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A NONHOMOGENEOUS BOUNDARY-VALUE PROBLEM FOR THE MODIFIED ANISOTROPIC HEISENBERG SPIN CHAIN POSED ON A FINITE DOMAIN

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Abstract This article is concerned with the Modified Anisotropic Heisenberg Spin Chain. We prove that the associated nonhomogeneous initial-boundary value problem has a unique globally smooth solution in $H^{2k+1}(m,n)$ for $k \geq 1$. Our main new ingredient is a technique of spatial difference and crucial uniformed estimates of the step-size h. Meanwhile, to prove the global existence, we overcome drawbacks which are not exist in corresponding Cauchy problem.

Keywords The Modified Anisotropic Heisenberg Spin Chain, globally smooth solution, initial-boundary value problem, spatial difference method.

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1. Introduction

The Heisenberg Spin Chain occurs in the domain of solid state physics [8] [10]. It is a strongly coupled nonlinear degenerate parabolic equation with high nonlinearity. We know that many such integrable equations do posses a similar but slightly different integrable form known as the modified form such as the KdV and MKdV cases [3]. At the same time, there is a connection between the usual and the modified form of the KdV equation: a phenomenon known as deformation [12]. In this paper, we consider the initial-boundary value problem of the Modified Anisotropic Heisenberg Spin Chain, which was not previously known and has not been well resolved. The modified anisotropic Heisenberg spin chain [1]:

$$Z_t = -Z \times Z_{xx} - \frac{1}{2}\{Z(Z, BZ)\}_x + (\alpha + B)Z_x, \tag{1.1}$$

posed on $\{(x,t) \in \Omega \times \mathbb{R}^+\}$, where $\Omega = (m,n)$ is an interval in $\mathbb{R}$, α is an arbitrary constant. The magnetization $Z(x,t)$ is a three-vector $\{Z_1(x,t), Z_2(x,t), Z_3(x,t)\}$ coupled by the constraint (magnetically saturated condition):

$$Z_1^2(x,0) + Z_2^2(x,0) + Z_3^2(x,0) = 1, \tag{1.2}$$

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where $x \in (m, n)$, $B = diag(b_1, b_2, b_3)$, and $b_i (i = 1, 2, 3)$ is constant. (a, b) denotes the scalar product, and $(a \times b)$ denotes the cross product in $\mathbb{R}^3$. The modified system(1.1) can be rewritten as the usual anisotropic Heisenberg spin chain when $\alpha = 0, B = 0$.

The initial and boundary value conditions of system (1.1) are given by (without loss of generality, we choose $[m, n] = [0, 1]$):

$$Z(x, 0) = \varphi(x), \qquad \text{for} \quad x \in [0, 1] \tag{1.3}$$

$$Z(0, t) = g_0(t), Z(1, t) = g_1(t), \qquad \text{for} \quad t \in \mathbb{R}^+ \tag{1.4}$$

where $\varphi(t), g_0(t), g_1(t)$ are given function. The initial value condition $\varphi(x)$ and boundary value conditions $g_0(t), g_1(t)$ satisfy the magnetically saturated condition(1.2).

In a contrast, the so-called Landau-Lifshitz equation for the isotropic Heisenberg spin chain has fared better than that of modified anisotropic Heisenberg spin chain [4,5,13–15,17]. Zhou, Guo and Tan proved in [18] that the initial value problem with periodic boundary condition and the Cauchy problem associated with the system of anisotropic Heisenberg spin chain with the Gilbert damping term existed a unique smooth solution in $H^k(\mathbb{R}^1)$ for $k \geq 4$. Alouges and Soyeur [2] obtained the globally weak solutions in $\mathbb{R}^3$, the non-uniqueness of weak solutions was demonstrated in [2] as well. Both results were proved by using the technique of spatial difference and crucial a priori estimates of high-order derivatives in Sobolev spaces. The local existence and uniqueness with small energy initial date for strong solution in $\mathbb{R}^3$ was shown in [6]. The local existence and uniqueness of strong solution on a bounded domain was proved in [7].

The case of modified anisotropic Heisenberg spin chain was treated by Tan in [16]. They obtained the global existence and uniqueness of smooth solutions to the modified anisotropic Heisenberg spin chain with periodic boundary conditions. But little is known for the initial-boundary value problem about the modified anisotropic Heisenberg spin chain. Because we can easily get maximum bound and gradient estimate in L^2 without any difficulty for the usual system of Heisenberg spin chain. However, it is much more difficult to get correspondent estimates for the initial-boundary value problem of modified anisotropic Heisenberg spin chain.

The objective of this article is to prove new globally smooth solutions results. In this direction, we obtain the globally smooth solutions in $H^{2k+1}(m, n)$ for $k \geq 1$. Next are our main results.

Lemma 1.1. *Let ϵ be any positive number, $g_0(t), g_1(t) \in C^{2k+1}(\mathbb{R}^+; \mathbb{S}^2), \varphi(x) \in H^{2k+1}((0, 1); \mathbb{S}^2)$. Then the initial-boundary value for two associated systems* (2.1)-(2.3)*and* (2.5)-(2.7) *with Gilbert damping term have local smooth solution $Z(x, t) \in W^{s,\infty}(0, T_0; H^{2(k-s)+1})$, where $s \in [0, k]$, $k \geq 0$, $T_0 > 0$.*

Remark 1.1. The two associated systems (2.1)-(2.3)and (2.5)-(2.7) with Gilbert damping term are give in Section2.1.

Lemma 1.2. *Let $g_0(t), g_1(t) \in C^{2k+1}(\mathbb{R}^+; \mathbb{S}^2), \varphi(x) \in H^{2k+1}((0, 1); \mathbb{S}^2)$. Then the initial-boundary value* (1.1)-(1.4) *has a unique global smooth solution $Z(x, t) \in L^\infty(\mathbb{R}^+; H^{2k+1})$ such that $|Z(x, t)| = 1$ for any $k \geq 1$.*

This paper is organized as follows: in the next section we introduce two associated systems with Gilbert damping term and derive crucial uniformed estimates of the step-size h. The main new ingredient in the proof of Theorem 1.1 is a technique of spatial difference. In Section 3, we establish a priori estimates and overcome some difficulties that do not exist for corresponding Cauchy problem. Meanwhile, we get the global existence of smooth solution to the problem (1.1)-(1.4).

2. Local smooth solutions to the system with Gilbert damping term

In this section, we obtain the local smooth solution to the Modified Anisotropic Heisenberg Spin Chain with Gilbert damping term.

2.1. The associated systems with Gilbert damping term.

Firstly, we introduce two associated systems with Gilbert damping term. we Replacing (1.1)-(1.4) by the following viscosity approximation

$$Z_t = -\epsilon Z \times (Z \times Z_{xx}) - Z \times Z_{xx} - \frac{1}{2}\{Z(Z,BZ)\}_x + (\alpha + B)Z_x, \tag{2.1}$$

$$Z(x,0) = \varphi(t), \qquad \text{for} \quad x \in [0,1], \tag{2.2}$$

$$Z(0,t) = g_0(t), Z(1,t) = g_1(t), \qquad \text{for} \quad t \in \mathbb{R}^+. \tag{2.3}$$

where ϵ is a non-negative number. The initial date satisfies magnetically saturated condition: $|Z(x,0)|^2 = 1$, then we get the following property.

