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# DYNAMIC BEHAVIOR OF A SEVEN-ORDER FUZZY DIFFERENCE EQUATION

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**Abstract** In this paper, we explore the qualitative features of a seven-order fuzzy difference equation

$$x_{n+1}=\frac{Ax_{n-1}}{B+Cx_{n-2}^{p}x_{n-4}^{q}x_{n-6}^{r}}, n=0,1,2,\cdots,$$

here the parameters $A,B,C\in R_f^+$, $p,q,r\in R^+$ and the initial values $x_{-6},\cdots,x_{-1}$, $x_0\in R_f^+$. Utilizing the fuzzy sets theory, linearization method, mathematical induction and inequality technique, we obtain some sufficient condition on the qualitative features including the boundedness of the positive solution of the equation and the stability and instability of the equilibrium point of the equation. Moreover, two simulation examples are presented to verify the effectiveness of our proposed results.

**Keywords** Fuzzy difference equation, equilibrium point, boundedness, stability, unstability.

**MSC(2010)** 39A10.

## 1. Introduction and Preliminaries

Difference equations are usually expressed in the form of recursive sequences, which are widely used in various practical problems. It is worth noting that the difference equation model has been important developed and applied in the fields of infectious disease dynamics [29], computer [30], engineering [16], biology [18], finance [6]. A large number of studies show that the dynamic behavior of difference equations with order greater than 1 is relatively complex, but can be better applied to practical problems. Therefore, it has attracted the interest of many experts and scholars, and also produced a large number of meaningful results [1,3,7,8,10,12,13,19]. In 2009, Ahmed [4] discussed the global asymptotic behavior and periodic characteristics of the solution for the rational difference equation

$$x_{n+1}=\frac{bx_{n-1}}{A+Bx_n^p x_{n-2}^q}, n=0,1,2\cdots, \tag{1.1}$$

where the parameters $b,A,B,p,q$ and the initial condition $x_{-2}$,$x_{-1}$,$x_0$ are any non-negative real number. In 2012, Stević [24] studied the dynamical behavior of the following higher-order difference equation

$$x_n=\frac{x_{n-k}}{b+cx_{n-1}\cdots x_{n-k}}, n\in\mathbb{N}_0, \tag{1.2}$$

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where $k \in \mathbb{N}$, the parameters $b, c$ and the initial values $x_{-k},\cdots,x_{-1}$ are real numbers. In 2015, Abo-Zeid [2] discussed the global asymptotic stability of all solutions for the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B + Cx_n x_{n-1} x_{n-2}}, n = 0, 1, \cdots, \tag{1.3}$$

where the parameters $A, B, C$ and the initial values $x_{-2},x_{-1},x_0$ are positive real number. In 2016, Erdogan [9] studied the global behavior for the following difference equation

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma x_n x_{n-1} x_{n-2} x_{n-3}}, n = 0, 1, \cdots, \tag{1.4}$$

where the parameters $\alpha, \beta$ an $\gamma$ are positive real numbers and the initial conditions $x_{-3},x_{-2},x_{-1}$ and $x_0$ are real numbers. In 2019, Belhannach [5] studied the boundedness, oscillation and global behavior of positive solutions for a class of nonlinear difference equations

$$y_{n+1} = \frac{a + b_0 y_{n-1} + b_1 y_{n-3}}{c + d y_n^{p_0} y_{n-2}^{p_1}}, n = 0, 1, \cdots, \tag{1.5}$$

where the parameters $a, b_i (i = 0, 1)$ and the initial conditions are non-negative real numbers, the parameters $c, d$ are positive real numbers, and the parameters $p_i (i = 0, 1)$ are positive integers. In 2021, Saleh et al. [20] obtained some results on the dynamics and bifurcations of the special case for the following second-order rational difference equations with quadratic terms

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx_n^2 + Cx_{n-1}}, n = 0, 1, 2, \cdots, \tag{1.6}$$

where $\alpha, \beta, A, B, C$ are positive real numbers and the initial values $x_{-1},x_0$ are non-negative real numbers.

In fact, with the development of set theory, the phenomena and concepts that cannot be described by classical set theory make it impossible for people to avoid ambiguity, thus prompting the generation of fuzzy sets, resulting in fuzzy mathematics. Therefore, with the continuous development of fuzzy mathematics, some scholars apply fuzzy numbers to difference equations to form the fuzzy difference equations [14,15,17,22,23,25–27,32,33], so that the established models can describe actual phenomena more appropriately.

In this paper, we study the following seven-order fuzzy difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + Cx_{n-2}^p x_{n-4}^q x_{n-6}^r}, n = 0, 1, 2, \cdots, \tag{1.7}$$

where the parameters $p, q, r$ are positive real number, the parameters $A, B, C$ and the initial conditions $x_{-6},\cdots,x_{-1},x_0$ are positive fuzzy numbers. When the parameters $p = q = r = 1$,$A, B, C$ and initial conditions are positive real numbers, Abo-Zeid [2] studied the global asymptotic stability of solutions to the above ordinary difference equation (1.7).

Next, we give some definitions and preliminary results, which can be founded in [11,21,31].

**Definition 1.1.** Let $X$ be a non-empty set, assume $T$ is a mapping from $X$ to $[0, 1]$ , a.e. $T{:}X \to [0, 1]$, $x \to T(x)$,$x \in X$, then we say $T$ is a fuzzy set on $X$, $T(x)$ be called a membership function on a fuzzy set $T$.

**Definition 1.2.** For a set $X$, we denote by $\overline{X}$ the closure of $X$. Assume $T$ is a fuzzy set and $\alpha \in (0,1]$, the $\alpha - cuts$ of $T$ on $R$ is defined as $[T]_\alpha = \{x \in R : T(x) \geq \alpha\}$ and $[T]_0 = \text{supp}(T)$. It is clear that the $[T]_\alpha$ is a bounded closed interval in $R$ for any specific $\alpha \in [0,1]$.

