Journal of Applied Analysis and Computation
Volume 13, Number 1, February 2023, 502–525
Website:http://www.jaac-online.com
DOI:10.11948/20220442

EXISTENCE AND UNIQUENESS OF PERIODIC WAVES FOR A PERTURBED SEXTIC GENERALIZED BBM EQUATION

Yanfei Dai[1,†] and Minzhi Wei[2]

Abstract This paper is devoted to the existence and uniqueness of periodic waves for a perturbed sextic generalized BBM equation with weak backward diffusion and dissipation effects. By applying geometric singular perturbation theory and analyzing the perturbations of a Hamiltonian system with a hyper-elliptic Hamiltonian of degree seven, we prove the existence and uniqueness of periodic wave solutions with each wave speed in an open interval. It is also proved that the periodic wave solution persists for any energy parameter h in an open interval and sufficiently small perturbation parameter. Furthermore, we prove that the wave speed $c_0(h)$ is strictly monotonically increasing with respect to h by analyzing Abelian integral having three generating elements. Moreover, the upper and lower bounds of the limiting wave speed are obtained.

Keywords BBM equation, hyper-elliptic Hamiltonian system, geometric singular perturbation theory, periodic waves, Abelian integral.

MSC(2010) 34C25, 34C60, 37C27.

1. Introduction

Traveling waves in nonlinear wave equations can describe many nonlinear complex phenomenon in physics, chemistry, biology, mechanics, optics, etc. Over the past few decades, the existence, uniqueness and bifurcations of traveling wave solutions in various shallow water wave models including Korteweg-de Vries (KdV) equation [25], Benjamin-Bona-Mahony equation [2], Green-Naghdi equation [16] and Camassa-Holm equation [5, 28, 44], have attracted great attention. The classical, one dimensional KdV equation

$$u_t + \alpha u u_x + \beta u_{xxx} = 0 \tag{1.1}$$

was first proposed by Korteweg and de Vries [25], where α and β are two parameters. This nonlinear partial differential equation (NPDE) has played an important role in describing the traveling of shallow water waves with small amplitudes as well as other physical and biological problems. The Benjamin-Bona-Mahony (BBM) equation given by

$$u_t + u_x + u u_x - u_{xxt} = 0 \tag{1.2}$$

†The corresponding author. Email: gnsydyf@126.com (Y. Dai)
[1]Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, 321004, China
[2]School of Mathematics and Quantitative Economics, Guangxi University of Finance and Economics, Nanning, Guangxi, 530003, China

is a nonlinear, dispersive equation which describes unidirectional propagation of weakly long surface waves in the presence of dispersion [2]. It also covers cases of the following type: surface waves of long wavelength in liquids, hydromagnetic waves in cold plasma, acoustic-gravity waves in compressible fluids, acoustic waves in anharmonic crystals, etc. Compared with the well-known KdV equation, the BBM equation has technical advantages, so it is a superior model for investigation on long waves and attracts much attention from researchers in many fields. It is worth to point out that the BBM equation is not an evolution equation, although it is usually considered to be a nonlinear partial differential equation of evolution type. Micu [30] presented that the BBM equation is approximately controllable but not spectrally controllable and proved a finite controllability result. Singh et al. [34] investigated the symmetries of the BBM equation with variable coefficients by Lie-group method. Besse et al. [3] considered various approximations of artificial boundary conditions for linearized BBM equation and proved consistency, stability and convergence of the numerical scheme. Biswas [4] obtained the solitary wave solutions of BBM equation with dual power law nonlinearity by inverse scattering transform.

In 2005, Wazwaz [38] analyzed the physical structures of some nonlinear dispersive generalized forms of the BBM equation

$$(u^m)_t + \alpha(u^n)_x + \beta(u^l)_{xxx} = 0, \tag{1.3}$$

and found their compaction solutions, where α and β are real parameters and m, n and l are positive integers. The dynamics of the generalized BBM equation have been extensively studied, see for instance [33, 46, 47].

However, in the real world, weak influences due to the existence of uncertainties and perturbations are inevitable, e.g., when shallow water waves travel in nonlinear dissipative media and dispersive media. After taking small perturbations from diffusion and dissipation into account, one gets the perturbed KdV equation, the perturbed BBM equation, etc. The question of how bounded solutions, such as periodic wave and solitary wave solutions, may be affected by weak backward diffusion and dissipation effects naturally arises. In particular, some specific cases of the perturbed BBM equation with weak backward diffusion and dissipation effects given by

$$(u^m)_t + (u^n)_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0 \tag{1.4}$$

have been considered to some extent in numerous literature, where m and n are positive integers, $0 < \varepsilon \ll 1$ is a perturbation parameter, and u_{xx} and u_{xxxx} represent the backward diffusion and dissipation, respectively. The traveling wave solutions for Eq. (1.4) with $(m, n) = (1, 2)$, which belongs to the KdV equation (1.1) when $\varepsilon = 0$, were investigated in [11, 32]. Eq. (1.4) with $(m, n) = (1, 3)$ was investigated in [12, 43], where the authors proved the persistence of some solitary wave and periodic wave solutions with certain wave speeds under small perturbation. The authors in [6,45,48] established the existence of solitary waves or periodic waves for Eq. (1.4) with $(m, n) = (2, 3)$. Chen et al. [7] detected the existence of kink waves and periodic waves for the case when $(m, n) = (3, 1)$. The existence of periodic wave and solitary wave solutions for Eq. (1.4) with $m = 1$ and $n \in \mathbb{Z}^+$ was established by Yan et al. in [39] and Zhuang et al. in [49], respectively. Sun and Yu [35] studied the existence of periodic waves for Eq. (1.4) with $(m, n) = (3, 4)$. The existence of periodic wave and solitary wave solutions for Eq. (1.4) with $(m, n) = (3, 5)$ was investigated by Guo and Zhao in [17] and Wang et al. in [36], respectively. Recently, Wang et al. [37] established the existence of solitary wave solutions for Eq. (1.4) with $m, n \in \mathbb{Z}^+$ restricted on the case when its associated ODE system possesses homoclinic orbits to hyperbolic saddles. More recently, Dai et al. [9] studied the existence of isolated periodic wave solution with each wave speed in an open interval for Eq. (1.4) with $(m, n) = (4, 5)$. The geometric singular perturbation theory established by Fenichel [13] plays an important role in all these research works.

Motivated by the works above, one would ask the question spontaneously: whether the periodic wave and solitary wave solutions persist or vanish when the order of the first two terms in the perturbed generalized BBM equation (1.4) becomes higher? To this end, we consider the BBM equation (1.4) with $(m, n) = (5, 6)$, described by

$$(u^5)_t + (u^6)_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{1.5}$$

where $\varepsilon > 0$ is a perturbation parameter. The associated ODE of (1.5) with $\varepsilon = 0$ has a more degenerate singularity, in fact a nilpotent saddle of order 2. The aim of this paper is to investigate the existence of periodic waves for Eq. (1.5). The main mathematical tools used in this paper are based on the relatively new theory of weak Hilbert's 16th problem and bifurcation theory.

The rest of this paper is organized as follows. In Section 2, we give some notations and state our main results. Section 3 is devoted to some preliminary results including geometric singular perturbation theory, theory of Chebyshev system and method of discriminant function. In Section 4, we investigate the existence of periodic wave solutions for system (1.5) and give the proof of our main results. It will be shown that our method is more effective compared with the Picard-Fuchs Equation method used in [7, 17, 39]. With the help of software Maple-17, all results are proved by real analysis and symbolic computation. Finally, this paper ends with a conclusion.

2. Some notations and main results

In this section, we give some notations and present our main results for system (1.5).

The aim of this paper is to seek travelling wave solutions of Eq. (1.5). They will be solutions of Eq. (1.5) that are functions of the single variable $\xi = x - ct$, where $c > 0$ is the wave speed. Then the wave $u = u(\xi)$ must satisfy the following ordinary differential equation (ODE)

$$-5cu^4(\xi)u'(\xi) + 6u^5(\xi)u'(\xi) + u'''(\xi) + \varepsilon\left(u''(\xi) + u''''(\xi)\right) = 0, \tag{2.1}$$

where prime denotes the derivative with respect to ξ. Integrating this equation once with respect to ξ and omitting the integral constant, we get

$$-cu^5(\xi) + u^6(\xi) + u''(\xi) + \varepsilon\left(u'(\xi) + u'''(\xi)\right) = 0. \tag{2.2}$$

Consider the changes of variables

$$\mu(\tau) = \frac{u(\xi)}{c}, \quad \tau = c^{\frac{5}{2}}\xi, \tag{2.3}$$

then we have

$$u(\xi) = c\mu(\tau),\ u'(\xi) = c^{\frac{7}{2}}\mu'(\tau),\ u''(\xi) = c^6\mu''(\tau),\ u'''(\xi) = c^{\frac{17}{2}}\mu'''(\tau).$$

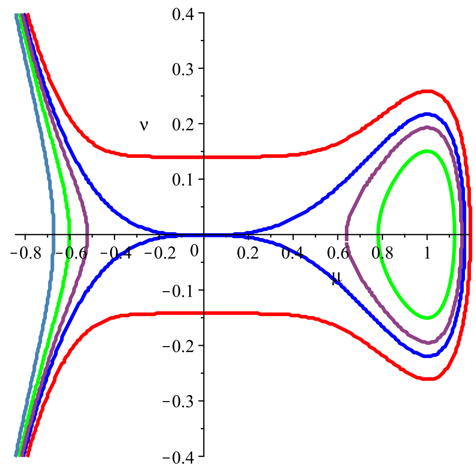

Figure 1. The phase portrait of system (2.5).

Taking the transformation (2.3) into the equation (2.2) yields

$$-\mu^5(\tau)+\mu^6(\tau)+\mu''(\tau)+\varepsilon\left(c^{-\frac{5}{2}}\mu'(\tau)+c^{\frac{5}{2}}\mu'''(\tau)\right)=0. \tag{2.4}$$

The discussions above tell us that if $\mu(\tau)$ a solution of Eq. (2.4) for some $\varepsilon>0$ and $c>0$, then $u(\xi)=c\mu(\tau)$ is a solution of Eq. (2.1), which corresponds to the travelling wave solution of the original equation (1.5).

System (1.5) when $\varepsilon=0$, i.e., $-\mu^5(\tau)+\mu^6(\tau)+\mu''(\tau)=0$, is called the corresponding unperturbed system, which is equivalent to the following two-dimensional system by setting $\nu=\mu'(\tau)$

$$\begin{cases}\dfrac{d\mu}{d\tau}=\nu,\\[2mm] \dfrac{d\nu}{d\tau}=\mu^5-\mu^6.\end{cases} \tag{2.5}$$

Clearly, system (2.5) is a Hamiltonian system with a Hamiltonian function of degree seven

$$H(\mu,\nu)=\frac{\nu^2}{2}-\frac{\mu^6}{6}+\frac{\mu^7}{7}. \tag{2.6}$$

It is well known that the global dynamics of system (2.5) is determined by its potential energy function and its equilibria. Obviously, system (2.5) has only two equilibria $(1,0)$ and $(0,0)$. By Section 3.4 of [19], it is not difficult to verify that the origin $(0,0)$ is a nilpotent saddle of order 2 and the equilibrium $(1,0)$ is a center. A direct calculation shows that $H(0,0)=H(7/6,0)=0$, $H(1,0)=-\frac{1}{42}$. The function $H(\mu,\nu)=h$ for $h\in(-1/42,0)$ and $\mu\in(0,7/6)$, depicted in Figure 1, shows a family of closed orbits surrounded by a homoclinic loop connecting with a nilpotent saddle of order 2 at the origin.

