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# SOLUTION AND CONSTRUCTION OF INVERSE PROBLEM FOR STURM-LIOUVILLE EQUATIONS WITH FINITELY MANY POINT $\delta$ -INTERACTIONS

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**Abstract** The goal of this paper is to reconstruct a one kind of Sturm-Liouville problem from its spectral properties. We considered the inverse spectral problems for a Sturm-Liouville equations with finitely many delta-interactions. We obtain the effective method and its steps to find the solution of the inverse problems and then we will give an example to illustrate this method.

**Keywords** Inverse problems, Sturm-Liouville equation, Point $\delta$-interactions.

**MSC(2010)** 34A55, 34B24, 47E05.

## 1. Introduction

In this paper, we shall study inverse problems for the following boundary value problem

$$L\left(q\left(x\right),h,H,a_s,\alpha_s,s=\overline{1,m}:=1,2,...m\right)$$

generated by the differential equation

$$ly:=-y''+q\left(x\right)y=\lambda y,\qquad x\in\bigcup_{s=0}^{m}\left(a_s,a_{s+1}\right)\ \ \left(0=a_0,a_{m+1}=\pi\right) \tag{1.1}$$

with the boundary conditions

$$U\left(y\right):=y'\left(0\right)-hy\left(0\right)=0,\quad V\left(y\right):=y'\left(\pi\right)+Hy\left(\pi\right)=0, \tag{1.2}$$

and the transmission conditions at the points $x=a_s,\ s=\overline{1,m}$,

$$I\left(y\right):=\begin{cases} y\left(a_s+0\right)=y\left(a_s-0\right)\equiv y\left(a_s\right) \\ y'\left(a_s+0\right)-y'\left(a_s-0\right)=\alpha_s y\left(a_s\right). \end{cases} \tag{1.3}$$

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Where $q(x)$ is a reel-valued function in $L_2(0,\pi)$; $h, H$ and $\alpha_s$'s are real numbers, and $\lambda$ is a spectral parameter.

Notice that, we can think the problem (1.1) together with (1.3) as studying the equation with Dirac potential:

$$y'' + \left(\lambda - \sum_{s=1}^{m} \alpha_s \delta(x-a_s) - q(x)\right) y = 0, \quad x \in (0,\pi), \tag{1.4}$$

where $\delta(x)$ is the Dirac Delta function (see [1]). Spectral problems are examined in two main subjects in the literature. These are direct and inverse investigations. While direct problems investigate and find some spectral properties of a differential operator, inverse problems aim to recover an operator using it's spectral characteristics. These characteristics could be one, two or more spectra, the spectral function, the normalized constants or Weyl function. Direct and inverse spectral studies for various Sturm-Liouville operators have been investigated in books [5, 12, 18] with many references. Some of these are, for instance, direct and inverse problems for discontinuous boundary value problems have been studied in [6–9, 16, 17, 19, 21, 22, 24]. Also, inverse problems for Sturm-Liouville operators which are not self-adjoint and includes discontinuity inside an interval have been investigated in papers [13, 15, 20, 23].

However, there are many problems where the coefficients must be considered as generalized functions in mathematical physics . For example, one of the first applications of quantum theory to the problem of an ideal crystal was based on the use of the proposed Kronig-Penny model. Kronig and Penny (see, [10]), studying the quantum-mechanical behavior of an electron in a crystal lattice, presented the potential of the crystal in terms of a linear set of rectangular wells, which they then converted into a chain of wells in terms of the Dirac-function such that the area of each well remained unchanged.

The most crucial point in this study is the more sharpenetion of the asymptotic formulas of eigenvalues and eigenfuctions. Sharpenetion here means that more precisely estimation of the asymptotic formulas. Thereby, we can see the contribution of transmission condition in (1.3) i.e. Delta potential in (1.4) to asymptotic formulas. In addition, we will obtain steps for finding solution of inverse spectral problems of Sturm-Liouville equations with finitely many point $\delta$-interactions.

## 2. Some properties of the spectral characteristics of operator $L$

In this section, we will provide the some spectral characteristics of $L$ and present the relationship among these spectral characteristics. The technique employed is similar to those used in [5].

