Journal of Applied Analysis and Computation
Volume 13, Number 1, February 2023, 553–574
Website:http://www.jaac-online.com
DOI:10.11948/20220476

PARTIAL PRACTICAL STABILITY AND ASYMPTOTIC STABILITY OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY NOISE WITH A GENERAL DECAY RATE

Shuaishuai Lu[1] and Xue Yang[1,†]

Abstract In this paper, we mainly study the almost sure partial practical stability of stochastic differential equations driven by Lévy noise with a general decay rate. By establishing a suitable Lyapunov function and using Exponential Martingale inequality and Borel-Cantelli theorem, giving some sufficient conditions that can guarantee the almost sure partial practical stability of equations. At the same time, we also study general conditions that guarantee the almost sure asymptotic stability of the equation. Finally, we also give two examples to illustrate our theoretical results.

Keywords Partial practical stability, Lévy noise, stochastic differential equations, Lyapunov functions, exponential martingale inequality.

MSC(2010) 60H10, 60G51.

1. Introduction

Stochastic differential equations have been used to express the motion laws in the real world. Their advantage is that they take into account the uncertainty of the environment. Stochastic differential equations be widely used in physics, mechanics, control engineering, financial market, economics and other scientific fields, attracting the attention of many researchers. Stability, as an important property of stochastic differential equations, has always been a popular research direction. Many achievements have been made on various stability problems of stochastic differential equations, see [12, 15, 23, 24, 28, 30–32, 36].

Stochastic differential equations driven by Lévy noise be an important kind of stochastic equations. It plays an important role in financial economics, biology, quantum field theory, stochastic control, and stochastic filtering, etc. The theory of Lévy process originated in 1930s, and after more than 90 years of development, the basic theory of Lévy process has been widely studied, see [1, 5, 11, 14, 19]. Stochastic differential equations driven by Lévy noise have the following important properties: the Lévy process has a random process of steady and independent increment; even though the Lévy process is not continuous, and the sample paths are right continuous; some random jump discontinuities occur at random time at each finite time interval. Therefore, compared with stochastic differential equations with Markov

[†]The corresponding author. Email: xueyang@jlu.edu.cn(X. Yang)
[1]College of Mathematics, Jilin University, Changchun 130012, China

switching, it is more reasonable and accurate to use stochastic differential equations driven by Lévy noise to describe random disturbances in some real systems. There have been many studies on the various stability of such equations. The p-moment stability of stochastic differential equations with Lévy noise has been studied in depth in [3, 37–39]. In particular, in [3], Applebaum et al. derived sufficient conditions to guarantee the almost sure stability and the p-moment exponential stability of solutions of stochastic differential equations (SDEs) driven by Lévy noise. In general, there is no obvious relationship between moment exponential stability and almost sure stability. However, in [3], almost sure stability is derived with some additional conditions when the equations are guaranteed to be moment exponential stable, and it is worth noting that the paper not only proves the stability of the system, but also gives an explicit convergence rate. In the paper [34], Xu et al. studied the stochastic stability of nonlinear systems driven by Lévy noise. And in [35], Xu et al. again proposed an equivalent linear method to reduce and simplify the original system, and studied the stochastic stability of nonlinear systems driven by Lévy processes based on the Lyapunov exponent. In [16], Li et al. studied the exponential and polynomial stability of a neutral stochastic time-lag differential equations driven by Lévy noise with a general decay rate using the Lyapunov function method and convergence theorem for nonnegative semimartingales, and also obtained some criteria for determining that the system is almost sure exponentially stable using M-matrix theory.

Lyapunov functions are used to prove the stability of differential systems. A function is called a Lyapunov candidate function if it has the possibility to prove the stability of the differential system at an equilibrium. With the development of Lyapunov's first and second methods, more and more work is based on Lyapunov methods to study the stability of differential systems, see [6,13,20,25]. In [4], Arnold et al. developed Lyapunov's second method in stochastic dynamical systems and random sets, and gave the concepts of attractor and stability accordingly. In [17], Li et al. obtained the stability of the system through some inherited properties of Lyapunov functions, and also obtained the existence of almost periodic solutions of the distribution under appropriate conditions other than Lyapunov functions.