For convenience, we use the following notations: Γ_h denoting the curve in the μ-ν plane defined by $H(\mu,\nu)=h$; $\mu(\tau,h)$ being the μ-component of Γ_h; $\mu(\tau,h,c,\varepsilon)$ representing the traveling wave solutions of system (2.4). Denote $\mu_0(\tau)\triangleq\mu(\tau,0)$, which is the orbit of system (2.5) corresponding to Γ_0, where Γ_0 incudes a homoclinic orbit to the origin. Denote $c_0 = \frac{3}{7}\sqrt[5]{\frac{2717}{24}} \approx 1.104$, then our main results are as follows.

Theorem 2.1. *For the perturbed BBM equation* (1.5)*, the following statements hold.*

(i) For any given $c \in (1, c_0)$, there exists $\varepsilon_0(c) > 0$ such that when $0 < \varepsilon < \varepsilon_0(c)$, then Eq. (1.5) *has a unique isolated periodic wave solution with the wave speed c, given by $u = c\mu(\tau, h, c, \varepsilon)$, satisfying*

$$\lim_{\varepsilon \to 0} \mu(\tau, h, c, \varepsilon) = \mu(\tau, h), \quad \lim_{\substack{(c,\varepsilon)\to(1,0)\\ 0<\varepsilon<\varepsilon_0(c)}} \mu(\tau, h, c, \varepsilon) \to 1,$$

$$\lim_{\substack{(c,\varepsilon)\to(c_0,0)\\ 0<\varepsilon<\varepsilon_0(c)}} \mu(\tau, h, c, \varepsilon) \to \mu_0(\tau),$$

and

$$\frac{\partial}{\partial \tau}\mu(0, h, c, \varepsilon) = 0, \quad \frac{\partial^2}{\partial \tau^2}\mu(0, h, c, \varepsilon) > 0.$$

(ii) For any $h \in (-1/42, 0)$, there exists $\varepsilon^*(h) > 0$ such that when $0 < \varepsilon < \varepsilon^*(h)$, there exists a smooth function $c(h, \varepsilon)$ in h and ε such that Eq. (1.5) *has one unique isolated periodic wave solution in a sufficiently small neighborhood of Γ_h, given by $u = c(h, \varepsilon)\mu(\tau, h, c(h, \varepsilon), \varepsilon)$, where $\mu(\tau, h, c, \varepsilon)$ satisfies the same properties of the case* (*i*) *above. Furthermore, $c(h, \varepsilon)$ satisfies*

$$\lim_{\varepsilon \to 0} c(h, \varepsilon) = c(h), \quad \frac{\partial c(h, \varepsilon)}{\partial h} > 0,$$

where $c(h)$ is a strictly increasing function in h satisfying $1 < c(h) < c_0$.

3. Some preliminary results

In this section, we introduce some preliminary results, which will be useful for the proof of our main results.

3.1. Geometric singular perturbation theory

Firstly, we introduce the geometric singular perturbation theory which comes from Fenichel [13]. One can consult [23, 42] and references therein for details.

Consider the singularly perturbed differential system in $\mathbb{R}^{k+l}$,

$$\dot{x} = f(x, y, \lambda, \varepsilon), \quad \varepsilon\dot{y} = g(x, y, \lambda, \varepsilon), \quad (x, y) \in \Omega \subset \mathbb{R}^k \times \mathbb{R}^l, \tag{3.1}$$

with $k, l \in \mathbb{Z}^+$ the set of positive integers and Ω an open and connected subset of $\mathbb{R}^{k+l}$, where the dot denotes the derivative with respect to the time t, $\lambda \in \mathbb{R}^m$ are m-dimensional real parameters and $\varepsilon > 0$ is a sufficiently small real parameter. When $\varepsilon \neq 0$, with a change of time scaling $t = \varepsilon\tau$, system (3.1) can be rewritten as

$$x' = \varepsilon f(x, y, \lambda, \varepsilon), \quad y' = g(x, y, \lambda, \varepsilon), \quad (x, y) \in \Omega, \tag{3.2}$$

where $' = \frac{d}{d\tau}$. The independent variables t and τ are called *slow time* and *fast time*, respectively. Correspondingly, system (3.1) is called *the slow system*, while system

(3.2) is called *the fast system*. Systems (3.1) and (3.2) when $\varepsilon = 0$ are respectively the *reduced* system

$$\dot{x} = f(x, y, \lambda, 0), \quad 0 = g(x, y, \lambda, 0), \quad (x, y) \in \Omega, \tag{3.3}$$

and the *layer* system

$$x' = 0, \quad y' = g(x, y, \lambda, 0), \quad (x, y) \in \Omega. \tag{3.4}$$

The set

$$\{(x, y) \in \Omega \subset \mathbb{R}^{k+l} \,|g(x, y, \lambda, 0) = 0\}$$

is called a *critical manifold*, denoted by M_0. Note that the critical manifold is formed by the singularities of the layer system. According to (3.4), the variable y will vary while x will remain constant. Thus x is called the *slow variabl*e, whereas y is called the *fast variable*.

The critical manifold M_0 is *normally hyperbolic* if the layer system (3.4) has l eigenvalues with nonzero real parts at all its singularities. We shall compile various hypotheses about the system (3.1), which are denoted by the letter H.

(H1) The set M_0 is a compact manifold, possibly with boundary, and is normally hyperbolic relative to (3.4).

(H2) The set M_0 has an explicit expression (at least locally) $y = \varphi(x, \lambda), x \in \mathrm{K} \subset \mathbb{R}^l$, with φ a smooth function and K being a compact, simply connected domain in $\mathbb{R}^l$ whose boundary is an $(l-1)$-dimensional C^∞ submanifold.

Next we introduce the following results on invariant manifolds which is due to Fenichel [13].

Lemma 3.1. *Under the hypothesis (H1), if $\varepsilon > 0$ is sufficiently small, then there exists a manifold M_ε that lies within $O(\varepsilon)$ of M_0 and is diffeomorphic to M_0. Moreover it is locally invariant under the flow of* (3.1)*, and C^r in x, y and ε, for any $0 < r < +\infty$.*

Lemma 3.2. *Under the hypotheses (H1)-(H2), if $\varepsilon > 0$ is sufficiently small, then there is a function $y = \varphi^\varepsilon(x, \lambda)$, defined for $x \in \mathrm{K}$, so that the graph*

$$M_\varepsilon = \{(x, y)|y = \varphi^\varepsilon(x, \lambda)\},$$

is locally invariant under the flow of (3.1)*. Moreover, $\varphi^\varepsilon(x, \lambda)$ is C^r, for any $0 < r < +\infty$, jointly in x and ε.*

Under the hypothesis (H2), restricted to the critical manifold one has a k-dimensional differential system

$$\dot{x} = f(x, \varphi(x, \lambda), \lambda, 0), \quad x \in \mathrm{K}.$$

3.2. Theory of Chebyshev system

In this subsection, we introduce the theory of Chebyshev system. Let us recall some definitions about Chebyshev system, see for instance [15, 29]. One can also consult [24, 31] and references therein for details.

Definition 3.1. Let $f_0(x), f_1(x), \cdots, f_{n-1}(x)$ be analytic functions defined on an interval $\mathbb{I} \subset \mathbb{R}$.

(i) The family of sets $\{f_0(x), f_1(x), \cdots, f_{n-1}(x)\}$ is a Chebyshev system (T-system for short) on $\mathbb{I}$ if any nontrivial linear combination

$$\alpha_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}(x)$$

has at most $n-1$ isolated zeros on $\mathbb{I}$.

(ii) The family of sets $\{f_0(x), f_1(x), \cdots, f_{n-1}(x)\}$ is an Extended Chebyshev system (ET-system for short) on $\mathbb{I}$ if any nontrivial linear combination

$$\alpha_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}(x)$$

has at most $n-1$ isolated zeros on $\mathbb{I}$ counted with multiplicities.

(iii) An ordered set of n functions $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is a complete Chebyshev system (CT-system for short) on $\mathbb{I}$ if $\{f_0(x), f_1(x), \cdots, f_{k-1}(x)\}$ is a T-system on $\mathbb{I}$ for all $k = 1, 2, \cdots, n$.

(iv) An ordered set of n functions $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is an Extended Complete Chebyshev system (ECT-system for short) on $\mathbb{I}$ if for all $k = 1, 2, \cdots, n$, any nontrivial linear combination

$$\alpha_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on $\mathbb{I}$ counted with multiplicities.

(Here in these abbreviations "T" stands for Tchebycheff, which in some sources is the transcription of the Russian name Chebyshev.)

Definition 3.2. Let $f_0(x), f_1(x), \cdots, f_{k-1}(x)$ be analytic functions defined on an interval $\mathbb{I} \subset \mathbb{R}$. The continuous Wronskian of $(f_0(x), f_1(x), \cdots, f_{k-1}(x))$ at $x \in \mathbb{I}$ is

$$W[f_0, f_1, \cdots, f_{k-1}](x) = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{k-1}(x) \\ f_0'(x) & f_1'(x) & \cdots & f_{k-1}'(x) \\ \vdots & \vdots & & \vdots \\ f_0^{(k-1)}(x) & f_1^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix},$$

where $f'(x)$ is the first order derivative of $f(x)$ and $f^{(i)}(x)$ is the i-th order derivative of $f(x)$ with respect to x, $i \geq 2$.

The following lemma is well known, see [24].

Lemma 3.3. *An ordered set of n smooth functions $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is an ECT-system on an interval $\mathbb{I}$ if and only if for each $k = 0, 1, \cdots, n-1$, the continuous Wronskian $W[f_0, f_1, \cdots, f_k](x) \neq 0$ for all $x \in \mathbb{I}$.*

This lemma not only shows the relation between an ECT-system and its continuous Wronskian, but also provides an easy-to-operate method to determine whether $(f_0(x), f_1(x), \cdots, f_n(x))$ is an ECT-system on $\mathbb{I}$.

3.3. Method of discriminant function

In this subsection, we introduce the method of discriminant function, which transforms the problem of Chebyshev property of some Abelian integrals into studying that of rational polynomial functions of the same number.

