sigma)} > \sigma\tilde{c}_{k,k}^{(\alpha,\sigma)}$.

4.4. H2N2-B formula

For $t_k = \log\left(e^{t_0} + k\widetilde{\tau}\right)$ and $\tilde{t}_{k-\frac{1}{2}} = \log\left(e^{t_k} - \frac{1}{2}\widetilde{\tau}\right)$, the H2N2-B formula for the fractional derivative (1.3) with $\alpha \in (1,2)$ at $t = \tilde{t}_{k-\frac{1}{2}}$ $(1 \le k \le N)$ is given by

$${}_{Ce}\mathbb{D}_{a,t}^{\alpha} f^{k-\frac{1}{2}} = \frac{2(\widetilde{\tau})^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{k} \tilde{c}_{j,k}^{(\alpha)} \left(f^j - f^{j-1}\right) - \frac{2(\widetilde{\tau})^{1-\alpha}}{\Gamma(3-\alpha)} \tilde{a}_{0,k}^{(\alpha)}\, \delta_e f(t_0),$$

where

$$\tilde{c}_{j,k}^{(\alpha)} = \begin{cases} \tilde{a}_{0,k}^{(\alpha)} - \tilde{a}_{1,k}^{(\alpha)}, \quad j=1, \\ \tilde{a}_{j-1,k}^{(\alpha)} - \tilde{a}_{j,k}^{(\alpha)}, \quad 2 \le j \le k-1, \\ \tilde{a}_{k-1,k}^{(\alpha)}, \quad j=k, \end{cases} \tag{4.13}$$

$$\tilde{a}_{j,k}^{(\alpha)} = \begin{cases} \left(k-\frac{1}{2}\right)^{2-\alpha} - (k-1)^{2-\alpha}, \quad j=0, \\ \frac{1}{2}\left[(k-j)^{2-\alpha} - (k-j-1)^{2-\alpha}\right], \quad 1 \le j \le k-1. \end{cases}$$

Remark 4.3. In particular, when $k=1$ and $k=2$, one has

$$\tilde{c}_{1,1}^{(\alpha)} = \tilde{a}_{0,1}^{(\alpha)}, \quad \tilde{c}_{1,2}^{(\alpha)} = \tilde{a}_{0,2}^{(\alpha)} - \tilde{a}_{1,2}^{(\alpha)}, \quad \tilde{c}_{2,2}^{(\alpha)} = \tilde{a}_{1,2}^{(\alpha)}.$$

Theorem 4.7. *Let $\delta_e^3 f(t) \in C[a,T]$ and $\alpha \in (1,2)$. Denote*

$$R^{k-\frac{1}{2}} = {}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)\big|_{t=\tilde{t}_{k-\frac{1}{2}}} - {}_{Ce}\mathbb{D}_{a,t}^{\alpha} f^{k-\frac{1}{2}}, \ 1 \le k \le N.$$

For $t_k = \log\left(e^{t_0} + k\widetilde{\tau}\right)$ and $\tilde{t}_{k-\frac{1}{2}} = \log\left(e^{t_k} - \frac{1}{2}\widetilde{\tau}\right)$, the following truncation errors hold

$$\left|R^{k-\frac{1}{2}}\right| \le \frac{1}{2^{2-\alpha}\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_e^3 f(t)\right| \widetilde{\tau}^{3-\alpha}, \ k=1,$$
$$\left|R^{k-\frac{1}{2}}\right| \le \max_{t_0 \le t \le t_k} \left|\delta_e^3 f(t)\right| \left\{\frac{5}{3\Gamma(2-\alpha)} + \frac{1}{\Gamma(3-\alpha)}\right\} \widetilde{\tau}^{3-\alpha}, \ k \ge 2.$$

Theorem 4.8. *For $\alpha \in (1,2)$, coefficients $\tilde{c}_{j,k}^{(\alpha)}$ $(1 \le j \le k,\ 1 \le k \le N)$ given by (4.13) satisfy*

(1) $\tilde{c}_{k,k}^{(\alpha)} > 0, \quad \tilde{c}_{j,k}^{(\alpha)} < 0\,(1 \le j \le k-1)$,

(2) $\tilde{c}_{1,k}^{(\alpha)} > \tilde{c}_{2,k}^{(\alpha)} > \cdots > \tilde{c}_{k-1,k}^{(\alpha)}\,(k \ge 3)$,

(3) $\left|\tilde{c}_{k-1,k}^{(\alpha)}\right| < \tilde{c}_{k,k}^{(\alpha)}$.