**Definition 1.3.** We say that a fuzzy set $T$ is a fuzzy number if it satisfies the following properties:

(i) $T$ is a normal fuzzy set, i.e., there exists $x \in R$ such that $T(x) = 1$;

(ii) $T$ is a fuzzy convex set, i.e., $T(ax + (1-a)y) \geq \min\{T(x), T(y)\}, \forall a \in (0,1), x, y \in R$;

(iii) $T$ is upper semicontinuous on $R$;

(iv) $T$ is compactly supported, i.e.,$\text{supp}(T) = \overline{U_{\alpha \in (0,1]}[T]_\alpha} = \overline{\{x \in R \mid T(x) > 0\}}$ is compact.

Let us denote by $R_f$ the set of all fuzzy numbers. A fuzzy number $T$ is positive if $\text{supp}(T) \subset (0, +\infty)$ , we denote by $R_f^+$ the space of all positive fuzzy numbers. Similarly, a fuzzy number $T$ is negative if $\text{supp}(u) \subset (-\infty, 0)$ we denote by $R_f^-$ the space of all negative fuzzy numbers. If $T$ is a positive real number, then $T$ is also a positive fuzzy number with $[T]_\alpha = [T, T]$,$\alpha \in [0,1]$, and we say that $T$ is a trivial fuzzy number.

**Definition 1.4.** For any $u, v \in R_f$, $[u]_\alpha = [u_{l,\alpha}, u_{r,\alpha}]$ ,$[v]_\alpha = [v_{l,\alpha}, v_{r,\alpha}]$ , and $\lambda \in R$, the sum $u + v$, the scalar product $\lambda u$, multiplication $uv$ and division $\frac{u}{v}$ in the standard interval arithmetic (SIA) setting are defined by

$$
\begin{aligned}
&[u+v]_\alpha = [u]_\alpha + [v]_\alpha, [\lambda u]_\alpha = \lambda [u]_\alpha, \quad \forall \alpha \in [0,1] \\
&[uv]_\alpha = \Big[ \min\{u_{l,\alpha}v_{l,\alpha}, u_{l,\alpha}v_{r,\alpha}, u_{r,\alpha}v_{l,\alpha}, u_{r,\alpha}v_{r,\alpha}\}, \\
&\qquad\qquad \max\{u_{l,\alpha}v_{l,\alpha}, u_{l,\alpha}v_{r,\alpha}, u_{r,\alpha}v_{l,\alpha}, u_{r,\alpha}v_{r,\alpha}\} \Big], \\
&\left[\frac{u}{v}\right]_\alpha = \left[ \min\left\{ \frac{u_{l,\alpha}}{v_{l,\alpha}}, \frac{u_{l,\alpha}}{v_{r,\alpha}}, \frac{u_{r,\alpha}}{v_{l,\alpha}}, \frac{u_{r,\alpha}}{v_{r,\alpha}} \right\}, \max\left\{ \frac{u_{l,\alpha}}{v_{l,\alpha}}, \frac{u_{l,\alpha}}{v_{r,\alpha}}, \frac{u_{r,\alpha}}{v_{l,\alpha}}, \frac{u_{r,\alpha}}{v_{r,\alpha}} \right\} \right], 0 \notin [v]_\alpha.
\end{aligned}
$$

**Definition 1.5.** Let $u, v$ be fuzzy numbers with $[u]_\alpha = [u_{l,\alpha}, u_{r,\alpha}]$, $[v]_\alpha = [v_{l,\alpha}, v_{r,\alpha}]$, $\alpha \in [0,1]$, then the metric of fuzzy numbers set is defined as follows

$$
D(u,v) = \sup_{\alpha \in [0,1]} \max\{|u_{l,\alpha} - v_{l,\alpha}|, |u_{r,\alpha} - v_{r,\alpha}|\}.
$$

And the norm on fuzzy numbers set is defined as follows

$$
\|u\| = \sup_{\alpha \in [0,1]} \max\{|u_{l,\alpha}|, |u_{r,\alpha}|\},
$$

then $(R_f, D)$ is a complete metric space. For the convenience of application in the future, we define $\hat{0} \in R_f$ as

$$
\hat{0} = \begin{cases} 1, x = 0, \\ 0, x \neq 0. \end{cases}
$$

Thus $[\hat{0}]_\alpha = [0,0], \alpha \in [0,1]$.

**Definition 1.6.** Let $x_n$ be a sequence of fuzzy numbers, if there exists a positive real number $N$ such that supp$x_n \subset (0, N]$ ,$n = 1, 2, \cdots$, then the fuzzy sequence $x_n$ is bounded.

Consider the system of difference equations

$$\begin{cases} x_{n+1} = f\left(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-l}\right), \\ y_{n+1} = g\left(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-l}\right), \end{cases} \quad n = 0, 1, \cdots, \tag{1.8}$$

where $I_x$,$I_y$ are the interval of real numbers, $f : I_x^{k+1} \times I_y^{l+1} \to I_x, g : I_x^{k+1} \times I_y^{l+1} \to I_y$ are continuous function.

**Definition 1.7.** The equilibrium point of the system of difference equations (1.8) refers to the solution of the following rational equations

$$x = f(x, x, \cdots, x, y, y, \cdots, y), y = g(x, x, \cdots, x, y, y, \cdots, y),$$

expressed as ($\overline{x}$,$\overline{y}$), and the equations (1.9) is called the equilibrium equations of equations (1.8). The equilibrium point of equations (1.8) refers to the solution $(x_n, y_n) = (\overline{x}, \overline{y})$ satisfying for $n \geq 0$ , which is also known as the equilibrium solution.

**Definition 1.8.** Assume that ($\overline{x}$,$\overline{y}$) be an equilibrium point of the difference system (1.8). Then, we have

(i) An equilibrium point ($\overline{x}$,$\overline{y}$) is called locally stable if for any $\epsilon > 0$, there exits $\delta(\epsilon) > 0$ such that for any initial conditions $(x_i, y_i) \in I_x \times I_y, (i = -k, \cdots, 0, j = -l, \cdots, 0)$ with $\sum_{i=-k}^{0} |x_i - \bar{x}| < \delta, \sum_{j=-l}^{0} |y_j - \bar{y}| < \delta$, we have $|x_n - \overline{x}| < \delta, |y_n - \overline{y}| < \delta|$ for any $n > 0$.

