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FRACTIONAL DISSIPATIVE STURM-LIOUVILLE PROBLEMS WITH DISCONTINUITY AND EIGEN-DEPENDENT BOUNDARY CONDITIONS

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Abstract This paper studies two main fractional discontinuous dissipative Sturm-Liouville type boundary-value problems with boundary conditions and transmission conditions. In both types of research, with the aid of the operator theory, we define different classes of new inner products by combining the parameters in the boundary and transmission conditions, then the boundary value problems are transferred to operators in the Hilbert spaces such that the eigenvalues and eigenfunctions of the main problem coincide with those of operators. And we prove those of operators are dissipative. Moreover, the fundamental solutions are constructed and the uniqueness of the solutions of the problem is also given.

Keywords Fractional Sturm-Liouville problem, dissipative operator, eigenparameter dependent boundary conditions, transmission conditions.

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1. Introduction

As is well known Sturm-Liouville(S-L) type boundary-value problems play a vital role in many fields such as science, engineering and mathematics. After Sturm and Liouville created S-L theory, quite a few mathematicians have studied it that made the S-L theory much more complete on this basis. Up to now there are plenty of researches on the regular S-L problems and the singular S-L problems which are very systematic [8, 12, 20, 27, 28]. When solving a large number of mathematical models in practical applications, we often encounter discontinuous differential equations models, and also meet that the eigenparameter appears in the boundary conditions. Specific research on such issues can be seen concretely in [1–3, 9, 17–19, 21]. Fractional calculus is the theory of arbitrary calculus, which is the extension of integral calculus. The historical background of fractional calculus can be found in the literature [13, 23]. There are many different types of definitions of fractional derivatives and integrals, in this paper we use the Riemann-Liouville(R-L) fractional derivatives and integrals and the Caputo fractional derivatives and integrals. Fernandez and Ustaoğlu [10] obtained special functions such as hypergeometric and Appell's functions by the fractional derivatives and integrals, which greatly enriched the

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mathematical theory of fractional calculus. Ozturk [22] applied the discrete fractional calculus operators to solve the solutions of the radial Schrödinger equations for some physical potentials such as pseudonarmonic and Mie-type potentials.

In recent years, in order to solve the problem in the application field, a growing number of researchers pay attention to fractional S-L type boundary value problems [4,5,7,15,16,24–26,29]. In [16], Li and Qi proved that the system of eigenfunctions of fractional S-L problem forms a completely orthogonal system by using the spectral theory of compact self-adjoint operators in Hilbert space. Rivero, Trujillo and Velasco [25] proved the applicability of fractional S-L theory by studying the properties of S-L problems corresponding to each fractional operator. In [29], the eigenfunctions and eigenvalues of two kinds of regular fractional S-L problems were studied and the authors extended the results to two new kinds of singular fractional S-L problems. In [5, 26], Akdogăn and his colleagues used the same approach to transform discontinuous fractional S-L problems into operator problems, and also investigated the corresponding eigenvalues and eigenfunctions. Besides, the authors in [11] discussed a fractional discontinuous S-L type boundary value problem by using operator theory, which provides new ideas for solving the S-L problem.

Generally speaking, in the study of the spectral properties of boundary value problems, if the operators associated with the problems are self-adjoint operators, the eigenvalues of the problems are real. On the other hand, there is an important class of non-self-adjoint operators composed of dissipative operators. In order to solve the problems in semiconductor physics, the authors [14] discussed the one-dimensional Schrödinger type operator and its eigenfunction expansion. Furthermore, Baleanu and Uğurlu [6] presented two kinds of fractional dissipative boundary value problems.

In this paper, we generalize the results of [6] to two classes of discontinuous fractional dissipative boundary value problems. One is the boundary conditions without spectral parameters, the other is the boundary condition with spectral parameters. In the latter study, using operator theories and analytical skills, we define a new inner product depended on the boundary conditions' coefficients and transmission conditions' coefficients by introducing a new Hilbert space. We prove that both kinds of operators are dissipative operators and investigate their eigenvalues and eigenfunctions. At the end of the paper, we prove the uniqueness of the solution of the boundary value problem.

The paper is organized as follows: In Section 2, we give some basic theoretical knowledge about R-L and Caputo fractional calculus for latter use. In Section 3, we investigate the discontinuous fractional S-L problems without eigenparameter dependent boundary conditions. In Section 4, in the same way, we investigate the discontinuous fractional S-L problems with eigenparameter dependent boundary conditions. In Section 5, we prove the uniqueness of the solution of the problem.

2. Preliminaries

In this section, we shall recall some basic definitions and properties of fractional calculus which are necessary for the development of the paper. In addition, we shall introduce some lemmas and give their proofs if needed.

Definition 2.1 (c.f. [13]. Left and right Riemann-Liouville (R-L) fractional integrals)**.** Let $[a, b] \subset \mathbb{R}, Re(\alpha) > 0$ and $y \in L^1[a, b]$. Then the left and right Riemann-

Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order $\alpha \in \mathbb{C}$ are given by

$$I^{\alpha}_{a+}y(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{y(t)dt}{(x-t)^{1-\alpha}},\ x \in (a,b],$$
$$I^{\alpha}_{b-}y(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{y(t)dt}{(t-x)^{1-\alpha}},\ x \in [a,b),$$

respectively.