Lemma 2.1. *Let ϵ be any positive number, $g_0(t), g_1(t) \in C^{2k+1}(\mathbb{R}^+;\mathbb{S}^2), \varphi(x) \in H^{2k+1}((0,1);\mathbb{S}^2)$. The smooth solution of the initial-boundary value* (2.1)-(2.3) *satisfies $W^{s,\infty}(0,T_0;H^{2(k-s)+1})$, where $s \in [0,k],\ k \geq 0,\ T_0 > 0$. Then we have*

$$|Z(x,t)| = 1, \qquad x \in [0,1], \qquad t \geq 0. \tag{2.4}$$

Proof. Multiplying (2.1) by $Z(x,t)$ and integrating over $[0,1]$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 |Z|^2 dx = &-\frac{1}{4}\int_0^1 (|Z|^2)_x (Z,BZ)dx - \frac{1}{2}\int_0^1 |Z|^2 (Z,BZ)_x dx \\ &+ \frac{\alpha}{2}\int_0^1 (|Z|^2)_x dx + \frac{1}{2}\int_0^1 (Z,BZ)_x dx.\end{aligned}$$

Set $V = |Z|^2 - 1$, then (2.1)-(2.3) can be rewritten as

$$\begin{aligned}&V_t = -\frac{1}{2}V_x(Z,BZ) - V(Z,BZ)_x + \alpha V_x, \\ &V(x,0) = 0, && \text{for} \quad x \in [0,1], \\ &V(0,t) = g_0^2(t) - 1, V(1,t) = g_1^2(t) - 1, && \text{for} \quad t \in \mathbb{R}^+.\end{aligned}$$

Multiplying the above identity by $V(x,t)$ and integrating over $[0,1]$, by integration by parts, we have

$$\begin{aligned}\frac{d}{dt}\|V\|_2^2 &= -\frac{1}{2}[V^2(Z,BZ)|_{x=0}^1 - \int_0^1 V^2(Z,BZ)_x dx] - 2\int_0^1 V^2(Z,BZ)_x dx + \int_0^1 |V^2|_x dx \\ &= -\frac{3}{2}\int_0^1 V^2(Z,BZ)_x dx \leq C(\alpha, g_0, g_1)\|(Z,BZ)_x\|_\infty \|V\|_2^2.\end{aligned}$$

Thus, we get (2.4) by using Gronwall inequality. □

Then in the classical sense problem (2.1)-(2.3) is equivalent to the following problem:

$$Z_t = \epsilon Z_{xx} - Z \times Z_{xx} - \frac{1}{2}\{Z(Z, BZ)\}_x + (\alpha + B)Z_x + \epsilon|Z_x|^2 Z, \tag{2.5}$$

$$Z(x,0) = \varphi(t), \quad \text{for} \quad x \in [0,1], \tag{2.6}$$

$$Z(0,t) = g_0(t), Z(1,t) = g_1(t), \quad \text{for} \quad t \in \mathbb{R}^+. \tag{2.7}$$

As described in the following Lemma:

Lemma 2.2. *Assuming that ϵ is any positive number, $Z_0(x) \in H^k, |Z_0(x)| = 1$. Then in the classical sense, Z is a solution of problem* (2.1)-(2.3) *if and only if Z is a solution of the problem* (2.5)-(2.7).

Proof. Let $Z(x,t)$ be a classical solution of the problem (2.1)-(2.3). From Lemma 2.1, we know that $|Z(x,t)| = 1, \forall (x,t) \in [0,1] \times [0,T]$. Combining the formula $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, we obtain

$$\begin{aligned} -\epsilon Z \times (Z \times Z_{xx}) &= \epsilon |Z|^2 Z_{xx} - \epsilon (Z \cdot Z_{xx}) Z \\ &= \epsilon Z_{xx} + \epsilon |Z_x|^2 Z - \epsilon (Z \cdot Z_x)_x Z_x \\ &= \epsilon Z_{xx} + \epsilon |Z_x|^2 Z. \end{aligned}$$

On the other hand, let $Z(x,t)$ be a classical solution of problem (2.5)-(2.7). Multiplying (2.5) by $Z(x,t)$, we get

$$\frac{d}{dt}|Z|^2 = \epsilon |Z^2|_{xx} + 2\epsilon |Z_x|^2 (Z^2 - 1) - Z^2 (Z, BZ)_x - \frac{1}{2}|Z^2|_x (Z, BZ) + \alpha |Z^2|_x + (Z, BZ)_x,$$

set $u(x,t) = |Z(x,t)|^2$, (2.5)-(2.7) can be rewritten as

$$u_t = \epsilon u_{xx} + 2\epsilon |Z_x|^2 (u - 1) - u(Z, BZ)_x - \frac{1}{2} u_x (Z, BZ) + \alpha u_x + (Z, BZ)_x, \tag{2.8}$$

$$u(x,0) = 1,$$

$$u(0,t) = g_0^2(t) = 1, u(1,t) = g_1^2(t) = 1.$$

Set $W(x,t) = u(x,t) - 1 = |Z(x,t)|^2 - 1$, we get

$$W_t = \epsilon W_{xx} + 2\epsilon |Z_x|^2 W - W(Z, BZ)_x - (Z, BZ)_x - \frac{1}{2} W_x (Z, BZ) + \alpha W_x + (Z, BZ)_x, \tag{2.9}$$

$$W(x,0) = 0, \tag{2.10}$$

$$W(0,t) = g_0^2(t) - 1, W(1,t) = g_1^2(t) - 1. \tag{2.11}$$

Multiplying (2.9) by $W(x,t)$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 |W|^2 dx + \epsilon \int_0^1 |W_x|^2 dx \\ =& \epsilon W_x W|_{x=0}^1 + 2\epsilon \int_0^1 |Z_x|^2 |W|^2 dx - \int_0^1 |W|^2 (Z, BZ)_x \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{4}\int_0^1 |W^2|_x(Z,BZ)dx+\frac{\alpha}{2}\int_0^1 |W^2|_x dx \\
\leq & C(\alpha,B,\epsilon,g_0,g_1)(max_{x,t}|Z_x|^2+max_{x,t}|Z|^2)\int_0^1 |W|^2dx.
\end{aligned}
$$

Thus, we get $W(x,t)\equiv 0$ by using Gronwall inequality, where $(x,t)\in[0,1]\times[0,T]$. Namely, the classical solution of problem (2.5)-(2.7) satisfies $|Z(x,t)|^2\equiv 1$. So we come to the following conclusion

$$
\begin{aligned}
Z_{xx}+|Z_x|^2Z &= |Z|^2Z_{xx}-(Z\cdot Z_{xx})Z+(Z\cdot Z_x)_xZ \\
&= -Z\times(Z\times Z_{xx}).
\end{aligned}
$$