(ii) An equilibrium point ($\overline{x}$,$\overline{y}$) is called locally attractor if $\lim_{n\to\infty} x_n = \bar{x}, \lim_{n\to\infty} y_n = \bar{y}$ for any initial conditions $(x_i, y_i) \in I_x \times I_y, (i = -k, \cdots, 0, j = -l, \cdots, 0)$.

(iii) An equilibrium point ($\overline{x}$,$\overline{y}$) is called asymptotically stable if it is stable and attractor.

(iv) An equilibrium point ($\overline{x}$,$\overline{y}$) is called unstable if it is not locally stable.

**Definition 1.9.** Let ($\overline{x}$,$\overline{y}$) be an equilibrium point of the vector map $F = (f, x_n, \cdots, x_{n-k}, g, y_n, \cdots, y_{n-l})$, where $f$ and $g$ are continuously differential functions at ($\overline{x}$,$\overline{y}$). The linearized system of symmetric system (1.8) about the equilibrium ($\overline{x}$,$\overline{y}$) point is $X_{n+1} = F(X_n) = F_j \cdot X_n$, where $F_j$ is the Jacobian matrix of the system (1.8) about ($\overline{x}$,$\overline{y}$) and $X_n = (x_n, \cdots, x_n - k, y_n, \cdots, y_{n-k})^T$.

**Lemma 1.1.** *Let $f : R^+ \times R^+ \times R^+ \to R^+$ be a continuous function and $A, B, C$ be fuzzy number, then*

$$[f(A, B, C)]_\alpha = f\left([A]_\alpha, [B]_\alpha, [C]_\alpha\right), \alpha \in (0, 1].$$

**Lemma 1.2.** *Let $I_x, I_y$ be some intervals of real numbers and let $f : I_x^{k+1} \times I_y^{l+1} \to I_x$,$g : I_x^{k+1} \times I_y^{l+1} \to I_x$ be continuously differentiable functions. Then for every set of initial conditions $(x_i, y_j) \in I_x \times I_y, (i = -k, -k+1, \cdots, 0, j = -l, -l+1, \cdots, 0)$, the difference equations (1.8) has a unique solution $\{(x_i, y_j)\}_{i=-k,j=-l}^{+\infty,+\infty}$.*

**Lemma 1.3.** *Assume that $X(n+1) = F(X(n)), n = 0, 1, \cdots,$ is a system of difference equations and $\overline{X}$ is the equilibrium point of this system i.e.,$F(\overline{X}) = \overline{X}$. Then we have*

*(i) If all eigenvalues of the Jacobian matrix $J_F$ about $\overline{X}$ lie inside the open unit disk, i.e.,$|\lambda| < 1$ , then $\overline{X}$ is locally asymptotically stable.*

*(ii) If one of eigenvalues of the Jacobian matrix $J_F$ about $\overline{X}$ has norm greater than one, then overline$X$ is unstable.*

**Lemma 1.4** (Theorem 1, [28])**.** *For the fuzzy difference equation (1.7), if the parameters $p, q, r$ are positive real number, the parameters $A, B, C$ and initial conditions $x_{-6}, \cdots, x_{-1}, x_0$ are positive fuzzy numbers, then for any initial conditions $x_{-6}, \cdots, x_{-1}, x_0$ there exists a unique positive fuzzy solution $\{x_n\}$ to the fuzzy difference equation (1.7).*

## 2. Main results and proofs

In this section, by using fuzzy set theory, iterative method, inequality technology and mathematical induction as well as the above **Lemmas**, the dynamic properties of the seven-order exponential fuzzy difference equation (1.7) are studied.

If $\{x_n\}$ is the solution of equation (1.7) with initial conditions $x_{-6}$, $x_{-5}$, $x_{-4}$, $x_{-3}$, $x_{-2}$, $x_{-1}$, $x_0$ and satisfies $[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0, 1], n = 0, 1, \cdots$ , then from **Lemma 1.1** $(L_{n,\alpha}, R_{n,\alpha})$ satisfy the following difference equations

$$\begin{cases} L_{n+1,\alpha} = \dfrac{A_{l,\alpha} L_{n-1,\alpha}}{B_{r,\alpha} + C_{r,\alpha} R^p_{n-2,\alpha} R^q_{n-4,\alpha} R^r_{n-6,\alpha}}, \\ R_{n+1,\alpha} = \dfrac{A_{r,\alpha} R_{n-1,\alpha}}{B_{l,\alpha} + C_{l,\alpha} L^p_{n-2,\alpha} L^q_{n-4,\alpha} L^r_{n-6,\alpha}}, \end{cases} \alpha \in (0, 1], n = 0, 1, \cdots . \tag{2.1}$$

In order to study the asymptotic properties of the solution for fuzzy difference equation (1.7), the above equations can be abbreviated as the following rational difference equations

$$y_{n+1} = \frac{a y_{n-1}}{c + e z^p_{n-2} z^q_{n-4} z^r_{n-6}}, z_{n+1} = \frac{b z_{n-1}}{d + f y^p_{n-2} y^q_{n-4} y^r_{n-6}}, n = 0, 1, \cdots , \tag{2.2}$$

where the parameter $a, b, c, d, e, f, p, q, r$ are positive real numbers and satisfies $a \le b$, $c \le d$, $e \le f$, initial conditions $y_{-6}$, $y_{-5}$, $y_{-4}$, $y_{-3}$, $y_{-2}$, $y_{-1}$, $y_0$, $z_{-6}$, $z_{-5}$, $z_{-4}$, $z_{-3}$, $z_{-2}$, $z_{-1}$, $z_0$ are also positive real numbers.

