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NORMAL FORMS OF NILPOTENT SYSTEM IN $\mathbb{C}^2 \times \mathbb{C}^2$

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Abstract In this paper, we consider the following nilpotent system:

$$\dot{\theta} = \omega + \Theta(\theta, u), \quad \dot{u} = Au + f(\theta, u),$$

where $\theta \in \mathbb{C}^2$, $u \in \mathbb{C}^2$, $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$, $A = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$, Θ and f are analytic functions and $2\pi-$periodic in each component of the vector θ, $\Theta = O(|u|)$ and $f = O(|u|^2)$ as $u \to 0$. Two kinds of normal forms are presented based on the different small-divisor conditions.

Keywords Normal forms, nilpotent system, small-divisor.

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1. Introduction

Normal form theory originated from the works by Poincaré [13] and provides one of the fundamental tools in the study of nonlinear dynamical systems, in particular in local stability and bifurcation analysis. For Limit cycle bifurcations near a double homoclinic loop, the reader may find the normal form theory in the recent works of Yang et al. [15]. Wang and Ren [14] extend the Siegel's theorem for analytically reducing periodic difference system to a linear one based on normal form theory. There are many researchers who have made contributions to normal form of different situations. Cushman and Sanders [3, 4] use the representation theory of $sl_2(\mathbb{R})$ to treat the nilpotent case. Elphick et al. [9] introduced an inner product on the space of homogeneous vector polynomials. Iooss [11] and Chen and Della Dora [2] extended the work of Elphick et al. [9] to the situation of computing the normal form of a vector field in the neighborhood of a periodic orbit and the situation of computing the normal form of a map near a fixed point. Chow et al. [5] presented a normal form theory with a diagonalizable matrix, Li and Lu [12] and Chen and Zhang [6] applied this theory to random dynamical systems and planar switching systems respectively. Recently, Du et al. [7] gave the normal form formulas of double-Hopf bifurcation; Barreira and Valls [1] studied normal forms for equivariant differential equations; Diez and Rudolph [8] established a convenient normal form for a large class of nonlinear differential equations with symmetries; Guo [10] introduced equivariant normal forms of semilinear functional differential equations in general

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Banach spaces. The main purpose of this paper is to investigate the normal form of nilpotent system in $\mathbb{C}^2\times\mathbb{C}^2$.

Consider the following nilpotent system

$$\begin{cases}\dot\theta=\omega+\Theta(\theta,u),\\ \dot u=Au+f(\theta,u),\end{cases}\tag{1.1}$$

where $\theta\in\mathbb{C}^2,\ u\in\mathbb{C}^2,\ \omega=(\omega_1,\omega_2)\in\mathbb{R}^2,\ A=\begin{pmatrix}\lambda & 1\\ & \lambda\end{pmatrix}$, Θ and f are analytic functions and $2\pi-$periodic in each component of the vector θ, $\Theta=O(|u|)$ and $f=O(|u|^2)$ as $u\to 0$.

For the sake of statement, we denote by $\langle\ ,\rangle$ the scalar product in $\mathbb{C}^2$,

$$\mathbb{N}^2=\{\alpha=(\alpha_1,\alpha_2)|\alpha_j\in\mathbb{N}^+,\ j=1,2\},\quad \mathbb{Z}^2=\{\alpha=(\alpha_1,\alpha_2)|\alpha_j\in\mathbb{Z},\ j=1,2\},$$

and $|k|=|k_1|+|k_2|$, $u^k=u_1^{k_1}u_2^{k_2}$ for $k\in\mathbb{Z}^2$.

We will obtain the following two results in this paper.

(1) Assume that $\lambda=0$ and the following small-divisor conditions are satisfied

(A1) $|\mathbf{i}\langle\omega,k\rangle|\ge\frac{C_0}{|k|^\mu}$, where $k\in\mathbb{Z}^2,k\neq 0$, and C_0 and μ are positive constants.

Then system (1.1) can be changed to

$$\begin{cases}\dot\vartheta=\omega+\sum\limits_{1\le|\alpha|\le m}a_\alpha x^\alpha+o(|x|^m),\\ \dot x=(B+\eta\tilde B)x+\sum\limits_{1\le|\alpha|\le m}b_\alpha x^\alpha+o(|x|^m)\end{cases}\tag{1.2}$$

by an analytic transformation, where $m\in\mathbb{N}^+,a_\alpha$ and b_α are constant vectors, $\eta>0$ is a constant,

$$B=\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix},\tilde B=\begin{pmatrix}0 & 1\\ & 0\end{pmatrix}.$$

(2) Assume that $\lambda\neq 0$ and the following small-divisor conditions hold for a fixed $m\in\mathbb{N}^+$,

(A2) $|\mathbf{i}\langle\omega,k\rangle+\lambda(|\alpha|-\varepsilon)|\ge\frac{C_0}{|k|^\mu}$, where $k\in\mathbb{Z}^2,\ k\neq 0,\alpha\in\mathbb{N}^2,1+\varepsilon\le|\alpha|\le m$, $\varepsilon=0$ or 1, C_0 and μ are positive constants.

Then system (1.1) can be changed to

$$\begin{cases}\dot\vartheta=\omega+o(|x|^m),\\ \dot x=(B+\eta\tilde B)x+o(|x|^m)\end{cases}\tag{1.3}$$

by an analytic transformation, where $\eta>0$ is a constant,

$$B=\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix},\tilde B=\begin{pmatrix}0 & 1\\ & 0\end{pmatrix}.$$

The rest of this paper is organized as follows. In the next section, we recall some notations. The main results will be stated and proved in section 3.

2. Preliminaries

In this section we recall some notations (see [5] for details).

For $r \in (0,1]$, set

$$D_r = \{(\theta, u) \in \mathbb{C}^2 \times \mathbb{C}^2 | \ |\mathrm{Im}\theta_1| < r, |\mathrm{Im}\theta_2| < r, |u| < r\}.$$

Let $F(\theta, u) : D_r \to \mathbb{C}^2$ be a bounded analytic function which is $2\pi-$periodic in each component of θ, then

$$F(\theta, u) = \sum_{\alpha \in \mathbb{N}^2} \sum_{k \in \mathbb{Z}^2} F_{\alpha^{p_1}, \ldots, \alpha^{p_{|\alpha|}}, k} e^{\mathbf{i}\langle \theta, k\rangle} u^\alpha = F^0(\theta, u) + F^k(\theta, u), \tag{2.1}$$

where

$$F^0(\theta, u) = \sum_{\alpha \in \mathbb{N}^2} F^0_{\alpha^{p_1}, \ldots, \alpha^{p_{|\alpha|}}} u^\alpha,$$
$$F^k(\theta, u) = \sum_{\alpha \in \mathbb{N}^2} \sum_{\substack{k \in \mathbb{Z}^2 \\ k \neq 0}} F^K_{\alpha^{p_1}, \ldots, \alpha^{p_{|\alpha|}}, k} e^{\mathbf{i}\langle \theta, k\rangle} u^\alpha,$$

$1 \le p_1 \le \cdots \le p_{|\alpha|} \le 2, p_{s'} \in \mathbb{N}^+, \alpha^{p_{s'}} = 1$ for $1 \le s' \le |\alpha|$ and $s' \in \mathbb{N}^+$, $\alpha_{p_{s_1}} = \alpha^{p_{s_1}} + \alpha^{p_{s_1+1}} + \cdots + \alpha^{p_{s_1+s_2}}$ if $p_{s_1} = p_{s_1+1} = \cdots = p_{s_1+s_2}$ for $1 \le s_1, s_1 + s_2 \le |\alpha|$ and $s_1, s_2 \in \mathbb{Z}$, $F_{\alpha^{p_1}, \ldots, \alpha^{p_{|\alpha|}}}$ are independent of the order of $\alpha^{p_1}, \ldots, \alpha^{p_{|\alpha|}}$.

Define

$$\breve{F}(\theta, u) = \sum_{\alpha \in \mathbb{N}^2} \sum_{k \in \mathbb{Z}^2} \frac{F_{\alpha,k}}{C_{\alpha,k}} e^{\mathbf{i}\langle \theta, k\rangle} u^\alpha, \tag{2.2}$$

where $C_{\alpha,k}$ are constants satisfying

$$|C_{\alpha,k}| \ge \frac{C_0}{(|k| + |\alpha|)^\mu}$$

for some positive constants C_0 and μ, then it is clearly that $\breve{F}$ is an analytic function, and $2\pi-$periodic in each component of θ in D_r.

3. Main results

3.1. Normal forms

Make the following transformation of variables

$$u = Cz, \tag{3.1}$$

where $C = \begin{pmatrix} 1 & \\ & \eta \end{pmatrix}$, $\eta > 0$ is a constant, then system (1.1) can be rewritten as

$$\begin{cases} \dot\theta = \omega + \Theta(\theta, Cz), \\ \dot z = (B + \eta\tilde{B})z + C^{-1} f(\theta, Cz), \end{cases} \tag{3.2}$$

where

$$B=\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}, \tilde{B}=\begin{pmatrix}0 & 1\\ & 0\end{pmatrix}, C^{-1}=\begin{pmatrix}1 & \\ & \eta^{-1}\end{pmatrix},$$

$Cz=(z_1,\eta z_2)$ and $C^{-1}f=(f^1,\eta^{-1}f^2)$, $\Theta(\theta,Cz)=O(|Cz|)$ and $C^{-1}f=O(|Cz|^2)$ as $Cz\to 0$.

Theorem 3.1. *Assume $\lambda=0$ and (A1) holds, then system (3.2) can be changed to*

$$\begin{cases}\dot{\vartheta}=\omega+\sum\limits_{1\le|\alpha|\le m}a_\alpha(Cy)^\alpha+o(|Cy|^m),\\ \dot{y}=\eta\tilde{B}y+\sum\limits_{1\le|\alpha|\le m}C^{-1}b_\alpha(Cy)^\alpha+o(|Cy|^m)\end{cases}\tag{3.3}$$

by a transformation

$$\theta=\vartheta+\Phi(\vartheta,Cy),\quad z=y+C^{-1}\Psi(\vartheta,Cy),\tag{3.4}$$

where $m\in\mathbb{N}^+$, $a_\alpha=(a^1_\alpha,a^2_\alpha)$ and $b_\alpha=(b^1_\alpha,b^2_\alpha)$ are constant vectors satisfying $a^j_\alpha=b^j_\alpha=0$ if $\alpha_1\neq|\alpha|$, and $b_\alpha=0$ if $|\alpha|=1$, $\Phi(\vartheta,Cy)$ and $\Psi(\vartheta,Cy)$ are analytic functions and $2\pi-$periodic in each component of ϑ in D_{r_m}, $\Phi(\vartheta,Cy)=O(|Cy|)$, $\Psi(\vartheta,Cy)=O(|Cy|^2)$, here r_m is a positive constant.

Theorem 3.2. *Assume $\lambda\neq 0$ and (A2) holds for a fixed $m\in\mathbb{N}^+$, then system (3.2) can be changed to*

$$\begin{cases}\dot{\vartheta}=\omega+o(|Cy|^m),\\ \dot{y}=(B+\eta\tilde{B})y+o(|Cy|^m)\end{cases}\tag{3.5}$$

by a transformation

$$\theta=\vartheta+\Phi_\lambda(\vartheta,Cy),\quad z=y+C^{-1}\Psi_\lambda(\vartheta,Cy),\tag{3.6}$$

where $\Phi_\lambda(\vartheta,Cy)$ and $\Psi_\lambda(\vartheta,Cy)$ are analytic functions and 2π-periodic in each component of ϑ in D_{r_m}, $\Phi_\lambda(\vartheta,Cy)=O(|Cy|)$, $\Psi_\lambda(\vartheta,Cy)=O(|Cy|^2)$, where r_m is a positive constant.

Corollary 3.1. *Suppose that for a fixed $m\in\mathbb{N}^+$ with $m\ge 2$,*

(A3) $|\mathbf{i}\langle\omega,k\rangle+\lambda(|\alpha|-\varepsilon)|\ge\frac{C_0}{(|k|+|\alpha|)^\mu}$, where $k\in\mathbb{Z}^2,\alpha\in\mathbb{N}^2,1+\varepsilon\le|\alpha|\le m$, $\varepsilon=0$ or 1, and C_0 and μ are positive constants.

Then system (1.1) can be changed to

$$\dot{\vartheta}=\omega+o(|x|^m),\quad \dot{x}=(B+\eta\tilde{B})x+o(|x|^m)$$

by an analytic transformation, where $\eta>0$ is a constant,

$$B=\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}, \tilde{B}=\begin{pmatrix}0 & 1\\ & 0\end{pmatrix}.$$

Remark 3.1. Chow et al. [5] has been derived for the case that A is a diagonalizable matrix.

3.2. Proof of Theorem 3.1

The proof is by induction on m.