4.5. L2$_1$-B formula

For $t_k = \log\left(e^{t_0} + k\widetilde{\tau}\right)$ and $\tilde{t}_{k-\frac{1}{2}} = \log\left(e^{t_k} - \frac{1}{2}\widetilde{\tau}\right)$, the L2$_1$-B formula for the fractional derivative (1.3) with $\alpha \in (1,2)$ at $t = \tilde{t}_{k-\frac{1}{2}}$ $(1 \le k \le N)$ is given by

$$\begin{aligned} &{}_{Ce}\mathcal{D}_{a,t}^{\alpha} f^{k-\frac{1}{2}} \\ &= \frac{1}{\Gamma(3-\alpha)}\left\{\tilde{a}_{k-1,k}^{(\alpha)} \nabla_{\exp,t} f^{k-\frac{1}{2}} - \sum_{j=1}^{k-1}\left(\tilde{a}_{j-1,k}^{(\alpha)} - \tilde{a}_{j,k}^{(\alpha)}\right) \nabla_{\exp,t} f^{j-\frac{1}{2}} - \tilde{a}_{0,k}^{(\alpha)} \delta_e f(t_0)\right\}, \end{aligned}$$

where

$$\tilde{a}_{j,k}^{(\alpha)} = \begin{cases} 2\widetilde{\tau}^{1-\alpha}\left[\left(k-\frac{1}{2}\right)^{2-\alpha} - \left(k-1\right)^{2-\alpha}\right], \quad j=0, \\ \widetilde{\tau}^{1-\alpha}\left[\left(k-j\right)^{2-\alpha} - \left(k-j-1\right)^{2-\alpha}\right], 1 \le j \le k-1. \end{cases} \tag{4.14}$$

Theorem 4.9. *For $\alpha \in (1,2)$, $\delta_e^3 f(t) \in C[a,T]$ and $\tilde{t}_{k-\frac{1}{2}} = \log\left(e^{t_k} - \frac{1}{2}\widetilde{\tau}\right)$ the truncation errors*

$$R^{k-\frac{1}{2}} = {}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)\big|_{t=\tilde{t}_{k-\frac{1}{2}}} - {}_{Ce}\mathcal{D}_{a,t}^{\alpha} f^{k-\frac{1}{2}},\ 1 \le k \le N,$$

are bounded as follows,

$$\left|R^{k-\frac{1}{2}}\right| \le \frac{2^{\alpha-1}}{6\Gamma(3-\alpha)} \max_{t_0\le t\le t_1} \left|\delta_e^3 f(t)\right| \widetilde{\tau}^{3-\alpha}, \quad k=1,$$
$$\left|R^{k-\frac{1}{2}}\right| \le \left\{\frac{1}{4\Gamma(2-\alpha)} + \frac{7}{12\Gamma(3-\alpha)}\right\} \max_{t_0\le t\le t_k} \left|\delta_e^3 f(t)\right| \widetilde{\tau}^{3-\alpha}, \quad k \ge 2.$$

Theorem 4.10. *For $\alpha \in (1,2)$, the coefficients $\tilde{a}_{j,k}^{(\alpha)}$ $(0 \le j \le k-1)$ defined by (4.14) satisfy*

$$a_{k-1,k}^{(\alpha)} > a_{k-2,k}^{(\alpha)} > \cdots > a_{0,k}^{(\alpha)} > 0.$$

Remark 4.4. Through further calculations, one can find that H2N2-B formula and $L2_1$-B formula for exponential Caputo fractional derivatives are the same albeit different factors in the error bounds. But it is not the same if non-uniform partition on $[e^a, e^T]$ is used.

5. Numerical Examples

In the section, numerical examples are displayed to test the obtained numerical formulae for the exponential Caputo fractional derivatives with $\alpha \in (0,1)$ and $\alpha \in (1,2)$.

Example 5.1. Let $f(t) = e^{3t}$, $[a,T] = [1,2]$, $\alpha \in (0,1)$. Then one has

$$\begin{aligned}{}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t) =& \frac{3}{\Gamma(2-\alpha)} e^{2a}(e^t - e^a)^{1-\alpha} + \frac{6}{\Gamma(3-\alpha)} e^a (e^t - e^a)^{2-\alpha} \\ &+ \frac{6}{\Gamma(4-\alpha)} (e^t - e^a)^{3-\alpha}.\end{aligned}$$