According to Lemma 1.2, it can be known that for any given initial values, the difference equations (2.2) have a unique solution $(y_n, z_n)$, and it is easy to know that the equation system (2.2) has an equilibrium point $\overline{X}_1 = (0, 0)$. When $a > c, b > d$, the system (2.2) has 2nd equilibrium point

$$\overline{X}_2 = \left( \sqrt[p+q+r]{\tfrac{b-d}{f}}, \sqrt[p+q+r]{\tfrac{a-c}{e}} \right).$$

**Theorem 2.1.** *For the equilibrium point $\overline{X}_1$ of difference equation system (2.2), we have the following conclusions:*

*(i) When $a < c$ and $b < d$, the equilibrium point $\overline{X}_1$ is locally asymptotically stable.*

*(ii) When $a > c$ or $b > d$, the equilibrium point $\overline{X}_1$ is unstable.*

**Proof.** Define functions $F : (R^{+})^4 \to R^{+}, H : (R^{+})^4 \to R^{+}$ as

$$F\left(y_{n-1}, z_{n-2}, z_{n-4}, z_{n-6}\right) = \frac{a y_{n-1}}{c + e z_{n-2}^{p} z_{n-4}^{q} z_{n-6}^{r}},$$
$$H\left(z_{n-1}, y_{n-2}, y_{n-4}, y_{n-6}\right) = \frac{b z_{n-1}}{d + f y_{n-2}^{p} y_{n-4}^{q} y_{n-6}^{r}}.$$

Therefore, to find its partial derivative, we have

$$\begin{aligned}
F_{y_{n-1}} &= \frac{a}{c + e z_{n-2}^{p} z_{n-4}^{q} z_{n-6}^{r}}, & F_{z_{n-2}} &= -\frac{aep y_{n-1} z_{n-2}^{p-1} z_{n-4}^{q} z_{n-6}^{r}}{\left(c + e z_{n-2}^{p} z_{n-4}^{q} z_{n-6}^{r}\right)^2},\\
F_{z_{n-4}} &= -\frac{\mathrm{aeq}\, y_{n-1} z_{n-2}^{p} z_{n-4}^{q-1} z_{n-6}^{r}}{\left(c + e z_{n-2}^{p} z_{n-4}^{q} z_{n-6}^{r}\right)^2}, & F_{z_{n-6}} &= -\frac{aer y_{n-1} z_{n-2}^{p} z_{n-4}^{q} z_{n-6}^{r-1}}{\left(c + e z_{n-2}^{p} z_{n-4}^{q} z_{n-6}^{r}\right)^2},\\
H_{z_{n-1}} &= \frac{b}{d + f y_{n-2}^{p} y_{n-4}^{q} y_{n-6}^{r}}, & H_{y_{n-2}} &= -\frac{bfp z_{n-1} y_{n-2}^{p-1} y_{n-4}^{q} y_{n-6}^{r}}{\left(d + f y_{n-2}^{p} y_{n-4}^{q} y_{n-6}^{r}\right)^2},\\
H_{y_{n-4}} &= -\frac{bfq z_{n-1} y_{n-2}^{p} y_{n-4}^{q-1} y_{n-6}^{r}}{\left(d + f y_{n-2}^{p} y_{n-4}^{q} y_{n-6}^{r}\right)^2}, & H_{y_{n-6}} &= -\frac{bfr z_{n-1} y_{n-2}^{p} y_{n-4}^{q} y_{n-6}^{r-1}}{\left(d + f y_{n-2}^{p} y_{n-4}^{q} y_{n-6}^{r}\right)^2}
\end{aligned} \tag{2.3}$$

From (2.3), the linear equation of equation (2.2) about the equilibrium point $\overline{X}_1 = (0, 0)$ can be constructed as

$$\varphi_{n+1} = D_1 \varphi_n, \tag{2.4}$$

where

$$\Phi_n = \begin{bmatrix} y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ y_{n-4} \\ y_{n-5} \\ y_{n-6} \\ z_n \\ z_{n-1} \\ z_{n-2} \\ z_{n-3} \\ z_{n-4} \\ z_{n-5} \\ z_{n-6} \end{bmatrix}, \quad D_1 = \begin{bmatrix}
0 & \frac{a}{c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{b}{d} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.$$

Then the characteristic equation of equation (2.4) is

$$\left(\lambda^7 - \tfrac{a}{c}\lambda^5\right)\left(\lambda^7 - \tfrac{b}{d}\lambda^5\right) = 0.$$

Therefore, if $a < c$ and $b < d$ , then all $|\lambda| < 1$ can be obtained. Thus, from Lemma 1.3 the equilibrium point $\overline{X}_1$ is locally asymptotically stable. If $a > c$ or $b > d$, then the equilibrium point $\overline{X}_1$ is unstable by means of Lemma 1.3. □

**Theorem 2.2.** *When $a > c, b > d$, the difference equations (2.2) have a unique positive equilibrium $\overline{X}_2 = \left(\sqrt[p+q+r]{\frac{b-d}{f}}, \sqrt[p+q+r]{\frac{a-c}{e}}\right)$ and this equilibrium is unstable.*

**Proof.** According to (2.3), the linear equation of equations (2.2) about the equilibrium point $\overline{X}_1$ is

$$\varphi_{n+1} = D_2\varphi_n, \tag{2.5}$$

where

$$\Phi_n = \begin{bmatrix} y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ y_{n-4} \\ y_{n-5} \\ y_{n-6} \\ z_n \\ z_{n-1} \\ z_{n-2} \\ z_{n-3} \\ z_{n-4} \\ z_{n-5} \\ z_{n-6} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{epM}{a} & 0 & -\frac{eqM}{a} & 0 & -\frac{erM}{a} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{fpN}{b} & 0 & -\frac{fqN}{b} & 0 & -\frac{frN}{b} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