1) Suppose that the conclusion holds for $m = 1$, i.e., (3.4) transforms (3.2) to

$$\begin{cases} \dot{\vartheta} = \omega + h(\vartheta, Cy), \\ \dot{y} = \eta \tilde{B} y + C^{-1} g(\vartheta, Cy), \end{cases} \tag{3.7}$$

where $h = O(|Cy|)$ and $C^{-1}g = O(|Cy|^2)$, then $(\Phi, C^{-1}\Psi)$ satisfies

$$h + D_\vartheta \Phi \omega + D_\vartheta \Phi h + D_y \Phi \eta \tilde{B} y + D_y \Phi C^{-1} g - \Theta \circ (\vartheta + \Phi, Cy + \Psi) = 0, \tag{3.8}$$

$$\begin{aligned} &C^{-1} g + D_\vartheta (C^{-1}\Psi)\omega + D_\vartheta (C^{-1}\Psi) h + D_y (C^{-1}\Psi) \eta \tilde{B} y \\ &+ D_y (C^{-1}\Psi) C^{-1} g - \eta \tilde{B} (C^{-1}\Psi) - C^{-1} f \circ (\vartheta + \Phi, Cy + \Psi) = 0. \end{aligned} \tag{3.9}$$

Since $m = 1$, by (3.7), we can take $C^{-1}\Psi = 0$. Then (3.8), (3.9) can be reduced to

$$h + D_\vartheta \Phi \omega + D_\vartheta \Phi h + D_y \Phi \eta \tilde{B} y + D_y \Phi C^{-1} g - \Theta \circ (\vartheta + \Phi, Cy) = 0. \tag{3.10}$$

Write $h = h^1 + h^+$, where h^1 is the first order terms in Cy and h^+ is the higher order terms in Cy. Using a Taylor expansion for Θ, we have

$$\Theta(\vartheta + \Phi, Cy) = \Theta(\vartheta, Cy) + D_\vartheta \Theta(\vartheta, Cy)\Phi + R(\Phi).$$

Assume Φ is first order in Cy. Writing $\Theta(\vartheta, Cy) = \Theta^1 + \Theta^+$ as for h and comparing the order of y in (3.10), we have

$$D_\vartheta \Phi \omega + D_y \Phi \eta \tilde{B} y + h^1 - \Theta^1 = 0, \tag{3.11}$$

$$h^+ + D_\vartheta \Phi h + D_y \Phi C^{-1} g - \Theta^+ - D_\vartheta \Theta \Phi - R(\Phi) = 0. \tag{3.12}$$

Note the left hand of (3.11) is a first order homogeneous polynomial in y and that (3.12) is $O(|y|^2)$.

By (2.1) we can write (3.11) and (3.12) as the following equations

$$D_y \Phi^0 \eta \tilde{B} y + h^{0,1} - \Theta^{0,1} = 0, \tag{3.13}$$

$$h^{0,+} + D_y \Phi^0 C^{-1} g^0 - \Theta^{0,+} - R^0(\Phi) = 0 \tag{3.14}$$

and

$$D_\vartheta \Phi^K \omega + D_y \Phi^K \eta \tilde{B} y + h^{K,1} - \Theta^{K,1} = 0, \tag{3.15}$$

$$\begin{aligned} &h^{K,+} + D_\vartheta \Phi^K h + D_y \Phi C^{-1} g \\ &- D_y \Phi^0 C^{-1} g^0 - \Theta^{K,+} - D_\vartheta \Theta^K \Phi - R^K(\Phi) = 0. \end{aligned} \tag{3.16}$$

Let us first solve (3.13) by finding the Φ^0 which makes $h^{0,1}$ as simple as possible. Using Fourier expansions for Φ^0, $h^{0,1}$, $\Theta^{0,1}$, we have

$$\begin{aligned} \Phi^{0,j}(Cy) &= \sum_{|\alpha|=1} \eta^{p_1 - 1} \Phi^{0,j}_{\alpha^{p_1}} y^\alpha, \\ h^{0,1,j}(Cy) &= \sum_{|\alpha|=1} \eta^{p_1 - 1} h^{0,1,j}_{\alpha^{p_1}} y^\alpha, \\ \Theta^{0,1,j}(Cy) &= \sum_{|\alpha|=1} \eta^{p_1 - 1} \Theta^{0,1,j}_{\alpha^{p_1}} y^\alpha. \end{aligned} \tag{3.17}$$

Then by (3.13) and (3.17), we have

$$\sum_{|\alpha|=1}\eta^{p_1}\Phi^{0,j}_{\alpha^{p_1}}\alpha_1 y_1^{\alpha_1-1}y_2^{\alpha_2+1}+\sum_{|\alpha|=1}\eta^{p_1-1}h^{0,1,j}_{\alpha^{p_1}}y^\alpha-\sum_{|\alpha|=1}\eta^{p_1-1}\Theta^{0,1,j}_{\alpha^{p_1}}y^\alpha=0. \quad (3.18)$$

The equation (3.18) can be written as

$$(h^{0,1,j}_{\alpha^1}y_1-\Theta^{0,1,j}_{\alpha^1}y_1)+\eta(\Phi^{0,j}_{\alpha^1}y_2+h^{0,1,j}_{\alpha^1}y_2-\Theta^{0,1,j}_{\alpha^1}y_2)=0,$$

i.e.,

$$\sum_{|\alpha|=1}\eta^{p_1-1}(\Phi^{0,j}_{\alpha^{p_1-1}}y^\alpha+h^{0,1,j}_{\alpha^{p_1}}y^\alpha-\Theta^{0,1,j}_{\alpha^{p_1}}y^\alpha)=0, \quad (3.19)$$

where $\Phi^{0,j}_{\alpha^{p_1-p_1'}}=0$ if $p_1-p_1'\le 0$.

By equating the terms in (3.19) of the same order with respect to η^{p_1-1}, we get

$$\Phi^{0,j}_{\alpha^{p_1-1}}y^\alpha+h^{0,1,j}_{\alpha^{p_1}}y^\alpha-\Theta^{0,1,j}_{\alpha^{p_1}}y^\alpha=0,$$

i.e.,

$$(\Theta-h)^{0,1,j}_{\alpha^{p_1}}-\Phi^{0,j}_{\alpha^{p_1-1}}=0, \quad (3.20)$$

where $(\Theta-h)^{0,1,j}_{\alpha^{p_1}}=\Theta^{0,1,j}_{\alpha^{p_1}}-h^{0,1,j}_{\alpha^{p_1}}$ and $\Phi^{0,j}_{\alpha^{p_1-p_1'}}=0$ if $p_1-p_1'\le 0$. Let $h^{0,1,j}_{\alpha^{p_1}}=0$. By (3.20) we get

$$\Phi^{0,j}_{\alpha^{p_1}}=\begin{cases}0, & \alpha_2=1,\\ \Theta^{0,1,j}_{\alpha^2}, & \alpha_2\neq 1.\end{cases}$$

Hence

$$h^{0,1,j}_{\alpha^{p_1}}=\begin{cases}\Theta^{0,1,j}_{\alpha^1}, & \alpha_1=1,\\ 0, & \alpha_1\neq 1.\end{cases}$$

It is clearly that Φ^0 is analytic and 2π-periodic in ϑ_1,ϑ_2 in $D_{r-\delta}$ for $\delta\in(0,r)$ and $\Phi^0=O(|Cy|)$. Now we determine $h^{0,+}$. Write (3.14) as follows

$$h^{0,+}=-D_y\Phi^0C^{-1}g^0+\Theta^{0,+}+R^0(\Phi).$$

We can choose sufficiently small $r_1\in(0,r-\delta)$ such that $(\vartheta,y)+(\Phi^0,C^{-1}\Psi^0)\in D_r$ for $(\vartheta,y)\in D_{r_1}$.

Now let us solve (3.15) by finding Φ^K such that $h^{K,1}$ as simple as possible. Using Fourier expansions for Φ^K, $h^{K,1}$, $\Theta^{K,1}$, we have

$$\Phi^{K,j}(\vartheta,Cy)=\sum_{|\alpha|=1}\sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}}\eta^{p_1-1}\Phi^{K,j}_{\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha,$$

$$h^{K,1,j}(\vartheta,Cy)=\sum_{|\alpha|=1}\sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}}\eta^{p_1-1}h^{K,1,j}_{\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha, \quad (3.21)$$

$$\Theta^{K,1,j}(\vartheta, Cy) = \sum_{|\alpha|=1} \sum_{\substack{k\in\mathbb{Z}^2 \\ k\neq 0}} \eta^{p_1-1} \Theta^{K,1,j}_{\alpha^{p_1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha.$$

According to (3.15) and (3.21), we have

$$\sum_{|\alpha|=1} \sum_{\substack{k\in\mathbb{Z}^2 \\ k\neq 0}} \eta^{p_1-1} \Phi^{K,j}_{\alpha^{p_1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} \mathbf{i}\langle k,\omega\rangle y^\alpha + \sum_{|\alpha|=1} \sum_{\substack{k\in\mathbb{Z}^2 \\ k\neq 0}} \eta^{p_1} \Phi^{K,j}_{\alpha^{p_1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} \alpha_1 y_1^{\alpha_1-1} y_2^{\alpha_2+1}$$
$$+ \sum_{|\alpha|=1} \sum_{\substack{k\in\mathbb{Z}^2 \\ k\neq 0}} \eta^{p_1-1} h^{K,1,j}_{\alpha^{p_1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha - \sum_{|\alpha|=1} \sum_{\substack{k\in\mathbb{Z}^2 \\ k\neq 0}} \eta^{p_1-1} \Theta^{K,1,j}_{\alpha^{p_1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha = 0. \tag{3.22}$$

The equation (3.22) can be written as

$$\sum_{\substack{k\in\mathbb{Z}^2 \\ k\neq 0}} (\Phi^{K,j}_{\alpha^1,k} \mathbf{i}\langle k,\omega\rangle e^{\mathbf{i}\langle k,\vartheta\rangle} y_1 + h^{K,1,j}_{\alpha^1,k} e^{\mathbf{i}\langle k,\vartheta\rangle} y_1 - \Theta^{K,1,j}_{\alpha^1,k} e^{\mathbf{i}\langle k,\vartheta\rangle} y_1)$$
$$+ \sum_{\substack{k\in\mathbb{Z}^2 \\ k\neq 0}} \eta (\Phi^{K,j}_{\alpha^2,k} \mathbf{i}\langle k,\omega\rangle e^{\mathbf{i}\langle k,\vartheta\rangle} y_2 + \Phi^{K,j}_{\alpha^1,k} e^{\mathbf{i}\langle k,\vartheta\rangle} y_2 + h^{K,1,j}_{\alpha^1,k} e^{\mathbf{i}\langle k,\vartheta\rangle} y_2 - \Theta^{K,1,j}_{\alpha^1,k} e^{\mathbf{i}\langle k,\vartheta\rangle} y_2) = 0,$$

i.e.,

$$\sum_{|\alpha|=1} \sum_{\substack{k\in\mathbb{Z}^2 \\ k\neq 0}} \eta^{p_1-1} (\Phi^{K,j}_{\alpha^{p_1},k} \mathbf{i}\langle k,\omega\rangle e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha + \Phi^{K,j}_{\alpha^{p_1-1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha$$
$$+ h^{K,1,j}_{\alpha^{p_1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha - \Theta^{K,1,j}_{\alpha^{p_1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha) = 0, \tag{3.23}$$

where $\Phi^{K,j}_{\alpha^{p_1-p_1'},k} = 0$ if $p_1 - p_1' \leq 0$.

By equating the terms in (3.23) of the same order with respect to η^{p_1-1}, we get

$$\sum_{\substack{k\in\mathbb{Z}^2 \\ k\neq 0}} (\Phi^{K,j}_{\alpha^{p_1},k} \mathbf{i}\langle k,\omega\rangle e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha + \Phi^{K,j}_{\alpha^{p_1-1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha$$
$$+ h^{K,1,j}_{\alpha^{p_1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha - \Theta^{K,1,j}_{\alpha^{p_1},k} e^{\mathbf{i}\langle k,\vartheta\rangle} y^\alpha) = 0.$$

Comparing the coefficient of y^α, we obtain

$$\mathbf{i}\langle k,\omega\rangle \Phi^{K,j}_{\alpha^{p_1},k} = (\Theta - h)^{K,1,j}_{\alpha^{p_1},k} - \Phi^{K,j}_{\alpha^{p_1-1},k}, \tag{3.24}$$

where $(\Theta - h)^{K,1,j}_{\alpha^{p_1},k} = \Theta^{K,1,j}_{\alpha^{p_1},k} - h^{K,1,j}_{\alpha^{p_1},k}$ and $\Phi^{K,j}_{\alpha^{p_1-p_1'},k} = 0$ if $p_1 - p_1' \leq 0$.

By (3.24), we have

$$\Phi^{K,j}_{\alpha^{p_1},k} = \frac{1}{\mathbf{i}\langle k,\omega\rangle} (\Theta - h)^{K,1,j}_{\alpha^{p_1},k} - \frac{1}{\mathbf{i}\langle k,\omega\rangle} \Phi^{K,j}_{\alpha^{p_1-1},k}$$
$$= \frac{1}{\mathbf{i}\langle k,\omega\rangle} (\Theta - h)^{K,1,j}_{\alpha^{p_1},k} - \frac{1}{(\mathbf{i}\langle k,\omega\rangle)^2} (\Theta - h)^{K,1,j}_{\alpha^{p_1-1},k}, \tag{3.25}$$

where $(\Theta - h)^{K,1,j}_{\alpha^{p_1-p_1'},k} = 0$ if $p_1 - p_1' \leq 0$.

By (A1), the following choice is the simplest for $h^{K,1,j}$. Let $h^{K,1,j}_{\alpha^{p_1},k} = 0$, by (3.25)

$$\Phi^{K,j}_{\alpha^{p_1},k} = \sum_{q=0}^{p_1-1} (-1)^q (\mathbf{i}\langle k,\omega\rangle)^{-(q+1)} \Theta^{K,1,j}_{\alpha^{p_1-q},k}.$$

Hence $h^{K,1,j}_{\alpha^{p_1},k} = 0$.