Let $y(x)$ and $z(x)$ be continuously differentiable functions on the intervals $(a_i, a_{i+1})$, $i = \overline{0,m}$. Denote $\langle y, z\rangle := yz' - y'z$. If $y(x)$ and $z(x)$ satisfy the conditions (1.3), then

$$\langle y, z\rangle_{x=a_s-0} = \langle y, z\rangle_{x=a_s+0}, \; s = \overline{1,m}. \tag{2.1}$$

This means that $\langle y, z\rangle$ is continuous on the interval $(0,\pi)$. If $y(x,\lambda)$ and $z(x,\mu)$

are solutions of the equations $ly = \lambda y$ and $lz = \mu z$ respectively, then

$$\frac{d}{dx}\langle y, z\rangle = (\lambda - \mu)\, yz.$$

Let $\varphi(x,\lambda)$, $\psi(x,\lambda)$, $C(x,\lambda)$, $S(x,\lambda)$ be solutions of (1.1) under the conditions

$$\begin{aligned} &C(0,\lambda) = \varphi(0,\lambda) = S'(0,\lambda) = \psi(\pi,\lambda) = 1, \\ &C'(0,\lambda) = S(0,\lambda) = 0, \; \varphi'(0,\lambda) = h, \quad \psi'(\pi,\lambda) = -H, \end{aligned} \tag{2.2}$$

and the conditions (1.3). Then $U(\varphi) = V(\psi) = 0$.

Let's denote $\Delta(\lambda) := \langle \varphi(x,\lambda), \psi(x,\lambda)\rangle$. By virtue of Ostrogradskii-Liouville theorem (see [3, p83]) and equality (2.1), $\Delta(\lambda)$ that will be called the *characteristic function* of $L$, does not depend on $x$. Clearly,

$$\Delta(\lambda) = -V(\varphi) = U(\psi). \tag{2.3}$$

It can be seen that, the function $\Delta(\lambda)$ is entire in $\lambda$ and it has at most a countable set of zeros $\{\lambda_n\}$.

**Lemma 2.1.** *The eigenvalues $\{\lambda_n\}_{n\geq 1}$ of the boundary value problem $L$ and the zeros of the characteristic function are coincide. The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions of the boundary value problem $L$, and there exists a sequence $\{\beta_n\}$ such that*

$$\psi(x,\lambda_n) = \beta_n \varphi(x,\lambda_n), \quad \beta \neq 0.$$

Now let's denote

$$\gamma_n = \int_0^\pi \varphi^2(x,\lambda_n)\, dx. \tag{2.4}$$

We call set $\{\lambda_n, \gamma_n\}_{n\geq 1}$ the *spectral data* of $L$.

**Lemma 2.2.** *The following equality*

$$\dot{\Delta}(\lambda_n) = \beta_n \alpha_n$$

*holds. Here* $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$.

We omit the proofs of Lemma 2.1 and Lemma 2.2 since their proofs can be conducted in a similar way to that of the proof for the classical Sturm-Liouville operators (see [11]).

Now, consider the solution $\varphi(x,\lambda)$. Let $C_0(x,\lambda)$ and $S_0(x,\lambda)$ be smooth solutions of (1.1) on the interval $(0,\pi)$ that satisfy the initial conditions $C_0(0,\lambda) = S_0'(0,\lambda) = 1$, $C_0'(0,\lambda) = S_0(0,\lambda) = 0$. Then

$$C(x,\lambda) = C_0(x,\lambda), \quad S(x,\lambda) = S_0(x,\lambda), \quad a_0 < x < a_1, \tag{2.5}$$

$$\begin{cases} C(x,\lambda) = A_{2s-1}C_0(x,\lambda) + B_{2s-1}S_0(x,\lambda), \\ S(x,\lambda) = A_{2s}C_0(x,\lambda) + B_{2s}S_0(x,\lambda), \end{cases} \quad a_s < x < a_{s+1}, \quad s = \overline{1,m}, \tag{2.6}$$

where

$$\begin{aligned} &A_1 = 1 - \alpha_1 C_0(a_1,\lambda) S_0(a_1,\lambda), \quad B_1 = \alpha_1 C_0^2(a_1,\lambda), \\ &A_2 = -\alpha_1 S_0^2(a_1,\lambda), \quad B_2 = 1 + \alpha_1 C_0(a_1,\lambda) S_0(a_1,\lambda). \end{aligned} \tag{2.7}$$