In matrix $D_2$ above

$$M = \sqrt[p+q+r]{\frac{b-d}{f}}\left(\frac{a-c}{e}\right)^{\frac{p+q+r-1}{p+q+r}}, N = \sqrt[p+q+r]{\frac{a-c}{e}}\left(\frac{b-d}{f}\right)^{\frac{p+q+r-1}{p+q+r}},$$

let $\frac{efMN}{ab} = T$, then the characteristic equation of equation (2.5) is as follow

$$\lambda^{14} - 2\lambda^{12} + \lambda^{10} - p^2T\lambda^8 - 2pqT\lambda^6 - \left(2abprT + abq^2T\right)\lambda^4 - 2qrT\lambda^2 - r^2T = 0. \tag{2.6}$$

Let

$$f(\lambda) = \lambda^{14} - 2\lambda^{12} + \lambda^{10} - p^2T\lambda^8 - 2pqT\lambda^6 - \left(2abprT + abq^2T\right)\lambda^4 - 2qrT\lambda^2 - r^2T. \tag{2.7}$$

According to the above function (2.7), we have

$$f(1) = -p^2 - 2pqT - \big(2abprT + abq^2T\big) - 2qrT - r^2T < 0.$$

It is obvious that $\lim\limits_{\lambda\to+\infty} f(\lambda) = +\infty$, so the equation (2.6) has at least one root in the interval $(1, +\infty)$. According to Lemma 1.3, the equilibrium point $\overline{X}_2$ of equations (2.2) is unstable. □

**Theorem 2.3.** *Let $(y_n, z_n)$ be a positive solution of the difference equations (2.2), if $a \le b \le c \le d$, the positive solution of the equations (2.2) is bounded.*

**Proof.** To prove that the positive solution of the equations (2.2) is bounded, we only need to prove that the following inequalities hold.

$$0 \le y_n \le \left(\frac{a}{c}\right)^{m+1} y_{-1} \le y_{-1},\ \text{µ±}\ n = 2m+1, m = 0, 1, \cdots,$$
$$0 \le y_n \le \left(\frac{a}{c}\right)^{m} y_0 \le y_0,\ \text{µ±}\ n = 2m, m = 0, 1, \cdots,$$
$$0 \le z_n \le \left(\frac{b}{d}\right)^{m+1} z_{-1} \le z_{-1},\ \text{µ±}\ n = 2m+1, m = 0, 1, \cdots,$$

and

$$0 \le z_n \le \left(\tfrac{b}{d}\right)^{m} z_0 \le z_0,\ \text{µ±}\ n = 2m, m = 0, 1, \cdots.$$

When $m = 0$, the above inequalities obviously hold. Now suppose that the above inequalities also hold when $m = k$, namely

$$0 \le y_n \le \left(\frac{a}{c}\right)^{k+1} y_{-1} \le y_{-1},\ \text{µ±}\ n = 2k+1, k = 0, 1, \cdots,$$
$$0 \le y_n \le \left(\frac{a}{c}\right)^{k} y_0 \le y_0,\ \text{µ±}\ n = 2k, k = 0, 1, \cdots,$$
$$0 \le z_n \le \left(\frac{b}{d}\right)^{k+1} z_{-1} \le z_{-1},\ \text{µ±}\ n = 2k+1, k = 0, 1, \cdots,$$

and

$$0 \le z_n \le \left(\tfrac{b}{d}\right)^{k} z_0 \le z_0\ \text{£¬µ±}\ n = 2k, k = 0, 1, \cdots.$$

Thus, when $m = k+1$, from equations (2.2) we have

$$0 \le y_{2(k+1)+1} \le \frac{a}{c} y_{2(k+1)-1} \le \left(\frac{a}{c}\right)^{k+2} y_{-1} \le y_{-1},$$
$$0 \le y_{2(k+1)} \le \frac{a}{c} y_{2k} \le \left(\frac{a}{c}\right)^{k+1} y_0 \le y_0,$$
$$0 \le z_{2(k+1)+1} \le \frac{a}{c} z_{2(k+1)-1} \le \left(\frac{b}{d}\right)^{k+2} z_{-1} \le z_{-1},$$

and

$$0 \le z_{2(k+1)} \le \tfrac{a}{c} z_{2k} \le \left(\tfrac{b}{d}\right)^{k+1} z_0 \le z_0.$$

According to mathematical induction, the proof is completed. □

**Theorem 2.4.** *Consider fuzzy difference equation (1.7), where the parameters $A, B, C$ are positive fuzzy numbers, the parameters $p, q, r$ are positive real numbers, and the initial conditions $x_i, (i = -6, -5, \cdots, 0)$ are positive fuzzy numbers. If the parameters satisfy $A_{l,\alpha} \le A_{r,\alpha} \le B_{l,\alpha} \le B_{r,\alpha}$, then the positive solutions of fuzzy difference equation (1.7) are all bounded.*

**Proof.** Assume that $x_n$ is a positive solution of fuzzy difference equation (1.7) based on the initial condition $x_i, (i = -6, -5, \cdots)$, and the parameter $A, B, C$ are positive fuzzy numbers. Consider the $\alpha - cuts, \alpha \in (0, 1]$, then we have

$$[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}], [B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]$$
$$[C]_\alpha = [C_{l,\alpha}, C_{r,\alpha}], [x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], n = -6, -5, \cdots.$$

According to Lemma 1.4, $(L_{n,\alpha}, R_{n,\alpha}), \alpha \in (0, 1]$ satisfies equation (2.1), and from Theorem 2.3 there are $L_{2m+1,\alpha} \le L_{-1,\alpha}, L_{2m,\alpha} \le L_{0,\alpha}, R_{2m+1,\alpha} \le R_{-1,\alpha}, R_{2m,\alpha} \le R_{0,\alpha}, m = 0, 1, \cdots$, namely there are two positive real numbers $\mu v$ satisfying $0 \le L_{n,\alpha} \le \mu, 0 \le R_{n,\alpha} \le v$, where $\mu = \max\{L_{-1,\alpha}, L_{0,\alpha}\}, v = \max\{R_{-1,\alpha}, R_{0,\alpha}\}$. Therefore, set $N = max\{\mu, v\}$, we have

$$\operatorname{supp}(x_n) = \overline{\bigcup_{\alpha \in (0,1]} [x_n]_\alpha} = \bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}] \subset (0, N],$$

the proof is completed. □

**Theorem 2.5.** *Consider fuzzy difference equation (1.7), where the parameter $A$, $B$, $C$ are positive fuzzy numbers, the parameters $p, q, r$ are positive real numbers, and the initial conditions $x_i, i = -6, -5, \cdots, 0$ are positive fuzzy numbers. The equilibrium point $[\hat{0}]_\alpha = [0, 0]$ of equation (1.7) is globally asymptotically stable if the parameters satisfy $A_{r,\alpha} < B_{l,\alpha}$.*