Using (A1) we find that Φ^K is analytic and 2π-periodic in ϑ_1, ϑ_2 in $D_{r-\delta}$ for $\delta \in (0, r)$ and $\Phi^K = O(|Cy|)$. Now we determine $h^{K,+}$. Write (3.16) as follows

$$(I + D_\vartheta \Phi^K)h^{K,+} = - D_y\Phi C^{-1}g + D_y\Phi^0 C^{-1}g^0 - D_\vartheta\Phi^K h^{0,+} - D_\vartheta \Phi^K h^1 \\ + \Theta^{K,+} + D_\vartheta\Theta^K\Phi + R^K(\Phi).$$

Since $D_\vartheta\Phi^K = O(|Cy|)$, we can choose sufficiently small r_1, $0 < r_1 < r - \delta$ such that $(I + D_\vartheta\Phi^K)$ is invertible and $(\vartheta, y) + (\Phi^K, C^{-1}\Psi^K) \in D_r$ for $(\vartheta, y) \in D_{r_1}$. Hence

$$h^{K,+} = (I + D_\vartheta\Phi^K)^{-1}(-D_y\Phi C^{-1}g + D_y\Phi^0 C^{-1}g^0 - D_\vartheta\Phi^K h^{0,+} - D_\vartheta\Phi^K h^1 \\ + \Theta^{K,+} + D_\vartheta\Theta^K\Phi + R^K(\Phi)).$$

We can get that $\Phi^j = \Phi^{0,j} + \Phi^{K,j} = \sum\limits_{|\alpha|=1}\sum\limits_{k\in\mathbb{Z}^2} \Phi^j_{\alpha,k} e^{\mathbf{i}\langle k,\vartheta\rangle}(Cy)^\alpha$ is analytic and 2π-periodic in ϑ_1, ϑ_2 in $D_{r-\delta}$, $0 < \delta < r$. It is clear that $\Phi = O(|Cy|)$. Therefore the transformation $\theta = \vartheta + \Phi(\vartheta, Cy)$, $z = Cy$ changes system (3.2) to

$$\begin{cases} \dot\vartheta = \omega + \sum\limits_{|\alpha|=1} a_\alpha (Cy)^\alpha + \tilde\Theta(\vartheta, Cy), \\ \dot y = \eta\tilde B y + C^{-1}\tilde f(\vartheta, Cy), \end{cases} \tag{3.26}$$

where $\tilde\Theta = O(|Cy|^2)$, $C^{-1}\tilde f = O(|Cy|^2)$, $a_\alpha = (a^1_\alpha, a^2_\alpha)$,

$$a^j_\alpha = a^j_{\alpha^{p_1}} = \begin{cases} \Theta^{0,1,j}_{\alpha^1}, & \alpha_1 = 1, \\ 0, & \alpha_1 \neq 1. \end{cases}$$

2) Assume that the conclusion holds for $m = d \geq 1$. By the induction hypothesis there exists a transformation $\theta = \vartheta + \Phi^d(\vartheta, Cy), z = y + C^{-1}\Psi^d(\vartheta, Cy)$ which satisfies the requirements of the conclusion and changes system (3.2) to

$$\begin{cases} \dot\vartheta = \omega + \sum\limits_{1\le|\alpha|\le d} a_\alpha (Cy)^\alpha + \hat\Theta(\vartheta, Cy), \\ \dot y = \eta\tilde B y + \sum\limits_{1\le|\alpha|\le d} C^{-1}b_\alpha (Cy)^\alpha + C^{-1}\hat f(\vartheta, Cy), \end{cases} \tag{3.27}$$

where $\hat\Theta = O(|Cy|^{d+1})$ and $C^{-1}\hat f = O(|Cy|^{d+1})$ are analytic in both variables and $2\pi-$periodic in each component of ϑ in D_{r_d}, where $r_d > 0$ is a positive constant.

When $m = d + 1$, consider the transformation $\vartheta = \zeta + \hat\phi(\zeta, Cv), y = v + C^{-1}\hat\psi(\zeta, Cv)$ where $\hat\phi$ and $C^{-1}\hat\psi$ are $(d+1)$th order homogeneous polynomials in Cv and $2\pi-$periodic in each component of the vector ζ. This transformation changes system (3.27) to the following system

$$\begin{cases} \dot\zeta = \omega + \sum\limits_{1\le|\alpha|\le d} a_\alpha (Cv)^\alpha + \hat h(\zeta, Cv), \\ \dot v = \eta\tilde B v + \sum\limits_{1\le|\alpha|\le d} C^{-1}b_\alpha (Cv)^\alpha + C^{-1}\hat g(\zeta, Cv), \end{cases} \tag{3.28}$$

where $\hat{h} = O(|Cv|^{d+1})$ and $C^{-1}\hat{g} = O(|Cv|^{d+1})$ are determined later. As before, equivalently $\vartheta = \zeta + \hat{\phi}$ and $y = v + C^{-1}\hat{\psi}$ satisfy the following equations

$$\begin{aligned}
&\sum_{1\le|\alpha|\le d} a_\alpha(Cv+\hat{\psi})^\alpha + \hat{\Theta}(\zeta+\hat{\phi}, Cv+\hat{\psi}) \\
&= \sum_{1\le|\alpha|\le d} a_\alpha(Cv)^\alpha + D_\zeta\hat{\phi}\omega + D_v\hat{\phi}\eta\tilde{B}v + \hat{h} + D_\zeta\hat{\phi}\Big(\sum_{1\le|\alpha|\le d} a_\alpha(Cv)^\alpha + \hat{h}\Big) \\
&\quad + D_v\hat{\phi}C^{-1}\Big(\sum_{1\le|\alpha|\le d} b_\alpha(Cv)^\alpha + \hat{g}\Big)
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
&\eta\tilde{B}\hat{\psi} + \sum_{1\le|\alpha|\le d} C^{-1}b_\alpha(Cv+\hat{\psi})^\alpha + C^{-1}\hat{f}(\zeta+\hat{\phi}, Cv+\hat{\psi}) \\
&= \sum_{1\le|\alpha|\le d} C^{-1}b_\alpha(Cv)^\alpha + D_\zeta(C^{-1}\hat{\psi})\omega + D_\zeta(C^{-1}\hat{\psi})\Big(\sum_{1\le|\alpha|\le d} a_\alpha(Cv)^\alpha + \hat{h}\Big) \\
&\quad + D_v(C^{-1}\hat{\psi})\eta\tilde{B}v + C^{-1}\hat{g} + D_v(C^{-1}\hat{\psi})C^{-1}\Big(\sum_{1\le|\alpha|\le d} b_\alpha(Cv)^\alpha + \hat{g}\Big).
\end{aligned} \tag{3.30}$$

Since the functions we consider are analytic, we can classify system of equations (3.29) and (3.30) into two systems according to the order of v. One contains only $(d+1)$th order terms with respect to v in equations (3.29) and (3.30), the other contains higher order terms.

By an elementary calculation, we have

$$\begin{aligned}
\sum_{1\le|\alpha|\le d} a_\alpha(Cv+\hat{\psi})^\alpha &= \sum_{1\le|\alpha|\le d} a_\alpha(Cv)^\alpha + \hat{R}_1(\hat{\psi}), \\
\sum_{1\le|\alpha|\le d} C^{-1}b_\alpha(Cv+\hat{\psi})^\alpha &= \sum_{1\le|\alpha|\le d} C^{-1}b_\alpha(Cv)^\alpha + \hat{R}_2(\hat{\psi}),
\end{aligned}$$

where $\hat{R}_1(\hat{\psi}) = O(|Cv|^{d+2})$ and $\hat{R}_2(\hat{\psi}) = O(|Cv|^{d+2})$. Using a Taylor expansion for $\hat{\Theta}(\zeta+\hat{\phi}, Cv+\hat{\psi})$ and $C^{-1}\hat{f}(\zeta+\hat{\phi}, Cv+\hat{\psi})$, we have

$$\begin{aligned}
\hat{\Theta}(\zeta+\hat{\phi}, Cv+\hat{\psi}) &= \hat{\Theta}(\zeta, Cv) + \hat{R}_3(\hat{\phi},\hat{\psi}), \\
\hat{f}(\zeta+\hat{\phi}, Cv+\hat{\psi}) &= \hat{f}(\zeta, Cv) + \hat{R}_4(\hat{\phi},\hat{\psi}),
\end{aligned}$$

where $\hat{R}_3(\hat{\phi},\hat{\psi}) = O(|Cv|^{d+2})$ and $\hat{R}_4(\hat{\phi},\hat{\psi}) = O(|Cv|^{d+2})$.

We write

$$\begin{aligned}
&\hat{\Theta}(\zeta, Cv) = \hat{\Theta}^{d+1} + \hat{\Theta}^{+},\ C^{-1}\hat{f}(\zeta, Cv) = C^{-1}\hat{f}^{d+1} + C^{-1}\hat{f}^{+}, \\
&\hat{h}(\zeta, Cv) = \hat{h}^{d+1} + \hat{h}^{+}, \quad C^{-1}\hat{g}(\zeta, Cv) = C^{-1}\hat{g}^{d+1} + C^{-1}\hat{g}^{+},
\end{aligned}$$

where $\hat{\Theta}^{d+1}, \hat{h}^{d+1}, C^{-1}\hat{f}^{d+1}$ and $C^{-1}\hat{g}^{d+1}$ are $(d+1)$th order homogeneous polynomials in Cv and $\hat{\Theta}^{+} = O(|Cv|^{d+2})$, $C^{-1}\hat{f}^{+} = O(|Cv|^{d+2})$, $\hat{h}^{+} = O(|Cv|^{d+2})$ and $C^{-1}\hat{g}^{+} = O(|Cv|^{d+2})$. Hence we can write system of equations (3.29) and (3.30) as the following two systems

$$\hat{\Theta}^{d+1} - \hat{h}^{d+1} = D_\zeta\hat{\phi}\omega + D_v\hat{\phi}\eta\tilde{B}v, \tag{3.31}$$

$$C^{-1}\hat{f}^{d+1} - C^{-1}\hat{g}^{d+1} = D_\zeta(C^{-1}\hat{\psi})\omega + D_v(C^{-1}\hat{\psi})(B+\eta\tilde{B})v - \eta\tilde{B}C^{-1}\hat{\psi} \tag{3.32}$$

and

$$\begin{aligned}
&\hat{h}^+ + D_\zeta\hat{\phi}\hat{h}^+ + D_v\hat{\phi}C^{-1}\hat{g}^+ \\
&= \hat{\Theta}^+ - D_\zeta\hat{\phi}(\sum_{1\le|\alpha|\le d} a_\alpha(Cv)^\alpha + \hat{h}^{d+1}) - D_v\hat{\phi}C^{-1}(\sum_{1\le|\alpha|\le d} b_\alpha(Cv)^\alpha + \hat{g}^{d+1}) \\
&\quad + \hat{R}_1 + \hat{R}_3, \qquad (3.33)\\
&C^{-1}\hat{g}^+ + D_\zeta(C^{-1}\hat{\psi})\hat{h}^+ + D_v(C^{-1}\hat{\psi})C^{-1}\hat{g}^+ \\
&= C^{-1}\hat{f}^+ - D_\zeta(C^{-1}\hat{\psi})(\sum_{1\le|\alpha|\le d} a_\alpha(Cv)^\alpha + \hat{h}^{d+1}) \\
&\quad - D_v(C^{-1}\hat{\psi})C^{-1}(\sum_{1\le|\alpha|\le d} b_\alpha(Cv)^\alpha + \hat{g}^{d+1}) + \hat{R}_2 + \hat{R}_4. \qquad (3.34)
\end{aligned}$$

By (2.1) we can write the above systems as the following systems

$$\begin{aligned}
&\hat{\Theta}^{0,d+1} - \hat{h}^{0,d+1} = D_v\hat{\phi}^0\eta\tilde{B}v, \qquad (3.35)\\
&C^{-1}\hat{f}^{0,d+1} - C^{-1}\hat{g}^{0,d+1} = D_v(C^{-1}\hat{\psi}^0)\eta\tilde{B}v - \eta\tilde{B}C^{-1}\hat{\psi}^0, \qquad (3.36)\\
&\hat{h}^{0,+} + D_v\hat{\phi}^0C^{-1}\hat{g}^{0,+} = \hat{\Theta}^{0,+} + R_1 + R_3^0 \\
&\qquad\qquad - D_v\hat{\phi}^0C^{-1}(\sum_{1\le|\alpha|\le d} b_\alpha(Cv)^\alpha + \hat{g}^{0,d+1}), \qquad (3.37)\\
&C^{-1}\hat{g}^{0,+} + D_v(C^{-1}\hat{\psi}^0)C^{-1}\hat{g}^{0,+} = C^{-1}\hat{f}^{0,+} + R_2 + R_4^0 \\
&\qquad\qquad - D_v(C^{-1}\hat{\psi}^0)C^{-1}(\sum_{1\le|\alpha|\le d} b_\alpha(Cv)^\alpha + \hat{g}^{0,d+1}) \qquad (3.38)
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\Theta}^{K,d+1} - \hat{h}^{K,d+1} = D_\zeta\phi^K\omega + D_v\phi^K\eta\tilde{B}v, \qquad (3.39)\\
&C^{-1}\hat{f}^{K,d+1} - C^{-1}\hat{g}^{K,d+1} = D_\zeta(C^{-1}\hat{\psi}^K)\omega + D_v(C^{-1}\hat{\psi}^K)\eta\tilde{B}v - \eta\tilde{B}C^{-1}\hat{\psi}^K, \qquad (3.40)\\
&\hat{h}^{K,+} + D_\zeta\hat{\phi}^K\hat{h}^{K,+} + D_v\hat{\phi}C^{-1}\hat{g}^{K,+} \\
&= \hat{\Theta}^{K,+} - D_\zeta\hat{\phi}^K\hat{h}^{0,+} - D_\zeta\hat{\phi}(\sum_{1\le|\alpha|\le d} a_\alpha(Cv)^\alpha + \hat{h}^{d+1}) - D_v\hat{\phi}^KC^{-1}\hat{g}^{0,+} + R_3^K \\
&\quad - D_v\hat{\phi}^K(\sum_{1\le|\alpha|\le d} C^{-1}b_\alpha(Cv)^\alpha + C^{-1}\hat{g}^{d+1}) - D_v\hat{\phi}^0C^{-1}\hat{g}^{K,d+1}, \qquad (3.41)\\
&C^{-1}\hat{g}^{K,+} + D_\zeta(C^{-1}\hat{\psi}^K)\hat{h}^{K,+} + D_v(C^{-1}\hat{\psi})C^{-1}\hat{g}^{K,+} \\
&= C^{-1}\hat{f}^{K,+} - D_\zeta(C^{-1}\hat{\psi}^K)\hat{h}^{0,+} - D_\zeta(C^{-1}\hat{\psi}^K)(\sum_{1\le|\alpha|\le d} a_\alpha(Cv)^\alpha + \hat{h}^{d+1}) \\
&\quad - D_v(C^{-1}\hat{\psi}^K)C^{-1}\hat{g}^{0,+} + R_4^K - D_v(C^{-1}\hat{\psi}^K)(\sum_{1\le|\alpha|\le d} C^{-1}b_\alpha(Cv)^\alpha + C^{-1}\hat{g}^{d+1}) \\
&\quad - D_v(C^{-1}\hat{\psi}^0)C^{-1}\hat{g}^{K,d+1}. \qquad (3.42)
\end{aligned}$$