and for $s = \overline{2,m}$

$$\begin{aligned} &\begin{pmatrix} A_{2s-1} \\ B_{2s-1} \end{pmatrix} \\ &= \left[\prod_{k=2}^{s} \begin{pmatrix} 1 - \alpha_k C_0(a_k,\lambda) S_0(a_k,\lambda) & -\alpha_k [S_0(a_k,\lambda)]^2 \\ \alpha_k [C_0(a_k,\lambda)]^2 & 1 + \alpha_k C_0(a_k,\lambda) S_0(a_k,\lambda) \end{pmatrix}\right] \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \\ &\begin{pmatrix} A_{2s} \\ B_{2s} \end{pmatrix} \\ &= \left[\prod_{k=2}^{s} \begin{pmatrix} 1 - \alpha_k C_0(a_k,\lambda) S_0(a_k,\lambda) & -\alpha_k [S_0(a_k,\lambda)]^2 \\ \alpha_k [C_0(a_k,\lambda)]^2 & 1 + \alpha_k C_0(a_k,\lambda) S_0(a_k,\lambda) \end{pmatrix}\right] \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}. \end{aligned} \tag{2.8}$$

Let $\lambda = \rho^2, \quad \rho = \sigma + i\tau$. It is easy to see that $C_0(x,\lambda)$ satisfies the relations:

$$\begin{aligned} C_0(x,\lambda) &= \cos\rho x + \frac{\sin\rho x}{2\rho}\int_0^x q(t)\,dt + \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\,dt \\ &\quad + O\left(\frac{1}{\rho^2}\exp(|\tau|x)\right), \end{aligned} \tag{2.9}$$

$$\begin{aligned} C_0'(x,\lambda) &= -\rho\sin\rho x + \frac{\cos\rho x}{2}\int_0^x q(t)\,dt + \frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)\,dt \\ &\quad + O\left(\frac{1}{\rho}\exp(|\tau|x)\right). \end{aligned} \tag{2.10}$$

Analogously,

$$\begin{aligned} S_0(x,\lambda) &= \frac{\sin\rho x}{\rho} - \frac{\cos\rho x}{2\rho^2}\int_0^x q(t)\,dt + \frac{1}{2\rho^2}\int_0^x q(t)\cos\rho(x-2t)\,dt \\ &\quad + O\left(\frac{1}{\rho^3}\exp(|\tau|x)\right), \end{aligned} \tag{2.11}$$

$$\begin{aligned} S_0'(x,\lambda) &= \cos\rho x + \frac{\sin\rho x}{2\rho}\int_0^x q(t)\,dt - \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\,dt \\ &\quad + O\left(\frac{1}{\rho^2}\exp(|\tau|x)\right). \end{aligned} \tag{2.12}$$

By virtue of (2.7), (2.8) and (2.9)-(2.12),

$$\begin{aligned} &A_{2s-1} = 1 - \tfrac{1}{2\rho}\sum_{l=1}^{s}\alpha_l\sin 2\rho a_l + O\left(\tfrac{1}{\rho^2}\right), \quad B_{2s-1} = \tfrac{1}{2}\sum_{l=1}^{s}\alpha_l(1+\cos 2\rho a_l) + O\left(\tfrac{1}{\rho}\right), \\ &A_{2s} = O\left(\tfrac{1}{\rho^2}\right), \quad B_{2s} = 1 + O\left(\tfrac{1}{\rho}\right), \qquad s = \overline{1,m}. \end{aligned}$$

Since $\varphi(x,\lambda) = C(x,\lambda) + hS(x,\lambda)$, using (2.5)-(2.12), we calculate:

$$\varphi(x,\lambda) = \cos\rho x + \left(h + \frac{1}{2}\int_0^x q(t)\,dt\right)\frac{\sin\rho x}{\rho} + O\left(\frac{1}{\rho^2}\exp(|\tau|\,x)\right), \qquad a_0 < x < a_1, \tag{2.13}$$

$$\varphi(x,\lambda) = \cos\rho x + \left(h + \frac{1}{2}\sum_{l=1}^{s}\alpha_l + \frac{1}{2}\int_0^x q(t)\,dt\right)\frac{\sin\rho x}{\rho} - \frac{1}{2}\sum_{l=1}^{s}\alpha_l\frac{\sin\rho(2a_l - x)}{\rho} + O\left(\frac{1}{\rho^2}\exp(|\tau|\,x)\right), \quad a_s < x < a_{s+1}, \qquad s = \overline{1,m}, \tag{2.14}$$

$$\varphi'(x,\lambda) = -\rho\sin\rho x + \left(h + \frac{1}{2}\int_0^x q(t)\,dt\right)\cos\rho x + O\left(\frac{1}{\rho}\exp(|\tau|\,x)\right), \qquad a_0 < x < a_1, \tag{2.15}$$

$$\varphi'(x,\lambda) = -\rho\sin\rho x + \left(h + \frac{1}{2}\sum_{l=1}^{s}\alpha_l + \frac{1}{2}\int_0^x q(t)\,dt\right)\cos\rho x + \frac{1}{2}\sum_{l=1}^{s}\alpha_l\cos\rho(2a_l - x) + O\left(\frac{1}{\rho}\exp(|\tau|\,x)\right), \quad a_s < x < a_{s+1}, \qquad s = \overline{1,m}. \tag{2.16}$$