**Proof.** According to Theorem 2.1, if $A_{r,\alpha} < B_{l,\alpha}$, it is easy to know that the equilibrium point $[\hat{0}]_\alpha = [0, 0]$ of equation (1.7) is locally asymptotically stable. According to the system (2.1), we have

$$\begin{aligned} L_{n+1,\alpha} &= \frac{A_{l,\alpha} L_{n-1,\alpha}}{B_{r,\alpha} + C_{r,\alpha} R^p_{n-2,\alpha} R^q_{n-4,\alpha} R^r_{n-6,\alpha}} \le \frac{A_{l,\alpha} L_{n-1,\alpha}}{B_{r,\alpha}}, \\ R_{n+1,\alpha} &= \frac{A_{r,\alpha} R_{n-1,\alpha}}{B_{l,\alpha} + C_{l,\alpha} L^p_{n-2,\alpha} L^q_{n-4,\alpha} L^r_{n-6,\alpha}} \le \frac{A_{r,\alpha} R_{n-1,\alpha}}{B_{l,\alpha}}. \end{aligned} \tag{2.8}$$

Let $\frac{A_{l,\alpha}}{B_{r,\alpha}} = k$, from (2.8), we can get

$$L_{n+1,\alpha} \le k L_{n-1,\alpha}. \tag{2.9}$$

According to (2.9), we can get the following inequality

$$\begin{aligned} &L_{2m,\alpha} \le k L_{2(m-1),\alpha} \le k^2 L_{2(m-2),\alpha} \le \cdots \le k^m L_{0,\alpha}, n = 2m, m = 0, 1, 2, \cdots, \\ &L_{2m+1,\alpha} \le k L_{2(m-1)+1,\alpha} \le k^2 L_{2(m-2)+1,\alpha} \le \cdots \le k^m L_{1,\alpha}, n = 2m+1, m = 0, 1, 2, \cdots. \end{aligned}$$

According to the conditions of Theorem 2.5, we can get $0 < k < 1$ and $L_{0,\alpha}, L_{1,\alpha}$ are positive real numbers, so $\lim_{m \to +\infty} L_{2m,\alpha} = 0$, $\lim_{m \to +\infty} L_{2m+1,\alpha} = 0$, and thus get $\lim_{n \to +\infty} L_{n,\alpha} = 0$. In the same way, we can prove $\lim_{n \to +\infty} R_{n,\alpha} = 0$ ,and the proof is completed. □

## 3. Numerical example

In this section, two examples are given to verify the asymptotic stability and unstability of the equilibrium points of the fuzzy difference equation (1.7).

**Example 3.1.** Considering equation (1.7) when $p = 5, q = 5, r = 5$, that is the following fuzzy difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + Cx_{n-2}^5 x_{n-2}^5 x_{n-2}^5}, n = 0, 1 \cdots, \tag{3.1}$$

where $A, B, C$ are positive fuzzy numbers, according to Theorem 2.5, the parameters $A, B, C$ satisfying $A_{r,\alpha} < B_{l,\alpha}$ can be defined as

$$[A]_\alpha = [2+\alpha, 5-2\alpha], [B]_\alpha = [11+\alpha, 13-\alpha], [C]_\alpha = [2+2\alpha, 6-2\alpha].$$

The initial conditions are defined as

$$x_0(x) = \begin{cases} x-3, & 3 \le x \le 4, \\ -\frac{1}{2}x+3, & 4 \le x \le 6, \end{cases} \qquad x_{-1}(x) \begin{cases} \frac{1}{2}x-2, & 4 \le x \le 6, \\ -x+7, & 6 \le x \le 7, \end{cases}$$
$$x_{-2}(x) = \begin{cases} 5x-1.5, & 0.3 \le x \le 0.5, \\ -5x+3.5, & 0.5 \le x \le 0.7, \end{cases} \qquad x_{-3}(x) = \begin{cases} x-2, & 2 \le x \le 3, \\ -\frac{1}{3}x+2, & 3 \le x \le 6. \end{cases} \tag{3.2}$$

and

$$x_{-4}(x) = \begin{cases} 10x-2, & 0.2 \le x \le 0.3, \\ -10x+4, & 0.3 \le x \le 0.4, \end{cases} \qquad x_{-5}(x) = \begin{cases} \frac{1}{2}x-1, & 2 \le x \le 4, \\ -x+5, & 4 \le x \le 5, \end{cases}$$
$$x_{-6}(x) = \begin{cases} \frac{10}{3}x-\frac{1}{3}, & 0.1 \le x \le 0.4, \\ -10x+5, & 0.4 \le x \le 0.5. \end{cases} \tag{3.3}$$

From (3.2) and (3.3), we can obtain

$$[x_0]_\alpha = [3+\alpha, 6-2\alpha], [x_{-1}]_\alpha = [4+2\alpha, 7-\alpha], [x_{-2}]_\alpha = [0.3+0.2\alpha, 0.7-0.2\alpha]$$
$$[x_{-3}]_\alpha = [2+\alpha, 6-3\alpha], [x_{-4}]_\alpha = [0.2+0.1\alpha, 0.4-0.1\alpha], [x_{-5}]_\alpha = [2+2\alpha.5-\alpha]$$
$$[x_{-6}]_\alpha = [0.1+0.3\alpha, 0.5-0.1\alpha].$$