Definition 2.2 (c.f. [13]. Left and right Riemann-Liouville (R-L) fractional derivatives)**.** Let $[a,b] \subset \mathbb{R}, Re(\alpha) \in (0,1)$ and $y \in L^1[a,b]$. Then the left and right Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}$ of function f are defined as

$$D^{\alpha}_{a+}y(x) := DI^{1-\alpha}_{a+}y(x),\ x \in (a,b],$$
$$D^{\alpha}_{b-}y(x) := -DI^{1-\alpha}_{b-}y(x),\ x \in [a,b),$$

respectively, where $D = \frac{d}{dx}$ is the usual differential operator.

Definition 2.3 (c.f. [13]. Left and right Caputo fractional derivatives)**.** Let $[a,b] \subset \mathbb{R}, Re(\alpha) \in (0,1)$ and $y \in L^1[a,b]$. Then the left and right Caputo fractional derivatives of order $\alpha \in \mathbb{C}$ are

$${}^cD^{\alpha}_{a+}y(x) := I^{1-\alpha}_{a+}Dy(x), x \in (a,b],$$
$${}^cD^{\alpha}_{b-}y(x) := -I^{1-\alpha}_{b-}Dy(x), x \in [a,b),$$

respectively, where $D = \frac{d}{dx}$ is the usual differential operator.

Lemma 2.1 (c.f. [13])**.**

$$D^{\alpha}_{a+}I^{\alpha}_{a+}y(x) = y(x),$$
$$D^{\alpha}_{b-}I^{\alpha}_{b-}y(x) = y(x).$$

and

$$I^{\alpha}_{a+}D^{\alpha}_{a+}y(x) = y(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}I^{1-\alpha}_{a+}y(a),$$
$$I^{\alpha}_{b-}D^{\alpha}_{b-}y(x) = y(x) - \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}I^{1-\alpha}_{a+}y(b),$$

where $\alpha \in (0,1)$.

According to the above equations, we can see that R-L derivative is the left inverse of the R-L integral, but not the right inverse.

Lemma 2.2 (c.f. [13])**.**

$${}^cD^{\alpha}_{a+}I^{\alpha}_{a+}y(x) = y(x),$$
$${}^cD^{\alpha}_{b-}I^{\alpha}_{b-}y(x) = y(x).$$

and

$$I^{\alpha}_{a+}\ {}^cD^{\alpha}_{a+}y(x) = y(x) - y(a),$$
$$I^{\alpha}_{b-}\ {}^cD^{\alpha}_{b-}y(x) = y(x) - y(b),$$

where $\alpha \in (0,1)$.

Now, we state the following lemmas which will be used in the later sections.

Lemma 2.3 (c.f. [5]). *Assume that* $0 < \alpha < 1, y \in AC[a,b]$ *and* $z \in L^p(a,b)(1 \leq p \leq \infty)$. *Then the following equation holds:*

$$\int_a^b y(x) D_{a^+}^{\alpha} z(x) dx = \int_a^b z(x) {}^c D_{b^-}^{\alpha} y(x) dx + y(x) I_{a^+}^{1-\alpha} z(x)|_{x=a}^{x=b}.$$

3. Discontinuous fractional dissipative Sturm-Liouville problems with transmission conditions

In this section, we consider the following fractional S-L differential expression $\pounds_\alpha$ as follows

$$\pounds_\alpha = \begin{cases} {}^c D_{0^-}^{\alpha} p(x) D_{-1^+}^{\alpha} + q(x), & x \in [-1,0), \\ {}^c D_{1^-}^{\alpha} p(x) D_{0^+}^{\alpha} + q(x), & x \in (0,1]. \end{cases}$$

Then we shall consider the following fractional S-L problem on I, where $I = [-1,0) \cup (0,1]$,

$$\pounds_\alpha y + \lambda w_\alpha(x) y = 0, \tag{3.1}$$

with boundary conditions:

$$L_1(y) := \cos\beta I_{-1^+}^{1-\alpha} y(-1) + \sin\beta p_1 D_{-1^+}^{\alpha} y(-1) = 0, \tag{3.2}$$

$$L_2(y) := I_{0^+}^{1-\alpha} y(1) - h p_2 D_{0^+}^{\alpha} y(1) = 0, \tag{3.3}$$

and transmission conditions:

$$L_3(y) := r_1 I_{-1^+}^{1-\alpha} y(0-) - I_{0^+}^{1-\alpha} y(0+) = 0, \tag{3.4}$$

$$L_4(y) := D_{-1^+}^{\alpha} y(0-) - r_2 D_{0^+}^{\alpha} y(0+) = 0, \tag{3.5}$$

where $\frac{1}{2} \leq \alpha \leq 1, \lambda \in \mathbb{C}$ and λ is eigenparameter.

$$\theta = \frac{r_1}{r_2} > 0.$$

$$p(x) = \begin{cases} p_1, & x \in [-1,0), \\ p_2, & x \in (0,1], \end{cases}$$

$q(x)$ is real-valued and continuous in both $[-1,0)$ and $(0,1]$, $w_\alpha(x)$ is the real-valued function such that $w_\alpha(x) > 0$ on I, $r_1, r_2 \neq 0$, r_1, r_2 are real numbers, p_1, p_2 are all positive real numbers and h is a complex number such that $h = h_1 + \mathrm{i} h_2$ with $h_2 > 0$.