In many cases, stochastic differential equations considered may not satisfy sufficient conditions to the stability of solutions. But in many cases, it suffices to study the partial stability. Therefore, it is also important to find some of the variables in equations that admit stability. This problem is called "partial stability". Partial stability has proven to have powerful applications in many branches such as biotechnology, gimbal gyroscopes, electromagnetic, rotating mechanical vibration, inertial navigation systems. The reader is referred to [21, 27, 29], for more details.

When the origin is not a trivial solution, we can study the asymptotic stability of the solution to a small neighborhood of the origin. The goal is to analyze the asymptotic stability of a system whose solution behavior is a small ball of state space or close to it. It is guaranteed that almost all state trajectories are bounded and close to a sufficiently small neighborhood of the origin. In this sense, the limit boundedness of solutions of random systems, or the possibility of convergence of solutions often need to be analyzed on a ball centered on the origin, which is called "practical stability". In [7–10], Caraballo et al. gave some results on the practical stability of nontrivial solutions for several classes of stochastic systems.

Although the concept of partial practical stability was put forward, the sufficient condition of partial practical stability has not been considered. Therefore, this paper is based on the basis of Caraballo's research on the partial practical stability of stochastic differential equations with general decay rate, and in the present paper we study and analyze the partial practical stability of stochastic differential equations driven by Lévy noise with a general decay rate. With the help of stochastic analysis theory and Lyapunov methods, by establishing the appropriate Lyapunov function, using Exponential Martingale inequality and Borel-Cantelli theorem, we provide some sufficient conditions that can guarantee the almost sure partial practical stability of equations.

The main structure of this paper is as follows: The second part gives the basic form of stochastic differential equations driven by Lévy noise and related definitions, the definition of boundedness of solutions and the definition of partial practical stability. In the third part, we give the sufficient conditions ensuring the almost sure partial practical stability of stochastic differential equations driven by Lévy noise. At the same time, we also provide general conditions that guarantee the almost sure asymptotic stability of the equation. In the fourth part, we give two examples to illustrate our theoretical results. The last part summarizes the main results of this paper.

2. Stochastic differential equations driven by Lévy noise

Let's assume that $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space, $\{\mathscr{F}_t\}_{t\geq 0}$ is a filtration in the probability space. Then $\mathscr{F}_0$ is right continuous and contains all the $\mathbb{P}$-null test sets. We're going to use $\|\cdot\|$ for the Euclidean norm in $\mathbb{R}^n$. If A is a matrix or a vector, A^T is its transpose. If A is a matrix, the norm is expressed as $\|A\| = \sqrt{\operatorname{trace}(AA^T)}$, if A is an operator, the operator norm with $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$. We use $m \vee n$ to represent $\max\{m, n\}$ and $m \wedge n$ to represent $\min\{m, n\}$. Let χ_Σ denote the indicator function of the set Σ and $W(t)$ be a w dimensional Brownian motion in probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. We define N as a Poisson random measure on $\mathbb{R}_+ \times (\mathbb{R}^n - \{0\})$, and its associated compensator and intensity measure are $\tilde{N}$ and ν respectively, where we assume that ν is a Lévy measure satisfying $\int_{\mathbb{R}^n - \{0\}} (z^2 \wedge 1)\nu \mathrm{d}(z) < \infty$ and $\tilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)$. We usually refer to the pair (W, N) as a Lévy noise. Now consider the nonlinear stochastic differential equation driven by Lévy noise as follows:

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}W(t) + \int_{\|z\|<c} H(x(t-), z, t)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \tag{2.1}$$

where

$$f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n, \quad g : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^{n\times w}, \quad H : \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n,$$

$c \in (0, \infty]$ is the maximum allowable jump size.