It follows from (2.3), (2.14) and (2.16) that

$$\Delta(\lambda) = \rho\sin\rho\pi - \omega\cos\rho\pi - \frac{1}{2}\sum_{l=1}^{m}\alpha_l\cos\rho(2a_l - \pi) + O\left(\frac{1}{\rho}\right), \tag{2.17}$$

where

$$\omega = h + H + \frac{1}{2}\sum_{l=1}^{m}\alpha_l + \frac{1}{2}\int_0^x q(t)\,dt.$$

Using (2.17) and applying Rouche's theorem ( [4, p125]), and the well-known method given in [2] as $n \to \infty$

$$\rho_n = n - 1 + o(1).$$

Similarly, again using Rouche's theorem, it can be shown that for sufficiently large values of $n$, every circle $\sigma_n(\delta) = \{\rho : |\rho - n| \le \delta\}$ contains exactly one zero of $\Delta\left(\rho^2\right)$. Since $\delta > 0$ is arbitrary, we have

$$\rho_n = n - 1 + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \to \infty. \tag{2.18}$$

Since $\rho_n$ are zeros of $\Delta\left(\rho^2\right)$, from (2.17) we get

$$n.\sin\varepsilon_n\pi - \omega\cos\varepsilon_n\pi - \frac{1}{2}\left(\sum_{l=1}^{m}\alpha_l\cos 2na_l\cos 2a_l\varepsilon_n\right).\cos\varepsilon_n\pi + \sigma_n = 0, \tag{2.19}$$

where $\sigma_n = \varepsilon_n \sin \varepsilon_n \pi - \frac{1}{2}\left(\sum\limits_{l=1}^{m} \alpha_l \sin 2\left(n+\varepsilon_n\right) a_l\right). \sin \varepsilon_n \pi + o\left(\exp\left|\tau_n\right| \pi\right)$ and $\tau_n = Im\rho_n$. Hence $\sin \varepsilon_n \pi = O\left(\frac{1}{n}\right)$, that is, $\varepsilon_n = O\left(\frac{1}{n}\right)$. Using (2.19) we get more precisely

$$\varepsilon_n = \frac{1}{\pi n}\left(\omega + \frac{1}{2}\sum_{l=1}^{m} \alpha_l \cos 2na_l\right) + o\left(\frac{1}{n}\right). \tag{2.20}$$

Substituting (2.20) into (2.18), we get

$$\rho_n = n - 1 + \frac{1}{\pi n}\left(\omega + \frac{1}{2}\sum_{l=1}^{m} \alpha_l \cos 2na_l\right) + o\left(\frac{1}{n}\right). \tag{2.21}$$

Since the boundary value problem $L$ is self-adjoint (see [14]), all eigenvalues $\{\lambda_n\}_{n\geq 1}$ are real and simple.

At last, using (2.4), (2.13), (2.14) and (2.21) one can calculate

$$\gamma_n = \frac{\pi}{2} + \frac{\omega_n}{n} + o\left(\frac{1}{n}\right), \tag{2.22}$$

where

$$\omega_n = -\frac{1}{2}\sum_{s=1}^{m} \left(a_{s+1} - a_s\right)\sum_{l=1}^{s} \alpha_l . \sin 2na_l .$$

If $q\left(x\right)$ is a smooth function, we can obtain sharper asymptotics for the spectral data.

## 3. Solution and construction of the inverse problem

In this section, we first study the inverse problems to recover the boundary value problem $L$. To do this, we will use the method of spectral mappings together with Cauchy's integral formula and Residue theorem. Then by using the solution of the main equation, we construct the algorithms for the solution of the inverse problems.

For this purpose we agree that together with $L$ we consider a boundary value problem $\widetilde{L}$ of the same form but with different coefficients $\widetilde{q}\left(x\right)$, $\widetilde{h}$, $\widetilde{H}$; $\widetilde{a}_s$ and $\widetilde{\alpha}_s$, $s = \overline{1,m}$. Everywhere below if a certain symbol $e$ denotes an object related to $L$, then the corresponding symbol $\widetilde{e}$ with tilde denotes the analogous object related to $\widetilde{L}$.