We define the following inner product in the Hilbert space $L^2_{w_\alpha}(I)$ by

$$\langle y, z \rangle = \frac{\theta}{p_1} \int_{-1}^{0} y(x) \overline{z(x)} w_\alpha(x) dx + \frac{1}{p_2} \int_0^1 y(x) \overline{z(x)} w_\alpha(x) dx, \tag{3.6}$$

for arbitrary $y, z \in L^2_{w_\alpha}(I)$. In the Hilbert space $L^2_{w_\alpha}(I)$, consider the operator $\mathcal{L}$ which is defined by

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \to L^2_{w_\alpha}(I),$$

where the domain $\mathcal{D}(\mathcal{L})$ consists of those functions y such that ${}^c D_{b^-}^{\alpha}(p(x) D_{a^+}^{\alpha} y)$ is meaningful satisfying (3.2),(3.3) and $\frac{1}{w_\alpha(x)} \pounds_\alpha y \in L^2_{w_\alpha}(I)$ with the rule $\mathcal{L} y = \frac{1}{w_\alpha(x)} \pounds_\alpha y$, $y \in \mathcal{D}(\mathcal{L})$. Then $\mathcal{L} y = \lambda y$ coincides with the problem (3.1)-(3.5).

Theorem 3.1. *The operator $\mathcal{L}$ is dissipative in $L^2_{w_\alpha}(I)$.*

Proof. For any $y \in \mathcal{D}(\mathcal{L})$.

$$< \mathcal{L}y, y >_{{}^2_{w_\alpha}(I)} =< \frac{1}{w_\alpha(x)} \pounds_\alpha y, y > .$$

$$< \frac{1}{w_\alpha(x)} \pounds_\alpha y, y >= \frac{\theta}{p_1} \int_{-1}^{0} ({}^c D^\alpha_{0-} p_1 D^\alpha_{-1+} y(x))\overline{y(x)} dx + \frac{\theta}{p_1} \int_{-1}^{0} q(x) y(x) \overline{y(x)} dx$$
$$+ \frac{1}{p_2} \int_{0}^{1} ({}^c D^\alpha_{1-} p_2 D^\alpha_{0+} y(x))\overline{y(x)} dx + \frac{1}{p_2} \int_{0}^{1} q(x) y(x) \overline{y(x)} dx.$$

By Lemma 2.3, we get

$$< \frac{1}{w_\alpha(x)} \pounds_\alpha y, y >= \theta(\int_{-1}^{0} D^\alpha_{-1+} y(x) D^\alpha_{-1+} \overline{y(x)} dx - D^\alpha_{-1+} y(x) I^{1-\alpha}_{-1+} \overline{y(x)}|^0_{-1})$$
$$+ \frac{\theta}{p_1} \int_{-1}^{0} q(x) y(x) \overline{y(x)} dx$$
$$+ \int_{0}^{1} D^\alpha_{0+} y(x) D^\alpha_{0+} \overline{y(x)} dx - D^\alpha_{0+} y(x) I^{1-\alpha}_{0+} \overline{y(x)}|^1_0$$
$$+ \frac{1}{p_2} \int_{0}^{1} q(x) y(x) \overline{y(x)} dx.$$

Similarly,

$$< y, \mathcal{L}y >_{L^2_{w_\alpha}(I)} =< y, \frac{1}{w_\alpha(x)} \pounds_\alpha y > .$$

$$< y, \frac{1}{w_\alpha(x)} \pounds_\alpha y >= \frac{\theta}{p_1} \int_{-1}^{0} ({}^c D^\alpha_{0-} p_1 D^\alpha_{-1+} \overline{y(x)}) y(x) dx + \frac{\theta}{p_1} \int_{-1}^{0} q(x) \overline{y(x)} y(x) dx$$
$$+ \frac{1}{p_2} \int_{0}^{1} ({}^c D^\alpha_{1-} p_2 D^\alpha_{0+} \overline{y(x)}) y(x) dx + \frac{1}{p_2} \int_{0}^{1} q(x) \overline{y(x)} y(x) dx.$$

By Lemma 2.3, we get

$$< y, \frac{1}{w_\alpha(x)} \pounds_\alpha y >= \theta(\int_{-1}^{0} D^\alpha_{-1+} \overline{y(x)} D^\alpha_{-1+} y(x) dx - D^\alpha_{-1+} \overline{y(x)} I^{1-\alpha}_{-1+} y(x)|^0_{-1})$$
$$+ \frac{\theta}{p_1} \int_{-1}^{0} q(x) \overline{y(x)} y(x) dx$$
$$+ \int_{0}^{1} D^\alpha_{0+} \overline{y(x)} D^\alpha_{0+} y(x) dx - D^\alpha_{0+} \overline{y(x)} I^{1-\alpha}_{0+} y(x)|^1_0$$
$$+ \frac{1}{p_2} \int_{0}^{1} q(x) \overline{y(x)} y(x) dx,$$

Hence, we have

$$< \mathcal{L}y, y >_{L^2_{w_\alpha}(I)} - < y, \mathcal{L}y >_{L^2_{w_\alpha}(I)}$$
$$= \theta(D^\alpha_{-1+} \overline{y(x)} I^{1-\alpha}_{-1+} y(x)|^0_{-1} - D^\alpha_{-1+} y(x) I^{1-\alpha}_{-1+} \overline{y(x)}|^0_{-1})$$
$$+ D^\alpha_{0+} \overline{y(x)} I^{1-\alpha}_{0+} y(x)|^1_0 - D^\alpha_{0+} y(x) I^{1-\alpha}_{0+} \overline{y(x)}|^1_0.$$