Let $x \in \mathbb{R}^n, x = (x_1, x_2)^T$ and $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, where $n_1 + n_2 = n$. At this point, we define the norm of x to be $\|x\| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$. In this paper we assume that Poisson random measure N and Brownian motion W are independent each other.

In order to ensure the existence and uniqueness of the above system solutions, the following assumptions are made:

(H1) f, g and H satisfy the following conditions:

1) There exists a nonnegative function $\eta_1(t)$, for all $x \in \mathbb{R}^n$, such that

$$\|f(x,t)\|^2 + \|\tilde{g}(x,x,t)\| + \int_{\|z\|<c} \|H(x,z,t)\|^2 \nu d(z) \leq \eta_1(t)(1+\|x\|^2),$$

where $\tilde{g}(x,y,t) = g(x,t)g^T(y,t)$ represents an $n \times n$ matrix;

2) There exists a nonnegative function $\eta_2(t)$, for all $x, y \in \mathbb{R}^n$, such that

$$\begin{aligned}&\|f(x,t)-f(y,t)\|^2 + \|\tilde{g}(x,x,t) - 2\tilde{g}(x,y,t) + \tilde{g}(y,y,t)\| \\ &+ \int_{\|z\|<c} \|H(x,z,t)-H(y,z,t)\|^2 \nu d(z) \\ \leq &\eta_2(t)\|x-y\|^2.\end{aligned}$$

It is known ([2] or [33]) that if system (2.1) satisfies the above conditions **(H1)**, then for any standard initial condition, there exists a unique solution interval $[t_0, T)$ that defines initial values $x_0 \in \mathbb{R}^n$. Since we study the asymptotic behavior of solutions, we assume $T = +\infty$. In addition, the solution process $x(t)$ is adapted and right continuous with a left limit. And for any $t > t_0$, if $\mathbb{E}\|x(0)\|^2 < \infty$, then

$$\mathbb{E}\|x(t)\|^2 < \infty. \tag{2.2}$$

For the convenience of description, we will give the following definitions:

Definition 2.1. Let's say that $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ denotes the family of all nonnegative functions $V(x,t)$ on $\mathbb{R}^n \times \mathbb{R}_+$ which are continuously twice differentiable in x and once differentiable in t. Suppose $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ has the following:

$$V_t = \frac{\partial V(x,t)}{\partial t}, \quad V_x = (\frac{\partial V(x,t)}{\partial x_1}, \frac{\partial V(x,t)}{\partial x_2}), \quad V_{xx} = (\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j})_{n\times n}.$$

Then we define an operator L that operates on $V(x,t)$ and $LV : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, where

$$\begin{aligned}LV(x,t) =& V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}\text{trace}\left[g^T(x,t)V_{xx}(x,t)g(x,t)\right] \\ &+ \int_{\|z\|<c} (V(x+H(x,z,t),t) - V(x,t) - H(x,z,t)V_x(x,t))\nu(\mathrm{d}z).\end{aligned} \tag{2.3}$$

The general attenuation function $\alpha(t)$ is used to define the convergence of the solutions towards a small ball centered at the origin, then the definition of globally uniformly practical stability is given. Mao [22] first introduced the concept of stability with polynomial decay rate. Then, as more and more people pay attention to the stability of stochastic differential systems and other related problems, the concept is gradually extended to stability with general decay rate. In the following, we will give the relevant concepts of partial practical stability of systems with general decay rates, and the specific contents can also be found in [7, 10]:

Definition 2.2. Let $\alpha(t) > 0$, such that $\alpha(t) \to +\infty$ as $t \to +\infty$. A non-trivial solution $x(t) = (x_1(t), x_2(t))$ of equation (2.1) is deemed to converge to the ball $\mathcal{B}_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}, r > 0$ with respect to x_1 with attenuation function $\alpha(t)$

and order at least $\gamma > 0$, if its generalized Lyapunov exponent is less than or equal to $-\gamma$ with probability one, *i.e.*,

$$\lim_{t\to+\infty} \sup \frac{\ln(\|x_1(t)\| - r)}{\ln \alpha(t)} \leq -\gamma, a.s. \tag{2.4}$$

Definition 2.3.