According to equation (3.1), an ordinary difference equations with parameter $\alpha$ is established as

$$\begin{aligned} L_{n+1,\alpha} &= \frac{A_{l,\alpha} L_{n-1,\alpha}}{B_{r,\alpha} + C_{r,\alpha} R_{n-2,\alpha}^5 R_{n-4,\alpha}^5 R_{n-6,\alpha}^5}, \\ R_{n+1,\alpha} &= \frac{A_{r,\alpha} R_{n-1,\alpha}}{B_{l,\alpha} + C_{l,\alpha} L_{n-2,\alpha}^5 L_{n-4,\alpha}^5 L_{n-6,\alpha}^5}. \end{aligned} \tag{3.4}$$

From Theorem 2.4, we have that every positive solution $x_n$ of the fuzzy difference equation (3.1) is bounded. In addition, from Theorem 2.5, the zero equilibrium point $x = \hat{0}$ of equation (3.1) is globally asymptotically stable with respect to $D$. (See Figure 1-Figure 4). From the difference equations (2.1), it can be known that when $A_{l,\alpha} > B_{r,\alpha}$, the fuzzy difference equation (1.7) has a unique positive equilibrium point

$$[\bar{x}]_\alpha = [\bar{L}_\alpha, \bar{R}_\alpha] = \left[ \sqrt[p+q+r]{\frac{A_{r,\alpha} - B_{l,\alpha}}{C_{l,\alpha}}}, \sqrt[p+q+r]{\frac{A_{l,\alpha} - B_{r,\alpha}}{C_{r,\alpha}}} \right],$$

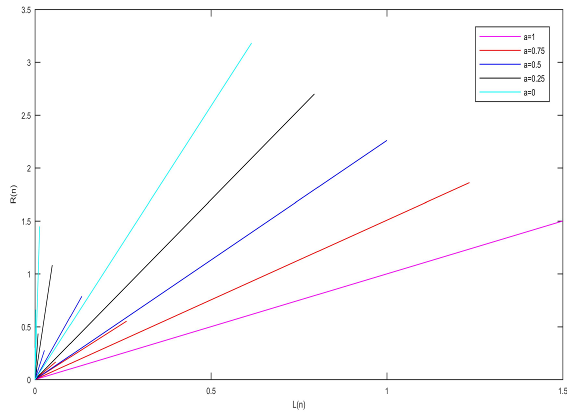


**Figure 1.** The dynamics of system (3.4).

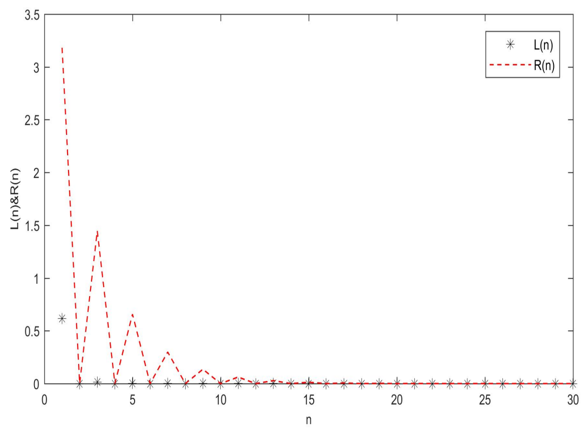


**Figure 2.** The solution of system (3.4) at $\alpha = 0$.

and a zero equilibrium point $x = \hat{0}$. According to Theorem 2.1 and Theorem 2.2, the two equilibrium points are all unstable. An example is given below to demonstrate the instability of the positive equilibrium point $[\overline{x}]_\alpha$ and zero equilibrium point $x = \hat{0}$ of the fuzzy difference equation (1.7).

**Example 3.2.** Consider the fuzzy difference equation (1.7) in $p = 1, q = 1, r = 1$, that is the following fuzzy difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + Cx_{n-2}x_{n-4}x_{n-6}}, n = 0, 1 \cdots . \tag{3.5}$$

where $A, B, C$ are positive fuzzy numbers, according to Theorem 2.2, the parameters $A, B, C$ satisfying $A_{l,\alpha} > B_{r,\alpha}$ can be defined as

$$[A]_\alpha = [2.1 + 0.1\alpha, 2.3 - 0.1\alpha], [B]_\alpha = [1.8 + 0.1\alpha, 2 - 0.1\alpha],$$
$$[C]_\alpha = [14 + 0.1\alpha, 14.2 - 0.1\alpha].$$