Let $H(x,y) = \Phi(x) + y^2$ be an analytic function in some open subset of $\mathbb{R}^2$. Assume that $H(0,0) = 0$ and $\Phi''(0) > 0$, which implies that $H(x,y)$ has a local minimum at the origin. It follows that there exists a punctured neighborhood $\mathcal{P}$ of the origin foliated by ovals $\Gamma_h \subseteq \{(x,y) : H(x,y) = h, h \in (0,h_0), h_0 = H(\partial\mathcal{P})\}$. The projection of $\mathcal{P}$ on the x-axis is an interval (x_l, x_r) with $x_l < x < x_r$. Under these assumptions, it is easy to verify that $x\Phi'(x) > 0$ for all $x \in (x_l, x_r) \setminus \{0\}$, and $\Phi(x)$ has a zero of even multiplicity at $x = 0$. Then there exists a unique analytic involution function $z(x)$ such that

$$\Phi(x) = \Phi(z(x)),\ \ z(x) \neq x,\ \ \text{for}\, x \in (x_l, x_r).$$

Let

$$\mathcal{I}_i(h) = \oint_{\Gamma_h} f_i(x) y^{2s-1} dx,\ \ \text{for}\, h \in (0, h_0), \tag{3.5}$$

where $f_i(x)$, $i = 0, 1, \cdots, n-1$, are analytic functions on (x_l, x_r) and $s \in \mathbb{N}$. Further, define a new analytic function in the interval (x_l, x_r) as follows

$$l_i = \frac{f_i(x)}{\Phi'(x)} - \frac{f_i(z(x))}{\Phi'(z(x))},$$

which is called the discriminant function of f_i, $i = 0, 1, \cdots, n-1$. Then, the following lemma holds for $\mathbb{I} = (x_l, 0)$ or $(0, x_r)$, see [15].

Lemma 3.4. *Under the assumptions above, $\{\mathcal{I}_0, \mathcal{I}_1, \cdots, \mathcal{I}_{n-1})\}$ is an ECT-system on $(0, h_0)$ if $\{l_0, l_1, \cdots, l_{n-1}\}$ is an ECT-system on $\mathbb{I}$ and $s > n-2$.*

4. Analysis of systems (2.4)

Now we go back to study system (2.4), which is equivalent to

$$\begin{cases} \dfrac{d\mu}{d\tau} = \nu, \\ \dfrac{d\nu}{d\tau} = \omega, \\ \varepsilon\dfrac{d\omega}{d\tau} = c^{-\frac{5}{2}}\left(\mu^5 - \mu^6 - \omega\right) - c^{-5}\nu\varepsilon, \end{cases} \tag{4.1}$$

via the transformations $\nu = \frac{d\mu}{d\tau}$ and $\omega = \frac{d^2\mu}{d\tau^2}$. Next we will prove the persistence of periodic wave solutions of system (4.1) for sufficiently small $\varepsilon > 0$ by using geometric singular perturbation theory. Taking the time scaling $\sigma = \frac{\tau}{\varepsilon}$ into (4.1) yields

$$\begin{cases} \dfrac{d\mu}{d\sigma} = \varepsilon\nu, \\ \dfrac{d\nu}{d\sigma} = \varepsilon\omega, \\ \dfrac{d\omega}{d\sigma} = c^{-\frac{5}{2}}\left(\mu^5 - \mu^6 - \omega\right) - c^{-5}\nu\varepsilon. \end{cases} \tag{4.2}$$

As mentioned in Section 3.1, the time scales τ and σ are the slow time and fast time, respectively. Correspondingly, system (4.1) is the slow system, while system

(4.2) is the fast system. Note that the two systems are equivalent when $\varepsilon \neq 0$. In (4.2) letting $\varepsilon \to 0$, we obtain the layer system

$$\begin{cases} \dfrac{d\mu}{d\sigma} = 0, \\ \dfrac{d\nu}{d\sigma} = 0, \\ \dfrac{d\omega}{d\sigma} = c^{-\frac{5}{2}} \left(\mu^5 - \mu^6 - \omega\right). \end{cases} \tag{4.3}$$

Therefore, the correspondence with our notation is: $x = (\mu, \nu)^T \in \mathbb{R}^2$ is the slow variable, and $y = \omega \in \mathbb{R}^1$ is the fast variable. In (4.1) letting $\varepsilon \to 0$, we get the reduced system

$$\begin{cases} \dfrac{d\mu}{d\tau} = \nu, \\ \dfrac{d\nu}{d\tau} = \omega, \\ \mu^5 - \mu^6 - \omega = 0. \end{cases} \tag{4.4}$$

The critical manifold M_0 is given by the condition $\omega = \mu^5 - \mu^6$ suitably restricted to any compact domain K of (μ, ν) space. The Jacobian matrix of the layer system (4.3) is

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^{-\frac{5}{2}}(5\mu^4 - 6\mu^5) & 0 & -c^{-\frac{5}{2}} \end{pmatrix},$$

which always has three eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -c^{-\frac{5}{2}}$, with λ_1 and λ_2 being on the imaginary axis. This follows that the Jacobian matrix F has exactly one eigenvalue with nonzero real part at all its singularities. Thus, the set M_0 is normally hyperbolic relative to (4.3), in fact attracting. Notice that the set M_0 is given as the graph of the C^∞ function $\omega = \varphi(\mu, \nu, c) = \mu^5 - \mu^6$ for $(\mu, \nu) \in \mathbb{R}^2$ and $c \in \mathbb{R}_+$. It is easy to see that the hypotheses (H1)-(H2) in Section 3.1 hold. From Lemma 3.1, it follows that for sufficiently small $\varepsilon > 0$, there exists a two-dimensional submanifold M_ε in $\mathbb{R}^3$, which is locally invariant under the flow of (4.1), within the Hausdorff distance ε of M_0. In addition, M_ε is C^r in μ, ν and ε, for any $0 < r < +\infty$. Furthermore, it follows from Lemma 3.2 that M_ε can be represented by the graph of a function $\omega = \varphi^\varepsilon(\mu, \nu, c)$, which is C^r in μ, ν and ε, for any $0 < r < +\infty$. By smoothness, this function can be expanded in ε, so that

$$\omega = \mu^5 - \mu^6 + g_1(\mu, \nu, c)\varepsilon + O(\varepsilon^2). \tag{4.5}$$

Next we calculate the term $g_1(\mu, \nu, c)$. The only remaining information about M_ε is the local invariance relative to system (4.2) and this can be used to evaluate $g_1(\mu, \nu, c)$. To this end, we differentiate (4.5) to σ yields

$$\omega' = 5\mu^4\mu' - 6\mu^5\mu' + \left(\frac{\partial g_1}{\partial \mu}\mu' + \frac{\partial g_1}{\partial \nu}\nu'\right)\varepsilon + O(\varepsilon^2), \tag{4.6}$$

where $' = \frac{d}{d\sigma}$. Substituting the expressions of μ', ν' and ω', from (4.2), and also the expression of ω, given by (4.5), into (4.6), we get

$$c^{-\frac{5}{2}} \left(-g_1\varepsilon + O(\varepsilon^2)\right) - c^{-5}\nu\varepsilon = \left(5\mu^4\nu - 6\mu^5\right)\nu\varepsilon + O(\varepsilon^2). \tag{4.7}$$

Equating the term of $O(\varepsilon)$ in (4.7), we obtain

$$g_1(\mu,\nu,c) = -c^{\frac{5}{2}}(5\mu^4 - 6\mu^5)\nu - c^{-\frac{5}{2}}\nu. \tag{4.8}$$

Therefore, the slow system (4.1) restricted on M_ε is given by

$$\begin{cases} \dfrac{d\mu}{d\tau} = \nu, \\ \dfrac{d\nu}{d\tau} = \mu^5 - \mu^6 + \left[-c^{\frac{5}{2}}(5\mu^4 - 6\mu^5)\nu - c^{-\frac{5}{2}}\nu\right]\varepsilon + O(\varepsilon^2), \end{cases} \tag{4.9}$$

which can be regarded as a regular perturbed system. For any given $h \in (-\frac{1}{42}, 0)$, $H(\mu,\nu) = h$ contains a periodic orbit Γ_h of (2.5) (or the system (4.9) with $\varepsilon = 0$), where $H(\mu,\nu)$ is defined by (2.6). Let $A(\alpha(h), 0)$ denote the intersection point of Γ_h and the positive μ-axis, and T denote the period of Γ_h. For $\varepsilon > 0$ sufficiently small, let $\Gamma_{h,\varepsilon}$ be the positive orbit of (4.9) starting from the point $A(\alpha(h), 0)$ at time $\tau = 0$, satisfying $a(h) \in (0,1)$, and $B(\beta(h,\varepsilon), 0)$ be the first intersection point of the positive orbit $\Gamma_{h,\varepsilon}$ returns to the positive μ-axis at time $\tau = \tau^*(\varepsilon)$. The displacement function between $B(\beta(h,\varepsilon), 0)$ and $A(\alpha(h), 0)$ is defined by

$$\begin{aligned} d(h,c,\varepsilon) &= H(B) - H(A) = \int_{\widehat{AB}} dH = \int_{\widehat{AB}} H_\mu d\mu + H_\nu d\nu \\ &= \int_{\widehat{AB}} (-\mu^5 + \mu^6) d\mu + \nu d\nu \\ &= \varepsilon \int_0^{\tau^*(\varepsilon)} \left[\left(-c^{\frac{5}{2}}(5\mu^4 - 6\mu^5)\nu - c^{-\frac{5}{2}}\nu\right) + O(\varepsilon)\right] \nu d\tau \\ &\triangleq \varepsilon \mathcal{F}(h,c,\varepsilon). \end{aligned}$$

By continuousness theorem, we have

$$\lim_{\varepsilon\to 0} \Gamma_{h,\varepsilon} = \Gamma_h,\ \lim_{\varepsilon\to 0} \beta(h,\varepsilon) = \alpha(h),\ \lim_{\varepsilon\to 0} \tau^*(\varepsilon) = T(h).$$

Thus,

$$\mathcal{F}(h,c,\varepsilon) = c^{-\frac{5}{2}} M(h,c) + O(\varepsilon),$$

where

$$M(h,c) = c^{\frac{5}{2}} \mathcal{F}(h,c,0) = \oint_{\Gamma_h} \left[c^5(-5\mu^4 + 6\mu^5) - 1\right] \nu d\mu. \tag{4.10}$$

The function $M(h,c)$ is called Abelian integral or Melnikov function [22].

To investigate the persistence of periodic waves for the perturbation problem, we will consider the zeros of the displacement function $d(h,\varepsilon)$ and their distributions. According to the analysis above, it follows from (4.10) and the Poincaré bifurcation theory [18,19] that it suffices to consider the Abelian integral $M(h,c)$.

Remark 4.1. It is worth to mention that the Picard-Fuchs equation method is hard to study the monotonicity of Abelian integral having three generating elements. Moreover, it has been shown that our approach developed in this paper is much simpler than the Picard-Fuchs equation method (eg. see [7,17,39]).

In the following, we study the Abelian integral $M(h,c)$. Let

$$J_n(h) = \oint_{\Gamma_h} \mu^n \nu d\mu, \tag{4.11}$$

where $n \in \mathbb{N}$. Then,

$$M(h,c) = c^5(-5J_4(h) + 6J_5(h)) - J_0(h). \tag{4.12}$$

We first give a lemma about the property of $J_0(h)$.