The truncation errors of L1-A, L1-2-A and L2-1_σ-A formulas are defined by

$$\text{Error} = \left|{}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)\big|_{t=t_N} - {}_{Ce}\mathscr{D}_{a,t}^{\alpha} f^N\right|,$$

$$\text{Error} = \left|{}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)\big|_{t=t_N} - {}_{Ce}\mathbb{D}_{a,t}^{\alpha} f^N\right|,$$

and

$$\text{Error} = \left|{}_{Ce}\mathrm{D}_{a,t}^{\alpha} f(t)\big|_{t=t_{N-1+\sigma}} - {}_{Ce}\mathfrak{D}_{a,t}^{\alpha} f^{N-1+\sigma}\right|.$$

The L1-A formula, L1-2-A formula and L2-1_σ-A formula with $\alpha = 0.2, 0.5, 0.8$ and $N = 100, 200, 300, 400$ are used to get some numerical results displayed in Table 2. The results show that the convergence order of errors is near to $(2-\alpha)$-th order for L1-A formula and $(3-\alpha)$-th order for the L1-2-A formula and L2-1_σ-A formula, which is in agreement with the theoretical analysis. Similarly, one can also check the error and convergence order of L1-B, L1-2-B and L2-1_σ-B formulas in Table 3, which is consistent with the theoretical result.

Table 2. Errors and convergence rates for L1-A, L1-2-A and L2-1_σ-A formulas

Formula	α / N	0.2 Error	Rate	0.5 Error	Rate	0.8 Error	Rate
L1-A	100	3.6004E-02	–	1.9490E-01	–	7.2844E-01	–
	200	1.0994E-02	1.7115	7.0397E-02	1.4692	3.1946E-01	1.1892
	300	5.4630E-03	1.7248	3.8672E-02	1.4774	1.9692E-01	1.1933
	400	3.3195E-03	1.7317	2.5254E-02	1.4813	1.3963E-01	1.1950
L1-2-A	100	1.9742E-04	–	1.1363E-03	–	4.5702E-03	–
	200	3.0079E-05	2.7243	2.0539E-04	2.4768	1.0049E-03	2.1931
	300	9.9488E-06	2.7342	7.5221E-05	2.4825	4.1331E-04	2.1956
	400	4.5286E-06	2.7397	3.6836E-05	2.4853	2.1990E-04	2.1967
L2-1_σ-A	100	1.8527E-04	–	8.5274E-04	–	2.6793E-03	–
	200	2.8216E-05	2.7150	1.5437E-04	2.4657	5.8856E-04	2.1866
	300	9.3374E-06	2.7274	5.6597E-05	2.4747	2.4203E-04	2.1916
	400	4.2526E-06	2.7339	2.7737E-05	2.4791	1.2876E-04	2.1937

Table 3. Errors and convergence rates for L1-B, L1-2-B and L2-1_σ-B formulas

Formula	α / N	0.2 Error	Rate	0.5 Error	Rate	0.8 Error	Rate
L1-B	100	1.7480E-02	–	1.0146E-01	–	4.2508E-01	–
	200	5.2400E-03	1.7380	3.6243E-02	1.4852	1.8545E-01	1.1967
	300	2.5811E-03	1.7464	1.9817E-02	1.4890	1.1410E-01	1.1979
	400	1.5597E-03	1.7509	1.2905E-02	1.4908	8.0826E-02	1.1985
L1-2-B	100	6.6642E-05	–	3.8328E-04	–	1.7031E-03	–
	200	9.7965E-06	2.7661	6.7896E-05	2.4970	3.7067E-04	2.2000
	300	3.1863E-06	2.7701	2.4661E-05	2.4977	1.5191E-04	2.2000
	400	1.4352E-06	2.7723	1.2020E-05	2.4981	8.0674E-05	2.2000
L2-1_σ-B	100	3.1329E-05	–	1.1333E-04	–	2.9864E-04	–
	200	4.7974E-06	2.7072	2.0380E-05	2.4754	6.5147E-05	2.1967
	300	1.5918E-06	2.7208	7.4510E-06	2.4816	2.6722E-05	2.1979
	400	7.2621E-07	2.7280	3.6458E-06	2.4846	1.4197E-05	2.1984

Example 5.2. Let $f(t) = e^{3t}$, $[a, T] = [1, 2]$, $\alpha \in (1, 2)$. Then one can get

$$ {}_{Ce}\mathrm{D}^{\alpha}_{a,t} f(t) = \frac{6}{\Gamma(3-\alpha)} e^a (e^t - e^a)^{2-\alpha} + \frac{6}{\Gamma(4-\alpha)} (e^t - e^a)^{3-\alpha}. $$