Let us solve (3.35) and (3.36) by finding the $\hat{\phi}^0$ and $C^{-1}\hat{\psi}^0$ which makes $\hat{h}^{0,d+1}$ and $\hat{g}^{0,d+1}$ as simple as possible respectively. Using Fourier expansions for the function $\hat{\Theta}^{0,d+1}$, $C^{-1}\hat{f}^{0,d+1}$, $\hat{h}^{0,d+1}$, $C^{-1}\hat{g}^{0,d+1}$, $\hat{\phi}^0$ and $C^{-1}\hat{\psi}^0$, we have

$$\begin{aligned}
&\hat{\Theta}^{0,d+1,j}(Cv) = \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{\Theta}^{0,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha,\\
&\eta^{-(j-1)}\hat{f}^{0,d+1,j}(Cv) = \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-j-d} \hat{f}^{0,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha,\\
&\hat{h}^{0,d+1,j}(Cv) = \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{h}^{0,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha,\\
&\eta^{-(j-1)}\hat{g}^{0,d+1,j}(Cv) = \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-j-d} \hat{g}^{0,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha, \qquad (3.43)\\
&\hat{\phi}^{0,j}(Cv) = \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{\phi}^{0,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha,\\
&\eta^{-(j-1)}\hat{\psi}^{0,j}(Cv) = \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-j-d} \hat{\psi}^{0,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha.
\end{aligned}$$

Then by (3.35), (3.36), (3.43) and the above equations, we have

$$\begin{aligned}
&\sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-d-1} \hat{\Theta}^{0,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha - \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-d-1} \hat{h}^{0,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha\\
&= \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-d} \hat{\phi}^{0,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} \alpha_1 v_1^{\alpha_1-1} v_2^{\alpha_2+1}, \qquad (3.44)\\
&\sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{f}^{0,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha - \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{g}^{0,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha\\
&= \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-d-j+1} \hat{\psi}^{0,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} \alpha_1 v_1^{\alpha_1-1} v_2^{\alpha_2+1}\\
&\quad - \sum_{|\alpha|=d+1} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{\psi}^{0,j+1}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} v^\alpha, \qquad (3.45)
\end{aligned}$$

where $\psi^{0,3}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}}} = 0$. The equations (3.44) and (3.45) can be written as

$$\begin{aligned}
&(\hat{\Theta}^{0,d+1,j}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d+1}} v_1^{d+1} - \hat{h}^{0,d+1,j}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d+1}} v_1^{d+1})\\
&\quad + \eta(\hat{\Theta}^{0,d+1,j}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d},\alpha^2} v_1^d v_2 - \hat{h}^{0,d+1,j}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d},\alpha^2} v_1^d v_2 - (d+1)\hat{\phi}^{0,j}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d+1}} v_1^d v_2)\\
&\quad + \cdots + \eta^{d+1}(\hat{\Theta}^{0,d+1,j}_{\underbrace{\alpha^2,\dots,\alpha^2}_{d+1}} v_2^{d+1} - \hat{h}^{0,d+1,j}_{\underbrace{\alpha^2,\dots,\alpha^2}_{d+1}} v_2^{d+1} - \hat{\phi}^{0,j}_{\alpha^1,\underbrace{\alpha^2,\dots,\alpha^2}_{d}} v_2^{d+1}) = 0,\\
&\eta^{1-j}(\hat{f}^{0,d+1,j}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d+1}} v_1^{d+1} - \hat{g}^{0,d+1,j}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d+1}} v_1^{d+1} + \hat{\psi}^{0,j+1}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d+1}} v_1^{d+1}) + \eta^{2-j}(\hat{f}^{0,d+1,j}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d},\alpha^2} v_1^d v_2\\
&\quad - \hat{g}^{0,d+1,j}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d},\alpha^2} v_1^d v_2 + \hat{\psi}^{0,j+1}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d},\alpha^2} v_1^d v_2 - (d+1)\hat{\psi}^{0,j}_{\underbrace{\alpha^1,\dots,\alpha^1}_{d+1}} v_1^d v_2) + \cdots\\
&\quad + \eta^{d-j}(\hat{f}^{0,d+1,j}_{\underbrace{\alpha^2,\dots,\alpha^2}_{d+1}} v_2^{d+1} - \hat{g}^{0,d+1,j}_{\underbrace{\alpha^2,\dots,\alpha^2}_{d+1}} v_2^{d+1} + \hat{\psi}^{0,j+1}_{\underbrace{\alpha^2,\dots,\alpha^2}_{d+1}} v_2^{d+1} - \hat{\psi}^{0,j}_{\alpha^1,\underbrace{\alpha^2,\dots,\alpha^2}_{d}} v_2^{d+1}) = 0,
\end{aligned}$$

i.e.,

$$\sum_{|\alpha|=2} \eta^{p_1+\cdots+p_{d+1}-(d+1)}(\hat{\Theta}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} v^\alpha - \hat{h}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} v^\alpha - \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\phi}^{0,j}_{\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}}} v^\alpha) = 0, \tag{3.46}$$

$$\sum_{|\alpha|=2} \eta^{p_1+\cdots+p_{d+1}-d-j}(\hat{f}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} v^\alpha - \hat{g}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} v^\alpha - \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\psi}^{0,j}_{\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}}} v^\alpha + \hat{\psi}^{0,j+1}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} v^\alpha) = 0, \tag{3.47}$$

where $\phi^{0,j}_{\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}}} = \psi^{0,j}_{\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}}} = 0$ if there is at least one $s'(1 \leq s' \leq d+1)$ such that $p_{s'} - p'_{s'} \leq 0$.

By equating all terms in (3.46) and (3.47) of the same order with respect to $\eta^{p_1+\cdots+p_{d+1}-(d+1)}$ and $\eta^{p_1+\cdots+p_{d+1}-d-j}$ respectively, we get

$$\hat{\Theta}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} v^\alpha - \hat{h}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} v^\alpha - \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\phi}^{0,j}_{\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}}} v^\alpha = 0,$$

$$\hat{f}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} v^\alpha - \hat{g}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} v^\alpha - \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\psi}^{0,j}_{\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}}} v^\alpha + \hat{\psi}^{0,j+1}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} v^\alpha = 0.$$

Comparing the coefficient of v^α, we have

$$0 = (\hat{\Theta} - \hat{h})^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} - \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\phi}^{0,j}_{\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}}}, \tag{3.48}$$

$$0 = (\hat{f} - \hat{g})^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} + \hat{\psi}^{0,j+1}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} - \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\psi}^{0,j}_{\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}}}, \tag{3.49}$$

where $(\hat{\Theta} - \hat{h})^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = \hat{\Theta}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} - \hat{h}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}}$, $(\hat{f} - \hat{g})^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = \hat{f}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} - \hat{g}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}}$ and $\hat{\phi}^{0,j}_{\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}}} = \hat{\psi}^{0,j}_{\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}}} = 0$ if there is at least one $s'(1 \leq s' \leq d+1)$ such that $p_{s'} - p'_{s'} \leq 0$.

By (3.48), (3.49), we have

$$(d+1)\hat{\phi}^{0,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1}} = (\hat{\Theta} - \hat{h})^{0,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2},$$

$$(d+1)\hat{\psi}^{0,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1}} = (\hat{f} - \hat{g})^{0,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2} + \hat{\psi}^{0,j+1}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2},$$

$$\cdots \tag{3.50}$$

$$\hat{\phi}^{0,j}_{\alpha^1,\underbrace{\alpha^2,\ldots,\alpha^2}_{d}} = (\hat{\Theta} - \hat{h})^{0,d+1,j}_{\underbrace{\alpha^2,\ldots,\alpha^2}_{d+1}},$$

$$\hat{\psi}^{0,j}_{\alpha^1,\underbrace{\alpha^2,\ldots,\alpha^2}_{d}} = (\hat{f} - \hat{g})^{0,d+1,j}_{\underbrace{\alpha^2,\ldots,\alpha^2}_{d+1}} + \hat{\psi}^{0,j+1}_{\underbrace{\alpha^2,\ldots,\alpha^2}_{d+1}},$$

where $(\hat{\Theta} - \hat{h})^{0,j}_{\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}}} = (\hat{f} - \hat{g})^{0,j}_{\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}}} = 0$ if there is at least one $s'(1 \le s' \le d+1)$ such that $p_{s'} - p'_{s'} \le 0$.

Our purpose is to find $\hat{\phi}^0$, $C^{-1}\hat{\psi}^0$ such that $\hat{h}^{0,d+1}$ and $C^{-1}\hat{g}^{0,d+1}$ have the simplest form. Let $\hat{h}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = 0$ and $\hat{g}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = 0$. By considering (3.50), we get

$$\hat{\phi}^{0,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = \begin{cases} 0, & \alpha_2 = d+1, \\ \frac{1}{\alpha_1}\hat{\Theta}^{0,d+1,j}_{\alpha^{p_1+1},\alpha^{p_2},\ldots,\alpha^{p_{d+1}}}, & \alpha_2 \neq d+1, \end{cases}$$

$$\hat{\psi}^{0,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = \begin{cases} 0, & \alpha_2 = d+1, \\ \hat{\psi}^{0,j}_{0,\alpha^{p_1},\ldots,\alpha^{p_{d+1}}}, & \alpha_2 \neq d+1, \end{cases}$$

where

$$\hat{\psi}^{0,j}_{0,\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = \frac{1}{\alpha_1}\hat{f}^{0,d+1,j}_{\alpha^{p_1+1},\alpha^{p_2},\ldots,\alpha^{p_{d+1}}} + \frac{1}{\alpha_1(\alpha_1-1)}\hat{f}^{0,d+1,j+1}_{\alpha^{p_1+1},\alpha^{p_2+1},\alpha^{p_3},\ldots,\alpha^{p_{d+1}}},$$

$\tilde{f}^{0,2,j}_{\alpha^{p_1+p'_1},\ldots,\alpha^{p_{d+1}+p'_{d+1}}} = 0$ if $j \ge 3$ and $\hat{\Theta}^{0,d+1,j}_{\alpha^{p_1+p'_1},\ldots,\alpha^{p_{d+1}+p'_{d+1}}} = \hat{f}^{0,d+1,j}_{\alpha^{p_1+p'_1},\ldots,\alpha^{p_{d+1}+p'_{d+1}}} = 0$ if there is at least one $s'(1 \le s' \le d+1)$ such that $p_{s'} + p'_{s'} \ge 3$. Then

$$\hat{h}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = \begin{cases} \hat{\Theta}^{0,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1}}, & \alpha_1 = d+1, \\ 0, & \alpha_1 \neq d+1, \end{cases}$$

$$\hat{g}^{0,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = \begin{cases} \hat{f}^{0,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1}} + \frac{1}{d+1}\hat{f}^{0,d+1,j+1}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2}, & \alpha_1 = d+1, \\ 0, & \alpha_1 \neq d+1, \end{cases}$$

where $\hat{f}^{0,d+1,3}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2} = 0$.

It is clearly that $\hat{\phi}^0$ and $C^{-1}\hat{\psi}^0$ are analytic and 2π-periodic in ζ in $D_{r_d-\delta}$ for $\delta \in (0, r_{d+1})$, $\hat{\phi}^0 = O(|Cv|^{d+1})$ and $C^{-1}\hat{\psi}^0 = O(|Cv|^{d+1})$. Moreover, $\hat{h}^{0,d+1}$, $C^{-1}\hat{g}^{0,d+1}$, $\hat{\phi}^0$ and $C^{-1}\hat{\psi}^0$ are solutions of system of equations (3.35) and (3.36).

We choose r_{d+1} sufficiently small such that $0 < r_{d+1} < r_d - \delta$,

$$\begin{bmatrix} Id & D_v\hat{\phi}^0 \\ 0 & Id + D_v(C^{-1}\hat{\psi}^0) \end{bmatrix}$$

has an inverse and $(\zeta, v) + (\hat{\phi}^0, C^{-1}\hat{\psi}^0) \in D_{r_d}$ for $(\zeta, v) \in D_{r_{d+1}}$. Hence we can solve system of equations (3.37) and (3.38) for $\hat{h}^{0,+}$, $C^{-1}\hat{g}^{0,+}$.