Lemma 4.1. *For $h \in (-\frac{1}{42}, 0)$, we have $J_0'(h) > 0$ and $J_0(h) > 0$.*

Proof. In the equation $H(\mu,\nu) = h$, we can think of ν as a function of μ and h, that is, $\nu = \nu(\mu, h)$. Taking the derivative on both sides of this equation with respect to h yields

$$\frac{\partial \nu}{\partial h} = \frac{1}{\nu}.$$

Thus,

$$J_n'(h) = \oint_{\Gamma_h} \mu^n \frac{\partial \nu}{\partial h} d\mu = \oint_{\Gamma_h} \frac{\mu^n}{\nu} d\mu, \tag{4.13}$$

which follows that

$$J_0'(h) = \oint_{\Gamma_h} \frac{1}{\nu} d\mu = \int_0^{T(h)} \frac{1}{\nu} \nu d\tau = \int_0^{T(h)} d\tau = T(h) > 0,$$

where the prime denotes the derivative with respect to h and $T(h)$ is the period of Γ_h. This implies that $J_0(h)$ is strictly increasing for $h \in (-\frac{1}{42}, 0)$.

When $h \to -\frac{1}{42}$, Γ_h approaches to the center $(1,0)$, implying that $\nu \to 0$. Thus, we have

$$J_0\left(-\frac{1}{42}\right) = \lim_{h \to -\frac{1}{42}} \oint_{\Gamma_h} \nu d\mu = \lim_{h \to -\frac{1}{42}} \int_0^{T(h)} \nu^2 d\tau = 0.$$

This together with $J_0'(h) > 0$ shows that $J_0(h) > 0$ for all $h \in (-\frac{1}{42}, 0)$. Thus, the proof is finished. $\square$

From Lemma 4.1, we know that the following ratio is well defined,

$$P(h) = -5\frac{J_4}{J_0} + 6\frac{J_5}{J_0}. \tag{4.14}$$

This shows that $M(h,c)$ in (4.12) can be rewritten as

$$M(h,c) = J_0\left(c^5 P(h) - 1\right). \tag{4.15}$$

To prove Theorem 2.1, we will study the monotonicity and range of the function $P(h)$ for $h \in (-\frac{1}{42}, 0)$, which can be stated as the following proposition.

Proposition 4.1. *For $h \in (-\frac{1}{42}, 0)$, we have $P'(h) < 0$ and*

$$\frac{134456}{220077} < P(h) < 1, \quad \lim_{h \to -\frac{1}{42}} P(h) = 1, \quad \lim_{h \to 0} P(h) = \frac{134456}{220077}.$$

Before proving this proposition, we first give some lemmas about the limit values of $P(h)$ at both ends and the property of ratios of two Abelian integrals.

Lemma 4.2. *Let $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$, $p > 0, q > 0$ be the Beta function. Then we have*

$$J_0(0) = \frac{2\sqrt{3}}{3}\left(\frac{7}{6}\right)^4 B\left(\frac{3}{2}, 4\right), \quad J_4(0) = \frac{2\sqrt{3}}{3}\left(\frac{7}{6}\right)^8 B\left(\frac{3}{2}, 8\right),$$

$$J_5(0) = \frac{2\sqrt{3}}{3}\left(\frac{7}{6}\right)^9 B\left(\frac{3}{2}, 9\right).$$

Moreover, the following ratio values at $h = 0$ hold:

$$\frac{J_4(0)}{J_0(0)} = \frac{134456}{196911}, \ \frac{J_5(0)}{J_0(0)} = \frac{7529536}{11223927}, \ \frac{J_5(0)}{J_4(0)} = \frac{56}{57}.$$

Proof. When $h = 0$, it follows from (2.6) that $\nu = \pm\sqrt{2\mu^6(\frac{1}{6} - \frac{\mu}{7})}$. Then we have

$$J_n(0) = \oint_{\Gamma_h} \mu^n \nu d\mu = 2\int_0^{\frac{7}{6}} \mu^n \sqrt{2\mu^6\left(\frac{1}{6} - \frac{\mu}{7}\right)} d\mu = \frac{2\sqrt{3}}{3}\int_0^{\frac{7}{6}} \mu^{n+3}\left(1 - \frac{6}{7}\mu\right)^{\frac{1}{2}} d\mu.$$

Taking the change of variable $\phi = \frac{6}{7}\mu$ into the integral above, we get

$$J_n(0) = \frac{2\sqrt{3}}{3}\int_0^1 \left(\frac{7}{6}\right)^{n+4} \phi^{n+3}(1 - \phi)^{\frac{1}{2}} d\phi = \frac{2\sqrt{3}}{3}\left(\frac{7}{6}\right)^{n+4} B\left(\frac{3}{2}, n+4\right),$$

in the last equality we have used the fact that $B(p, q) = B(q, p)$. This proves the first part of the lemma by setting $n = 0, 4, 5$.

Next we will prove the second part. Noting that

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \ \Gamma(s+1) = s\Gamma(s),$$

where $\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}dx$ $(s > 0)$ is the Gamma function, we have

$$\begin{aligned} \frac{J_4(0)}{J_0(0)} &= \frac{\left(\frac{7}{6}\right)^8 B(\frac{3}{2}, 8)}{\left(\frac{7}{6}\right)^4 B(\frac{3}{2}, 4)} = \left(\frac{7}{6}\right)^4 \frac{\Gamma(\frac{3}{2})\Gamma(8)}{\Gamma(\frac{19}{2})} \times \frac{\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(4)} \\ &= \left(\frac{7}{6}\right)^4 \times \frac{7 \times 6 \times 5 \times 4 \times \Gamma(4) \times \Gamma(\frac{11}{2})}{\frac{17}{2} \times \frac{15}{2} \times \frac{13}{2} \times \frac{11}{2} \times \Gamma(\frac{11}{2}) \times \Gamma(4)} = \frac{134456}{196911}. \end{aligned} \tag{4.16}$$

Similarly, it is not difficult to verify that the following equalities hold.

$$\frac{J_5(0)}{J_0(0)} = \frac{7529536}{11223927}, \ \frac{J_5(0)}{J_4(0)} = \frac{56}{57}.$$

Thus, the proof is completed. □

Lemma 4.3. *The following ratio values at $h = -\frac{1}{42}$ hold:*

$$\begin{aligned} \frac{J_4(-\frac{1}{42})}{J_0(-\frac{1}{42})} &= \lim_{h \to -\frac{1}{42}} \frac{J_4(h)}{J_0(h)} = 1, \\ \frac{J_5(-\frac{1}{42})}{J_0(-\frac{1}{42})} &= \lim_{h \to -\frac{1}{42}} \frac{J_5(h)}{J_0(h)} = 1, \\ \frac{J_5(-\frac{1}{42})}{J_4(-\frac{1}{42})} &= \lim_{h \to -\frac{1}{42}} \frac{J_5(h)}{J_4(h)} = 1. \end{aligned}$$

Proof. Taking the following change of variables

$$\begin{cases} \mu = 1 + r\cos\theta, \\ \nu = r\sin\theta, \end{cases} \tag{4.17}$$

and denoting $\rho = \sqrt{h + \frac{1}{42}}$, then the equation $H(\mu, \nu) - h = 0$ becomes

$$G(r,\rho,\theta) \triangleq \frac{1}{7}r^7\cos^7\theta + \frac{5}{6}r^6\cos^6\theta + 2r^5\cos^5\theta + \frac{5}{2}r^4\cos^4\theta + \frac{5}{3}r^3\cos^3\theta + \frac{1}{2}r^2 - \rho^2 = 0. \tag{4.18}$$

This equation can be rewritten as

$$\frac{r^2}{2}\left(1 + g(r,\theta)\right) = \rho^2, \tag{4.19}$$

where

$$g(r,\theta) = \frac{2}{7}r^5\cos^7\theta + \frac{5}{3}r^4\cos^6\theta + 4r^3\cos^5\theta + 5r^2\cos^4\theta + \frac{10}{3}r\cos^3\theta.$$

Since $g(0,\theta) = 0$ and $g(r,\theta) \in C^\infty$, there exists $r_1 > 0$ such that $1 + g(r,\theta) > 0$ holds for $0 < r < r_1$. Noting that $r \geq 0$ and $\rho \geq 0$, Eq. (4.19) is equivalent to

$$\frac{r}{\sqrt{2}}\sqrt{1 + g(r,\theta)} = \rho,$$

for $0 < r < r_1$, which can be rewritten as

$$\widetilde{F}(r,\rho,\theta) \triangleq r\sqrt{1 + g(r,\theta)} - \sqrt{2}\rho = 0. \tag{4.20}$$

When $h \to -\frac{1}{42}$, then $\rho \to 0$. It is not difficult to verify that

$$\widetilde{F}(0,0,\theta) = 0, \text{ and } \left.\frac{\partial\widetilde{F}(r,\rho,\theta)}{\partial r}\right|_{(r,\rho)=(0,0)} = 1 \neq 0.$$

Applying the implicit function theorem to the equation (4.20) for (r,ρ) in a neighborhood of $(0,0)$, one gets its solution $r = r(\rho,\theta)$, which is an analytic function of ρ near 0. This together with the fact that $r(0,\theta) = 0$ shows that $r(\rho,\theta)$ can be expanded as a Taylor's series with respect to ρ, i.e.,

$$r(\rho,\theta) = a_1(\theta)\rho + a_2(\theta)\rho^2 + a_3(\theta)\rho^3 + a_4(\theta)\rho^4 + O(\rho^5). \tag{4.21}$$

Substituting (4.21) into (4.20) and letting all coefficients of ρ^n, $n = 1,2,3,4$, be zero, one can get a unique group of solution

$$\begin{aligned} &a_1(\theta) = \sqrt{2}, a_2(\theta) = -\frac{10}{3}\cos^3\theta, a_3(\theta) = \frac{5}{9}\sqrt{2}\cos^4\theta\left(25\cos^2\theta - 9\right), \\ &a_4(\theta) = -\frac{4}{27}\cos^5\theta\left(1000\cos^4\theta - 675\cos^2\theta + 54\right). \end{aligned}$$

Then $r(\rho,\theta)$ can be written as

$$\begin{aligned} r(\rho,\theta) = &\sqrt{2}\rho - \frac{10}{3}\cos^3\theta\rho^2 + \frac{5}{9}\sqrt{2}\cos^4\theta\left(25\cos^2\theta - 9\right)\rho^3 \\ &- \frac{4}{27}\cos^5\theta\left(1000\cos^4\theta - 675\cos^2\theta + 54\right)\rho^4 + O(\rho^5). \end{aligned} \tag{4.22}$$

Notice that the direction of curve Γ_h is clockwise. By Green's formula and the variable transformation (4.17), we obtain

$$J_n(h) = \oint_{\Gamma_h} \mu^n \nu d\mu = \iint\limits_{int\Gamma_h} \mu^n d\mu d\nu = \int_0^{2\pi} d\theta \int_0^{r(\rho,\theta)} (1 + r\cos\theta)^n r dr. \tag{4.23}$$