The definitions of truncation errors for the H2N2-A and L2$_1$-A formulas are given as follows, respectively,

$$ \text{Error} = \left| {}_{Ce}\mathrm{D}^{\alpha}_{a,t} f(t)\big|_{t=t_{N-\frac{1}{2}}} - {}_{Ce}\mathbb{D}^{\alpha}_{a,t} f^{N-\frac{1}{2}} \right|, $$

and

$$ \text{Error} = \left| {}_{Ce}\mathrm{D}^{\alpha}_{a,t} f(t)\big|_{t=\tilde{t}_{N-\frac{1}{2}}} - {}_{Ce}\mathcal{D}^{\alpha}_{a,t} f^{N-\frac{1}{2}} \right|. $$

Some numerical results presented in Table 4 are obtained by H2N2-A and L2$_1$-A formulae with $\alpha = 1.2, 1.5, 1.8$ and $N = 200, 300, 400, 500$. Through observing the numerical results, one can know that the convergence order of errors coincides with theoretical order $3-\alpha$. Furthermore, comparing the results of H2N2-A formula and $L2_1$-A formula, one can find that convergence order of $L2_1$-A is closer to the theoretical order $3-\alpha$ than that of H2N2-A under the same parameters. In addition, one also checks the error and convergence order of H2N2-B/$L2_1$-B formula (See Remark 4.4) in Table 5, which is in agreement with the theoretical result.

Table 4. Errors and convergence rates for H2N2-A, $L2_1$-A formulas

	α	1.2		1.5		1.8	
Formula	N	Error	Rate	Error	Rate	Error	Rate
H2N2-A	200	8.8149E-04	–	9.1168E-03	–	4.2921E-02	–
	300	4.7008E-04	1.5506	5.0534E-03	1.4553	2.6514E-02	1.1880
	400	2.9787E-04	1.5859	3.3171E-03	1.4634	1.8822E-02	1.1911
	500	2.0810E-04	1.6072	2.3903E-03	1.4683	1.4423E-02	1.1928
$L2_1$-A	200	1.2465E-03	–	9.3823E-03	–	4.3104E-02	–
	300	6.3262E-04	1.6728	5.1716E-03	1.4690	2.6596E-02	1.1909
	400	3.8939E-04	1.6869	3.3836E-03	1.4747	1.8868E-02	1.1933
	500	2.6670E-04	1.6959	2.4329E-03	1.4781	1.4453E-02	1.1946

Table 5. Errors and convergence rates for H2N2-B/$L2_1$-B formula

α	1.2		1.5		1.8	
N	Error	Rate	Error	Rate	Error	Rate
200	6.5026E-04	–	4.9161E-03	–	2.5146E-02	–
300	3.2315E-04	1.7246	2.6867E-03	1.4901	1.5462E-02	1.1993
400	1.9637E-04	1.7315	1.7492E-03	1.4918	1.0950E-02	1.1995
500	1.3330E-04	1.7360	1.2537E-03	1.4928	8.3786E-03	1.1996

6. Conclusion and Remark

This article is concerned with constructing typical numerical discrete formulas for the exponential Caputo fractional derivatives on the two kinds of partitions. In general, for fractional order $\alpha \in (0,1)$, L1 formula is of $(2-\alpha)$-th order convergence and L1-2, L2-1_σ formulas are of $(3-\alpha)$-th order convergence. For order $\alpha \in (1,2)$ H2N2 and $L2_1$ formulae have $(3-\alpha)$-th order convergence. Finally, some numerical examples are given to verify the correctness of the derived formulas.

One may be curious about naming rules of formulas. In fact, there are two naming rules in this article. On one hand, the symbol L1 is from [14], where L1 is used for the left fractional derivative with order in $(0,1)$. $L2_1$ is for the left fractional derivative with order in $(1,2)$ while the subscript 1 means that the original derivative order is reduced to derivative order in $(0,1)$ by the function replacement. On the other hand, the symbols L1-2, L2-1_σ and H2N2 are named from the types of interpolation polynomials, for example, L1-2 means that the linear interpolation (in the sense of exponential function) is used in the first subinterval and the quadratic interpolations (in the sense of exponential function) in the following subintervals. It seems not to be necessary to unite the naming rules in order to avoid confusion because they have already existed in the references available.