For the sake definiteness, we just consider the inverse problem of recovering $L$ from the spectral data $\{\lambda_n, \gamma_n\}_{n\geq 1}$. Let boundary value problems $L$ and $\widetilde{L}$ be such that

$$a_s = \widetilde{a}_s, \quad s = \overline{1,m}, \qquad \sum_{n=1}^{\infty} \xi_n \left|\lambda_n\right| < \infty, \tag{3.1}$$

where $\xi_n := \left|\lambda_n - \widetilde{\lambda}_n\right| + \left|\gamma_n - \widetilde{\gamma}_n\right|$. Denote

$$\lambda_{n0} = \lambda_n, \quad \lambda_{n1} = \widetilde{\lambda}_n, \quad \gamma_{n0} = \gamma_n, \quad \gamma_{n1} = \widetilde{\gamma}_n,$$

$$\varphi_{ni}(x)=\varphi(x,\lambda_{ni}), \quad \widetilde{\varphi}_{ni}(x)=\widetilde{\varphi}(x,\lambda_{ni}),$$
$$Q_{kj}(x,\lambda)=\frac{\langle\varphi(x,\lambda),\varphi_{kj}(x)\rangle}{\gamma_{kj}(\lambda-\lambda_{kj})}=\frac{1}{\gamma_{kj}}\int_0^x\varphi(t,\lambda)\,\varphi_{kj}(t)\,dt,$$
$$Q_{ni,kj}(x)=Q_{kj}(x,\lambda_{ni})$$

for $i,j=0,1$ and $n,k=1,2,...$, where $\widetilde{\varphi}(x,\lambda)$ is the solution of (1.4) with the potential $\widetilde{q}$ under the initial conditions $\widetilde{\varphi}(0,\lambda)=1,\ \widetilde{\varphi}'(0,\lambda)=\widetilde{h}$. Similarly, we can define $\widetilde{Q}_{kj}(x,\lambda)$ by replacing $\varphi$ with $\widetilde{\varphi}$ in the above definition.

Using Schwarz's lemma ( [4, p130]) and (2.13)-(2.16), (2.21) we obtain the following asymptotic estimates:

$$\left|\varphi_{nj}^{(v)}(x)\right|\le C(n+1)^{v}, \tag{3.2}$$

$$|Q_{ni,kj}(x)|\le\frac{C}{|n-k|+1}, \qquad \left|Q_{ni,kj}^{(v+1)}(x)\right|\le C(n+k+1)^{v}, \tag{3.3}$$

where $n,k=1,2,...,\ i,j,v=0,1$ and $C$ is a positive constant. Analogous estimates are also valid for $\widetilde{\varphi}_{ni}(x),\ \widetilde{Q}_{ni,kj}(x)$.

**Lemma 3.1.** *Let $\varphi_{nj}(x)$ and $Q_{ni,kj}(x)$ be defined as above. Then the following representations hold for $i,j=0,1$ and $n,k=1,2,...$:*

$$\widetilde{\varphi}_{ni}(x)=\varphi_{ni}(x)+\sum_{k=1}^{\infty}\left(\widetilde{Q}_{ni,k0}(x)\,\varphi_{k0}(x)-Q_{ni,k1}(x)\,\varphi_{k1}(x)\right). \tag{3.4}$$

*The series in (3.4) converge absolutely and uniformly with respect to $x\in(0,\pi)\backslash\{a_s\}_{s=1}^{m}$.*

The proof of this lemma is similar to that of the lemma given in [15] and hence is omitted.

From all arguments mentioned above, it is seen that, for each fixed $x\in(0,\pi)\backslash\{a_s\}_{s=1}^{m}$, the relation (3.4) can be thought as a system of linear equations with respect to $\varphi_{ni}(x)$ for $n=1,2,...$ and $i=0,1$. But the series in (3.4) converges only "with brackets". So, it is not convenient to use (3.4) as a main equation of the inverse problem.