Noticing $\rho = \sqrt{h + \frac{1}{42}}$ and substituting (4.22) into (4.23) yields

$$\begin{aligned}
J_0(h) &= 2\pi\left(h + \frac{1}{42}\right) + \frac{40}{3}\pi\left(h + \frac{1}{42}\right)^2 + O\left(\left(h + \frac{1}{42}\right)^3\right),\\
J_1(h) &= 2\pi\left(h + \frac{1}{42}\right) + \frac{25}{3}\pi\left(h + \frac{1}{42}\right)^2 + O\left(\left(h + \frac{1}{42}\right)^3\right),\\
J_2(h) &= 2\pi\left(h + \frac{1}{42}\right) + \frac{13}{3}\pi\left(h + \frac{1}{42}\right)^2 + O\left(\left(h + \frac{1}{42}\right)^3\right),\\
J_3(h) &= 2\pi\left(h + \frac{1}{42}\right) + \frac{4}{3}\pi\left(h + \frac{1}{42}\right)^2 + O\left(\left(h + \frac{1}{42}\right)^3\right),\\
J_4(h) &= 2\pi\left(h + \frac{1}{42}\right) - \frac{2}{3}\pi\left(h + \frac{1}{42}\right)^2 + O\left(\left(h + \frac{1}{42}\right)^3\right),\\
J_5(h) &= 2\pi\left(h + \frac{1}{42}\right) - \frac{5}{3}\pi\left(h + \frac{1}{42}\right)^2 + O\left(\left(h + \frac{1}{42}\right)^3\right),
\end{aligned}$$

for $0 < h + \frac{1}{42} \ll 1$. Therefore,

$$\frac{J_i(-\frac{1}{42})}{J_j(-\frac{1}{42})} = \lim_{h\to -\frac{1}{42}} \frac{J_i(h)}{J_j(h)} = 1,\ \ i, j = 0, 1, \cdots, 5.$$

This completes the proof. $\square$

Lemma 4.4. $J_n(h) = \sum_{i=0}^{n} (-1)^i C_n^i I_i(h)$, *where* $I_i(h) = \oint_{H^*(\widetilde{\mu},\widetilde{\nu})=h} \widetilde{\mu}^i \widetilde{\nu} d\widetilde{\mu}$, *in which* $\widetilde{\mu} = 1 - \mu, \widetilde{\nu} = -\nu$, *and*

$$H^*(\widetilde{\mu},\widetilde{\nu}) = H(1 - \widetilde{\mu}, -\widetilde{\nu}). \tag{4.24}$$

In particular,

$$\begin{aligned}
&J_0(h) = I_0(h),\\
&J_4(h) = I_4(h) - 4I_3(h) + 6I_2(h) - 4I_1(h) + I_0(h),\\
&J_5(h) = -I_5(h) + 5I_4(h) - 10I_3(h) + 10I_2(h) - 5I_1(h) + I_0(h).
\end{aligned} \tag{4.25}$$

Proof. Taking the change of variables

$$\begin{cases} \widetilde{\mu} = 1 - \mu, \\ \widetilde{\nu} = -\nu, \end{cases} \tag{4.26}$$

into $J_n(h)$ yields

$$\begin{aligned} J_n(h) &= \oint_{\Gamma_h} \mu^n \nu d\mu = \oint_{H(1-\widetilde{\mu},-\widetilde{\nu})=h} (1-\widetilde{\mu})^n(-\widetilde{\nu})d(1-\widetilde{\mu}) \\ &= \oint_{H^*(\widetilde{\mu},\widetilde{\nu})=h} \left[\sum_{i=0}^{n} C_n^i(-1)^i\widetilde{\mu}^i\widetilde{\nu}\right] d\widetilde{\mu} = \sum_{i=0}^{n} C_n^i(-1)^i \oint_{H^*(\widetilde{\mu},\widetilde{\nu})=h} \widetilde{\mu}^i\widetilde{\nu}d\widetilde{\mu} \\ &= \sum_{i=0}^{n} C_n^i(-1)^i I_i(h). \end{aligned}$$

Then, we can obtain (4.25) by setting $n = 0, 4, 5$, respectively. □

Lemma 4.5. *For $h \in (-1/42, 0)$, $\frac{J_4(h)}{J_0(h)}$ is strictly decreasing from 1 to $\frac{134456}{196911}$, and $\frac{J_5(h)}{J_4(h)}$ is strictly decreasing from 1 to $\frac{56}{57}$.*

Proof. By Lemmas 4.2 and 4.3, we only need to prove that both $\frac{J_4(h)}{J_0(h)}$ and $\frac{J_5(h)}{J_4(h)}$ are strictly monotonous on the interval $(-1/42, 0)$. This is equivalent to proving that both of any nontrivial linear combinations $\alpha_1 J_0(h) + \alpha_2 J_4(h)$ and $\alpha_1^* J_4(h) + \alpha_1^* J_5(h)$ have at most one zero on $(-1/42, 0)$ counted with multiplicities, i.e., both $\{J_0(h), J_4(h)\}$ and $\{J_4(h), J_5(h)\}$ are *ECT*-systems on $(-1/42, 0)$. Next we will use the method of discriminant function introduced in Section 3.3 to solve these problems.

Under the variable transformation (4.26), the expression of (4.24) becomes

$$H^*(\widetilde{\mu},\widetilde{\nu}) = H(1-\widetilde{\mu},-\widetilde{\nu}) = -\frac{1}{42}+\frac{\widetilde{\nu}^2}{2}+\frac{1}{2}\widetilde{\mu}^2-\frac{5}{3}\widetilde{\mu}^3+\frac{5}{2}\widetilde{\mu}^4-2\widetilde{\mu}^5+\frac{5}{6}\widetilde{\mu}^6-\frac{1}{7}\widetilde{\mu}^7. \quad (4.27)$$

Denoting $\widetilde{h} = h + \frac{1}{42}$, then $H^*(\widetilde{\mu},\widetilde{\nu}) = h, h \in (-1/42, 0)$ becomes $\widetilde{H}(\widetilde{\mu},\widetilde{\nu}) = \widetilde{h}, \widetilde{h} \in (0, 1/42)$, with

$$\widetilde{H}(\widetilde{\mu},\widetilde{\nu}) = H^*(\widetilde{\mu},\widetilde{\nu}) + \frac{1}{42} = \Phi(\widetilde{\mu}) + \frac{\widetilde{\nu}^2}{2}, \quad (4.28)$$

where

$$\Phi(\widetilde{\mu}) = \frac{1}{2}\widetilde{\mu}^2 - \frac{5}{3}\widetilde{\mu}^3 + \frac{5}{2}\widetilde{\mu}^4 - 2\widetilde{\mu}^5 + \frac{5}{6}\widetilde{\mu}^6 - \frac{1}{7}\widetilde{\mu}^7.$$

It is easy to verify that $\widetilde{H}$ satisfies the assumptions as mentioned in Section 3.3. Then for any given $\widetilde{\mu} \in (-\frac{1}{6}, 1)$, there exists a unique analytic involution function $z(\widetilde{\mu}) \in (-\frac{1}{6}, 1)$ such that

$$\Phi(\widetilde{\mu}) = \Phi(z(\widetilde{\mu})), \quad z(\widetilde{\mu}) \neq \widetilde{\mu}. \quad (4.29)$$

Let

$$f_0(\widetilde{\mu})=1,\ f_4(\widetilde{\mu})=\widetilde{\mu}^4-4\widetilde{\mu}^3+6\widetilde{\mu}^2-4\widetilde{\mu}+1,\ f_5(\widetilde{\mu})=-\widetilde{\mu}^5+5\widetilde{\mu}^4-10\widetilde{\mu}^3+10\widetilde{\mu}^2-5\widetilde{\mu}+1.$$

By Lemma 4.4, we have

$$\widetilde{J}_i\left(\widetilde{h}\right) \triangleq J_i(h) = J_i\left(\widetilde{h}+\frac{1}{42}\right) = \oint_{H^*=\widetilde{h}+\frac{1}{42}} \mu^i \nu d\mu = \oint_{\widetilde{H}=\widetilde{h}} f_i(\widetilde{\mu})\widetilde{\nu}d\widetilde{\mu},\ i = 0, 1, \cdots, 5.$$

Then, the discriminant function associated to $f_i(\widetilde{\mu})$ $(i = 0, 4, 5)$ is

$$l_i(\widetilde{\mu}) = \left(\frac{f_i}{\Phi'}\right)(\widetilde{\mu}) - \left(\frac{f_i}{\Phi'}\right)(z(\widetilde{\mu})). \quad (4.30)$$

Factorizing $\Phi(\widetilde{\mu})-\Phi(z)$, we obtain $\Phi(\widetilde{\mu})-\Phi(z)=-\frac{1}{42}(\widetilde{\mu}-z)\varphi(\widetilde{\mu},z)$, where

$$\begin{aligned}\varphi(\widetilde{\mu},z)=&6\sum_{i=0}^{6}\widetilde{\mu}^iz^{6-i}-35\sum_{i=0}^{5}\widetilde{\mu}^iz^{5-i}+84\sum_{i=0}^{4}\widetilde{\mu}^iz^{4-i}-105\sum_{i=0}^{3}\widetilde{\mu}^iz^{3-i}\\&+70\sum_{i=0}^{2}\widetilde{\mu}^iz^{2-i}-21\sum_{i=0}^{1}\widetilde{\mu}^iz^{1-i}.\end{aligned}$$

From (4.29), it follows that $z(\widetilde{\mu})$ is defined implicitly by the equation $\varphi(\widetilde{\mu},z)=0$. Then, we have

$$\frac{dz}{d\widetilde{\mu}}=-\frac{\partial\varphi(\widetilde{\mu},z)}{\partial\widetilde{\mu}}\Big/\frac{\partial\varphi(\widetilde{\mu},z)}{\partial z}$$

and

$$\frac{d}{d\widetilde{\mu}}l_i(\widetilde{\mu})=\frac{d}{d\widetilde{\mu}}\left(\frac{f_i}{\Phi'}(\widetilde{\mu})\right)-\frac{d}{dz}\left[\left(\frac{f_i}{\Phi'}\right)(z(\widetilde{\mu}))\right]\frac{dz}{d\widetilde{\mu}}.$$