Therefore we obtain this Lemma. □

2.2. Uniformed estimates of the step-size.

In this subsection, the existence of smooth solution to the problem (2.5)-(2.7) is proved by the spatial difference method and a priori estimates in the Sobolev space. We now consider the following ordinary differential-difference equations:

$$
\begin{aligned}
\frac{dZ_j}{dt} &= \epsilon\frac{\Delta_+\Delta_-Z_j}{h^2}-Z_j\times\frac{\Delta_+\Delta_-Z_j}{h^2}+(\alpha+B)\frac{\Delta_+Z_j}{h} \\
&\quad -\frac{1}{2}\frac{\Delta_+\{Z_j(Z_j,BZ_j)\}}{h}+\epsilon|\frac{\Delta_+Z_j}{h}|^2Z_j, && (2.12)\\
Z_j|_{t=0} &= \varphi(x), \qquad \text{for}\quad x\in(0,1), && (2.13)\\
Z_0 &= g_0(t), Z_J=g_1(t), \qquad \text{for}\quad t\in\mathbb{R}^+, && (2.14)
\end{aligned}
$$

where $0\leq j\leq J-1, h=\frac{1}{J}, J>0, j=0,1,2,...,J.x_1=h, x_J=1.\Delta_+,\Delta_-$ represent the forward and backward difference operators, respectively. We define discrete functions $Z_j(t)=Z(x_j,t)$ on grid point (x_j,t) satisfies $\Delta_+Z_j=Z_{j+1}-Z_j, \Delta_-Z_j=Z_{j+1}-Z_j$. It follows from the standard theory on ordinary differential-difference equations that the problem (2.12)-(2.14) admits unique local smooth solution $Z_h=\{Z_j(t), j=0,...,J\}$. To verify the local existence of the smooth solution to question (2.5)-(2.7), we need only prove that $Z_h(t)$maintains the bounds of $h\to 0$ and $t\in[0,T^*]$, in which T^* is a constant independent of h. In addition, Z_h converges to $Z(x,t)$ as $h\to 0$ such that we can easily see that $Z(x,t)$ is the local smooth solution of the initial-boundary value problem (2.5)-(2.7). By convention, we denote

$$
\begin{aligned}
\|\delta^k u_h\|_p &= (\sum_{j=0}^{J-k}|\frac{\triangle_+^k u_j}{h^k}|^p h)^{\frac{1}{p}},\\
\|\delta^k u_h\|_\infty &= max_{0\leq j\leq J-k}|\frac{\triangle_+^k u_j}{h^k}|,\\
\|u_h\|_{\widetilde{H}^k} &= (\sum_{i=0}^{k}\|\delta^i u_h\|_2^2)^{\frac{1}{2}},
\end{aligned}
$$

where $u_h=\{u_j|j=0,1,...,J.\}, h=\frac{1}{J}, 0\leq k<J$. Meanwhile, we recall some well-known Sobolev and Caliarrdo-Nirenberg inequalities which were given in [9,19].

Lemma 2.3. *Assuming that $j, m \in \mathbb{N} \cup \{0\}, q, r \in \mathbb{R}^+,\ 0 \leq j < m, 1 \leq q, r \leq \infty$. Then*

$$\|D^j u\|_p \leq C\|D^m u\|_r^a \|u\|_q^{1-a}, \tag{2.15}$$

for $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, where $\Omega \subseteq \mathbb{R}^1$, $\frac{j}{m} \leq \alpha \leq 1$, $\frac{1}{p} = j + a(\frac{1}{r} - m) + (1-a)\frac{1}{q}$.

Lemma 2.4. *Assuming that $p \in \mathbb{R}^1, j, m \in \mathbb{N} \cup \{0\},\ 2 \leq p \leq \infty, 0 \leq j < m$. Then*

$$\|\delta^j u_h\|_p \leq C\|u_h\|_2^{1-a}(\|\delta^n u_h\|_2 + (2D)^{-m}\|u_h\|_2)^a, \tag{2.16}$$

for $a = \frac{1}{m}(j + \frac{1}{2} - \frac{1}{p})$, where $u_h = \{u_j = u(x_j)\}, j = 0, 1, 2, ..., J\}, h = \frac{2D}{J}$.

Moreover, we get the following Lemma, which can be proved easily by using the definition of the forward and backward difference operators.

Lemma 2.5. *Assume that $u_h = \{u_j = u(x_j), j = 0, 1, 2, ..., J\}, v_h = \{v_j = v(x_j), j = 0, 1, 2, ..., J\}$. Then we have*

$$(i) \qquad \sum_{j=1}^{J} v_j \Delta_- u_j = -\sum_{j=0}^{J-1} u_j \Delta_+ v_j - u_0 v_0 + u_J v_J,$$

$$(ii) \qquad \Delta_+ \Delta_- u_j = \Delta_- \Delta_+ u_j = \Delta_+^2 u_{j-1},$$

$$(iii) \qquad \Delta_+(u_j v_j) = u_{j+1}\Delta_+ v_j + v_j \Delta_+ u_j,$$

where Δ_+, Δ_- represent the forward and backward difference operators, respectively.

Now we derive crucial uniformed estimates of the step-size h for the solutions of the problem (2.12)-(2.14).

Lemma 2.6. *Assuming that $g_0(t), g_1(t) \in \mathbb{C}_1(\mathbb{R}^+; \mathbb{S}^2), \varphi(x) \in H^1((0,1); \mathbb{S}^2)$. Then for the smooth solution $Z_h(t)$ of the problem (2.12)-(2.14) we have the following estimates*

$$sup_{0 \leq t \leq T_0} \|Z_h(t)\|_{\widetilde{H}^1} \leq C, \tag{2.17}$$

where the constant C is independent of h.

Proof. Multiplying (2.12) by hZ_j, summing up the products for $j = 1, 2, ..., J-1$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Z_h\|_2^2 + \epsilon\|\delta Z_h\|_2^2 =& \epsilon g_0(t)\frac{\triangle_+ Z_0}{h} - \epsilon g_1(t)\frac{\triangle_+ Z_J}{h} + (\alpha + B)\sum_{j=1}^{J-1} h\frac{\triangle_+ Z_j}{h} Z_j \\ & - \frac{1}{2}\sum_{j=1}^{J-1} hZ_j \frac{\triangle_+\{Z_j(Z_j, BZ_j)\}}{h},\end{aligned}$$

then we have

$$\frac{1}{2}\frac{d}{dt}\|Z_h\|_2^2 + \epsilon\|\delta Z_h\|_2^2 \leq (\alpha + B)\|\delta Z_h\|_2\|Z_h\|_2 + \|\delta Z_h\|_\infty\|Z_h\|_3^3 + 2\epsilon\|\delta Z_h\|_\infty. \tag{2.18}$$