B-type formulas can be obtained via the transform $t' = e^t - e^a$ which changes $t \in [a, T]$ into $t' \in [0, e^T - e^a]$ and which changes $f(t)$, $t \in [a, T]$ into $\tilde{f}_a(t') = f(\log(t' + e^a))$, $t' \in [0, e^T - e^a]$. But nodes $t_k = \log(e^{t_0} + k\widetilde{\tau})$ and stepsize $\widetilde{\tau} = \frac{e^T - e^a}{N}(1 \leq k \leq N)$ (see (2.2)) involve the exponential and/or logarithmic functions which may bring a bit bigger rounding errors in the real calculations. In this sense, A-type formulas may be a bit more applicable. One one hand, numerical methods for Caputo fractional derivatives problems have been extensively and intensively studied, for example see [1,2,4,10] and references cites therein, such a kind of Caputo-type derivative may attract intention due to its possible applications in describing exponential asymptotics in nonlocal problems. On the other hand, the method and technique derived in this paper may be applied to ψ-type fractional derivatives [8,17,20]. Hope numerical studies in this respect will be appeared somewhere in the future.

References

[1] A. A. Alikhanov, *A new difference scheme for the time fractional diffusion equation,* J. Comput. Phys., 2015, 280, 424–438.

[2] E. G. M. Elmahdi and J. Huang, *Efficient numerical solution of two-dimensional time-space fractional nonlinear diffusion-wave equations with initial singularity,* J. Appl. Anal. Comput., 2022, 12(2), 831–849.

[3] G. Gao, Z. Sun and H. Zhang, *A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications,* J. Comput. Phys., 2014, 259, 33–50.

[4] C. Guo and S. Fang, *Crank-Nicolson difference scheme for the derivative nonlinear Schrödinger equation with the riesz space fractional derivative,* J. Appl. Anal. Comput., 2021, 11(3), 1074–1094.

[5] J. Guo and T. Wang, *Fractional Hermite degenerate kernel method for linear Fredholm integral equations involving endpoint weak singularities,* J. Appl. Anal. Comput., 2020, 10(5), 1918–1936.

[6] C. Li and Z. Li, *On blow-up for a time-space fractional partial differential equation with exponential kernel in temporal derivative,* J. Math. Sci., DOI: 10.1007/s10958-022-05894-w.

[7] C. Li and Z. Li, *Stability and logarithmic decay of the solution to Hadamard-type fractional differential equation,* J. Nonlinear Sci., 2021, 31(2), 31.

[8] C. Li and Z. Li, *Stability and ψ-algebraic decay of the solution to ψ-fractional differential system,* Int. J. Nonlinear Sci. Numer. Simul., DOI: 10.1515/ijnsns-2021-0189.

[9] C. Li, Z. Li and C. Yin, *Which kind of fractional partial differential equations has solution with exponential asymptotics?* In: A. Dzielinski, D. Sierociuk, P. Ostalczyk (eds), Proceedings of the International Conference on Fractional Differentiation and its Applications (ICFDA'21), ICFDA 2021, Lecture Notes in Networks and Systems, vol 452, pp. 112–117, Springer, Cham, 2022.

[10] C. Li and F. Zeng, *Numerical Methods for Fractional Calculus,* CRC Press, Boca Raton, 2015.

[11] Y. Lin and C. Xu, *Finite difference/spectral approximations for the time-fractional diffusion equation,* J. Comput. Phys., 2007, 225(2), 1533–1552.

[12] Y. Ma, F. Zhang and C. Li, *The asymptotics of the solutions to the anomalous diffusion equations,* Comput. Math. Appl., 2013, 66(5), 682–692.

[13] D. Matignon, *Stability results for fractional differential equations with applications to control processing,* Proceedings of the IMACS-SMC, 1996, 2(1), 963–968.

[14] K. B. Oldham and J. Spanier, *The Fractional Calculus,* Academic Press, New York, 1974.

[15] I. Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, 1999.

[16] D. Qian, C. Li, R. P. Agarwal and P. J. Y. Wong, *Stability analysis of fractional differential system with Riemann-Liouville derivative,* Math. Comput. Model., 2010, 52(5–6), 862–847.

[17] S. Sadiq and M. ur Rehman, *ψ-shifted operational matrix scheme for fractional partial differential equations,* J. Appl. Anal. Comput., 2022, 12(2), 497–516.

[18] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications,* Gordon and Breach, Switzerland, 1993.

[19] J. Shen, C. Li and Z. Sun, *An H2N2 interpolation for Caputo derivative with order in (1,2) and its application to time fractional wave equation in more than one space dimension,* J. Sci. Comput., 2020, 83(2), 38.

[20] J. V. C. Sousa, L. S. Tavares and C. E. T. Ledesma, *A variational approach for a problem involving a ψ-Hilfer fractional operator,* J. Appl. Anal. Comput., 2021, 11(3), 1610–1630.

[21] Z. Sun and X. Wu, *A fully discrete difference scheme for a diffusion-wave system,* Appl. Numer. Math., 2006, 56, 193–209.