It follows from Lemma 3.4 that we only need to prove that both $\{l_0,l_4\}$ and $\{l_4,l_5\}$ are ECT-systems on $(0,1)$. From Lemma 3.3, it suffices to prove that the four Wronskians $W[l_0(\widetilde{\mu})]$, $W[l_0(\widetilde{\mu}),l_4(\widetilde{\mu})]$, $W[l_4(\widetilde{\mu})]$ and $W[l_4(\widetilde{\mu}),l_5(\widetilde{\mu})]$ are all non-vanishing on $(0,1)$. With aids of Maple-2017, a direct computation shows that

$$\begin{aligned}W[l_0(\widetilde{\mu})]&=\frac{(\widetilde{\mu}-z)w_1(\widetilde{\mu},z)}{\widetilde{\mu}z(\widetilde{\mu}-1)^5(z-1)^5},\ W[l_0(\widetilde{\mu}),l_4(\widetilde{\mu})]=\frac{(\widetilde{\mu}-z)^3w_2(\widetilde{\mu},z)}{\widetilde{\mu}^2z^2(\widetilde{\mu}-1)^7(z-1)^7w_0(\widetilde{\mu},z)},\\W[l_4(\widetilde{\mu})]&=\frac{(\widetilde{\mu}-z)(\widetilde{\mu}+z-1)}{\widetilde{\mu}z(\widetilde{\mu}-1)(z-1)},\ W[l_4(\widetilde{\mu}),l_5(\widetilde{\mu})]=\frac{(\widetilde{\mu}-z)^3w_3(\widetilde{\mu},z)}{\widetilde{\mu}^2z^2(\widetilde{\mu}-1)^2(z-1)^2w_0(\widetilde{\mu},z)},\end{aligned}$$

where

$$\begin{aligned}w_0(\widetilde{\mu},z)=&6\widetilde{\mu}^5+12\widetilde{\mu}^4z+18\widetilde{\mu}z^2+24\widetilde{\mu}^2z^3+30\widetilde{\mu}z^4+36z^5-35\widetilde{\mu}^4-70\widetilde{\mu}^3z\\&-105\widetilde{\mu}^2z^2-140\widetilde{\mu}z^3-175z^4+84\widetilde{\mu}^3+168\widetilde{\mu}^2z+252\widetilde{\mu}z^2+336z^3\\&-105\widetilde{\mu}^2-210\widetilde{\mu}z-315z^2+70\widetilde{\mu}+140z-21,\\w_1(\widetilde{\mu},z)=&\sum_{i=0}^{5}\widetilde{\mu}^iz^{5-i}-5\sum_{i=0}^{4}\widetilde{\mu}^iz^{4-i}+10\sum_{i=0}^{3}\widetilde{\mu}^iz^{3-i}-10\sum_{i=0}^{2}\widetilde{\mu}^iz^{2-i}\\&+5\sum_{i=0}^{1}\widetilde{\mu}^iz^{1-i}-1,\\w_2(\widetilde{\mu},z)=&-144\widetilde{\mu}^{11}-552\widetilde{\mu}^{10}z-1104\widetilde{\mu}^9z^2-1680\widetilde{\mu}^8z^3-2160\widetilde{\mu}^7z^4-2424\widetilde{\mu}^6z^5\\&-2424\widetilde{\mu}^5z^6-2160\widetilde{\mu}^4z^7-1680\widetilde{\mu}^3z^8-1104\widetilde{\mu}^2z^9-552\widetilde{\mu}z^{10}-144z^{11}\\&+1708\widetilde{\mu}^{10}+6272\widetilde{\mu}^9z+12204\widetilde{\mu}^8z^2+18016\widetilde{\mu}^7z^3+22220\widetilde{\mu}^6z^4+23724\widetilde{\mu}^5z^5\\&+22220\widetilde{\mu}^4z^6+18016\widetilde{\mu}^3z^7+12204\widetilde{\mu}^2z^8+6272\widetilde{\mu}z^9+1708z^{10}-9232\widetilde{\mu}^9\\&-32365\widetilde{\mu}^8z-60911\widetilde{\mu}^7z^2-86382\widetilde{\mu}^6z^3-101190\widetilde{\mu}^5z^4-101190\widetilde{\mu}^4z^5\\&-86382\widetilde{\mu}^3z^6-60911\widetilde{\mu}^2z^7-32365\widetilde{\mu}z^8-9232z^9+30017\widetilde{\mu}^8+99969\widetilde{\mu}^7z\\&+180383\widetilde{\mu}^6z^2+242866\widetilde{\mu}^5z^3+266160\widetilde{\mu}^4z^4+242866\widetilde{\mu}^3z^5+180383\widetilde{\mu}^2z^6\\&+99969\widetilde{\mu}z^7+30017z^8-65218\widetilde{\mu}^7-204805\widetilde{\mu}^6z-350037\widetilde{\mu}^5z^2-440300\widetilde{\mu}^4z^3\\&-440300\widetilde{\mu}^3z^4-350037\widetilde{\mu}^2z^5-204805\widetilde{\mu}z^6+65218z^7+99352\widetilde{\mu}^6\end{aligned}$$

$$
\begin{aligned}
&+290927\widetilde{\mu}^5 z+463069\widetilde{\mu}^4 z^2+530840\widetilde{\mu}^3 z^3+463069\widetilde{\mu}^2 z^4+290927\widetilde{\mu} z^5\\
&+99352z^6-108122\widetilde{\mu}^5-290367\widetilde{\mu}^4 z-419615\widetilde{\mu}^3 z^2-419615\widetilde{\mu}^2 z^3\\
&-290367\widetilde{\mu} z^4-108122z^5+83832\widetilde{\mu}^4+201257\widetilde{\mu}^3 z+252882\widetilde{\mu}^2 z^2\\
&+201257\widetilde{\mu} z^3+83832z^4-45178\widetilde{\mu}^3-92862\widetilde{\mu}^2 z-92862\widetilde{\mu} z^2-45178z^3\\
&+15995\widetilde{\mu}^2+25830\widetilde{\mu} z+15995z^2-3304\widetilde{\mu}-3304z+294,\\
w_3(\widetilde{\mu},z)=&36\widetilde{\mu}^6+66\widetilde{\mu}^5 z+84\widetilde{\mu}^4 z^2+90\widetilde{\mu}^3 z^3+84\widetilde{\mu}^2 z^4+66\widetilde{\mu} z^5+36z^6-211\widetilde{\mu}^5\\
&-375\widetilde{\mu}^4 z-457\widetilde{\mu}^3 z^2-457\widetilde{\mu}^2 z^3-375\widetilde{\mu} z^4-211z^5+511\widetilde{\mu}^4+868\widetilde{\mu}^3 z\\
&+987\widetilde{\mu}^2 z^2+868\widetilde{\mu} z^3+511z^4-651\widetilde{\mu}^3-1029\widetilde{\mu}^2 z-1029\widetilde{\mu} z^2-651z^3\\
&+455\widetilde{\mu}^2+630\widetilde{\mu} z+455z^2-161\widetilde{\mu}-161z+21.
\end{aligned}
$$

Computing the resultant of $\widetilde{\mu}+z-1$ and $\varphi(\widetilde{\mu},z)$ with respect to z yields

$$\mathrm{Res}(\widetilde{\mu}+z-1,\varphi,z)=6\widetilde{\mu}^6-18\widetilde{\mu}^5+33\widetilde{\mu}^4-36\widetilde{\mu}^3+17\widetilde{\mu}^2-2\widetilde{\mu}-1\triangleq\psi(\widetilde{\mu}). \quad (4.31)$$

By Sturm's theorem in Appendix, the polynomial $\psi(\widetilde{\mu})$ has no real root for $\widetilde{\mu}\in(0,1)$. This implies that $\psi(\widetilde{\mu})$ is sign-definite for $\widetilde{\mu}\in(0,1)$. Noting that $\psi(0)=-1$, we know that $\psi(\widetilde{\mu})<0$ for all $\widetilde{\mu}\in(0,1)$. From Lemma 5.2 in Appendix, it follows that $\varphi(\widetilde{\mu},z)$ and $\widetilde{\mu}+z-1$ have no common real root for $\widetilde{\mu}\in(0,1)$. This combining with $\varphi(\widetilde{\mu},z)=0$ shows that $\widetilde{\mu}+z-1\neq0$ and hence $W[l_4(\widetilde{\mu})]\neq0$ for $\widetilde{\mu}\in(0,1)$.

Similarly, we can also conclude that $W[l_0(\widetilde{\mu})]\neq0$, $W[l_0(\widetilde{\mu}),l_4(\widetilde{\mu})]\neq0$ and $W[l_4(\widetilde{\mu}),l_54(\widetilde{\mu})]\neq0$ for $\widetilde{\mu}\in(0,1)$ by verifying the resultants $\mathrm{Res}(w_j,\varphi,z)\neq0$, $j=0,1,2,3$. This implies that both $\{J_0(h),J_4(h)\}$ and $\{J_4(h),J_5(h)\}$ are *ECT*-systems on $(-1/42,0)$. Thus, $\frac{J_4(h)}{J_0(h)}$ and $\frac{J_5(h)}{J_4(h)}$ are strictly monotonic on $(-1/42,0)$. By Lemmas 4.2 and 4.3, the assertion in this lemma is proved. □

Proposition 4.1 is easy to prove by Lemmas 4.2–4.5.

Proof of Proposition 4.1. From Lemma 4.5, it is easy to see that

$$\left(\frac{J_4(h)}{J_0(h)}\right)'<0,\ \frac{134456}{196911}<\frac{J_4(h)}{J_0(h)}<1$$

and

$$\left(\frac{J_5(h)}{J_4(h)}\right)'<0,\ \frac{56}{57}<\frac{J_5(h)}{J_4(h)}<1$$

hold for $h\in(-\frac{1}{42},0)$. This follows that

$$\frac{17}{19}<-5+6\frac{J_5(h)}{J_4(h)}<1.$$

Thus, for $h\in(-\frac{1}{42},0)$, we have

$$P'(h)=\left(\frac{J_4(h)}{J_0(h)}\right)'\left(-5+6\frac{J_5(h)}{J_4(h)}\right)+6\frac{J_4(h)}{J_0(h)}\left(\frac{J_5(h)}{J_4(h)}\right)'<0,$$

and hence

$$\frac{134456}{220077}=\lim_{h\to0}P(h)<P(h)<\lim_{h\to-\frac{1}{42}}P(h)=1.$$

This completes the proof of Proposition 4.1. □

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. It follows from Proposition 4.1 that for $h \in (-\frac{1}{42}, 0)$, we have

$$P(h) \in \left(\frac{134456}{220077}, 1\right), \ (P(h))^{-\frac{1}{5}} \in (1, c_0),$$

where $c_0 = \frac{3}{7}\sqrt[5]{\frac{2717}{24}}$. For any $c \in (1, c_0)$, it follows from (4.15) that $M(h,c)$ has a unique solution at $h = P^{-1}\left(c^{-5}\right) \triangleq h^*(c)$, where P^{-1} is the inverse function of P. Notice that

$$d(h,c,\varepsilon) = \varepsilon \mathcal{F}(h,c,\varepsilon) = \varepsilon\left(c^{-\frac{5}{2}} M(h,c) + O(\varepsilon)\right). \tag{4.32}$$

From Analytic Implicit Function Theorem and Theorem 2.1 of [20], it follows that for any $c \in (1, c_0)$, there exists $\varepsilon_0(c) > 0$ such that $d(h,c,\varepsilon)$ has a unique isolated zero at $h = h(c,\varepsilon) = h^*(c) + O(\varepsilon)$ when $0 < \varepsilon < \varepsilon_0(c)$, which means that system (4.9) has a unique limit cycle near $\Gamma_{h^*(c)}$, or system (1.5) has a unique isolated periodic wave solution given by $u = c\mu(\tau,h,c,\varepsilon)$, where $\mu(\tau,h,c,\varepsilon) = \mu(\tau,h^*(c)) + O(\varepsilon)$. By continuousness theorem, it is not difficult to obtain that when $(c,\varepsilon) \to (1,0)$ with $0 < \varepsilon < \varepsilon_0(c)$, then $\Gamma_{h,\varepsilon} \to (1,0)$ and hence $\mu(\tau,h,c,\varepsilon) \to 1$. Similarly, $\Gamma_{h,\varepsilon} \to \Gamma_0$ when $(c,\varepsilon) \to (c_0,0)$ with $0 < \varepsilon < \varepsilon_0(c)$, which gives $\mu(\tau,h,c,\varepsilon) \to \mu_0(\tau)$.