Multiplying (2.12) by $\frac{1}{h}\triangle_+\triangle_- Z_j$, summing up the products for $j = 1, 2, ..., J-1$, we get

$$-\frac{1}{2}\frac{d}{dt}\|\delta Z_h\|_2^2 + g_1'(t)\frac{\triangle_+ Z_J}{h} - g_0'(t)\frac{\triangle_+ Z_0}{h} - \epsilon\|\delta^2 Z_h(t)\|_2^2$$

$$=\frac{\alpha+B}{h}\sum_{j=1}^{J-1}\triangle_+^2 Z_{j-1}\frac{\triangle_+ Z_j}{h}-\frac{B}{2}\sum_{j=1}^{J-1}|Z_j|^2 h\frac{\triangle_+ Z_J}{h}\frac{\triangle_+\triangle_- Z_J}{h^2},$$

then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta Z_h\|_2^2+\epsilon\|\delta^2 Z_h\|_2^2 \le &(|g_0'(t)|+|g_1'(t)|)\|\delta Z_h\|_\infty+\|\delta^2 Z_h\|_2\|\delta Z_h\|_2\\ &+\frac{B}{2}\|Z_h\|_2^2\|\delta Z_h\|_\infty\|\delta^2 Z_h\|_\infty. \end{aligned}\tag{2.19}$$

Adding (2.18)(2.19) together, one gets

$$\frac{d}{dt}\|Z_h\|^2_{\widetilde{H}^1}+\epsilon\|Z_h\|^2_{\widetilde{H}^2}\le C(1+\|Z_h\|^3_{\widetilde{H}^1}),$$

where C is given by $\epsilon,\alpha,B,g_i(t),i=0,1$. C is independent of h. □

Lemma 2.7. *Under conditions of Lemma 2.6, for the smooth solution $Z_h(t)$ of the problem* (2.12)-(2.14)*, there exists a constant $T_0>0$ such that*

$$sup_{0\le t\le T_0}\|\delta Z_{ht}\|_2+\int_0^{T_0}\|\delta^2 Z_{ht}(t)\|_2^2 dt\le C \tag{2.20}$$

where the constant C is independent of h.

Proof. Differentiating (2.12) with respect to t, we have

$$\begin{aligned}Z_{jtt}=&\epsilon\frac{\triangle_+\triangle_- Z_{jt}}{h^2}-Z_j\times\frac{\triangle_+\triangle_- Z_{jt}}{h^2}-Z_{jt}\frac{\triangle_+\triangle_- Z_j}{h^2}+(\alpha+B)\frac{\triangle_+ Z_{jt}}{h}\\ &-\frac{B}{2}(|Z_j|^2\frac{\triangle_+ Z_j}{h})_t\end{aligned}$$

multiplying above equation by $\frac{\triangle_+\triangle_- Z_{jt}}{h}$, summing up the products for $j=1,2,...,J-1$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\delta Z_{ht}\|_2^2+\epsilon\|\delta^2 Z_{ht}\|_2^2\\ =&g_1''(t)\frac{\triangle_+ Z_{J,t}}{h}-g_0''(t)\frac{\triangle_+ Z_{0,t}}{h}-h\sum_{j=1}^{J-1}\triangle_+\triangle_- Z_{jt}(Z_{jt}\times\frac{\triangle_+\triangle_- Z_j}{h^2})\\ &+\frac{B}{2h}\sum_{j=1}^{J-1}(|Z_j|^2\frac{\triangle_+ Z_j}{h})_t\triangle_+\triangle_- Z_{jt}+\frac{\alpha+B}{h}\sum_{j=1}^{J-1}\triangle_+^2 Z_{j-1,t}\frac{\triangle_+ Z_{jt}}{h}.\end{aligned}$$

Meanwhile, we have $\|Z_{ht}\|_2\sim\|\delta^2 Z_h\|_2$, therefore

$$\frac{d}{dt}\|Z_{ht}\|^2_{\widetilde{H}_1}+\epsilon\|Z_{ht}\|^2_{\widetilde{H}_2}\le C(1+\|Z_{ht}\|^3_{\widetilde{H}_1}),$$

where C is independent of h. □

According to Lemma 2.6 and 2.7, we can easily prove the following result by generalizing k.

Lemma 2.8. *Assuming that* $k \geq 0$, $g_0(t)$, $g_1(t) \in C^{2k+1}(\mathbb{R}^+; \mathbb{S}^2)$, $\varphi(x) \in H^{2k+1}((0,1); \mathbb{S}^2))$, *for the smooth solution* $Z_h(t)$ *of the problem* (2.12)-(2.14)*, there exists constants* C, T_0 *such that*

$$sup_{0 \leq t \leq T_0} \|\delta Z_{ht^k}(t)\|_2 + \int_0^{T_0} \|\delta^2 Z_{ht^k}(t)\|_2^2 dt \leq C, \tag{2.21}$$

where constants C, T_0 *are independent of* h*. And*

$$sup_{0 \leq t \leq T_0} \|Z_{ht}(t)\|_{\widetilde{H}^{2k+1}} + \int_0^{T_0} \|Z_{ht}(t)\|_{\widetilde{H}^{2k+2}}^2 dt \leq C, \tag{2.22}$$

where constants C, T_0 *are independent of* h*.*

From Lemma 2.8 we conclude that the solution Z_h of problem (2.12)-(2.14) for $t \in [0, T_0]$ is uniformly bounded in the space $W^{s,\infty}(0, T_0; H^{2(k-s)+1})$ relative to the step-size $h \to 0$. One could therefore repeat the same procedure in [11] and then achieve a smooth solution to the local existence of problem (2.5)-(2.7). Therefore, we propose Theorem 1.1.

3. Global smooth solutions to the system without Gilbert damping term

In this section, we consider the global existence of smooth solution to the problem (1.1)-(1.4). A common approach is to establish a priori estimates which are not dependent on α, t in the Sobolev space. However there is a serious drawback to this approach, that is, we can not get the estimate $\|Z(\cdot,t)\|_{H^1}$ as followes:

Lemma 3.1. *Under conditions of Theorem 1.1, for the smooth solution* $Z(x,t)$ *of the problem* (2.5)-(2.7) *, there exists constants* $C_1 = C(\|\varphi\|_{H^1(\Omega)})$, $C_2 = C(T, \|g\|_{0,\infty})$, $C_3 = C(T, \|g\|_{1,\infty})$, $C_4 = C(T, \alpha, B)$ *such that*

$$\|Z_x(\cdot,t)\|_2^2 \leq C_1 + C_3 \int_0^t \|Z_x(\cdot,\tau)\|_\infty d\tau + C(C_2, C_4) \int_0^t \|Z_x(\cdot,\tau)\|_\infty^2 d\tau, \tag{3.1}$$

where $t \in [0,T], \alpha > 0$*.*

Proof. Multiplying (2.5) by Z_{xx} and integrating over $[0,1]$, we get

$$\int_0^1 Z_{xx} Z_t dx = \epsilon \int_0^1 |Z_{xx}|^2 dx - \int_0^1 Z_{xx} \cdot \{\frac{1}{2}[Z(Z,BZ)]_x - (\alpha + B) Z_x\} dx,$$