Let $V$ be a set of indexes $u=(n,i)$, $n=1,2,...$ and $i=0,1$. For each fixed $x\in(0,\pi)\setminus\{a_s\}_{s=1}^{m}$, we define the vectors

$$\phi(x)=[\phi_u(x)]_{u\in V}=\begin{bmatrix}\phi_{n0}(x)\\ \phi_{n1}(x)\end{bmatrix}_{n=1,2,...}$$
$$\widetilde{\phi}(x)=\left[\widetilde{\phi}_u(x)\right]_{u\in V}=\begin{bmatrix}\widetilde{\phi}_{n0}(x)\\ \widetilde{\phi}_{n1}(x)\end{bmatrix}_{n=1,2,...}$$

by the formulas

$$\begin{bmatrix}\phi_{n0}(x)\\ \phi_{n1}(x)\end{bmatrix}=\begin{bmatrix}\xi_n^{-1} & -\xi_n^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varphi_{n0}(x)\\ \varphi_{n1}(x)\end{bmatrix},$$
$$\begin{bmatrix}\widetilde{\phi}_{n0}(x)\\ \widetilde{\phi}_{n1}(x)\end{bmatrix}=\begin{bmatrix}\xi_n^{-1} & -\xi_n^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\widetilde{\varphi}_{n0}(x)\\ \widetilde{\varphi}_{n1}(x)\end{bmatrix}. \tag{3.5}$$

If $\xi_n = 0$ for a certain $n$, then we put $\phi_{n0}(x) = \widetilde{\phi}_{n0}(x) = 0$.

Further, we define the block matrix

$$H(x) = [H_{u,v}(x)]_{u,v\in V} = \begin{bmatrix} H_{no,ko}(x) & H_{no,k1}(x) \\ H_{n1,ko}(x) & H_{n1,k1}(x) \end{bmatrix}_{n,k=1,2,\dots},$$

where $u = (n, i)$, $v = (k, j)$ and

$$\begin{bmatrix} H_{no,ko}(x) & H_{no,k1}(x) \\ H_{n1,ko}(x) & H_{n1,k1}(x) \end{bmatrix} = \begin{bmatrix} \xi_n^{-1} & -\xi_n^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q_{no,ko}(x) & Q_{no,k1}(x) \\ Q_{n1,ko}(x) & Q_{n1,k1}(x) \end{bmatrix} \begin{bmatrix} \xi_k & \xi_k \\ 0 & -1 \end{bmatrix}.$$

Analogously we define $\widetilde{\phi}(x)$, $\widetilde{H}(x)$ by replacing in the previous definitions $\varphi_{ni}(x)$, $Q_{ni,kj}(x)$ by $\widetilde{\varphi}_{ni}(x)$, $\widetilde{Q}_{ni,kj}(x)$, respectively. It follows from (2.13)-(2.16), (2.21), (2.22), (3.2), (3.3) and Schwarz's lemma that

$$\begin{aligned} &\left|\phi_{nj}^{(v)}(x)\right|, \ \left|\widetilde{\phi}_{nj}^{(v)}(x)\right| \le C(n+1)^v, \quad v = 0, 1, \\ &|H_{ni,kj}(x)|, \ \left|\widetilde{H}_{ni,kj}(x)\right| \le \frac{C\xi_k}{|n-k|+1}, \end{aligned} \tag{3.6}$$

$$\left|H_{ni,kj}^{(v+1)}(x)\right|, \ \left|\widetilde{H}_{ni,kj}^{(v+1)}(x)\right| \le C(n+k+1)^v, \quad v = 0, 1. \tag{3.7}$$

Let us consider the Banach space $B$ of bounded sequences $\alpha = [\alpha_u]_{u\in V}$ with the norm $\|\alpha\|_B = \sup_{u\in V} |\alpha_u|$ and define an operator $E + \widetilde{H}(x)$ from $B$ to $B$. Here $E$ is the identity operator. It follows from (3.6), (3.7) that for each fixed $x$, this operator is linear bounded operator, and

$$\left\|\widetilde{H}(x)\right\| \le C \sup_n \sum_{k=1}^{\infty} \frac{1}{|n-k|+1} < \infty.$$

Now we are ready to give the main result of the section.