By boundary conditions (3.2) and (3.3)and transmission conditions (3.4) and (3.5), we get

$$< \mathcal{L}y, y >_{L^2_{w_\alpha}(I)} - < y, \mathcal{L}y >_{L^2_{w_\alpha}(I)} = 2\mathrm{i}p_2 Imh|D^\alpha_{0^+}y(1)|^2.$$

Hence we see that $\mathcal{L}$ is dissipative in $L^2_{w_\alpha}(I)$. □

Corollary 3.1. *Let λ be an eigenvalue of the operator $\mathcal{L}$. Then $Im\lambda \geq 0$.*

4. Discontinuous fractional dissipative Sturm-Liouville problems with eigen-dependent boundary and transmission conditions

In this section, we also consider the following fractional S-L differential expression $\pounds_\alpha$ as follows

$$\pounds_\alpha = \begin{cases} {}^cD^\alpha_{0^-}p(x)D^\alpha_{-1^+} + q(x), & x \in [-1, 0), \\ {}^cD^\alpha_{1^-}p(x)D^\alpha_{0^+} + q(x), & x \in (0, 1]. \end{cases}$$

Then we shall consider the following fractional S-L problem on I, where $I = [-1, 0) \cup (0, 1]$,

$$\pounds_\alpha y + \lambda w_\alpha(x)y = 0 \tag{4.1}$$

with boundary conditions:

$$\begin{aligned} L_1(y) :=& \gamma_1 I^{1-\alpha}_{-1^+}y(-1) - \gamma_2 p_1 D^\alpha_{-1^+}y(-1) \\ & - \lambda(\gamma_1' I^{1-\alpha}_{-1^+}y(-1) - \gamma_2' p_1 D^\alpha_{-1^+}y(-1)) = 0, \end{aligned} \tag{4.2}$$

$$L_2(y) := I^{1-\alpha}_{0^+}y(1) - hp_2 D^\alpha_{0^+}y(1) = 0, \tag{4.3}$$

and transmission conditions:

$$L_3(y) := r_1 I^{1-\alpha}_{-1^+}y(0-) - I^{1-\alpha}_{0^+}y(0+) = 0, \tag{4.4}$$

$$L_4(y) := D^\alpha_{-1^+}y(0-) - r_2 D^\alpha_{0^+}y(0+) = 0, \tag{4.5}$$

where $\frac{1}{2} \leq \alpha \leq 1, \lambda \in \mathbb{C}$ and λ is eigenparameter.

$$\gamma = \gamma_1'\gamma_2 - \gamma_1\gamma_2' > 0; \theta = \frac{r_1}{r_2} > 0.$$

$$p(x) = \begin{cases} p_1, & x \in [-1, 0), \\ p_2, & x \in (0, 1], \end{cases}$$

$q(x)$ is real-valued and continuous in both $[-1, 0)$ and $(0, 1]$, $w_\alpha(x)$ is the real-valued function such that $w_\alpha(x) > 0$ on I, $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$ are real number, $r_1, r_2 \neq 0$, r_1, r_2 are real numbers, p_1, p_2 are all positive real numbers and h is a complex number such that $h = h_1 + \mathrm{i}h_2$ with $h_2 > 0$.

We define the following inner product in the Hilbert space $L^2_{w_\alpha}(I)$ by

$$\langle y, z\rangle_1 = \frac{\theta}{p_1}\int_{-1}^0 y(x)\overline{z(x)}w_\alpha(x)dx + \frac{1}{p_2}\int_0^1 y(x)\overline{z(x)}w_\alpha(x)dx, \tag{4.6}$$

for arbitrary $y, z \in L^2_{w_\alpha}(I)$. Obviously, $H_1 = (L^2_{w_\alpha}(I), \langle\cdot,\cdot\rangle_1)$ is a Hilbert space. Then we define the following inner product in the Hilbert space $H := H_1 \oplus \mathbb{C}$ as

$$\langle Y, Z\rangle_H = \langle y, z\rangle_1 + \frac{\theta}{\gamma p_1} y_1 \overline{z}_1. \tag{4.7}$$

for $Y = (y(x), y_1), Z = (z(x), z_1) \in H,\ y(x), z(x) \in H_1,\ y_1, z_1 \in \mathbb{C}$.

In the Hilbert space H, consider the operator $\mathcal{L}_\lambda$ which is defined by

$$\mathcal{L}_\lambda : \mathcal{D}(\mathcal{L}_\lambda) \to H,$$

where the domain $\mathcal{D}(\mathcal{L}_\lambda)$ is defined as follows

$$\begin{aligned}\mathcal{D}(\mathcal{L}_\lambda) = \{&(y(x), y_1) \in H | y(x), D^\alpha_{-1^+} y(x), {}^c D^\alpha_{1^-} y(x) \in AC([-1, 0) \cup (0, 1];\\ &y(0\pm), D^\alpha_{-1^+} y(0\pm), I^{1-\alpha}_{-1^+} y(0\pm)\text{all have finite limits;}\\ &L_i y = 0, i = 2, 3, 4;\\ &y_1 = \gamma'_1 I^{1-\alpha}_{-1^+} y(-1) - \gamma'_2 p_1 D^\alpha_{-1^+} y(-1)\}\\ \mathcal{L}_\lambda Y = \mathcal{L}_\lambda&(y(x), y_1) = (\frac{1}{w_\alpha x}\pounds_\alpha y, \gamma_1 I^{1-\alpha}_{-1^+} y(-1) - \gamma_2 p_1 D^\alpha_{-1^+} y(-1)),\end{aligned}$$

for

$$Y = (y(x), \gamma'_1 I^{1-\alpha}_{-1^+} y(-1) - \gamma'_2 p_1 D^\alpha_{-1^+} y(-1)).$$