noting that

$$\int_0^1 Z_{xx} \cdot \{Z(Z,BZ)\}_x dx = -\frac{3}{2} \int_0^1 |Z_x|^2 (Z,BZ)_x dx + \frac{1}{2} \{|Z_x|^2 (Z,BZ)\}|_{x=0}^1,$$

where we have used the integration by parts and identities:

$$Z \cdot Z_x = 0, \qquad Z \cdot Z_{xx} = -|Z_x|^2.$$

Hence, we have

$$\frac{d}{dt} \int_0^1 |Z_x|^2 dx + 2\epsilon \int_0^1 |Z_{xx}|^2 dx$$

$$= -\frac{3}{2}\int_0^1 |Z_x|^2(Z,BZ)_x dx + \{2Z_xZ_t + |Z_x|^2(Z,BZ) + (\alpha+B)|Z_x|^2\}|_{x=0}^1,$$

thus

$$\begin{aligned}
&\|Z_x(\cdot,t)\|_2^2 \\
\le&\|Z_x(\cdot,0)\|_2^2 + \frac{3}{2}\int_0^t \{(Z,BZ)|_{x=0}^1 \|Z_x\|_\infty^2\}d\tau + 2\int_0^t [Z_x(1,\tau)Z_t(1,\tau) \\
&- Z_x(0,\tau)Z_t(0,\tau)]d\tau + \int_0^t [|Z_x(1,\tau)|^2(Z(1,\tau),BZ(1,\tau)) - |Z_x(0,\tau)|^2 \\
&(Z(0,\tau),BZ(0,\tau))]d\tau + (\alpha+B)\int_0^t [|Z_x(1,\tau)|^2 - |Z_x(0,\tau)|^2]d\tau
\end{aligned}$$

then we yield (3.1). □

Noting that this disadvantage dose not exist in corresponding Cauchy problem. Our approach to solve the this problem is to combine (3.1) with H^2- estimate. Then we can get $\|Z(\cdot,t)\|_{H^2}$ is not dependent on α. Before that, straightforward computations give the following Lemma.

Lemma 3.2. *Under conditions of Theorem 1.1, $Z(x,t)$ is a smooth solution of the problem* (2.5)-(2.7). *Set*

$$\begin{aligned}
A_i(t) &= Z_{xx}(i,t), \\
B_i(t) &= Z(i,t) \times Z_{xx}(i,t), \\
C_i(t) &= Z(i,t)\cdot(Z_x(i,t)\times Z_{xx}(i,t)), \\
D_i(t) &= Z_x(i,t)\cdot Z_{xx}(i,t), \\
E_i(t) &= Z_x(i,t)\cdot Z_{xxt}(i,t),
\end{aligned}$$

where $i=0,1$. Then we get

$$\begin{aligned}
A_i(t) =& \frac{1}{\epsilon^2+1}(g_i\times g_i') - |Z_x(i,t)|^2 g_i + \frac{\epsilon}{2(\epsilon^2+1)}Z_x(i,t)(g_i,Bg_i) - \frac{\epsilon(\alpha+B)}{\epsilon^2+1}Z_x(i,t) \\
&- \frac{1}{2(\epsilon^2+1)}(g_i,Bg_i)Z_x(i,t)\times g_i + \frac{\alpha+B}{\epsilon^2+1}Z_x(i,t)\times g_i + \frac{\epsilon}{\epsilon^2+1}g_i' - \frac{2\epsilon}{\epsilon^2+1}g_i\cdot g_i',
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
B_i(t) =& \frac{\epsilon}{\epsilon^2+1}g_i\times g_i' - \frac{1}{2(\epsilon^2+1)}Z_x(i,t)(g_i,Bg_i) + \frac{\alpha+B}{\epsilon^2+1}Z_x(i,t) - \frac{1}{\epsilon^2+1}g_i' \\
&- \frac{\epsilon}{2(\epsilon^2+1)}(g_i,Bg_i)Z_x(i,t)\times g_i + \frac{\epsilon(\alpha+B)}{\epsilon^2+1}Z_x(i,t)\times g_i + \frac{2}{\epsilon^2+1}g_i\cdot g_i',
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
C_i(t) =& \frac{1}{\epsilon^2+1}g_i'\cdot Z_x(i,t) - \frac{\epsilon}{\epsilon^2+1}(g_i\times g_i')\cdot Z_x(i,t) \\
&+ \frac{1}{2(\epsilon^2+1)}|Z_x(i,t)|^2(g_i,Bg_i) + \frac{\alpha+B}{\epsilon^2+1}|Z_x(i,t)|^2,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
D_i(t) =& \frac{\epsilon}{\epsilon^2+1}g_i'\cdot Z_x(i,t) + \frac{1}{\epsilon^2+1}(g_i\times g_i')\cdot Z_x(i,t) \\
&+ \frac{\epsilon}{2(\epsilon^2+1)}|Z_x(i,t)|^2(g_i,Bg_i) + \frac{\epsilon(\alpha+B)}{\epsilon^2+1}|Z_x(i,t)|^2,
\end{aligned} \tag{3.5}$$

$$E_i(t) = \frac{1}{\epsilon} g_i'' \cdot Z_x(i,t), \tag{3.6}$$

where $i = 0, 1$.

Proof. Multiply (2.5) with $Z(x,t)$, we get

$$-2g_i \cdot g_i' = (Z(i,t), BZ(i,t))_x,$$

moreover, (2.5) can be rewritten as

$$-\epsilon A_i(t) + B_i(t) = -\frac{1}{2} Z_x(i,t)(g_i, Bg_i) + (\alpha + B) Z_x(i,t) + \epsilon |Z_x(i,t)|^2 g_i.$$

Cross $Z(x,t)$ to both sides of (2.5), we get

$$-g_i \times g_i' = -\epsilon B_i(t) - A_i(t) - |Z_x(i,t)|^2 g_i - \frac{1}{2}(g_i, Bg_i) Z_x(i,t) \times g_i + (\alpha + B) Z_x(i,t) \times g_i.$$

(3.2)(3.3) can be proved above. On the other hand, multiply (2.5) with $Z_x(i,t)$, we get

$$g_i' \cdot Z_x(i,t) = \epsilon D_i + C_i - \frac{1}{2}|Z_x(i,t)|^2 (g_i, Bg_i) + (\alpha + B)|Z_x(i,t)|^2,$$

multiply (2.5) with $(Z(i,t) \times Z_x(i,t))$, we have

$$-Z_x(i,t) \cdot (g_i \times g_i') = \epsilon C_i - D_i,$$

so we get (3.4)(3.5). Similarly, both sides of the equation compute derives about t, we get

$$\begin{aligned} g_i'' = \epsilon Z_{xxt}(i,t) - g_i' \times Z_{xx}(i,t) - Z \times Z_{xxt}(i,t) - \frac{1}{2}\{Z(i,t)(Z(i,t), BZ(i,t))\}_{xt} \\ + (\alpha + B) Z_{xt}(i,t) + 2\epsilon Z_x(i,t) \cdot Z_{xt}(i,t) \cdot Z(i,t) + \epsilon |Z_x(i,t)|^2 g_i' \end{aligned}$$