**Theorem 3.1.** *For each fixed $x \in (0,\pi) \setminus \{a_s\}_{s=1}^m$, the vector $\phi(x) \in B$ satisfies the equation*

$$\widetilde{\phi}(x) = \left(E + \widetilde{H}(x)\right)\phi(x), \tag{3.8}$$

*in the Banach space $B$. Moreover, the operator $E + \widetilde{H}(x)$ has a bounded inverse operator, i.e. the equation (3.8) is uniquely solvable.*

**Proof.** Using the notation $\widetilde{\phi}(x)$, we rewrite (3.4) as

$$\widetilde{\phi}_{ni}(x) = \phi_{ni}(x) + \sum_{k,j} \widetilde{H}_{ni,kj}(x)\phi_{kj}(x), \quad (n,i) \in V, \quad (k,j) \in V,$$

which is equivalent to (3.4). Interchanging places for $L$ and $\widetilde{L}$, we obtain analogously

$$\phi(x) = (E - H(x))\widetilde{\varphi}(x), \quad (E - H(x))\left(E + \widetilde{H}(x)\right) = E.$$

Hence the operator $\left(E+\widetilde{H}(x)\right)^{-1}$ exists, and it is a linear bounded operator. □

Let,

$$M(\lambda)=\frac{\Delta^1(\lambda)}{\Delta(\lambda)}, \tag{3.9}$$

where $\Delta^1(\lambda)=\psi(0,\lambda)=V(S)$ is the characteristic function of the boundary value problem $L_1$, which is equation (1.4) with the boundary conditions $U(y)=0,\ y(\pi)=0$. Let $\{\mu_n\}_{n\geq 1}$ be zeros of $\Delta^1(\lambda)$, in other words the eigenvalues of $L_1$.

Equation (3.8) is named as *basic equation* of the inverse problem. Solving (3.8) we find the vector $\phi(x)$, and hence, the functions $\varphi_{ni}(x)$. Thus, we get the following algorithms to find the solution of inverse problems.

**Algorithm 3.1.** *Given the spectral data* $\{\lambda_n,\gamma_n\}_{n\geq 1}$*, to construct* $q(x)$ *and* $h$, $H$; $a_s$, $\alpha_s$, $s=\overline{1,m}$.

$(i)$ *Choose* $\widetilde{L}$ *and find* $\widetilde{\phi}(x)$ *and* $\widetilde{H}(x)$*;*

$(ii)$ *Find* $\phi(x)$ *by solving the equation (3.8) and calculate* $\varphi_{n0}(x)$ *via (3.5);*

$(iii)$ *Choose some (e.g.,* $n=0$*) and construct* $q(x)$ *and* $h,\ H;\ a_s,\ \alpha_s,\ s=\overline{1,m}$ *by following formulas*

$$q(x)=\frac{\varphi''_{n0}(x)}{\varphi_{n0}(x)}+\lambda_n,\quad h=\varphi'_{n0}(0),\quad H=-\frac{\varphi'_{n0}(\pi)}{\varphi_{n0}(\pi)},$$
$$\varphi_{n0}(a_s+0)=\varphi_{n0}(a_s-0),\quad \alpha_s=\frac{\varphi'_{n0}(a_s+0)-\varphi'_{n0}(a_s-0)}{\varphi_{n0}(a_s-0)},\quad s=\overline{1,m}.$$

**Algorithm 3.2.** *Given* $M(\lambda)$*, to construct* $q(x)$ *and* $h,\ H;\ a_s,\ \alpha_s,\ s=\overline{1,m}$.

$(i)$ *According to* $M(\lambda)=\sum\limits_{k=1}^{\infty}\frac{1}{\gamma_k(\lambda-\lambda_k)}$ *construct the spectral data* $\{\lambda_n,\gamma_n\}_{n\geq 1}$*;*

$(ii)$ *Using Algorithm 3.1, construct* $q(x)$ *and* $h,\ H;\ a_s,\ \alpha_s,\ s=\overline{1,m}$.

**Algorithm 3.3.** *Given two spectra* $\{\lambda_n,\mu_n\}_{n\geq 1}$*, to construct* $q(x)$ *and* $h,\ H;\ a_s,\ \alpha_s,\ s=\overline{1,m}$.

$(i)$ *Using (3.9) find* $M(\lambda)$*;*

$(ii)$ *Using Algorithm 3.2, construct* $q(x)$ *and* $h,\ H;\ a_s,\ \alpha_s,\ s=\overline{1,m}$.

# 4. Example

In this section we give an example that exhibits the algorithm obtained above. Here we construct $q(x)$ and $h,\ H;\ a_s,\ \alpha_s,\ s=\overline{1,m}$ when the spectral data set $\{\lambda_n,\gamma_n\}_{n\geq 1}$ is given.