For simplicity, let

$$\begin{aligned}N(y) &= \gamma_1 I^{1-\alpha}_{-1^+} y(-1) - \gamma_2 p_1 D^\alpha_{-1^+} y(-1),\\ N'(y) &= \gamma'_1 I^{1-\alpha}_{-1^+} y(-1) - \gamma'_2 p_1 D^\alpha_{-1^+} y(-1).\end{aligned}$$

Now, we can rewrite the considered problem (4.1)-(4.5) in operator form

$$\mathcal{L}_\lambda Y = \lambda Y. \tag{4.8}$$

Obviously, the following lemma holds.

Lemma 4.1. *(i) The eigenvalues of boundary value problem (4.1)-(4.5) consist with those of operator $\mathcal{L}_\lambda$.*

(ii) The eigenfunctions of boundary value problem (4.1)-(4.5) are the first component of corresponding eigen element of operator $\mathcal{L}_\lambda$.

Theorem 4.1. *The operator $\mathcal{L}_\lambda$ is dissipative in H.*

Proof. For any $Y \in \mathcal{D}(\mathcal{L}_\lambda), Y = (y(x), N'(y)), \mathcal{L}_\lambda Y = (\frac{1}{w_\alpha(x)}\pounds_\alpha y, N(y))$.

$$< \mathcal{L}_\lambda Y, Y >_H = < \frac{1}{w_\alpha(x)}\pounds_\alpha y, y >_1 + \frac{\theta}{\gamma p_1} N(y)\overline{N'(y)}.$$

$$\begin{aligned}< \frac{1}{w_\alpha(x)}\pounds_\alpha y, y >_1 = &\frac{\theta}{p_1}\int_{-1}^0 ({}^c D^\alpha_{0^-} p_1 D^\alpha_{-1^+} y(x))\overline{y(x)}dx + \frac{\theta}{p_1}\int_{-1}^0 q(x)y(x)\overline{y(x)}dx\\ &+ \frac{1}{p_2}\int_0^1 ({}^c D^\alpha_{1^-} p_2 D^\alpha_{0^+} y(x))\overline{y(x)}dx + \frac{1}{p_2}\int_0^1 q(x)y(x)\overline{y(x)}dx.\end{aligned}$$

By Lemma 2.3, we get

$$\begin{aligned}< \frac{1}{w_\alpha(x)}\pounds_\alpha y, y >_1 =& \theta(\int_{-1}^{0} D^\alpha_{-1^+} y(x) D^\alpha_{-1^+}\overline{y(x)}dx - D^\alpha_{-1^+} y(x) I^{1-\alpha}_{-1^+}\overline{y(x)}|^0_{-1})\\ &+ \frac{\theta}{p_1}\int_{-1}^{0} q(x)y(x)\overline{y(x)}dx\\ &+ \int_0^1 D^\alpha_{0^+} y(x) D^\alpha_{0^+}\overline{y(x)}dx - D^\alpha_{0^+} y(x) I^{1-\alpha}_{0^+}\overline{y(x)}|^1_0\\ &+ \frac{1}{p_2}\int_0^1 q(x)y(x)\overline{y(x)}dx,\end{aligned}$$

Similarly,

$$< Y, \mathcal{L}_\lambda Y >_H =< y, \frac{1}{w_\alpha(x)}\pounds_\alpha y >_1 + \frac{\theta}{\gamma p_1} N'(y)\overline{N(y)}.$$

$$\begin{aligned}< y, \frac{1}{w_\alpha(x)}\pounds_\alpha y >_1 =& \frac{\theta}{p_1}\int_{-1}^{0} ({}^cD^\alpha_{0^-} p_1 D^\alpha_{-1^+}\overline{y(x)})y(x)dx + \frac{\theta}{p_1}\int_{-1}^{0} q(x)\overline{y(x)}y(x)dx\\ &+ \frac{1}{p_2}\int_0^1 ({}^cD^\alpha_{1^-} p_2 D^\alpha_{0^+}\overline{y(x)})y(x)dx + \frac{1}{p_2}\int_0^1 q(x)\overline{y(x)}y(x)dx.\end{aligned}$$

By Lemma 2.3, we get

$$\begin{aligned}< y, \frac{1}{w_\alpha(x)}\pounds_\alpha y >_1 =& \theta(\int_{-1}^{0} D^\alpha_{-1^+}\overline{y(x)} D^\alpha_{-1^+} y(x)dx - D^\alpha_{-1^+}\overline{y(x)} I^{1-\alpha}_{-1^+} y(x)|^0_{-1})\\ &+ \frac{\theta}{p_1}\int_{-1}^{0} q(x)\overline{y(x)}y(x)dx\\ &+ \int_0^1 D^\alpha_{0^+}\overline{y(x)} D^\alpha_{0^+} y(x)dx - D^\alpha_{0^+}\overline{y(x)} I^{1-\alpha}_{0^+} y(x)|^1_0\\ &+ \frac{1}{p_2}\int_0^1 q(x)\overline{y(x)}y(x)dx.\end{aligned}$$