In what follows we solve (3.39) and (3.40) by finding $\hat{\phi}^K$ and $C^{-1}\hat{\psi}^K$ which makes $\hat{h}^{K,d+1}$ and $C^{-1}\hat{g}^{K,d+1}$ as simple as possible respectively. Using Fourier expansions for the function $\hat{\Theta}^{K,d+1}$, $C^{-1}\hat{f}^{K,d+1}$, $\hat{h}^{K,d+1}$, $C^{-1}\hat{g}^{K,d+1}$, $\hat{\phi}^K$ and $C^{-1}\hat{\psi}^K$, we have

$$\begin{aligned}
&\hat{\Theta}^{K,d+1,j}(\zeta, Cv)\\
&= \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{\Theta}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{i\langle\zeta,k\rangle} v^\alpha,\\
&\eta^{-(j-1)}\hat{f}^{K,d+1,j}(\zeta, Cv)\\
&= \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{f}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{i\langle\zeta,k\rangle} v^\alpha,\\
&\hat{h}^{K,d+1,j}(\zeta, Cv)\\
&= \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{h}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{i\langle\zeta,k\rangle} v^\alpha,\\
&\eta^{-(j-1)}\hat{g}^{K,d+1,j}(\zeta, Cv)\\
&= \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{g}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{i\langle\zeta,k\rangle} v^\alpha,\\
&\hat{\phi}^{K,j}(\zeta, Cv)\\
&= \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{\phi}^{K,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{i\langle\zeta,k\rangle} v^\alpha,\\
&\eta^{-(j-1)}\hat{\psi}^{K,j}(\zeta, Cv)\\
&= \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{\psi}^{K,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{i\langle\zeta,k\rangle} v^\alpha.
\end{aligned} \tag{3.51}$$

Then by (3.39), (3.40) and (3.51), we have

$$\begin{aligned}
&\sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{\Theta}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^\alpha\\
&\quad - \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{h}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^\alpha\\
&= \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{\phi}^{K,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} \mathbf{i}\langle k,\omega\rangle v^\alpha\\
&\quad + \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-d} \hat{\phi}^{K,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} \alpha_1 v_1^{\alpha_1-1} v_2^{\alpha_2+1},\\
&\sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{f}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^\alpha\\
&\quad - \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{g}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^\alpha
\end{aligned} \tag{3.52}$$

$$
\begin{aligned}
&= \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{\psi}^{K,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} \mathbf{i}\langle k,\omega\rangle v^{\alpha} \\
&\quad + \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-d-j+1} \hat{\psi}^{K,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} \alpha_1 v_1^{\alpha_1-1} v_2^{\alpha_2+1} \\
&\quad - \sum_{|\alpha|=d+1} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{\psi}^{K,j+1}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^{\alpha},
\end{aligned} \tag{3.53}
$$

where $\hat{\psi}^{K,3}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}=0$. The equations (3.52) and (3.53) can be written as

$$
\begin{aligned}
&\sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} e^{\mathbf{i}\langle k,\zeta\rangle} (\hat{\Theta}^{K,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1},k} v_1^{d+1} - \hat{h}^{K,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1},k} v_1^{d+1} - \hat{\phi}^{K,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1},k} \mathbf{i}\langle k,\omega\rangle v_1^{d+1}) \\
&+ \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} e^{\mathbf{i}\langle k,\zeta\rangle} \eta (\hat{\Theta}^{K,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2,k} v_1^{d} v_2 - \hat{h}^{K,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2,k} v_1^{d} v_2 - (d+1)\hat{\phi}^{K,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1},k} v_1^{d} v_2 \\
&- \hat{\phi}^{K,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2,k} \mathbf{i}\langle k,\omega\rangle v_1^{d} v_2) + \cdots + \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} e^{\mathbf{i}\langle k,\zeta\rangle} \eta^{d+1} (\hat{\Theta}^{K,d+1,j}_{\underbrace{\alpha^2,\ldots,\alpha^2}_{d+1},k} v_2^{d+1} \\
&- \hat{h}^{K,d+1,j}_{\underbrace{\alpha^2,\ldots,\alpha^2}_{d+1},k} v_2^{d+1} - \hat{\phi}^{K,j}_{\underbrace{\alpha^2,\ldots,\alpha^2}_{d+1},k} \mathbf{i}\langle k,\omega\rangle v_2^{d+1} - \hat{\phi}^{K,j}_{\alpha^1,\underbrace{\alpha^2,\ldots,\alpha^2}_{d},k} v_2^{d+1}) = 0, \\
&\sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} e^{\mathbf{i}\langle k,\zeta\rangle} \eta^{1-j} (\hat{f}^{K,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1},k} v_1^{d+1} - \hat{g}^{K,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1},k} v_1^{d+1} - \hat{\psi}^{K,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1},k} \mathbf{i}\langle k,\omega\rangle v_1^{d+1} \\
&+ \hat{\psi}^{K,j+1}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1},k} v_1^{d+1}) + \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} e^{\mathbf{i}\langle k,\zeta\rangle} \eta^{2-j} (\hat{f}^{K,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2,k} v_1^{d} v_2 - \hat{g}^{K,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2,k} v_1^{d} v_2 \\
&- \hat{\psi}^{K,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2,k} \mathbf{i}\langle k,\omega\rangle v_1^{d} v_2 + \hat{\psi}^{K,j+1}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2,k} v_1^{d} v_2 - (d+1)\hat{\psi}^{K,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1},k} v_1^{d} v_2) + \ldots \\
&+ \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} e^{\mathbf{i}\langle k,\zeta\rangle} \eta^{d-j} (\hat{f}^{K,d+1,j}_{\underbrace{\alpha^2,\ldots,\alpha^2}_{d+1},k} v_2^{d+1} - \hat{g}^{K,d+1,j}_{\underbrace{\alpha^2,\ldots,\alpha^2}_{d+1},k} v_2^{d+1} - \hat{\psi}^{K,j}_{\underbrace{\alpha^2,\ldots,\alpha^2}_{d+1},k} \mathbf{i}\langle k,\omega\rangle v_2^{d+1} \\
&+ \hat{\psi}^{K,j+1}_{\underbrace{\alpha^2,\ldots,\alpha^2}_{d+1},k} v_2^{d+1} - \hat{\psi}^{K,j}_{\alpha^1,\underbrace{\alpha^2,\ldots,\alpha^2}_{d},k} v_2^{d+1}) = 0,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\sum_{|\alpha|=2} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} e^{\mathbf{i}\langle k,\zeta\rangle} \eta^{p_1+\cdots+p_{d+1}-(d+1)} (\hat{\Theta}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} v^{\alpha} - \hat{h}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} v^{\alpha} \\
&- \hat{\phi}^{K,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} \mathbf{i}\langle k,\omega\rangle v^{\alpha} \\
&- \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\phi}^{K,j}_{\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}},k} v^{\alpha}) = 0, \qquad (3.54) \\
&\sum_{|\alpha|=2} \sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} e^{\mathbf{i}\langle k,\zeta\rangle} \eta^{p_1+\cdots+p_{d+1}-d-j} (\hat{f}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} v^{\alpha} - \hat{g}^{K,d+1,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} v^{\alpha} \\
&+ \hat{\psi}^{K,j+1}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} v^{\alpha} - \hat{\psi}^{K,j}_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} \mathbf{i}\langle k,\omega\rangle v^{\alpha}
\end{aligned}
$$

$$-\sum_{s'=1}^{d+1}\frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}}\hat{\psi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'}-1},\alpha^{p_{s'}-1},\alpha^{p_{s'}+1},\dots,\alpha^{p_{d+1}},k}v^{\alpha})=0. \tag{3.55}$$

By equating all terms in (3.54) and (3.55) of the same order with respect to $\eta^{p_1+\cdots+p_{d+1}-(d+1)}$ and $\eta^{p_1+\cdots+p_{d+1}-d-j}$ respectively, we get

$$\sum_{\substack{k\in\mathbb{Z}^2\\k\neq 0}}e^{\mathbf{i}\langle k,\zeta\rangle}(\hat{\Theta}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}v^{\alpha}-\hat{h}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}v^{\alpha}-\hat{\phi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}\mathbf{i}\langle k,\omega\rangle v^{\alpha}$$
$$-\sum_{s'=1}^{d+1}\frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}}\hat{\phi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'}-1},\alpha^{p_{s'}-1},\alpha^{p_{s'}+1},\dots,\alpha^{p_{d+1}},k}v^{\alpha})=0,$$
$$\sum_{\substack{k\in\mathbb{Z}^2\\k\neq 0}}e^{\mathbf{i}\langle k,\zeta\rangle}(\hat{f}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}v^{\alpha}-\hat{g}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}v^{\alpha}-\hat{\psi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}\mathbf{i}\langle k,\omega\rangle v^{\alpha}$$
$$-\sum_{s'=1}^{d+1}\frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}}\hat{\psi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'}-1},\alpha^{p_{s'}-1},\alpha^{p_{s'}+1},\dots,\alpha^{p_{d+1}},k}v^{\alpha}+\hat{\psi}^{K,j+1}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}v^{\alpha})=0.$$

Comparing the coefficient of v^{α}, we have

$$\mathbf{i}\langle k,\omega\rangle\hat{\phi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}=(\hat{\Theta}-\hat{h})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}-\sum_{s'=1}^{d+1}\frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}}\hat{\phi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'}-1},\alpha^{p_{s'}-1},\alpha^{p_{s'}+1},\dots,\alpha^{p_{d+1}},k}, \tag{3.56}$$

$$\mathbf{i}\langle k,\omega\rangle\hat{\psi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}=(\hat{f}-\hat{g})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}+\hat{\psi}^{K,j+1}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}-\sum_{s'=1}^{d+1}\frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}}\hat{\psi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'}-1},\alpha^{p_{s'}-1},\alpha^{p_{s'}+1},\dots,\alpha^{p_{d+1}},k}, \tag{3.57}$$

where $(\hat{\Theta}-\hat{h})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}=\hat{\Theta}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}-\hat{h}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}$, $(\hat{f}-\hat{g})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}=\hat{f}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}-\hat{g}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}$ and $\hat{\phi}^{K,j}_{\alpha^{p_1-p_1'},\dots,\alpha^{p_{d+1}-p_{d+1}'},k}=\hat{\psi}^{K,j}_{\alpha^{p_1-p_1'},\dots,\alpha^{p_{d+1}-p_{d+1}'},k}=0$ if there is at least one $s'(1\le s'\le d+1)$ such that $p_{s'}-p_{s'}'\le 0$.

By (3.56), (3.57), we have

$$\begin{aligned}
&\hat{\phi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}\\
&=\frac{1}{\mathbf{i}\langle k,\omega\rangle}(\hat{\Theta}-\hat{h})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}\\
&\quad-\frac{1}{\mathbf{i}\langle k,\omega\rangle}\sum_{s_1'=1}^{d+1}\frac{\alpha_{p_{s_1'}-1}+1}{\alpha_{p_{s_1'}}}\hat{\phi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{s_1'}-1},\alpha^{p_{s_1'}-1},\alpha^{p_{s_1'}+1},\dots,\alpha^{p_{d+1}},k}\\
&=\frac{1}{\mathbf{i}\langle k,\omega\rangle}(\hat{\Theta}-\hat{h})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}-\frac{1}{(\mathbf{i}\langle k,\omega\rangle)^2}\sum_{s_1'=1}^{d+1}\frac{\alpha_{p_{s_1'}-1}+1}{\alpha_{p_{s_1'}}}\\
&\quad(\hat{\Theta}-\hat{h})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{s_1'}-1},\alpha^{p_{s_1'}-1},\alpha^{p_{s_1'}+1},\dots,\alpha^{p_{d+1}},k}+\frac{1}{(\mathbf{i}\langle k,\omega\rangle)^2}\sum_{s_1'=1}^{d+1}\sum_{s_2'=1}^{d+1}\frac{\alpha_{p_{s_1'}-1}+1}{\alpha_{p_{s_1'}}}
\end{aligned}$$

$$\frac{\alpha_{p_{s'_2}-1}+1}{\alpha_{p_{s'_2}}}\hat{\phi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{s'_2}-1},\alpha^{p_{s'_2}-1},\alpha^{p_{s'_2}+1},\dots,\alpha^{p_{d+1}},k}$$

$$= \dots$$

$$= \frac{1}{\mathbf{i}\langle k,\omega\rangle}(\hat{\Theta}-\hat{h})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k} - \frac{1}{(\mathbf{i}\langle k,\omega\rangle)^2}\sum_{s'_1=1}^{d+1}\frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}}$$

$$(\hat{\Theta}-\hat{h})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{d+1}},k} + \dots + (-1)^{p_1+\dots+p_{d+1}-(d+1)}$$

$$\frac{1}{(\mathbf{i}\langle k,\omega\rangle)^{p_1+\dots+p_{d+1}-d}}\sum_{s'_1=1}^{d+1}\dots\sum_{s'_{d+1}=1}^{d+1}\frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}}\dots\frac{\alpha_{p_{s'_{d+1}}-1}+1}{\alpha_{p_{s'_{d+1}}}}$$

$$(\hat{\Theta}-\hat{h})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{s'_{d+1}}-1},\alpha^{p_{s'_{d+1}}-1},\alpha^{p_{s'_{d+1}}+1},\dots,\alpha^{p_{d+1}},k}, \tag{3.58}$$

$$\hat{\psi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}$$

$$= \frac{1}{\mathbf{i}\langle k,\omega\rangle}((\hat{f}-\hat{g})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k} + \hat{\psi}^{K,j+1}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k})$$

$$-\frac{1}{\mathbf{i}\langle k,\omega\rangle}\sum_{s'_1=1}^{d+1}\frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}}\hat{\psi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{d+1}},k}$$

$$= \frac{1}{\mathbf{i}\langle k,\omega\rangle}((\hat{f}-\hat{g})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k} + \hat{\psi}^{K,j+1}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}) - \frac{1}{(\mathbf{i}\langle k,\omega\rangle)^2}\sum_{s'_1=1}^{d+1}\frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}}$$

$$((\hat{f}-\hat{g})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{d+1}},k} + \hat{\psi}^{K,j+1}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{d+1}},k})$$

$$+\frac{1}{(\mathbf{i}\langle k,\omega\rangle)^2}\sum_{s'_1=1}^{d+1}\sum_{s'_2=1}^{d+1}\frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}}\frac{\alpha_{p_{s'_2}-1}+1}{\alpha_{p_{s'_2}}}$$

$$\hat{\psi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{s'_2}-1},\alpha^{p_{s'_2}-1},\alpha^{p_{s'_2}+1},\dots,\alpha^{p_{d+1}},k}$$

$$= \dots$$

$$= \frac{1}{\mathbf{i}\langle k,\omega\rangle}((\hat{f}-\hat{g})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k} + \hat{\psi}^{K,j+1}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}) - \frac{1}{(\mathbf{i}\langle k,\omega\rangle)^2}\sum_{s'_1=1}^{d+1}\frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}}$$

$$((\hat{f}-\hat{g})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{d+1}},k} + \hat{\psi}^{K,j+1}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{d+1}},k})$$

$$+\dots+(-1)^{p_1+\dots+p_{d+1}-(d+1)}\frac{1}{(\mathbf{i}\langle k,\omega\rangle)^{p_1+\dots+p_{d+1}-d}}\sum_{s'_1=1}^{d+1}\dots\sum_{s'_{d+1}=1}^{d+1}\frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}}\dots$$

$$\frac{\alpha_{p_{s'_{d+1}}-1}+1}{\alpha_{p_{s'_{d+1}}}}((\hat{f}-\hat{g})^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{s'_{d+1}}-1},\alpha^{p_{s'_{d+1}}-1},\alpha^{p_{s'_{d+1}}+1},\dots,\alpha^{p_{d+1}},k}$$

$$+\hat{\psi}^{K,j+1}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{s'_{d+1}}-1},\alpha^{p_{s'_{d+1}}-1},\alpha^{p_{s'_{d+1}}+1},\dots,\alpha^{p_{d+1}},k}), \tag{3.59}$$

where $(\hat{\Theta}-\hat{h})^{K,j}_{\alpha^{p_1-p'_1},\dots,\alpha^{p_{d+1}-p'_{d+1}},k} = (\hat{f}-\hat{g})^{K,j}_{\alpha^{p_1-p'_1},\dots,\alpha^{p_{d+1}-p'_{d+1}},k} = 0$ if there is at least one $s'(1\le s'\le d+1)$ such that $p_{s'}-p'_{s'}\le 0$.