1. The ball $\mathcal{B}_r$ is said to be uniformly stable in probability, if for each $\varepsilon \in [0,1]$ and $k > r$, there exists $T \geq t_0$ and $\delta = \delta(\varepsilon, k) > 0$, for all $\|x_0\| < \delta$ such that

$$\mathbb{P}(\|x(t;t_0,x_0)\| < k, \forall t \geq T) < 1 - \varepsilon. \tag{2.5}$$

2. The ball $\mathcal{B}_r$ is said to be globally uniformly stable in probability if it is uniformly stable in probability, and the solution of (2.1) is globally uniformly bounded in probability.

Definition 2.4. The ball $\mathcal{B}_r$ is said to be almost sure globally uniformly practically stable with respect to the decay function $\alpha(t)$, if for any initial value $x_0 \in \mathbb{R}^n$ where $x_0 \neq 0$ such that for all $t \geq t_0 \geq 0$, $\|x(t;t_0,x_0)\| - r > 0$, it holds that

$$\lim_{t\to+\infty} \sup \frac{\ln(\|x(t;t_0,x_0)\| - r)}{\ln \alpha(t)} \leq 0, \quad a.s. \tag{2.6}$$

The systems (2.1) is said to be almost sure globally uniformly practically stable with respect to the decay function $\alpha(t)$, if there exists $r > 0$ such that the ball $\mathcal{B}_r$ is said to be almost sure globally uniformly practically stable with respect to the decay function $\alpha(t)$. Similarly, a solution $x(t) = (x_1(t), x_2(t))$ to system (2.1) is said to be almost sure partial practical stable with respect to x_1 with decay function $\alpha(t)$, if $x_1(t)$ is almost sure globally uniformly practically stable with respect to the decay function $\alpha(t)$ and $x_2(t)$ is globally uniformly bounded in probability.

In addition, if 0 is a solution to the equation, then the zero solution is said to be almost suer practically asymptotically stable with respect to the decay function $\alpha(t)$ and order at least $\gamma > 0$, if any solution to system (2.1) tends to the boundary of the ball $\mathcal{B}_r$ with respect to the decay function $\alpha(t)$ and order at least $\gamma > 0$, for all $r > 0$ is small enough.

Remark 2.1. Clearly, in the above definition, replacing the decay function $\alpha(t)$ with $O(e^t)$ leads to almost sure uniformly practically asymptotic exponential stability, which is studied in [8].

Below we will give the definition of globally uniformly bounded with probability one about the solutions.

Definition 2.5. The solution of system (2.1) is globally uniformly bounded with probability one, if for every $\beta > 0$, there exists $G > 0$ (G only related to β and independent of t_0), for all $t_0 \geq 0$ and all $x_0 \in \mathbb{R}^n$, if $\|x_0\| \leq \beta$ such that

$$\sup\{\|x(t,t_0,x_0)\| : t \geq t_0\} \leq G, a.s.$$

See reference [8] for details.

The following lemmas will play important roles in our derivation, which is a generalization of Caraballo's work in the Lemma 3.1 of [10]. We will prove that, under some conditions, if $x(t_0) = x_0 \neq 0$, then almost all sample paths of the solution of system (2.1) can never reach the origin.

(H2) We suppose that $H(x,z,t)$ always satisfies:

$$\nu\{z \in \hat{\mathcal{B}}_c, t \geq t_0, \exists x \neq 0, \quad \text{such that} \quad x + H(x,z,t) = 0\} = 0,$$

where $\mathcal{B}_c$ represents the ball with the origin as the center and c as the radius, and $\hat{\mathcal{B}}_c = \mathcal{B}_c - \{0\}$. We require that assumption **(H2)** holds for the rest of this paper.