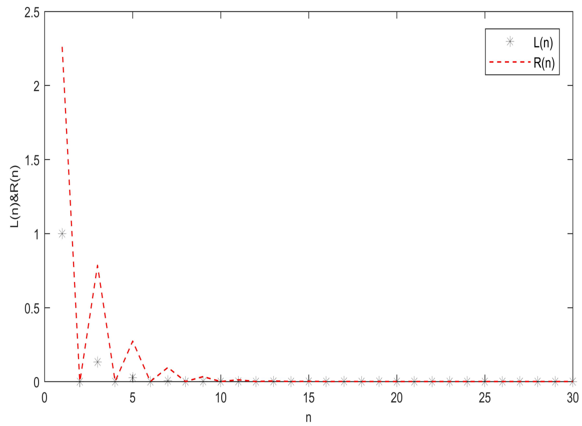


**Figure 3.** The solution of system (3.4) at $\alpha = 0.5$.

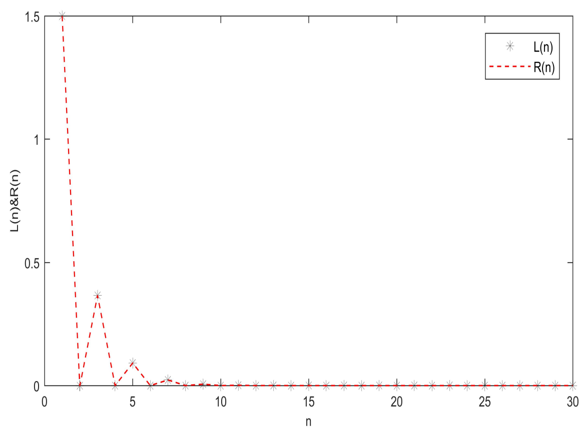


**Figure 4.** The solution of system (3.4) at $\alpha = 1$.

Take the initial values as follows

$$\begin{aligned}
&[x_0]_\alpha = [2.3 + 0.1\alpha, 2.5 - 0.1\alpha], [x_{-1}]_\alpha = [1.3 + 0.1\alpha, 1.5 - 0.1\alpha],\\
&[x_{-2}]_\alpha = [3.2 + 0.1\alpha, 3.4 - 0.1\alpha], [x_{-3}]_\alpha = [1.4 + 0.1\alpha, 1.7 - 0.1\alpha],\\
&[x_{-4}]_\alpha = [2.5 + 0.1\alpha, 2.7 - 0.1\alpha], [x_{-5}]_\alpha = [0.4 + 0.1\alpha, 0.6 - 0.1\alpha],\\
&[x_{-6}]_\alpha = [1.3 + 0.1\alpha, 1.5 - 0.1\alpha].
\end{aligned}$$

According to equation (3.5), an ordinary difference equations with parameter $\alpha$ is established as

$$\begin{aligned}
L_{n+1,\alpha} &= \frac{A_{l,\alpha} L_{n-1,\alpha}}{B_{r,\alpha} + C_{r,\alpha} R_{n-2} R_{n-4} R_{n-6}},\\
R_{n+1,\alpha} &= \frac{A_{r,\alpha} R_{n-1,\alpha}}{B_{l,\alpha} + C_{l,\alpha} L_{n-2} L_{n-4} L_{n-6}}.
\end{aligned} \tag{3.6}$$

It is easy to see that the equation (3.6) has a positive equilibrium point

$$[\bar{x}]_\alpha = \left[\bar{L}_\alpha, \bar{R}_\alpha\right] = \left[\sqrt[3]{\frac{A_{l,\alpha} - B_{r,\alpha}}{C_{r,\alpha}}}, \sqrt[3]{\frac{A_{r,\alpha} - B_{l,\alpha}}{C_{l,\alpha}}}\right] = \left(\sqrt[3]{\frac{0.1+0.2\alpha}{14.2-0.1\alpha}}, \sqrt[3]{\frac{0.5-0.2\alpha}{14+0.1\alpha}}\right),$$

and a zero equilibrium point. According to Theorem 2.1 and Theorem 2.2, the two equilibrium points are all unstable with respect to $D$. (See Figure 5-Figure 7).

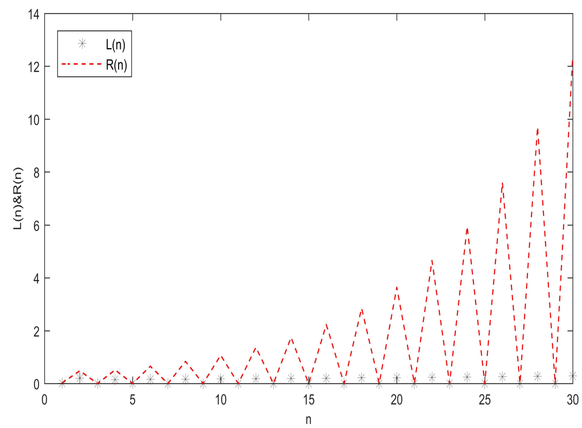


**Figure 5.** The solution of system (3.5) at $\alpha = 0$.

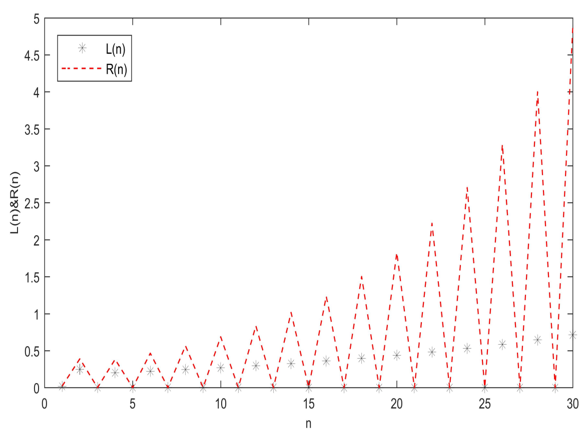


**Figure 6.** The solution of system (3.5) at $\alpha = 0.5$.

# 4. Conclusion

This paper mainly discusses the dynamic properties of a class of seventh-order fuzzy difference equations by utilizing the fuzzy sets theory, linearization method, mathematical induction and inequality technique. The main results are as follows:

(i) When $A_{r,\alpha} < B_{l,\alpha}$, the positive solution of the fuzzy difference equation (1.7) is bounded, and the unique zero equilibrium point of equation (1.7) is globally asymptotically stable with respect to $D$.

(ii) When $A_{l,\alpha} > B_{r,\alpha}$, the fuzzy difference equation (1.7) have two equilibrium points which are unstable with respect to $D$.

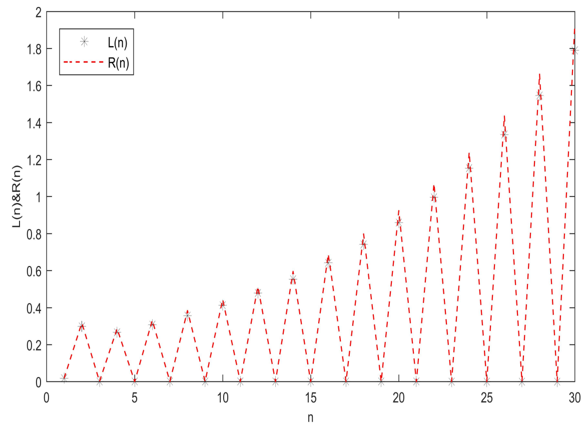


**Figure 7.** The solution of system (3.5) at $\alpha = 1$.

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