Cross $Z(x,t)$ and $Z_x(x,t)$ to both sides of the above equation, then multiply with $Z(x,t)$, we have

$$g_i'' \cdot Z_x(i,t) = -\epsilon(Z_x(i,t) \cdot Z_{xxt}(i,t))$$

thus we get (3.6). □

Lemma 3.3. *Under conditions of Theorem 1.1, for the smooth solution $Z(x,t)$ of the problem* (2.5)-(2.7) *, there exists constants $C_1 = C(\|\varphi\|_{H^1(\Omega)})$, $C_2 = C(T, \|g\|_{0,\infty})$, $C_3 = C(T, \|g\|_{1,\infty})$, $C_4 = C(T, \alpha, B)$, $C_5 = C(T, \|g\|_{2,\infty})$, $C_6 = C(\|\varphi\|_{H^2(\Omega)})$ such that*

$$\begin{aligned} \|Z_{xx}(\cdot,t)\|_2^2 - \frac{5}{4}\|Z_x(\cdot,t)\|_4^4 \le C_6 + C_2\|Z_x(\cdot,t)\|_\infty^2 + C(C_2, C_5, \epsilon)\int_0^t \|Z_x(\cdot,\tau)\|_\infty^4 d\tau \\ + C(C_2, C_5, \epsilon)\int_0^t \|Z_x(\cdot,\tau)\|_2^{10} d\tau + C_4 \int_0^t \|Z_{xx}(\cdot,\tau)\|_2^2 d\tau, \end{aligned} \tag{3.7}$$

where $t \in [0, T], \epsilon > 0$.

Proof. Using the integration by parts, we get

$$\frac{d}{dt}\int_0^1 |Z_{xx}(x,t)|^2 = 2\int_0^1 Z_{xx}Z_{xxt}dx = -2\int_0^1 Z_{xxx}Z_{xt}dx + 2Z_{xx}Z_{xt}|_{x=0}^1$$

which means

$$\|Z_{xx}(\cdot,t)\|_2^2 = \|\varphi_{xx}\|_2^2 - 2\int_0^t\int_0^1 Z_{xxx}Z_{xt}dxd\tau + 2\int_0^t Z_{xx}(x,\tau)Z_{xt}(x,\tau)d\tau|_{x=0}^1. \tag{3.8}$$

Next, we estimate (3.8) in two steps.
Step1. For the third term on the right of (3.8), we have

$$\begin{aligned}
&2\int_0^t Z_{xx}(x,\tau)Z_{xt}(x,\tau)d\tau|_{x=0}^1\\
&= -2\int_0^t Z_{xxt}(x,\tau)Z_x(x,\tau)d\tau|_{x=0}^1 + 2Z_{xx}(x,\tau)Z_x(x,\tau)|_{x=0}^1|_{\tau=0}^t\\
&= -2\int_0^t (E_1(\tau)-E_0(\tau))d\tau + 2(D_1(\tau)-D_0(\tau))|_{\tau=0}^t\\
&\le C(\|g\|_{2,\infty})\int_0^t \|Z_x\|_\infty d\tau + C(T,\|g\|_{0,\infty})\|Z_x\|_\infty^2.
\end{aligned}\tag{3.9}$$

Step2. For the second term on the right of (3.8), we know that $Z\cdot Z_{xxx} = -3Z_x\cdot Z_{xx}$, thus

$$\begin{aligned}
&-2\int_0^t\int_0^1 Z_{xxx}Z_{xt}dxd\tau\\
&= -2\int_0^t\int_0^1 Z_{xxx}\{\epsilon Z_{xx} - Z\times Z_{xx} - \frac{1}{2}\{Z(Z,BZ)\}_x + (\alpha+B)Z_x + \epsilon|Z_x|^2Z\}_x dxdt\\
&= -2\epsilon\int_0^t\int_0^1 |Z_{xxx}|^2dxd\tau + 2\int_0^t\int_0^1 Z_{xxx}\cdot(Z_x\times Z_{xx})dxd\tau - 2\epsilon\int_0^t\int_0^1 Z_{xxx}\cdot\{|Z_x|^2Z_x\\
&\quad + (|Z_x|^2)_xZ\}dxd\tau + \int_0^t\int_0^1 Z_{xxx}\cdot\{Z(Z,BZ)\}_{xx}dxd\tau - 2(\alpha+B)\int_0^t\int_0^1 Z_{xxx}\cdot Z_{xx}dxd\tau\\
&\le -2\epsilon\int_0^t\int_0^1 |Z_{xxx}|^2dxd\tau + 2\int_0^t\int_0^1 Z_{xxx}\cdot(Z_x\times Z_{xx})dxd\tau + 2\epsilon\int_0^t\{\|Z_{xxx}\|_2\|Z_x\|_6^3\\
&\quad + \|Z_{xx}\|_\infty^2\|Z_x\|_2^2\}d\tau + C(B)\int_0^t \|Z_{xx}\|_2\|Z_x\|_6^3d\tau + C(\alpha,B)\int_0^t\|Z_{xxx}\|_2\|Z_{xx}\|_2d\tau\\
&\le -C(\epsilon)\int_0^t\int_0^1 |Z_{xxx}|^2dxd\tau + 2\int_0^t\int_0^1 Z_{xxx}\cdot(Z_x\times Z_{xx})dxd\tau\\
&\quad + C(\alpha,B)\int_0^t \|Z_{xx}\|_2^2d\tau + C(\epsilon,B)\int_0^t \|Z_x\|_2^{10}d\tau.
\end{aligned}\tag{3.10}$$

Combining (3.8) with (3.9)(3.10), we get

$$\|Z_{xx}(\cdot,t)\|_2^2 \le C_6 + C_2\|Z_x\|_\infty^2 + C_5\int_0^t \|Z_x\|_\infty d\tau + C(\epsilon,C_5)\int_0^t \|Z_x\|_2^{10}d\tau$$

$$+ C(\alpha, B)\int_0^t \|Z_{xx}\|_2^2 d\tau + 2\int_0^t\int_0^1 Z_{xxx}\cdot(Z_x\times Z_{xx})dxd\tau. \quad (3.11)$$