Lemma 2.1. *Assume that, for any $\iota > 0$, there exist positive constants K_1 and K_2 such that for any $\|x\| < \iota$,*

$$\begin{aligned} &\|f(x,t)\| \vee \|g(x,t)\| \vee 2\int_{\|z\|<c} \|H(x,z,t)\| \left(\frac{\|x\| + \|H(x,z,t)\|}{\|x + H(x,z,t)\|}\right)\nu(dz) \\ \leq &K_1 \|x\| + K_2. \end{aligned} \tag{2.7}$$

If $x_0 \in \mathbb{R}^n$ and $x_0 \neq 0$, then

$$\mathbb{P}(x(t,t_0,x_0) \neq 0, \forall t \geq t_0) = 1.$$

This means that almost all sample paths of any solution beginning from a nonzero state will never arrive at the origin.

Proof. Suppose that the conclusion of the above lemma is false. It means that for some initial values $x_0 \neq 0$ of system (2.1), there will be a stopping time $\tau(\mathbb{P}\{\tau < +\infty\} > 0)$ when the solution of the system is zero for the first time:

$$\tau = \inf\{t \geq t_0 : x(t) = 0\}.$$

Let's simply write $x(t,t_0,x_0)$ as $x(t)$. Since the solution path of system (2.1) is almost sure right continuous, there exist $T > t_0$, $\zeta > 1$ such that $\mathbb{P}(\Theta) > 0$, where

$$\Theta = \{\omega \in \Omega : \tau \leq T \quad \text{and} \quad \|x(t)\| \leq \zeta - 1 \quad \text{for all} \quad t_0 \leq t \leq \tau\}.$$

According to the existing assumptions, there is a small normal number e such that for any $0 < e \leq \|x\| \leq \iota$, $t_0 \leq t \leq T$,

$$\|f(x,t)\| \vee \|g(x,t)\| \vee 2\int_{\|z\|<c} \|H(x,z,t)\| \left(\frac{\|x\| + \|H(x,z,t)\|}{\|x + H(x,z,t)\|}\right)\nu(\mathrm{d}z) \leq K_1 \|x\| + K_2.$$

Let $U(x) = \frac{1}{\|x\|}$. According to the definition of operator L and assumption **(H2)**, we have

$$\begin{aligned} LU(x) \leq &\frac{\|f(x,t)\|}{\|x\|^2} + \frac{\|g(x,t)\|^2}{\|x\|^3} \\ &+ 2\int_{\|z\|<c} \frac{\|H(x,z,t)\|}{\|x\|^2}\left(\frac{\|x\| + \|H(x,z,t)\|}{\|x + H(x,z,t)\|}\right)\nu(\mathrm{d}z) \\ \leq &\frac{2(K_1\|x\| + K_2)}{\|x\|^2} + \frac{(K_1\|x\| + K_2)^2}{\|x\|^3} \\ = &(2K_1 + K_1^2)U(x) + K_2\left(\frac{2}{\|x\|} + \frac{2K_1}{\|x\|} + \frac{K_2}{\|x\|^2}\right)U(x) \\ \leq &(2K_1 + K_1^2 + K_3)U(x), \end{aligned} \tag{2.8}$$

where $K_3 = K_2(\frac{2}{e} + \frac{2K_1}{e} + \frac{K_2}{e^2})$. Now, for any $\ell \in (0, \|x_0\|)$, define the stopping time