Our purpose is to find $\hat{\phi}^K$, $C^{-1}\hat{\psi}^K$ such that $\hat{h}^{K,d+1}$ and $C^{-1}\hat{g}^{K,d+1}$ have the simplest form. By (3.8), the following choice is the simplest for $\hat{h}^{K,d+1,j}$ and $\hat{g}^{K,d+1,j}$. Let $\hat{h}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k} = 0$ and $\tilde{g}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k} = 0$. By considering (3.58), (3.59), we get

$$\begin{aligned}
&\hat{\phi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}\\
&= \sum_{q=0}^{p_1+\cdots+p_{d+1}-(d+1)} \sum_{s'_1=1}^{d+1} \cdots \sum_{s'_q=1}^{d+1} (-1)^q (\mathbf{i}\langle k,\omega\rangle)^{-(q+1)} \frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}} \cdots \frac{\alpha_{p_{s'_q}-1}+1}{\alpha_{p_{s'_q}}}\\
&\hat{\Theta}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{s'_q}-1},\alpha^{p_{s'_q}-1},\alpha^{p_{s'_q}+1},\dots,\alpha^{p_{d+1}},k},\\
&\hat{\psi}^{K,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k}\\
&= \sum_{\tilde{j}=j}^{2}((\sum_{q_j=0}^{p_1+\cdots+p_{d+1}-(d+1)} \sum_{{}_j s'_1=1}^{d+1} \cdots \sum_{{}_j s'_{q_j}=1}^{d+1}) \dots (\sum_{q_{\tilde{j}}=0}^{p_1+\cdots+p_{d+1}-(d+1)-q_{\hat{j},\tilde{j}-1}+\tilde{j}-j} \sum_{{}_{\tilde{j}} s'_1=1}^{d+1}\\
&\cdots \sum_{{}_{\tilde{j}} s'_{q_{\tilde{j}}}=1}^{d+1})(-1)^{q_{\hat{j},\tilde{j}}} (\mathbf{i}\langle k,\omega\rangle)^{-q_{\hat{j},\tilde{j}}-(\tilde{j}-j+1)} (\frac{\alpha_{p_{{}_j s'_1}-1}+1}{\alpha_{p_{{}_j s'_1}}} \cdots \frac{\alpha_{p_{{}_j s'_{q_j}}-1}+1}{\alpha_{p_{{}_j s'_{q_j}}}})\\
&\dots (\frac{\alpha_{p_{{}_{\tilde{j}} s'_1}-1}+1}{\alpha_{p_{{}_{\tilde{j}} s'_1}}} \cdots \frac{\alpha_{p_{{}_{\tilde{j}} s'_{q_{\tilde{j}}}}-1}+1}{\alpha_{p_{{}_{\tilde{j}} s'_{q_{\tilde{j}}}}}})\\
&\hat{f}^{K,d+1,\tilde{j}}_{\alpha^{p_1},\dots,\alpha^{p_{-j s'_1}-1},\alpha^{p_{-j s'_1}-1},\alpha^{p_{-j s'_1}+1},\dots,\alpha^{p_{-\tilde{j} s'_{q_{\tilde{j}}}}-1},\alpha^{p_{-\tilde{j} s'_{q_{\tilde{j}}}}-1},\alpha^{p_{-\tilde{j} s'_{q_{\tilde{j}}}}+1},\dots,\alpha^{p_d},\alpha^{p_{d+1}},k}),
\end{aligned}$$

where

$$\begin{aligned}
&\hat{\Theta}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{s'_0}-1},\alpha^{p_{s'_0}-1},\alpha^{p_{s'_0}+1},\dots,\alpha^{p_{d+1}},k} = \hat{\Theta}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k},\\
&\hat{f}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\dots,\alpha^{p_{s'_0}-1},\alpha^{p_{s'_0}-1},\alpha^{p_{s'_0}+1},\dots,\alpha^{p_{d+1}},k} = \hat{f}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k},
\end{aligned}$$

$\sum_{s'_1=1}^{d+1} \cdots \sum_{s'_0=1}^{d+1} = \sum_{{}_j s'_1=1}^{d+1} \cdots \sum_{{}_j s'_0=1}^{d+1} = \sum_{{}_{\tilde{j}} s'_1=1}^{d+1} \cdots \sum_{{}_{\tilde{j}} s'_0=1}^{d+1} = 1$, $\frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}} \cdots \frac{\alpha_{p_{s'_0}-1}+1}{\alpha_{p_{s'_0}}} = \frac{\alpha_{p_{{}_j s'_1}-1}+1}{\alpha_{p_{{}_j s'_1}}} \cdots \frac{\alpha_{p_{{}_j s'_0}-1}+1}{\alpha_{p_{{}_j s'_0}}} = \frac{\alpha_{p_{{}_{\tilde{j}} s'_1}-1}+1}{\alpha_{p_{{}_{\tilde{j}} s'_1}}} \cdots \frac{\alpha_{p_{{}_{\tilde{j}} s'_0}-1}+1}{\alpha_{p_{{}_{\tilde{j}} s'_0}}} = 1$, $\tilde{f}^{K,2,j}_{\alpha^{p_1-p'_1},\dots,\alpha^{p_{d+1}-p'_{d+1}},k} = 0$ if $j \geq 3$ and $\hat{\Theta}^{K,d+1,j}_{\alpha^{p_1-p'_1},\dots,\alpha^{p_{d+1}-p'_{d+1}},k} = \hat{f}^{K,d+1,j}_{\alpha^{p_1-p'_1},\dots,\alpha^{p_{d+1}-p'_{d+1}},k} = 0$ if there is at least one $s'(1 \leq s' \leq d+1)$ such that $p_{s'} - p'_{s'} \leq 0, q_{\hat{j},\tilde{j}} = \sum_{\hat{j}=j}^{\tilde{j}} q_{\hat{j}}$. Then $\hat{h}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k} = 0$ and $\hat{g}^{K,d+1,j}_{\alpha^{p_1},\dots,\alpha^{p_{d+1}},k} = 0$.

Using (A1) we obtain that $\hat{\phi}^K$ and $C^{-1}\hat{\psi}^K$ are analytic and 2π-periodic in ζ in $D_{r_d-\delta}$, where $0 < \delta < r_{d+1}$. Moreover, $\hat{h}^{K,d+1}$, $C^{-1}\hat{g}^{K,d+1}$, $\hat{\phi}^K$ and $C^{-1}\hat{\psi}^K$ are solutions of system of equations (3.39) and (3.40). It is clear that $\hat{\phi}^K = O(|Cv|^{d+1})$ and $C^{-1}\hat{\psi}^K = O(|Cv|^{d+1})$.

We choose r_{d+1} sufficiently small such that $0 < r_{d+1} < r_d - \delta$,

$$\begin{bmatrix} Id + D_\zeta \hat{\phi}^K & D_v \hat{\phi} \\ D_\zeta (C^{-1}\hat{\psi}^K) & Id + D_v(C^{-1}\hat{\psi}) \end{bmatrix}$$

has an inverse and $(\zeta, v) + (\hat{\phi}^K, C^{-1}\hat{\psi}^K) \in D_{r_d}$ for $(\zeta, v) \in D_{r_{d+1}}$. Hence we can solve system of equations (3.41) and (3.42) for $\hat{h}^{K,+}$, $C^{-1}\hat{g}^{K,+}$.

We can get that $\hat{\phi}^j = \hat{\phi}^{0,j} + \hat{\phi}^{K,j} = \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \hat{\phi}^j_{\alpha,k} e^{\mathbf{i}\langle k,\zeta\rangle}(Cv)^\alpha$ and $\eta^{-(j-1)}\hat{\psi}^j = \eta^{-(j-1)}\hat{\psi}^{0,j} + \hat{\psi}^{-(j-1)}\psi^{K,j} = \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{-(j-1)}\hat{\psi}^j_{\alpha,k} e^{\mathbf{i}\langle k,\zeta\rangle}(Cv)^\alpha$ are analytic and 2π-periodic in ζ_1, ζ_2 in $D_{r-\delta}$, $0 < \delta < r$. It is clear that $\phi = O(|Cx|^{d+1})$ and $C^{-1}\psi = O(|Cx|^{d+1})$. Therefor the transformation $\vartheta = \zeta + \hat{\phi}(\zeta, Cv)$, $y = v + C^{-1}\hat{\psi}(\zeta, Cv)$ changes system (3.27) to

$$\begin{cases} \dot{\zeta} = \omega + \sum_{1\le|\alpha|\le d} a_\alpha (Cv)^\alpha + \sum_{|\alpha|=d+1} a_\alpha (Cv)^\alpha + \hat{h}^+(\zeta, Cv), \\ \dot{v} = (B + \eta\tilde{B})v + \sum_{1\le|\alpha|\le d} C^{-1} b_\alpha (Cv)^\alpha + \sum_{|\alpha|=d+1} C^{-1} b_\alpha (Cv)^\alpha + C^{-1}\hat{g}^+(\zeta, Cv), \end{cases}$$

where for $|\alpha| \le d$, a_α and b_α are the same as those in system (3.27), which is given by the induction hypotheses; and for $\alpha = d+1$, $a_\alpha = (a^1_\alpha, a^2_\alpha)$ and $b_\alpha = (b^1_\alpha, b^2_\alpha)$ are constant vectors given by

$$a^j_\alpha = a^j_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = \begin{cases} \hat{\Theta}^{0,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1}}, & \alpha_1 = d+1, \\ 0, & \alpha_1 \ne d+1, \end{cases}$$

$$b^j_\alpha = b^j_{\alpha^{p_1},\ldots,\alpha^{p_{d+1}}} = \begin{cases} \hat{f}^{0,d+1,j}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d+1}} + \frac{1}{d+1} \hat{f}^{0,|\alpha|,j+1}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2}, & \alpha_1 = d+1, \\ 0, & \alpha_1 \ne d+1, \end{cases}$$

$\hat{f}^{0,d+1,3}_{\underbrace{\alpha^1,\ldots,\alpha^1}_{d},\alpha^2} = 0$. $\hat{h}^+ = O(|Cv|^{d+2})$ and $C^{-1}\hat{g}^+ = O(|Cv|^{d+2})$.

Take

$$\begin{aligned} \theta &= \zeta + \hat{\phi} + \Phi^d(\zeta + \hat{\phi}, Cv + \hat{\psi}), \\ v &= v + C^{-1}\hat{\psi} + C^{-1}\Psi^d(\zeta + \hat{\phi}, Cv + \hat{\psi}). \end{aligned} \tag{3.60}$$

Then the transformation (3.60) changes system (3.2) to system (3.3) in which we recognize (ϑ, Cy) as (ζ, Cv). Hence the conclusion holds for $m = d+1$. This completes the proof of Theorem 3.1.

3.3. Proof of Theorem 3.2

We will prove the Theorem 3.2 by using the induction on m.

1) For $m=1$, suppose the transformation (3.6) transforms system (3.2) to

$$\begin{cases}\dot{\vartheta}=\omega+h_\lambda(\vartheta,Cy),\\ \dot{y}=(B+\eta\tilde{B})y+C^{-1}g_\lambda(\vartheta,Cy),\end{cases}\tag{3.61}$$

where $h_\lambda=O(|Cy|)$ and $C^{-1}g_\lambda=O(|Cy|^2)$.

Similar to the proof of Theorem 3.1, we can obtain directly

$$D_\vartheta\Phi_\lambda\omega+D_y\Phi_\lambda(B+\eta\tilde{B})y+h_\lambda^1-\Theta^1=0,\tag{3.62}$$
$$h_\lambda^++D_\vartheta\Phi_\lambda h_\lambda+D_y\Phi_\lambda C^{-1}g_\lambda-\Theta^+-D_\vartheta\Theta\Phi_\lambda-R(\Phi_\lambda)=0.\tag{3.63}$$

Noting that the left hand of (3.62) is a first order homogeneous polynomial in y and that (3.63) is $O(|y|^2)$.

Let us solve (3.62) by finding the Φ_λ which makes h_λ^1 as simple as possible. Using Fourier expansions for Φ_λ, h_λ^1, Θ^1, we have

$$\begin{aligned}\Phi_\lambda^j(\vartheta,Cy)&=\sum_{|\alpha|=1}\sum_{k\in\mathbb{Z}^2}\eta^{p_1-1}\Phi^j_{\lambda,\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha,\\ h_\lambda^{1,j}(\vartheta,Cy)&=\sum_{|\alpha|=1}\sum_{k\in\mathbb{Z}^2}\eta^{p_1-1}h^{1,j}_{\lambda,\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha,\\ \Theta^{1,j}(\vartheta,Cy)&=\sum_{|\alpha|=1}\sum_{k\in\mathbb{Z}^2}\eta^{p_1-1}\Theta^{1,j}_{\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha.\end{aligned}\tag{3.64}$$

Then by (3.62), (3.64), we have

$$\begin{aligned}&\sum_{|\alpha|=1}\sum_{k\in\mathbb{Z}^2}(\eta^{p_1-1}\Phi^j_{\lambda,\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}\mathbf{i}\langle k,\omega\rangle y^\alpha+\eta^{p_1-1}\Phi^j_{\lambda,\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}\lambda|\alpha|y^\alpha\\ &+\eta^{p_1}\Phi^j_{\lambda,\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}\alpha_1y_1^{\alpha_1-1}y_2^{\alpha_2+1}+\eta^{p_1-1}h^{1,j}_{\lambda,\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha\\ &-\eta^{p_1-1}\Theta^{1,j}_{\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha)=0.\end{aligned}\tag{3.65}$$

The equation (3.65) can be written as

$$\begin{aligned}&\sum_{|\alpha|=1}\sum_{k\in\mathbb{Z}^2}\eta^{p_1-1}(\Phi^j_{\lambda,\alpha^{p_1},k}(\mathbf{i}\langle k,\omega\rangle+\lambda|\alpha|)e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha+\Phi^j_{\lambda,\alpha^{p_1-1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha\\ &+h^{1,j}_{\lambda,\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha-\Theta^{1,j}_{\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha)=0,\end{aligned}\tag{3.66}$$

where $\Phi^j_{\lambda,\alpha^{p_1-p_1'},k}=0$ if $p_1-p_1'\le 0$.