Now, we consider the last term on the right of (3.11). Noticing that

$$Z\cdot Z_x = 0,\quad Z\cdot Z_{xx} = -|Z_x|^2,\quad Z\cdot Z_{xxx} = -3Z_x\cdot Z_{xx}, \quad (3.12)$$

and $(x,t)\in S = \{x\in[0,1], t\in[0,T]; |Z_x(x,t)|\neq 0\}$, then from the orthogonality of the three vectors $Z, Z_x, Z\times Z_x$, we know that

$$Z_{xx} = \alpha Z + \beta Z_x + \gamma Z\times Z_x \quad (3.13)$$

where $\alpha = -|Z_x|^2,\quad \beta = Z_x\cdot Z_{xx}/|Z_x|^2,\quad \gamma = (Z\times Z_x)\cdot Z_{xx}/|Z_x|^2$ then we get

$$\begin{aligned}
&2\int_0^t\int_0^1 Z_{xxx}\cdot(Z_x\times Z_{xx})dxd\tau\\
&= 2\int_0^t\int_0^1 Z_{xxx}\cdot\{Z_x\times(\alpha Z+\beta Z_x+\gamma Z\times Z_x)\}dxd\tau\\
&= 2\int_0^t\int_0^1 |Z_x|^2(Z\times Z_x)\cdot Z_{xxx}dxd\tau + 2\int_0^t\int_0^1\frac{(Z\times Z_x)\cdot Z_{xx}}{|Z_x|^2}\{Z_x\times(Z\times Z_x)\}\cdot Z_{xxx}dxd\tau\\
&= 5\int_0^t\int_0^1 |Z_x|^2(Z\times Z_x)\cdot Z_{xxx}dxd\tau + 3\int_0^t\{|Z_x(1,\tau)|^2C_1(\tau) - |Z_x(0,\tau)|^2C_0(\tau)\}d\tau\\
&= 5\int_0^t\int_0^1 |Z_x|^2(Z\times Z_x)\cdot Z_{xxx}dxd + C_3\int_0^t \|Z_x\|_\infty^3 d\tau + (\epsilon, C_2, C_5)\int_0^t\|Z_x\|_\infty^4 d\tau,
\end{aligned} \quad (3.14)$$

where

$$\begin{aligned}
&5\int_0^t\int_0^1 |Z_x|^2(Z\times Z_x)\cdot Z_{xxx}dxd\tau\\
&= 5\int_0^t\int_0^1 |Z_x|^2 Z_x\cdot\{Z_t - \epsilon Z_{xx} - \epsilon|Z_x|^2 Z + \frac{1}{2}\{Z(Z,BZ)\}_x - (\alpha+B)Z_x\}_x dxd\tau\\
&\le \frac{5}{4}\|Z_x(\cdot,t)\|_4^4 - \frac{5}{4}\|\varphi_x\|_4^4 + 5\epsilon\int_0^t\|Z_{xxx}\|_2\|Z_x\|_6^3 d\tau\\
&\quad + 15\epsilon\int_0^t \|Z_x\|_6^6 d\tau + C_4\int_0^t\|Z_x\|_6^3\|Z_{xx}\|_2 d\tau\\
&\le \frac{5}{4}\|Z_x(\cdot,t)\|_4^4 + \frac{C(\epsilon)}{2}\int_0^t\|Z_{xxx}\|_2^2 d\tau + \epsilon C_5\int_0^t\|Z_x\|_2^{10}d\tau + C_4\int_0^t\|Z_{xx}\|_2^2 d\tau + C.
\end{aligned}$$

Combining (3.11) with (3.14), we get (3.7). □

Lemma 3.4. *Under conditions of Theorem 1.1, for the smooth solution $Z(x,t)$ of the problem* (2.5)-(2.7) *, we have*

$$\|Z_{xx}(\cdot,t)\|_2 \le C, \quad (3.15)$$

where $t\in[0,T], \epsilon\in(0,\epsilon_0]$ and ϵ is a positive constant. The constant C is only depends on $T, \|\varphi\|_{H_2}, \|g\|_{2,\infty}$.

Proof. Setting $F(t) = \|Z_{xx}(\cdot,t)\|_2^2 + \lambda\|Z_x(\cdot,t)\|_2^2 + \lambda\|Z_x(\cdot,t)\|_2^5 + \lambda\|Z_x(\cdot,t)\|_2^6$, where $t \in [0.T]$, λ is a constant to be determined. Combining (3.1) with (3.7), we get

$$\begin{aligned}F(t) \leq & C_6 + C_2\|Z_x(\cdot,t)\|_\infty^2 + C(C_2,C_5,\epsilon)\int_0^t \|Z_x(\cdot,\tau)\|_\infty^4 d\tau \\ & + C(C_2,C_5,\epsilon)\int_0^t \|Z_x(\cdot,\tau)\|_2^{10} d\tau + C_4\int_0^t \|Z_{xx}(\cdot,\tau)\|_2^2 d\tau + \frac{5}{4}\|Z_x(\cdot,t)\|_4^4,\end{aligned}$$

using the Gagliardo-Nirenberg inequality and the Young inequality, we have

$$\begin{aligned}F(t) \leq & C + \frac{1}{2}\|Z_{xx}(\cdot,\tau)\|_2^2 + C_0(\|Z_x(\cdot,\tau)\|_2^2 + \|Z_x(\cdot,\tau)\|_2^6) \\ & + C\int_0^t F(\tau)d\tau + C\epsilon\int_0^t F^2(\tau)d\tau,\end{aligned}$$

where C_0 is independent of λ, ϵ, t. Setting $\lambda \geq 2C_0$, we get

$$F(t) \leq E_1 + E_2\int_0^t F(\tau)d\tau + \epsilon E_3\int_0^t F^2(\tau)d\tau$$

where E_1, E_2, E_3 are positive constants independent of ϵ. □

Lemma 3.5. *Under conditions of Theorem 1.1, for the smooth solution $Z(x,t)$ of the problem* (2.5)-(2.7) *, we have*

$$\|Z_{xt}(\cdot,t)\|_2 + \|Z_{xxx}(\cdot,t)\|_2 \leq C, \tag{3.16}$$

where $t \in [0,T], \epsilon \in (0,\epsilon_0]$ and ϵ is a positive constant. The constant C is only depends on $T, \|\varphi\|_{H_3}, \|g\|_{3,\infty}$.

Proof. Using the integration by parts, we get

$$\frac{d}{dt}\int_0^1 |Z_{xt}(x,t)|^2 dx = 2\int_0^1 Z_{xt}\cdot Z_{xtt}dx = -2\int_0^1 Z_{xxt}\cdot Z_{tt}dx + 2Z_{xt}\cdot Z_{tt}|_{x=0}^1,$$

which means

$$\|Z_{xt}(\cdot,t)\|_2^2 - \|Z_{xt}(0,t)\|_2^2 = -2\int_0^t\int_0^1 Z_{xxt}\cdot Z_{tt}dxd\tau + 2\int_0^t Z_{xt}\cdot Z_{tt}d\tau|_{x=0}^1, \tag{3.17}$$

where

$$\begin{aligned}Z_{xt}(x,0) = & [-\epsilon\varphi(x)\times(\varphi(x)\times\varphi''(x)) - \varphi(x)\times\varphi''(x) \\ & - \frac{1}{2}\{\varphi(x)(\varphi(x), B\varphi(x))\}_x + (\alpha+B)\varphi(x)]'\end{aligned}$$

and

$$\begin{aligned}& 2\int_0^t Z_{xt}\cdot Z_{tt}d\tau|_{x=0}^1 \\ = & -2\int_0^t Z_x\cdot Z_{ttt}d\tau|_{x=0}^1 + 2Z_x\cdot Z_{tt}|_{x=0}^1|_{\tau=0}^t\end{aligned}$$

$$
\begin{aligned}
&= -2\int_0^t (Z_x(1,\tau)\cdot g_1'''(\tau) - Z_x(0,\tau)\cdot g_0'''(\tau))d\tau + 2(Z_x(1,\tau)\cdot g_1''(\tau) - Z_x(0,\tau)\cdot g_0''(\tau))|_{\tau=0}^t \\
&\leq C.
\end{aligned}
$$