Next we will consider the signs of $\frac{\partial}{\partial\tau}\mu(0,h,c,\varepsilon)$ and $\frac{\partial^2}{\partial\tau^2}\mu(0,h,c,\varepsilon)$. Note that when $\varepsilon \neq 0$, $\mu(\tau,h,c,\varepsilon)$ is the travelling wave solution of system (2.4) and hence is the solution of system (4.1) (or system (4.2)). From the fact that $\Gamma_{h,\varepsilon}$ is the positive orbit of (4.9) starting from the point $A(a(h),0)$ at time $\tau = 0$ satisfying $a(h) \in (0,1)$, we know that $\mu(0,h,c,\varepsilon) \in (0,1)$ and $\nu(0,h,c,\varepsilon) = 0$. From (4.1), it is easy to see that $\frac{\partial}{\partial\tau}\mu(0,h,c,\varepsilon) = \nu(0,h,c,\varepsilon) = 0$.

Noting that $\mu(0,h,c,\varepsilon) \in (0,1)$, we have $\mu^5(0,h,c,\varepsilon) - \mu^6(0,h,c,\varepsilon) > 0$. By local inheriting order property, it follows from (4.9) and $\nu(0,h,c,\varepsilon) = 0$ that

$$\frac{\partial^2}{\partial\tau^2}\mu(0,h,c,\varepsilon) = \frac{\partial}{\partial\tau}\nu(0,h,c,\varepsilon) = \mu^5(0,h,c,\varepsilon) - \mu^6(0,h,c,\varepsilon) + O(\varepsilon^2) > 0$$

for sufficiently small $\varepsilon > 0$. This proves the first part of Theorem 2.1.

Now we are going to prove the second part of of Theorem 2.1. For any $h \in (-1/42, 0)$, it follows from (4.15) that $M(h,c)$ has a unique solution at $c = c(h) = (P(h))^{-\frac{1}{5}}$. From Analytic Implicit Function Theorem and Theorem 2.1 of [20], we know that for any $h \in (-1/42, 0)$, there exists $\varepsilon_1(h) > 0$ such that when $0 < \varepsilon < \varepsilon_1(h)$, then $d(h,c,\varepsilon)$ has a unique isolated zero at $c = c(h,\varepsilon) = c(h) + O(\varepsilon)$, which means that system (4.9) has a unique limit cycle near Γ_h, or system (1.5) has a unique isolated periodic wave solution given by $u = c(h,\varepsilon)\mu(\tau,h,c(h,\varepsilon),\varepsilon)$, where $\mu(\tau,h,c(h,\varepsilon),\varepsilon) = \mu(\tau,h) + O(\varepsilon)$. The proof for the properties of $\mu(\tau,h,c,\varepsilon)$ is similar as above. At this time, it is easy to see that $\lim\limits_{\varepsilon\to 0} c(h,\varepsilon) = c(h)$ and $c(h) \in (1, c_0)$. On the other hand, we have $c'(h,\varepsilon) = c'(h) + O(\varepsilon)$, where $'$ represents the derivative by h. From Proposition 4.1, it follows that

$$c'(h) = -\frac{1}{5}\left(P(h)\right)^{-\frac{6}{5}} P'(h) > 0,$$

for $h \in (-1/42, 0)$. Then for any $h \in (-1/42, 0)$, there exists $\varepsilon_2(h) > 0$ such that $c'(h,\varepsilon) > 0$ for $0 < \varepsilon < \varepsilon_2(h)$. Choose $\varepsilon^*(h) = \min\{\varepsilon_1(h), \varepsilon_2(h)\}$, then for any $h \in (-1/42, 0)$, the conclusions of (ii) of Theorem 2.1 hold for $0 < \varepsilon < \varepsilon^*(h)$. Thus, the proof of Theorem 2.1 is completed. □

Remark 4.2. Note that system (2.5) posses a homoclinic orbit to a nilpotent saddle of order 2. We have not discussed the existence of homoclinic orbit in system (4.9) (corresponds to solitary wave of system (1.5)) in the present paper.

5. Conclusion

In this paper, the existence of periodic waves in a perturbed sextic BBM equation with weak backward diffusion and dissipation effects has been established. By applying geometric singular perturbation theory and Melnikov theory, this problem can be reduced to the number of real zeros of hyper-elliptic Abelian integrals with three generating elements. We introduced the transformation (4.26) to use Chebyshev criteria to overcome the difficulty arising from higher-order degenerate singularities. It has been shown that periodic wave solutions with certain wave speeds persist under small perturbation. It has also been shown that periodic wave solutions persist for any energy parameter h in an open interval and sufficiently small perturbation parameter. In addition, the signs of several derivatives of the travelling wave solution with respect to time at the initial value, and the upper and lower bounds of the limiting wave speeds are obtained. Furthermore, it has been proven that the wave speed $c(h)$ is strictly monotonically increasing with respect to the energy parameter h.

One interesting problem that whether the solitary waves of system (1.5) persist or vanish remains unsolved, see Remark 4.2. This is a very challenging problem since it is difficult to guarantee the existence of invariant manifolds at the nilpotent saddle of order 2 connected by a homoclinic loop. It should be pointed out that the proof for the persistence of homoclinic orbit by letting Hamiltonian value h tends to the critical value 0, which corresponds to Hamiltonian value at homoclinic loop, is invalid (eg. see [6, 35, 39]). There are many different ways to discuss the persistence of homoclinic orbit in a perturbed Hamiltonian system under small perturbation with the corresponding Hamiltonian system having a homoclinic orbit to a hyperbolic saddle, such as Melnikov method [19] and the theory of generalized rotated vector field [42]. For Eq. (1.4), some researchers have established the existence of homoclinic orbits of the associated ODE for the case when the corresponding unperturbed system possesses homoclinic orbits to hyperbolic saddles, see for instance [36, 37, 43, 45, 48, 49]. The bifurcations near a homoclinic loop with a nilpotent singular point or near a nilpotent singular point have been studied extensively, see for example [1, 21, 26, 27, 40, 41]. To the best of our knowledge, at present, there is no general theory dealing with the persistence of homoclinic orbits in a perturbed Hamiltonian system under small perturbation with the corresponding unperturbed Hamiltonian system having a homoclinic orbit to an equilibrium point but not hyperbolic saddle type. This remains for future research.

Acknowledgements. The authors are grateful to the reviewers for their valuable constructive comments, which certainly improved the quality of original paper.

Appendix

In this appendix, we introduce some results about symbolic computation, including resultant elimination theory, Fourier-Budan theorem and Sturm's theorem.

Let $\mathbf{K}$ be an algebraically closed field. Given two polynomials $A(x_1, x_2, \cdots, x_n)$, $B(x_1, x_2, \cdots, x_n) \in \mathbf{K}[x_1, x_2, \cdots, x_n]$, $(x_1, x_2, \cdots, x_n) \in \mathbf{K}^n$, of the forms

$$A(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{k} A_i(x_1, x_2, \cdots, x_{n-1})x_n^i,$$
$$B(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{l} B_i(x_1, x_2, \cdots, x_{n-1})x_n^i,$$

where both k and l are positive integers. Denote $\mathrm{Res}(A, B, x_n)$ the Sylvester resultant of A and B with respect to x_n, see [14]. Then the following lemma holds, see Theorem 5 in [8].

Lemma 5.1. *Denote $C(x_1, x_2, \cdots, x_{n-1}) \triangleq Res(A, B, x_n)$. If $(a_1, a_2, \cdots, a_n)$ is a common zero of A and B, then we have $C(a_1, a_2, \cdots, a_{n-1}) = 0$. Conversely, if $C(a_1, a_2, \cdots, a_{n-1}) = 0$, then at least one of the following holds:*

(a) $A_k(a_1, a_2, \cdots, a_{n-1}) = \cdots = A_0(a_1, a_2, \cdots, a_{n-1}) = 0$,

(b) $B_l(a_1, a_2, \cdots, a_{n-1}) = \cdots = B_0(a_1, a_2, \cdots, a_{n-1}) = 0$,

(c) $A_k(a_1, a_2, \cdots, a_{n-1}) = B_l(a_1, a_2, \cdots, a_{n-1}) = 0$,

(d) For some $a_n \in \boldsymbol{K}$, $(a_1, a_2, \cdots, a_n)$ is a common zero of A and B.

Clearly, $C = 0$ is a necessary condition of $A = B = 0$, but not sufficient. This fact not only gives a criterion for the existence of common zeros for two polynomials, but also provides a method of finding the common zeros of multivariate polynomial systems [10]. In particular, setting $n = 2$, the following lemma is obvious.

Lemma 5.2. *If $Res(A, B, x_2)$ (resp. $Res(A, B, x_1)$) has no real root on an interval $\mathbb{I}$ (resp. $\mathbb{J}$), then $A(x_1, x_2)$ and $B(x_1, x_2)$ has no common real root on $\mathbb{I} \times \mathbb{R}$ (resp. $\mathbb{R} \times \mathbb{J}$). If $Res(A, B, x_2)$ has a unique real root on an interval $\mathbb{I}$ and $Res(A, B, x_1)$ has a unique real root on an interval $\mathbb{J}$, then $A(x_1, x_2)$ and $B(x_1, x_2)$ have at most one common real root on $\mathbb{I} \times \mathbb{J}$.*

Next we will introduce some known results about the number of real roots of univariate polynomial. Assume that $f(x)$ is a polynomial of degree n with real coefficients, $a < b$ are two real numbers, $f(a) \neq 0, f(b) \neq 0$, and the derivatives of $f(x)$ are

$$f(x),\ f'(x),\ f''(x),\ \cdots,\ f^{(n)}(x).$$

For a real series $\boldsymbol{c} = [c_0, c_1, \cdots, c_n]$, we denote by

$$\mathrm{var}(\boldsymbol{c}) = \mathrm{var}[c_0, c_1, \cdots, c_n]$$

the number of variations of $\boldsymbol{c}$ (skip zero(s), if it appears in this series). Let

$$\mathrm{sgn}(\boldsymbol{c}) = [\mathrm{sgn}(c_0), \mathrm{sgn}(c_1), \cdots, \mathrm{sgn}(c_n)]$$

denote the sign sequence of $\boldsymbol{c}$. Then $\mathrm{var}(\boldsymbol{c}) = \mathrm{var}(\mathrm{sgn}(\boldsymbol{c}))$. Denote $num(f, I)$ the number of real roots (counting the multiplicity) in the interval I of polynomial f in x and $num_+(f, I)$ the number of positive real roots (counting the multiplicity) in the interval I of polynomial f in x, respectively. To find the number of real roots of $f(x)$ for $x \in (a, b)$, the following two lemmas are well known.