By equating all terms in (3.66) of the same order with respect to η^{p_1-1}, we get

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^2}(\Phi^j_{\lambda,\alpha^{p_1},k}(\mathbf{i}\langle k,\omega\rangle+\lambda|\alpha|)e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha+\Phi^j_{\lambda,\alpha^{p_1-1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha\\ &+h^{1,j}_{\lambda,\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha-\Theta^{1,j}_{\alpha^{p_1},k}e^{\mathbf{i}\langle k,\vartheta\rangle}y^\alpha)=0.\end{aligned}$$

Comparing the coefficient of y^α, we obtain

$$(\mathbf{i}\langle k,\omega\rangle+\lambda|\alpha|)\Phi^j_{\lambda,\alpha^{p_1},k}=(\Theta-h)^{1,j}_{\lambda,\alpha^{p_1},k}-\Phi^j_{\lambda,\alpha^{p_1-1},k},\tag{3.67}$$

where $(\Theta-h)^{1,j}_{\lambda,\alpha^{p_1},k}=\Theta^{1,j}_{\alpha^{p_1},k}-h^{1,j}_{\lambda,\alpha^{p_1},k}$ and $\Phi^j_{\lambda,\alpha^{p_1-p_1'},k}=0$ if $p_1-p_1'\le 0$.

By (A2), the following choice is the simplest for $h_\lambda^{1,j}$. Let $h^{1,j}_{\lambda,\alpha^{p_1},k} = 0$. By considering (3.67), we get

$$\Phi^j_{\lambda,\alpha^{p_1},k} = \sum_{q=0}^{p_1-1} (-1)^q(\mathbf{i}\langle k,\omega\rangle + \lambda|\alpha|)^{-(q+1)}\Theta^{1,j}_{\alpha^{p_1-q},k}.$$

Hence $h^{1,j}_{\lambda,\alpha^{p_1},k} = 0$.

Using (A2) we find that Φ_λ is analytic and 2π-periodic in ϑ_1, ϑ_2 in $D_{r-\delta}$, $0 < \delta < r$. It is clear that $\Phi_\lambda = O(|Cy|)$. Now we determine h^+_λ. Write (3.63) as follows

$$(I + D_\vartheta\Phi_\lambda)h^+_\lambda = -D_y\Phi_\lambda C^{-1}g_\lambda - D_\vartheta\Phi_\lambda h^1_\lambda + \Theta^+ + D_\vartheta\Theta\Phi_\lambda + R(\Phi_\lambda).$$

Since $D_\vartheta\Phi_\lambda = O(|Cy|)$, we can choose sufficiently small r_1, $0 < r_1 < r - \delta$ such that $(I + D_\vartheta\Phi_\lambda)$ is invertible and $(\vartheta, y) + (\Phi_\lambda, C^{-1}\Psi_\lambda) \in D_r$ for $(\vartheta, y) \in D_{r_1}$. Hence

$$h^+ = (I + D_\vartheta\Phi_\lambda)^{-1}(-D_y\Phi_\lambda C^{-1}g_\lambda - D_\vartheta\Phi_\lambda h^1_\lambda + \Theta^+ + D_\vartheta\Theta\Phi_\lambda + R(\Phi_\lambda)).$$

Therefor the transformation $\theta = \vartheta + \Phi_\lambda(\vartheta, Cy)$, $z = Cy$ changes system (3.2) to

$$\begin{cases} \dot\vartheta = \omega + \tilde\Theta_\lambda(\vartheta, Cy), \\ \dot y = (B + \eta\tilde B)y + C^{-1}\tilde f_\lambda(\vartheta, Cy), \end{cases} \tag{3.68}$$

where $\tilde\Theta_\lambda = O(|Cy|^2)$, $C^{-1}\tilde f_\lambda = O(|Cy|^2)$.

2) Assume that the conclusion holds for $m = d \geq 1$, then there exists a transformation $\theta = \vartheta + \Phi^d_\lambda(\vartheta, Cy), z = y + C^{-1}\Psi^d_\lambda(\vartheta, Cy)$ which satisfies the requirements of the conclusion and changes system (3.2) to

$$\begin{cases} \dot\vartheta = \omega + \hat\Theta_\lambda(\vartheta, Cy), \\ \dot y = (B + \eta\tilde B)y + C^{-1}\hat f_\lambda(\vartheta, Cy), \end{cases} \tag{3.69}$$

where $\hat\Theta_\lambda = O(|Cy|^{d+1})$ and $C^{-1}\hat f_\lambda = O(|Cy|^{d+1})$ are analytic in both variables and $2\pi-$periodic in each component of ϑ in D_{r_d}, where $r_d > 0$ is a positive constant.

When $m = d + 1$, consider the transformation $\vartheta = \zeta + \hat\phi_\lambda(\zeta, Cv), y = v + C^{-1}\hat\psi_\lambda(\zeta, Cv)$ where $\hat\phi_\lambda$ and $C^{-1}\hat\psi_\lambda$ are $(d+1)$th order homogeneous polynomials in Cv and $2\pi-$periodic in each component of the vector ζ. This transformation changes system (3.69) to the following system

$$\begin{cases} \dot\zeta = \omega + \hat h_\lambda(\zeta, Cv), \\ \dot v = (B + \eta\tilde B)v + C^{-1}\hat g_\lambda(\zeta, Cv), \end{cases} \tag{3.70}$$

where $\hat h_\lambda = O(|Cv|^{d+1})$ and $C^{-1}\hat g_\lambda = O(|Cv|^{d+1})$ are determined later. As before, equivalently $\vartheta = \zeta + \hat\phi_\lambda$ and $y = v + C^{-1}\hat\psi_\lambda$ satisfy the following equations

$$\begin{aligned} &\hat\Theta_\lambda(\zeta + \hat\phi_\lambda, Cv + \hat\psi_\lambda) \\ &= D_\zeta\hat\phi_\lambda\omega + D_v\hat\phi_\lambda(B + \eta\tilde B)v + \hat h_\lambda + D_\zeta\hat\phi_\lambda\hat h + D_v\hat\phi_\lambda C^{-1}\hat g_\lambda \end{aligned} \tag{3.71}$$

and

$$(B + \eta\tilde B)\hat\psi_\lambda + C^{-1}\hat f_\lambda(\zeta + \hat\phi_\lambda, Cv + \hat\psi_\lambda)$$

$$= D_\zeta(C^{-1}\hat{\psi}_\lambda)\omega + D_v(C^{-1}\hat{\psi}_\lambda)(B+\eta\tilde{B})v + C^{-1}\hat{g}_\lambda + D_\zeta(C^{-1}\hat{\psi}_\lambda)\hat{h}_\lambda + D_v(C^{-1}\hat{\psi}_\lambda)C^{-1}\hat{g}_\lambda. \tag{3.72}$$

Since the functions we consider are analytic, we can classify system of equations (3.71) and (3.72) into two systems according to the order of v. One contains only $(d+1)$th order terms with respect to v in equations (3.71) and (3.72), the other contains higher order terms. Using a Taylor expansion for $\hat{\Theta}_\lambda(\zeta+\hat{\phi}_\lambda, Cv+\hat{\psi}_\lambda)$ and $C^{-1}\hat{f}_\lambda(\zeta+\hat{\phi}_\lambda, Cv+\hat{\psi}_\lambda)$, we have

$$\hat{\Theta}_\lambda(\zeta+\hat{\phi}_\lambda, Cv+\hat{\psi}_\lambda) = \hat{\Theta}_\lambda(\zeta, Cv) + \hat{R}_{\lambda,1}(\hat{\phi}_\lambda, \hat{\psi}_\lambda),$$
$$\hat{f}_\lambda(\zeta+\hat{\phi}_\lambda, Cv+\hat{\psi}_\lambda) = \hat{f}_\lambda(\zeta, Cv) + \hat{R}_{\lambda,2}(\hat{\phi}_\lambda, \hat{\psi}_\lambda),$$

where $\hat{R}_{\lambda,1}(\hat{\phi}_\lambda, \hat{\psi}_\lambda) = O(|Cv|^{d+2})$ and $\hat{R}_{\lambda,2}(\hat{\phi}_\lambda, \hat{\psi}_\lambda) = O(|Cv|^{d+2})$.

We write

$$\hat{\Theta}_\lambda(\zeta, Cv) = \hat{\Theta}_\lambda^{d+1} + \hat{\Theta}_\lambda^{+},\ C^{-1}\hat{f}_\lambda(\zeta, Cv) = C^{-1}\hat{f}_\lambda^{d+1} + C^{-1}\hat{f}_\lambda^{+},$$
$$\hat{h}_\lambda(\zeta, Cv) = \hat{h}_\lambda^{d+1} + \hat{h}_\lambda^{+},\quad C^{-1}\hat{g}_\lambda(\zeta, Cv) = C^{-1}\hat{g}_\lambda^{d+1} + C^{-1}\hat{g}_\lambda^{+},$$

where $\hat{\Theta}_\lambda^{d+1}, \hat{h}_\lambda^{d+1}, C^{-1}\hat{f}_\lambda^{d+1}$ and $C^{-1}\hat{g}_\lambda^{d+1}$ are $(d+1)$th order homogeneous polynomials in Cv and $\hat{\Theta}_\lambda^{+} = O(|Cv|^{d+2})$, $C^{-1}\hat{f}_\lambda^{+} = O(|Cv|^{d+2})$, $\hat{h}^{+} = O(|Cv|^{d+2})$ and $C^{-1}\hat{g}_\lambda^{+} = O(|Cv|^{d+2})$. Hence we can write system of equations (3.71) and (3.72) as the following two systems

$$\hat{\Theta}_\lambda^{d+1} - \hat{h}_\lambda^{d+1} = D_\zeta\hat{\phi}_\lambda\omega + D_v\hat{\phi}_\lambda(B+\eta\tilde{B})v, \tag{3.73}$$
$$C^{-1}\hat{f}_\lambda^{d+1} - C^{-1}\hat{g}_\lambda^{d+1} = D_\zeta(C^{-1}\hat{\psi}_\lambda)\omega + D_v(C^{-1}\hat{\psi}_\lambda)(B+\eta\tilde{B})v - (B+\eta\tilde{B})C^{-1}\hat{\psi}_\lambda \tag{3.74}$$

and

$$\hat{h}_\lambda^{+} + D_\zeta\hat{\phi}_\lambda\hat{h}_\lambda^{+} + D_v\hat{\phi}_\lambda C^{-1}\hat{g}_\lambda^{+} = \hat{\Theta}_\lambda^{+} + \hat{R}_{\lambda,1} - D_\zeta\hat{\phi}_\lambda\hat{h}_\lambda^{d+1} - D_v\hat{\phi}_\lambda C^{-1}\hat{g}_\lambda^{d+1}, \tag{3.75}$$
$$C^{-1}\hat{g}_\lambda^{+} + D_\zeta(C^{-1}\hat{\psi}_\lambda)\hat{h}_\lambda^{+} + D_v(C^{-1}\hat{\psi}_\lambda)C^{-1}\hat{g}_\lambda^{+} = C^{-1}\hat{f}_\lambda^{+} + \hat{R}_{\lambda,2} - D_\zeta(C^{-1}\hat{\psi}_\lambda)\hat{h}_\lambda^{d+1} - D_v(C^{-1}\hat{\psi}_\lambda)C^{-1}\hat{g}_\lambda^{d+1}. \tag{3.76}$$

Let us solve (3.73) and (3.74) by finding $\hat{\phi}_\lambda$ and $C^{-1}\hat{\psi}_\lambda$ which makes $\hat{h}_\lambda^{d+1}$ and $C^{-1}\hat{g}_\lambda^{d+1}$ as simple as possible respectively. Using Fourier expansions for the function $\hat{\Theta}_\lambda^{d+1}$, $C^{-1}\hat{f}_\lambda^{d+1}$, $\hat{h}_\lambda^{d+1}$, $C^{-1}\hat{g}_\lambda^{d+1}$, $\hat{\phi}_\lambda$ and $C^{-1}\hat{\psi}_\lambda$, we have

$$\hat{\Theta}_\lambda^{d+1,j}(\zeta, Cv) = \sum_{|\alpha|=d+1}\sum_{k\in\mathbb{Z}^2}\eta^{p_1+\cdots+p_{d+1}-(d+1)}\hat{\Theta}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}e^{\mathbf{i}\langle\zeta,k\rangle}v^\alpha,$$
$$\eta^{-(j-1)}\hat{f}_\lambda^{d+1,j}(\zeta, Cv) = \sum_{|\alpha|=d+1}\sum_{k\in\mathbb{Z}^2}\eta^{p_1+\cdots+p_{d+1}-d-j}\hat{f}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}e^{\mathbf{i}\langle\zeta,k\rangle}v^\alpha,$$
$$\hat{h}_\lambda^{d+1,j}(\zeta, Cv)$$

$$= \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{h}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle\zeta,k\rangle} v^{\alpha}, \tag{3.77}$$

$$\eta^{-(j-1)} \hat{g}^{d+1,j}_{\lambda}(\zeta, Cv)$$

$$= \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{g}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle\zeta,k\rangle} v^{\alpha},$$

$$\hat{\phi}^{j}_{\lambda}(\zeta, Cv)$$

$$= \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle\zeta,k\rangle} v^{\alpha},$$

$$\eta^{-(j-1)} \hat{\psi}^{j}_{\lambda}(\zeta, Cv)$$

$$= \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle\zeta,k\rangle} v^{\alpha}.$$