Hence, we have

$$\begin{aligned}< \mathcal{L}_\lambda Y, Y >_H - < Y, \mathcal{L}_\lambda Y >_H =& \theta(D^\alpha_{-1^+}\overline{y(x)} I^{1-\alpha}_{-1^+} y(x)|^0_{-1} - D^\alpha_{-1^+} y(x) I^{1-\alpha}_{-1^+}\overline{y(x)}|^0_{-1})\\ &+ D^\alpha_{0^+}\overline{y(x)} I^{1-\alpha}_{0^+} y(x)|^1_0 - D^\alpha_{0^+} y(x) I^{1-\alpha}_{0^+}\overline{y(x)}|^1_0\\ &+ \frac{\theta}{\gamma p_1}(N(y)\overline{N'(y)} - N'(y)\overline{N(y)}).\end{aligned}$$

By boundary conditions (4.2) and (4.3) and transmission conditions (4.4) and (4.5), we get

$$< \mathcal{L}_\lambda Y, Y >_H - < Y, \mathcal{L}_\lambda Y >_H = 2\mathrm{i}p_2 Imh|D^\alpha_{0^+} y(1)|^2.$$

Hence we see that $\mathcal{L}_\lambda$ is dissipative in H. □

Corollary 4.1. *Let λ be an eigenvalue of the operator $\mathcal{L}_\lambda$. Then $Im\lambda \geq 0$.*

5. Uniqueness of solutions for discontinuous fractional dissipative Sturm-Liouville problems

Lemma 5.1. *The equivalent integral form of equation (4.1) with fractional conditions (4.4)-(4.5) is given as*

$$y(x) = y_0(x) + \frac{1}{p_2} I_{0+}^{2\alpha}[N_y(x) + (-1)^{1-\alpha}(q(x) + \lambda w_\alpha(x))y(x)], \tag{5.1}$$

where $y_0(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}(r_1 I_{-1^+}^{1-\alpha} y(0-)) + I_{0+}^{\alpha}(\frac{1}{r_2} D_{-1^+}^{\alpha} y(0-))$.

Proof. The proof of this lemma can be referred to [11]. □

Next, we use the conclusion of Lemma 5.1 to construct $y_n(x,\lambda)$, and then discuss the successive approximations.

$$y_n(x,\lambda) = y_0(x,\lambda) + \frac{\int_0^x (x-t)^{2\alpha-1}[N_{y_{n-1}}(t) + (-1)^{1-\alpha}(\lambda w_\alpha(t) + q(t))y_{n-1}(t)]dt}{p_2\Gamma(2\alpha)}. \tag{5.2}$$

If $\alpha = 1$, the above problem becomes classical S-L problem.

Lemma 5.2. *Let*

$$Q := \max_{x\in(0,1]} |q(x)|,\ \ P_R := \max_{|\lambda|\le R} P(\lambda),\ \ P(\lambda) := \max_{x\in(0,1]} |y_0(x,\lambda)|.$$

Then, for any m*, the following estimate*

$$\|y_m(x,\lambda) - y_{m-1}(x,\lambda)\| \le P_R\{\frac{|\lambda w_\alpha(x)| + 2k_\alpha + Q}{p_2\Gamma(2\alpha+1)}\}^m \tag{5.3}$$

holds, where $k_\alpha := \frac{1}{(2-\alpha)\Gamma(1-\alpha)}$.

Proof. The proof of this lemma can be referred to [26]. □

For the following initial value problem

$$\begin{cases} {}^cD_{0^-}^{\alpha} p_1 D_{-1^+}^{\alpha} y(x) + (q(x) + \lambda w_\alpha(x))y(x) = 0,\ x \in [-1,0), \\ I_{-1^+}^{1-\alpha} y(-1) = (\gamma_2 - \lambda\gamma_2')p_1, \\ D_{-1^+}^{\alpha} y(-1) = \gamma_1 - \lambda\gamma_1'. \end{cases} \tag{5.4}$$

If we use a similar way in Lemma 5.1, we can get a corresponding integral equation of the problem (5.4) as follows:

$$y(x) = y_0(x) + \frac{1}{p_1} I_{-1^+}^{2\alpha}[N_y(x) + (-1)^{1-\alpha}(q(x) + \lambda w_\alpha(x))y(x)], \tag{5.5}$$

where $y_0(x) = \frac{(x+1)^{\alpha-1}}{\Gamma(\alpha)} p_1(\gamma_2 - \lambda\gamma_2') + \frac{(x+1)^{\alpha}}{\Gamma(\alpha+1)}(\gamma_1 - \lambda\gamma_1')$.

Lemma 5.3. *The initial value problem (5.4) has a unique solution on* $[-1,0)$ *provided that* $\frac{1}{p_1\Gamma(2\alpha+1)}(|\lambda w_\alpha(x)| + 2k_\alpha + Q) < 1$.