For the first term on the right of (3.17), we have

$$
\begin{aligned}
& -2\int_0^t\int_0^1 Z_{xxt}\cdot Z_{tt}dxd\tau \\
=& -2\int_0^t\int_0^1 Z_{xxt}\cdot\{\epsilon Z_{xx} - Z\times Z_{xx} - \frac{1}{2}\{Z(Z,BZ)\}_x + (\alpha+B)Z_x + \epsilon|Z_x|^2 Z\}_t \\
\leq& -2\epsilon\int_0^t \|Z_{xxt}(\cdot,\tau)\|_2^2 d\tau + 2\int_0^t\int_0^1 Z_{xxt}\cdot(Z_t\times Z_{xx})dxd\tau \\
&+ \int_0^t\int_0^1 Z_{xxt}\cdot\{Z(Z,BZ)\}_{xt}dxd\tau - 2(\alpha+B)\int_0^t\int_0^1 Z_{xxt}\cdot Z_{xt}dxd\tau \\
&- 2\epsilon\int_0^t\int_0^1 Z_{xxt}(|Z_x|^2)_t Z dxd\tau - 2\epsilon\int_0^t\int_0^1 Z_{xxt}(|Z_x|^2)Z_t dxd\tau \\
\leq& -2\epsilon\int_0^t \|Z_{xxt}(\cdot,\tau)\|_2^2 d\tau + 2\int_0^t\int_0^1 Z_{xxt}\cdot(Z_t\times Z_{xx})dxd\tau \\
&+ C(C_3,C_4)\int_0^t \|Z_{xt}(\cdot,\tau)\|_2^2 d\tau + C,
\end{aligned}
\tag{3.18}
$$

combining (3.17) with (3.18), we get

$$
\begin{aligned}
&\|Z_{xt}(\cdot,t)\|_2^2 + 2\epsilon\int_0^t \|Z_{xxt}(\cdot,t)\|_2^2 d\tau \\
\leq& 2\int_0^t\int_0^1 Z_{xxt}\cdot(Z_t\times Z_{xx})dxd\tau + C(C_3,C_4)\int_0^t \|Z_{xt}(\cdot,\tau)\|_2^2 d\tau + C.
\end{aligned}
\tag{3.19}
$$

Now, we consider the first term on the right of (3.19). Noticing the orthogonality of the three vectors $Z, Z_t, Z\times Z_t$, we get

$$
Z_{xxt} = \mu Z + \nu Z_t + \omega Z\times Z_t
$$

where $\mu = -2Z_x\cdot Z_{xt} - Z_t\cdot Z_{xx},\ \nu = Z_t\cdot Z_{xxt}/|Z_t|^2,\ \omega = (Z\times Z_t)\cdot Z_{xxt}/|Z_t|^2$, thus we have

$$
\begin{aligned}
& 2\int_0^t\int_0^1 Z_{xxt}\cdot(Z_t\times Z_{xx})dxd\tau \\
=& -2\int_0^t\int_0^1 \{(\mu Z + \nu Z_t + \omega Z\times Z_t)\times Z_t\}\cdot Z_{xx}dxd\tau \\
=& -2\int_0^t\int_0^1 (2Z_x\cdot Z_{xt} - Z_t\cdot Z_{xx})(Z\times Z_t)\cdot Z_{xx}dxd\tau \\
&+ 2\int_0^t\int_0^1 (Z\times Z_t)\cdot Z_{xxt}|Z_x|^2 dxd\tau \\
\leq& C\int_0^t (\|Z_{xt}\|_2\|Z_{xx}\|_2\|Z_t\|_\infty + \|Z_t\|_\infty^2\|Z_{xx}\|_2^2)d\tau - \int_0^t\int_0^1 Z_t\cdot(Z\times Z_{xx})_t|Z_x|^2 dxd\tau
\end{aligned}
$$

$$\leq C(1+\int_0^t \|Z_{xt}\|_2^2 d\tau) - 2\int_0^t\int_0^1 Z_t \cdot \{Z_t - \epsilon Z_{xx} - \epsilon|Z_x|^2 Z + \frac{1}{2}\{Z(Z,BZ)\}_x - (\alpha+B)Z_x\}_t |Z_x|^2 dx d\tau$$
$$\leq C + C\int_0^t \|Z_{xt}(\cdot,\tau)\|_2^2 d\tau + \epsilon\int_0^t \|Z_{xxt}(\cdot,\tau)\|_2^2 d\tau, \tag{3.20}$$

combining with (3.19), we get

$$\|Z_{xt}(\cdot,t)\|_2^2 + \epsilon\int_0^t \|Z_{xxt}(\cdot,t)\|_2^2 d\tau \leq C + C\int_0^t \|Z_{xt}(\cdot,\tau)\|_2^2 d\tau,$$

using the Gronwall inequality, we have

$$\|Z_{xt}(\cdot,t)\|_2^2 \leq C. \tag{3.21}$$

Meanwhile, we know that

$$Z_{xt} = \epsilon Z_{xxx} - (Z\times Z_{xx})_x + \epsilon(|Z_x|^2 Z)_x - \frac{1}{2}\{Z(Z,BZ)\}_{xx} + (\alpha+B)Z_{xx},$$

and

$$\begin{aligned}-(Z\times Z_t)_x =& -\epsilon(Z\times Z_{xx})_x - Z_{xxx} - (|Z_x|^2 Z)_x \\ &+ \frac{1}{2}(Z\times Z_x)(Z,BZ) - (\alpha+B)Z\times Z_{xx},\end{aligned}$$

we get

$$Z_{xxx} = \frac{1}{\epsilon^2+1}\{\epsilon Z_{xt} + (Z\times Z_t)_x + \frac{1}{2}(Z\times Z_x)(Z,BZ) - (\alpha+B)Z\times Z_{xx}\} + (|Z_x|^2 Z)_x \tag{3.22}$$

thus we can easily get (3.16). □

Then we repeat the same procedure in Lemma(3.4)(3.5) and easily prove the following result by induction.

Lemma 3.6. *Under conditions of Theorem 1.1, for the smooth solution $Z(x,t)$ of the problem* (2.5)-(2.7) *, we have*

$$\Sigma_{r+2s\leq k}\|Z_{x^r t^s}(\cdot,t)\|_2 \leq C, \tag{3.23}$$

where $t \in [0,T], \epsilon \in (0,\epsilon_0]$ and ϵ is a positive constant. The constant C is only depends on $T, \|\varphi\|_{H_k}, \|g\|_{k,\infty}$.

This completes the proof of Theorem1.2 by passing to the limit in equation(2.5) as $\epsilon \to 0$.

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