Then by (3.73), (3.74), (3.77), we have

$$\begin{aligned}
&\sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{\Theta}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^{\alpha} \\
&\quad - \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{h}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^{\alpha} \\
&= \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} \mathbf{i}\langle k,\omega\rangle v^{\alpha} \\
&\quad + \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-(d+1)} \lambda|\alpha| \hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^{\alpha} \\
&\quad + \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-d} \hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} \alpha_1 v_1^{\alpha_1-1} v_2^{\alpha_2+1},
\end{aligned} \tag{3.78}$$

$$\begin{aligned}
&\sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{f}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^{\alpha} \\
&\quad - \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{g}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^{\alpha} \\
&= \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} \mathbf{i}\langle k,\omega\rangle v^{\alpha} \\
&\quad + \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-d-j} \lambda|\alpha| \hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^{\alpha} \\
&\quad + \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-d-j+1} \hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} \alpha_1 v_1^{\alpha_1-1} v_2^{\alpha_2+1} \\
&\quad - \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-d-j} \lambda \hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^{\alpha} \\
&\quad - \sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} \eta^{p_1+\cdots+p_{d+1}-d-j} \hat{\psi}^{j+1}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} e^{\mathbf{i}\langle k,\zeta\rangle} v^{\alpha},
\end{aligned} \tag{3.79}$$

where $\hat{\psi}^{3}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} = 0$. The equations (3.78) and (3.79) can be written as

$$\sum_{|\alpha|=d+1} \sum_{k\in\mathbb{Z}^2} e^{\mathbf{i}\langle k,\zeta\rangle} \eta^{p_1+\cdots+p_{d+1}-(d+1)} (\hat{\Theta}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} v^{\alpha} - \hat{h}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} v^{\alpha}$$

$$
\begin{aligned}
&- \hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}(\mathbf{i}\langle k,\omega\rangle + \lambda|\alpha|)v^{\alpha}\\
&- \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}},k} v^{\alpha}) = 0, \qquad (3.80)\\
&\sum_{|\alpha|=d+1}\sum_{k\in\mathbb{Z}^2} e^{\mathbf{i}\langle k,\zeta\rangle}\eta^{p_1+\cdots+p_{d+1}-d-j}(\hat{f}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}v^{\alpha} - \hat{g}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}v^{\alpha}\\
&+ \hat{\psi}^{j+1}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}v^{\alpha} - \hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}(\mathbf{i}\langle k,\omega\rangle + \lambda(|\alpha|-1))v^{\alpha}\\
&- \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}},k} v^{\alpha}) = 0, \qquad (3.81)
\end{aligned}
$$

where $\phi^{j}_{\lambda,\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}},k} = \psi^{j}_{\lambda,\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}},k} = 0$ if there is at least one $s'(1\le s'\le d+1)$ such that $p_{s'} - p'_{s'} \le 0$.

By equating all terms in (3.80) and (3.81) of the same order with respect to $\eta^{p_1+\cdots+p_{d+1}-(d+1)}$ and $\eta^{p_1+\cdots+p_{d+1}-d-j}$ respectively, we get

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^2} e^{\mathbf{i}\langle k,\zeta\rangle}(\hat{\Theta}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}v^{\alpha} - \hat{h}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}v^{\alpha} - \hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}(\mathbf{i}\langle k,\omega\rangle\\
&+ \lambda|\alpha|)v^{\alpha} - \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}},k} v^{\alpha}) = 0,\\
&\sum_{k\in\mathbb{Z}^2} e^{\mathbf{i}\langle k,\zeta\rangle}(\hat{f}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}v^{\alpha} - \hat{g}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}v^{\alpha}\\
&- \hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}(\mathbf{i}\langle k,\omega\rangle + \lambda(|\alpha|-1))v^{\alpha}\\
&- \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}},k} v^{\alpha} + \hat{\psi}^{j+1}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}v^{\alpha}) = 0.
\end{aligned}
$$

Comparing the coefficient of v^{α}, we have

$$
\begin{aligned}
&(\mathbf{i}\langle k,\omega\rangle + \lambda|\alpha|)\hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}\\
&= (\hat{\Theta} - \hat{h})^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} - \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}},k}, \qquad (3.82)
\end{aligned}
$$

$$
\begin{aligned}
&(\mathbf{i}\langle k,\omega\rangle + \lambda(|\alpha|-1))\hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}\\
&= (\hat{f} - \hat{g})^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} + \hat{\psi}^{j+1}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}\\
&\quad - \sum_{s'=1}^{d+1} \frac{\alpha_{p_{s'}-1}+1}{\alpha_{p_{s'}}} \hat{\psi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{s'-1}},\alpha^{p_{s'}-1},\alpha^{p_{s'+1}},\ldots,\alpha^{p_{d+1}},k}, \qquad (3.83)
\end{aligned}
$$

where $(\hat{\Theta}-\hat{h})^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} = \hat{\Theta}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} - \hat{h}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}$, $(\hat{f}-\hat{g})^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} = \hat{f}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} - \hat{g}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}$.

Our purpose is to find $\hat{\phi}_{\lambda}$, $C^{-1}\hat{\psi}_{\lambda}$ such that $\hat{h}^{d+1}_{\lambda}$ and $C^{-1}\hat{g}^{d+1}_{\lambda}$ have the simplest form. By (A2), the following choice is the simplest for $\hat{h}^{d+1,j}_{\lambda}$ and $\hat{g}^{d+1,j}_{\lambda}$. Let $\hat{h}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} = 0$ and $\tilde{g}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} = 0$. By considering (3.82), (3.83), we get

$$\hat{\phi}^{j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}$$

$$= \sum_{q=0}^{p_1+\cdots+p_{d+1}-(d+1)} \sum_{s'_1=1}^{d+1} \cdots \sum_{s'_q=1}^{d+1} (-1)^q (\mathbf{i}\langle k,\omega\rangle + \lambda|\alpha|)^{-(q+1)} \frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}} \cdots \frac{\alpha_{p_{s'_q}-1}+1}{\alpha_{p_{s'_q}}}$$
$$\hat{\Theta}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\ldots,\alpha^{p_{s'_q}-1},\alpha^{p_{s'_q}-1},\alpha^{p_{s'_q}+1},\ldots,\alpha^{p_{d+1}},k},$$
$$\hat{\psi}^j_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k}$$
$$= \sum_{\tilde{j}=j}^{2} ((\sum_{q_j=0}^{p_1+\cdots+p_{d+1}-(d+1)} \sum_{{}_js'_1=1}^{d+1} \cdots \sum_{{}_js'_{q_j}=1}^{d+1}) \ldots (\sum_{q_{\tilde{j}}=0}^{p_1+\cdots+p_{d+1}-(d+1)-q_{\hat{j},\tilde{j}-1}+\tilde{j}-j}$$
$$\sum_{{}_{\tilde{j}}s'_1=1}^{d+1} \cdots \sum_{{}_{\tilde{j}}s'_{q_{\tilde{j}}}=1}^{d+1})(-1)^{q_{\hat{j},\tilde{j}}} (\mathbf{i}\langle k,\omega\rangle + \lambda(|\alpha|-1))^{-q_{\hat{j},\tilde{j}}-(\tilde{j}-j+1)}$$
$$\left(\frac{\alpha_{p_{{}_js'_1}-1}+1}{\alpha_{p_{{}_js'_1}}} \cdots \frac{\alpha_{p_{{}_js'_{q_j}}-1}+1}{\alpha_{p_{{}_js'_{q_j}}}}\right) \ldots \left(\frac{\alpha_{p_{{}_{\tilde{j}}s'_1}-1}+1}{\alpha_{p_{{}_{\tilde{j}}s'_1}}} \cdots \frac{\alpha_{p_{{}_{\tilde{j}}s'_{q_{\tilde{j}}}}-1}+1}{\alpha_{p_{{}_{\tilde{j}}s'_{q_{\tilde{j}}}}}}\right)$$
$$\hat{f}^{d+1,\tilde{j}}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{-j}s'_1-1},\alpha^{p_{-j}s'_1-1},\alpha^{p_{-j}s'_1+1},\ldots,\alpha^{p_{-\tilde{j}}s'_{q_{\tilde{j}}}-1},\alpha^{p_{-\tilde{j}}s'_{q_{\tilde{j}}}-1},\alpha^{p_{-\tilde{j}}s'_{q_{\tilde{j}}}+1},\ldots,\alpha^{p_d},\alpha^{p_{d+1}},k}),$$

where

$$\hat{\Theta}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\ldots,\alpha^{p_{s'_0}-1},\alpha^{p_{s'_0}-1},\alpha^{p_{s'_0}+1},\ldots,\alpha^{p_{d+1}},k} = \hat{\Theta}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k},$$
$$\hat{f}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}-1},\alpha^{p_{s'_1}+1},\ldots,\alpha^{p_{s'_0}-1},\alpha^{p_{s'_0}-1},\alpha^{p_{s'_0}+1},\ldots,\alpha^{p_{d+1}},k} = \hat{f}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k},$$

$\sum_{s'_1=1}^{d+1} \cdots \sum_{s'_0=1}^{d+1} = \sum_{{}_js'_1=1}^{d+1} \cdots \sum_{{}_js'_0=1}^{d+1} = \sum_{{}_{\tilde{j}}s'_1=1}^{d+1} \cdots \sum_{{}_{\tilde{j}}s'_0=1}^{d+1} = 1$, $\frac{\alpha_{p_{s'_1}-1}+1}{\alpha_{p_{s'_1}}} \cdots \frac{\alpha_{p_{s'_0}-1}+1}{\alpha_{p_{s'_0}}} = \frac{\alpha_{p_{{}_js'_1}-1}+1}{\alpha_{p_{{}_js'_1}}} \cdots \frac{\alpha_{p_{{}_js'_0}-1}+1}{\alpha_{p_{{}_js'_0}}} = \frac{\alpha_{p_{{}_{\tilde{j}}s'_1}-1}+1}{\alpha_{p_{{}_{\tilde{j}}s'_1}}} \cdots \frac{\alpha_{p_{{}_{\tilde{j}}s'_0}-1}+1}{\alpha_{p_{{}_{\tilde{j}}s'_0}}} = 1$, $\hat{f}^{d+1,j}_{\lambda,\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}},k} = 0$ if $j \geq 3$, $\hat{\Theta}^{d+1,j}_{\lambda,\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}},k} = \hat{f}^{d+1,j}_{\lambda,\alpha^{p_1-p'_1},\ldots,\alpha^{p_{d+1}-p'_{d+1}},k} = 0$ if there is at least one $s'(1 \leq s' \leq d+1)$ such that $p_{s'} - p'_{s'} \leq 0$ and $q_{\hat{j},\tilde{j}} = \sum_{\hat{j}=j}^{\tilde{j}} q_{\hat{j}}$. Then $\hat{h}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} = 0$ and $\hat{g}^{d+1,j}_{\lambda,\alpha^{p_1},\ldots,\alpha^{p_{d+1}},k} = 0$.

Using (A2) we obtain that $\hat{\phi}_\lambda$ and $C^{-1}\hat{\psi}_\lambda$ are analytic and 2π-periodic in ζ in $D_{r_d-\delta}$, where $0 < \delta < r_{d+1}$. Moreover, $\hat{h}^{d+1}_\lambda$, $C^{-1}\hat{g}^{d+1}_\lambda$, $\hat{\phi}_\lambda$ and $C^{-1}\hat{\psi}_\lambda$ are solutions of system of equations (3.73) and (3.74). It is clear that $\hat{\phi}_\lambda = O(|Cv|^{d+1})$ and $C^{-1}\hat{\psi}_\lambda = O(|Cv|^{d+1})$.

We choose r_{d+1} sufficiently small such that $0 < r_{d+1} < r_d - \delta$,

$$\begin{bmatrix} Id + D_\zeta\hat{\phi}_\lambda & D_v\hat{\phi}_\lambda \\ D_\zeta(C^{-1}\hat{\psi}_\lambda) & Id + D_v(C^{-1}\hat{\psi}_\lambda) \end{bmatrix}$$

has an inverse and $(\zeta,v)+(\hat{\phi}_\lambda, C^{-1}\hat{\psi}_\lambda) \in D_{r_d}$ for $(\zeta,v) \in D_{r_{d+1}}$. Hence we can solve system of equations (3.75) and (3.76) for $\hat{h}^+_\lambda$, $C^{-1}\hat{g}^+_\lambda$. Therefor the transformation

$\vartheta = \zeta + \hat{\phi}_\lambda(\zeta, Cv)$, $y = v + C^{-1}\hat{\psi}_\lambda(\zeta, Cv)$ changes system (3.69) to

$$\begin{cases} \dot{\zeta} = \omega + \hat{h}^+_\lambda(\zeta, Cv), \\ \dot{v} = (B + \eta\tilde{B})v + C^{-1}\hat{g}^+_\lambda(\zeta, Cv), \end{cases}$$

where $\tilde{h}^+_\lambda = O(|Cv|^{d+2})$ and $C^{-1}\tilde{g}^+_\lambda = O(|Cv|^{d+2})$.

Take

$$\begin{aligned} \theta &= \zeta + \hat{\phi}_\lambda + \Phi^d_\lambda(\zeta + \hat{\phi}_\lambda, Cv + \hat{\psi}_\lambda), \\ v &= v + C^{-1}\hat{\psi}_\lambda + C^{-1}\Psi^d_\lambda(\zeta + \hat{\phi}_\lambda, Cv + \hat{\psi}_\lambda). \end{aligned} \tag{3.84}$$

Then the transformation (3.84) changes system (3.2) to system (3.5) in which we recognize (ϑ, Cy) as (ζ, Cv). Hence the conclusion holds for $m = d + 1$. This completes the proof of Theorem 3.2.

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