Proof. The proof of this lemma can be referred to [11]. □

Next, for the following initial value problem

$$\begin{cases} {}^cD^{\alpha}_{1-}p_2D^{\alpha}_{0+}y(x)+(q(x)+\lambda w_{\alpha}(x))y(x)=0,\ x\in(0,1],\\ I^{1-\alpha}_{0^+}y(0+)=r_1I^{1-\alpha}_{-1^+}y(0-),\\ D^{\alpha}_{0^+}y(0+)=\dfrac{1}{r_2}D^{\alpha}_{-1^+}y(0-). \end{cases} \tag{5.6}$$

We establish the sequence $y_n(x,\lambda)$ for $x\in(0,1]$ (5.5) and $n=1,2,...$ Obviously, each of the functions $y_n(x,\lambda)$ is an entire function of λ for each $x\in(0,1]$. Now let us consider the series

$$y^*(x,\lambda)=\lim_{n\to\infty}(y_n(x,\lambda)-y_0(x,\lambda)=\sum_{j=1}^{\infty}(y_j(x,\lambda)-y_{j-1}(x,\lambda))). \tag{5.7}$$

According to the (5.3), for $x\in(0,1]$, the absolute value of its terms is less than the corresponding terms of the convergent numeric series $P_R\sum_{j=1}^{\infty}\{\frac{|\lambda w_{\alpha}(x)|+2k_{\alpha}+Q}{p_2\Gamma(2\alpha+1)}\}^j$ Hence, series (5.7) converges uniformly. Obviously, each term $(y_j(x,\lambda)-y_{j-1}(x,\lambda))$ of series (5.7) is continuous on $x\in(0,1]$. Therefore, the sum of series (5.7) is continuous on $x\in(0,1]$ and

$$\phi_2(x,\lambda)=\lim_{n\to\infty}y_n(x,\lambda)=y_0(x,\lambda)+y^*(x,\lambda)$$

is continuous on $x\in(0,1]$. The uniform convergency of the sequence $y_n(x,\lambda)$ allows us to take $n\to\infty$ in the relation (5.2). We can get the initial value problem (5.6) has a unique solution $\phi_2(x,\lambda)$ on (0,1] provided that $\frac{1}{p_2\Gamma(2\alpha+1)}(|\lambda w_{\alpha}(x)|+2k_{\alpha}+Q)<1$, in what follows, we will always assume that this condition holds.

For any $\lambda\in\mathbb{C}$, let $\phi_{1,\lambda}(x):=\phi_1(x,\lambda)$ be the solution of equation (4.1) on interval $[-1,0)$, and satisfies initial conditions:

$$\begin{cases} I^{1-\alpha}_{-1^+}y(-1)=(\gamma_2-\lambda\gamma_2')p_1,\\ D^{\alpha}_{-1^+}y(-1)=(\gamma_1-\lambda\gamma_1'). \end{cases} \tag{5.8}$$

$\phi_1(x,\lambda)$ is an entire function of λ for each $x\in[-1,0)$. By considering Lemma 5.3, the equation (4.1) with initial conditions (5.8) has a unique solution $\phi_1(x,\lambda)$. Let $\phi_{2,\lambda}(x):=\phi_2(x,\lambda)$ be the solution of equation (4.1) on interval $(0,1]$, and satisfy

$$\begin{cases} I^{1-\alpha}_{0^+}\phi_2(0+)=r_1I^{1-\alpha}_{-1^+}\phi_1(0-,\lambda),\\ D^{\alpha}_{0^+}\phi_2(0+)=\dfrac{1}{r_2}D^{\alpha}_{-1^+}\phi_1(0-,\lambda). \end{cases} \tag{5.9}$$

$\phi_2(x,\lambda)$ also is an entire function of λ for each $x\in(0,1]$.

Hence,we have the following theorem.

Theorem 5.1. *For any $\lambda\in C$, the differential equation $\pounds_{\alpha}y+\lambda w_{\alpha}(x)y=0$ has a unique solution*

$$\phi(x,\lambda)=\begin{cases}\phi_1(x,\lambda), & x\in[-1,0),\\ \phi_2(x,\lambda), & x\in(0,1],\end{cases}$$

satisfying the boundary conditions (4.2), and both transmission conditions (4.4) and (4.5) for each $x\in[-1,0)\cup(0,1]$.

Similarly, we see that the problem (4.1) with initial conditions:

$$\begin{cases} I_{0^+}^{1-\alpha} y(1) = hp_2, \\ D_{0^+}^{\alpha} y(1) = 1. \end{cases} \tag{5.10}$$

has a unique solution $\chi_2(x, \lambda)$, which is an entire function of the parameter λ for each fixed $x \in (0, 1]$. As the same as above discussion, we can define the solution $\chi_1(x, \lambda)$ of equation (4.1) by initial conditions:

$$\begin{cases} I_{-1^+}^{1-\alpha} y(0-) = \frac{1}{r_1} I_{0^+}^{1-\alpha} \chi_2(0+, \lambda), \\ D_{-1^+}^{\alpha} y(0-) = r_2 D_{0^+}^{\alpha} \chi_2(0+, \lambda). \end{cases} \tag{5.11}$$

$\chi_1(x, \lambda)$ is an entire function of the parameter λ for each fixed $x \in [-1, 0)$.

Theorem 5.2. *For any $\lambda \in C$, the differential equation $\pounds_\alpha y + \lambda w_\alpha(x) y = 0$ has a unique solution*

$$\chi(x, \lambda) = \begin{cases} \chi_1(x, \lambda), & x \in [-1, 0), \\ \chi_2(x, \lambda), & x \in (0, 1], \end{cases}$$

satisfying the boundary conditions (4.3), and both transmission conditions (4.4) and (4.5) for each $x \in [-1, 0) \cup (0, 1]$.

6. Conclusions

In this paper, we use the R-L fractional and Caputo fractional operator to research two classes of discontinuous dissipative S-L type boundary-value problem. We study the eigenvalues and eigenfunctions of fractional S-L problem and prove that fractional operator is dissipative, as well as the imaginary part of eigenvalues corresponding to different boundary value problems are greater than or equal to zero. In addition, we prove the uniqueness of the solutions of the problems. This conclusion provides a basis for getting the asymptotic formula of eigenvalues of fractional S-L problems in the forthcoming work.