$(i)$ Take such that $\widetilde{q}(x)=0,\ \widetilde{h}=\widetilde{H}=0;\ \widetilde{a}_1=\frac{\pi}{4},\ \widetilde{\alpha}_1=1\ (m=1)$. Let $\left\{\widetilde{\lambda}_n,\widetilde{\gamma}_n\right\}_{n\geq 1}$ be the spectral data of $\widetilde{L}$. Clearly,

$$\widetilde{\lambda}_1=1,\ \widetilde{\gamma}_1=\frac{5\pi+2}{16},\ \widetilde{\varphi}_{10}(x)=\cos x\quad\left(x<\frac{\pi}{4}\right),\ \ \widetilde{\varphi}_{10}=\frac{\sqrt{2}}{2}\sin\left(x+\frac{\pi}{4}\right)\quad\left(x>\frac{\pi}{4}\right).$$

Let $\lambda_n = \widetilde{\lambda}_n \quad (n \geq 1)$, $\gamma_n = \widetilde{\gamma}_n \quad (n \geq 2)$, and let $\gamma_1 > 0$ be an arbitrary positive number. Denote $Q := \frac{1}{\gamma_1} - \frac{1}{\widetilde{\gamma}_1}$. Then

$$\widetilde{H}(x) = Q \int_0^x \widetilde{\varphi}_{10}^2(t)\,dt = \begin{cases} \dfrac{Q}{4}(2x + \sin 2x), & x < \frac{\pi}{4}, \\ \dfrac{Q}{4}\left(x + \frac{\pi}{2} + 1 - \cos 2x\right), & x > \frac{\pi}{4}. \end{cases}$$

$(ii)$ Find $\phi(x)$ by solving the equation (3.8) and calculate $\varphi_{10}(x)$ via (3.5), that

$$\varphi_{10}(x) = \begin{cases} \left[1 + \dfrac{Q}{4}(2x + \sin 2x)\right]^{-1}, & x < \frac{\pi}{4}, \\ \left[1 + \dfrac{Q}{4}\left(x + \frac{\pi}{2} + 1 - \cos 2x\right)\right]^{-1}, & x > \frac{\pi}{4}. \end{cases}$$

$(iii)$ We calculate

$$q(x) = \begin{cases} \begin{array}{l} 2\left[1 + \frac{Q}{4}(2x + \sin 2x)\right]^{-2}\left(\frac{Q}{2} + \frac{Q}{2}\cos 2x\right)^2 \\ +Q.\sin 2x\left[1 + \frac{Q}{4}(2x + \sin 2x)\right]^{-1} + 1 \end{array}, & x < \frac{\pi}{4}, \\ \begin{array}{l} 2\left[1 + \frac{Q}{4}\left(x + \frac{\pi}{2} + 1 - \cos 2x\right)\right]^{-2}\left(\frac{Q}{4} + \frac{Q}{2}\cos 2x\right)^2 \\ +Q.\cos 2x\left[1 + \frac{Q}{4}\left(x + \frac{\pi}{2} + 1 - \cos 2x\right)\right]^{-1} + 1 \end{array}, & x > \frac{\pi}{4}, \end{cases}$$

$$h = -Q, \qquad H = \frac{Q}{4}\left(1 + \frac{3\pi}{8}Q\right)^{-1},$$

$$\alpha_1 = \frac{\left[1 + \frac{Q}{4}\left(\frac{\pi}{2} + 1\right)\right]^{-2}.\frac{Q}{2} - \left[1 + \frac{Q}{4}\left(\frac{3\pi}{4} + 1\right)\right]^{-2}.\frac{3\pi}{4}}{\left[1 + \frac{Q}{4}\left(\frac{\pi}{2} + 1\right)\right]^{-1}}.$$

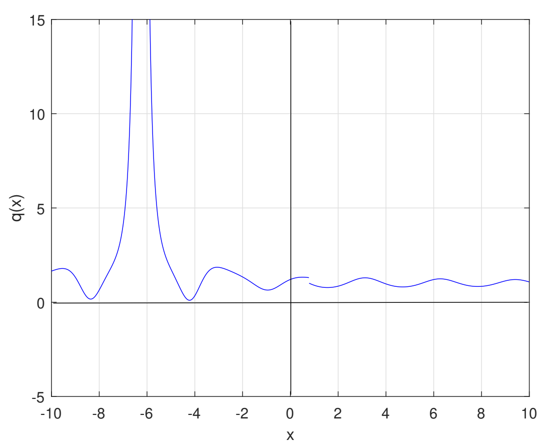


**Figure 1.** The graph of $q(x)$, which has a discontinuity point at $x = \pi/4$, constructed by the given spectral data set

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