Lemma 5.3 (Fourier-Budan theorem)**.** *If*

$$var[f(a), f'(a), \cdots, f^{(n)}(a)] = p,$$
$$var[f(b), f'(b), \cdots, f^{(n)}(b)] = q,$$

then $p \geq q$, and the number of real roots (counting the multiplicity) of $f(x)$ for $x \in (a, b)$ is equal to either $p - q$ or $p - q - r$, where r is a positive even integer. In particular, if $p = q$ (resp. $p = q + 1$), then $f(x)$ has no (resp. has a unique) real root in (a, b).

Lemma 5.4 (Sturm's theorem)**.** *Assume that $f(x)$ has no multiple root in (a, b), and we construct the series $[f_0(x), f_1(x), \cdots, f_s(x)]$ as follows: $f_0(x) = f(x), f_1(x) = f'(x)$. Divide $f_0(x)$ by $f_1(x)$, and take the remainder with negative sign as $f_2(x)$, then divide $f_1(x)$ by $f_2(x)$, and take the remainder with negative sign as $f_3(x), \cdots$, the last remainder with negative sign (a non-zero number) is $f_s(x)$. If*

$$var[f_0(a),\ f_1(a),\ f_2(a), \cdots, f_s(a)] = p,$$
$$var[f_0(b),\ f_1(b),\ f_2(b), \cdots, f_s(b)] = q,$$

then $p \geq q$, and $num(f, (a, b)) = p - q$.

References

[1] Y. An, M. Han, *On the number of limit cycles near a homoclinic loop with a nilpotent singular point*, J. Diff. Eqs., 2015, 258(9), 3194–3247.

[2] T. Benjamin, J. Bona, J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Phil. Trans. R. Soc. Lond. A, 1972, 272(1220), 47–78.

[3] C. Besse, B. Mésognon-Gireau, P. Noble, *Artificial boundary conditions for the linearized Benjamin-Bona-Mahony equation*, Numer. Math., 2018, 139(2), 281–314.

[4] A. Biswas, 1-*Soliton solution of Benjamin-Bona-Mahoney equation with dual-power law nonlinearity*, Commun. Non. Sci Numer. Simul., 2010, 15(10), 2744–2746.

[5] R. Camassa and D. Holm, *An integrable shallow wave equation with peaked solitons*, Phys. Rev. Lett., 1993, 71(11), 1661–1664.

[6] A. Chen, L. Guo, X. Deng, *Existence of solitary waves and periodic waves for a perturbed generalized BBM equation*, J. Diff. Equat., 2016, 261(10), 5324–5349.

[7] A. Chen, L. Guo and W. Huang, *Existence of kink waves and periodic waves for a perturbed defocusing mKdV equation*, Qual. Theory Dyn. Syst., 2018, 17(3), 495–517.

[8] G. Collins, *The calculation of multivariate polynomial resultants*, J. Assoc. Comput. Mach., 1971, 18(4), 515–532.

[9] Y. Dai, M. Wei, M. Han, *Periodic waves for a perturbed generalized BBM equation*, preprint.

[10] Y. Dai, Y. Zhao, B. Sang, *Four limit cycles in a predator-prey system of Leslie type with generalized Holling type III functional response*, Nonlinear Anal. Real World Appl., 2019, 50, 218-239.

[11] G. Derks and S. van Gils, *On the uniqueness of traveling waves in perturbed Korteveg-deVries equations*, Japan J. Indust. Appl. Math., 1993, 10(3), 413–430.

[12] X. Fan and L. Tian, *The existence of solitary waves of singularly perturbed mKdV-KS equation*, Chaos Solitons Fractals, 2005, 26(4), 1111–1118.

[13] N. Fenichel, *Geometric singular perturbation theory for ordinary differential equations*, J. Diff. Equat., 1979, 31(1), 53–98.

[14] I. Gelfand, M. Kapranov, A. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäser, Boston, 1994.

[15] M. Grau, F. Mañsas, J. Villadelprat, *A Chebyshev criterion for Abelian integrals*, Trans. Am. Math. Soc., 2011, 363(1), 109–129.

[16] A. Green and P. Naghdi, *A derivation of equations for wave propagation in water of variable depth*, J. Fluid. Mech., 1976, 78(02), 237–246.

[17] L. Guo and Y. Zhao, *Existence of periodic waves for a perturbed quintic BBM equation*, Discrete Contin. Dyn. Syst. Ser. B, 2020, 40(8), 46894703.

[18] M. Han, *Bifurcation theory and periodical solution of dynamic system*, Science Press, Beijing, 2002.

[19] M. Han, *Bifurcation theory of limit cycles*, Science Press, Beijing, 2013.

[20] M. Han, J. Yang, *The maximum number of zeros of functions with parameters and application to differential equations*, Journal of Nonlinear Modeling and Analysis, 2021, 3(1), 13–34.

[21] M. Han, J. Yang, D. Xiao, *Limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2012, 22(8), 1250189-1-33.

[22] M. Han, P. Yu, *Normal forms, Melnikov functions and bifurcations of limit cycles*, Springer, New York, 2012.

[23] C. Jones, *Geometric singular perturbation theory: Dynamic systems*, Lecture Notes in Math., Springer, Berlin, 1995, 1609, 44–118.

[24] S. Karlin, W. Studden, *Tchebycheff systems: with applications in analysis and statistics*, J. Amer. Statis. Assoc., 1967, 62(319), 1093.

[25] D. Korteweg and G. de Vries, *On the change of form of the long waves advancing in a rectangular canal, and on a new type of stationary waves*, Philos. Mag., 1895, 39(240), 422–443.

[26] F. Li, H. Li, Y. Liu, *New double bifurcation of nilpotent focus*, Int. J. Bifurcation Chaos, 2021, 31(4), 2150053.

[27] Y. Liu, F. Li, *Double bifurcation of nilpotent focus*, Int. J. Bifurcation Chaos, 2015, 25(03), 1550036.

[28] X. Lu, L. Lu, A. Chen, *New peakons and periodic peakons of the modified Camassa-Holm equation*, Journal of Nonlinear Modeling and Analysis, 2020, 2(3), 345–353.

[29] F. Mañosas, J. Villadelprat, *Bounding the number of zeros of certain Abelian integrals*, J. Diff. Equat., 2011, 251(6), 1656–1669.

[30] S. Micu, *On the controllability of the linearized Benjamin-Bona-Mahony equation*, SIAM J. Control. Optim., 2001, 39(6), 1677–1696.

[31] D. Novaes and J. Torregrosa, *On extended chebyshev systems with positive accuracy*, J. Math. Anal. Appl., 2017, 448(1), 171–186.

[32] T. Ogawa, *Traveling wave solutions to a perturbed Korteweg-de Vries equation*, Hiroshima Math. J., 1994, 24(2), 401–422.

[33] K. Omrani, *The convergence of fully discrete Galerkin approximations for the Benjamin-Bona-Mahony (BBM) equation*, Appl. Math. Comput., 2006, 180(2), 614–621.

[34] K. Singh, R. Gupta, S. Kumar, *Benjamin-Bona-Mahony (BBM) equation with variable coefficients: similarity reductions and Painlevé analysis*, Appl. Math. Comput., 2011, 217(16), 7021–7027.

[35] X. Sun, P. Yu, *Periodic travelling waves in a generalized BBM equation with weak backward diffusion and dissipation terms*, Discrete Contin. Dyn. Syst. Ser. B, 2019, 24(2), 965–987.

[36] J. Wang, M. Yuen, L. Zhang, *Persistence of solitary wave solutions to a singularly perturbed generalized mKdV equation*, Appl. Math. Lett., 2022, 124, 107668.

[37] J. Wang, L. Zhang, E. Shchepakina, et al., *Solitary waves of singularly perturbed generalized KdV equation with high order nonlinearity*, Discrete Contin. Dyn. Syst. Ser. S, 2022, doi: 10.3934/dcdss.2022124.

[38] A. Wazwaz, *Exact solution with compact and noncompact structures for the one-dimensional generalized Benjamin-Bona-Mahony equation*, Commun. Non. Sci. Numer. Simulat., 2005, 10(8), 855–867.

[39] W. Yan, Z. Liu, Y. Liang, *Existence of solitary waves and periodic waves to a perturbed generalized KdV equation*, Math. Model. Anal., 2014, 19(4), 537–555.

[40] P. Yu, F. Li, *Bifurcation of limit cycles in a cubic-order planar system around a nilpotent critical point*, J. Math. Anal. Appl., 2017, 453(2), 645–667.

[41] H. Zang, M. Han, D. Xiao, *On Melnikov functions of a homoclinic loop through a nilpotent saddle for planar near-Hamiltonian systems*, J. Differential Equations, 2008, 245(4), 1086–1111.

[42] X. Zhang, *Homoclinic, heteroclinic and periodic orbits of singularly perturbed systems*, Sci. China. Math., 2019, 62(9), 1687–1704.

[43] L. Zhang, M. Han, M. Zhang, et al. *A new type of solitary wave solution of the mKdV equation under singular perturbations*, Int. J. Bifurcation Chaos, 2020, 30(11), 2050162.

[44] L. Zhang, Y. Wang, C. Khalique, et al., *Peakon and cuspon solutions of a generalized Camassa-Holm-Novikov equation*, J. Appl. Anal. Comput., 2018, 8(6), 1938–1958.

[45] L. Zhang, Y. Wang, C. Khalique, et al., *New type of solitary wave solution with coexisting crest and trough for a perturbed wave equation*, Nonlinear Dyn., 2021, 106(4), 3479–3493.

[46] X. Zhao, W. Xu, S. Li, et al., *Bifurcations of traveling wave solutions for a class of the generalized Benjamin-Bona-Mahony equation*, Appl. Math. Comput., 2006, 175(2), 1760–1774.

[47] X. Zhao, H. Jia, H. Zhou, et al., *Bifurcations of travelling wave solutions in a non-linear dispersive equation*, Chaos Solitons Fractals, 2008, 37(2), 525–531.

[48] K. Zhu, Y. Wu, Z. Yu, et al., *New solitary wave solutions in a perturbed generalized BBM equation*, Nonlinear Dyn., 2019, 97(4), 2413–2423.

[49] K. Zhuang, Z. Du, X. Lin, *Solitary waves solutions of singularly perturbed higher-order KdV equation via geometric singular perturbation method*, Nonlinear Dyn., 2015, 80(1–2), 629–635.