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References

[1] Z. Akdoğan, M. Demirci and O. Sh. Mukhtarov, *Discontinuous Sturm-Liouville problem with eigen parameter-dependent boundary and transmission conditions*, Acta Applicandae Mathematica, 2005, 86, 329–344.

[2] Z. Akdoğan, M. Demirci and O. Sh. Mukhtarov, *Sturm-Liouville problem with eigen dependent boundary and transmissions conditions*, Acta Mathematica Scientia, 2005, 25B(4), 731–740.

[3] K. Aydemir1 and O. Sh. Mukhtarov, *Variational principles for spectral analysis of one Sturm-Liouville problem with transmission conditions*, Advances in Difference Equations, 2016, 2016, 1–14.

[4] Q. M. Al-Mdallal, *On the numerical solution of fractional Sturm-Liouville problem*, International Journal Computer Mathematics, 2010, 87(12), 2837–2845.

[5] Z. Akdoğan, A. Yakar and M. Demirci, *Discontinuous fractional Sturm-Liouville problems with transmission conditions*, Applied Mathematics and Computation, 2019, 350, 1–10.

[6] D. Baleanu and E. Uğurlu, *Regular fractional dissipative boundary value problems*, Advances in Difference Equation, 2016, 2016(1), 1–6.

[7] M. Dehghan and A. Mingarelli, *Fractional Sturm-Liouville eigenvalue problems, I*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales-Serie A. Matemáticas, 2020, 114(2), 1–15.

[8] W. Everitt and M. Marletta, *Inequalities and eigenvalues of Sturm-Liouville problems near a singular boundary*, Journal of inequalites and applications, 2001, 6, 405–413.

[9] C. T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1977, 77A, 293–308.

[10] A. Fernandez and C. Ustaoğlu, *On some analytic properties of tempered fractional calculus*, Journal of Computational and Applied Mathematics, 2020, 366, 1–14.

[11] J. Fu, X. Hao, K. Li and S. Yao, *Discontinuous fractional Sturm-Liouville problems with eigen-dependent boundary conditions*, Journal of Applied Analysis and Computation, 2021, 11(4), 2037–2051.

[12] G. J. Kynch, *Book Review: Linear Differential Operators–Part 1: M. A. Naimark*, The International Journal of Electrical Engineering Education, 1970, 8, 165.

[13] A. A. Kilbass, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.

[14] H. C. Kaiser, H. Neidhardt and J. Rehberg, *On 1-dimensional dissipative Schrödinger-type operators their dilations and eigenfunction expansions*, Mathematische Nachrichten, 2003, 252(1), 51–69.

[15] J. Li and J. Qi, *Note on a nonlocal Sturm-Liouville problem with both right and left fractional derivatives*, Applied Mathematics Letters, 2019, 97, 14–19.

[16] J. Li and J. Qi, *Eigenvalue problems for fractional differential equations with right and left fractional derivatives*, Applied Mathematics and Computation, 2015, 256, 1–10.

[17] O. Sh. Mukhtarov, *Discontinuous boundary-value problem with spectral parameter in boundary conditions*, Turkish Journal of Mathematics, 1994, 18, 183–192.

[18] O. Sh. Mukhtarov and K. Aydemir, *Minimization principle and generalized Fourier series for discontinuous Sturm-Liouville systems in direct sum spaces*, Journal of Applied Analysis and Computation, 2018, 8(5), 1511–1523.

[19] O. Sh. Mukhtarov and K. Aydemir, *Eigenfunction expansion for Sturm-Liouville problems with transmission conditions at one interior point*, Acta Mathematica Scientia, 2015, 35(3), 639–649.

[20] M. A. Naimark, *Linear Differential Operators Part II: Linear Differential Operators in Hilbert Space*, Frederick Ungar Publishing Company, 1968.

[21] H. Olğar and O. Sh. Mukhtarov, *Weak eigenfunctions of two-interval Sturm-Liouville problems together with interaction conditions*, Journal of Mathematical Physics, 2017, 58(4), 042201.

[22] O. Ozturk, *A study of* ∇*-discrete fractional calculus operator on the radial Schrödinger equation for some physical potentials*, Questiones Mathematicae, 2017, 40(7), 879–889.

[23] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[24] J. Qi and S. Chen, *Eigenvalue problems of the model from nonlocal continuum mechanics*, Journal of Mathematical Physics, 2011, 52(7), 537–546.

[25] M. Rivero, J. Trujillo and M. Velasco, *A fractional approach to the Sturm-Liouville problem*, Open Physics, 2013, 11(10), 1246–1254.

[26] A. Yakar and Z. Akdoğan, *On the fundamental solutions of a discontinuous fractional boundary value problem*, Advances in Difference Equations, 2017, 2017(1), 1–15.

[27] A. Zettl, *Sturm-Liouville Theory*, Mathematical Surveys and Monographs, American Mathematical Society, 2005.

[28] M. Zhang, J. Sun and A. Zettl, *The specturm of singular Sturm-Liouville problems with eigenparameter dependent boundary conditions and its approximation*, Results in Mathematics, 2013, 63(3), 1311–1330.

[29] M. Zayernouri and G. E. Karniadakis, *Fractional Sturm-Liouville eigenproblems: theory and numerical approximation*, Journal of Computational Physics, 2013, 252, 495–517.