$$\tau_\ell = \inf\{t \geq t_0 : \|x\| \notin (\ell, \zeta)\}.$$

Applying Itô formula for (2.1) yields

$$\begin{aligned}
&\mathbb{E}[e^{-(2K_1+K_1^2+K_3)(\tau_\ell \wedge T - t_0)} U(x(\tau_\ell \wedge T))] \\
=&\mathbb{E}U(x(t_0)) + \mathbb{E}[\int_{t_0}^{\tau_\ell \wedge T} e^{-(2K_1+K_1^2+K_3)s}[L_2 U(x(s)) - (2K_1 + K_1^2 + K_3)U(x(s))]\mathrm{d}s],
\end{aligned}$$

by (2.8)

$$\mathbb{E}[e^{-(2K_1+K_1^2+K_3)(\tau_\ell \wedge T - t_0)} U(x(\tau_\ell \wedge T))] \leq \mathbb{E}U(x(t_0)).$$

We note that if $\omega \in \Theta$, then $\tau_\ell \leq T$ and $x(\tau_\ell) = \ell$. Then we can get from the above inequality:

$$\mathbb{E}[e^{-(2K_1+K_1^2+K_3)(T-t_0)} \ell^{-1} \chi_\Theta] \leq U(x_0).$$

Hence

$$\mathbb{P}(\Theta) \leq \frac{\ell}{\|x_0\|} e^{(2K_1+K_1^2+K_3)(T-t_0)}.$$

Letting $\ell \to 0$, we obtain that $\mathbb{P}(\Theta) = 0$, but this contradicts the definition of Θ. In summary, the proof of Lemma 2.1 is complete. □

Before we present the Exponential Martingale inequality, we will introduce two spaces $\mathfrak{F}(T)$ and $\mathfrak{F}(T, A)$:

$$\mathfrak{F}(T) = \left\{ F : [0, T] \times \Omega \to \mathbb{R}^n | \mathbb{P}\left[\int_0^T \|F(t)\|^2 \mathrm{d}t < \infty\right] = 1 \right\};$$

$$\mathfrak{F}(T, A) = \left\{ H : [0, T] \times A \times \Omega \to \mathbb{R}^n | \mathbb{P}\left[\int_0^T \int_A \|H(s, z)\|^2 \nu(\mathrm{d}z)\mathrm{d}s < \infty\right] = 1 \right\},$$

where A is a Borel set in $\mathbb{R}^n - \{0\}$.

Lemma 2.2. *(Exponential Martingale inequality) Let $F(t) \in \mathfrak{F}(T)$, $H(t, z) \in \mathfrak{F}(T, A)$, and T, ϵ, η be any positive numbers. Then the following inequality holds*

$$\mathbb{P}\left[\sup\{Y(t) | 0 \leq t \leq T\} > \eta\right] \leq e^{-\epsilon\eta}, \tag{2.9}$$

where

$$\begin{aligned}
Y(t) = &\int_0^t F(s)dW(s) - \frac{\epsilon}{2}\int_0^t \|F(s)\|^2 ds + \int_0^t \int_{\|z\|<c} H(s, z)\tilde{N}(ds, dz) \\
&- \frac{1}{\epsilon}\int_0^t \int_{\|z\|<c} \left[e^{\epsilon H(s,z)} - 1 - \epsilon H(s, z)\right] \nu(dz)ds.
\end{aligned}$$

The details of the Exponential Martingale inequality can be found in Theorem 1.7.4 of [2].

With the above preparation, we aim to seeking sufficient conditions about the partial practical stability of stochastic differential equations driven by Lévy noise.

3. Partial practical stability of stochastic differential equations driven by Lévy noise

In recent years, ones are interested in noise-driven stochastic differential equations (SDEs) with discontinuous jumps. Various stability problems of this kind of equations have been well studied, such as p-moment stability, probabilistic stability and almost sure exponential stability. Based on the definition of partial practical stability, we explore the sufficient conditions to guarantee the partial practical stability of the system (2.1), mainly using Exponential Martingale inequality, Borel-Cantelli Lemma and so forth. We are in a position to state the first result:

Theorem 3.1. *Suppose besides (2.7) that there are continuous function $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $G(t) > 0$, $h_1(t) \in \mathbb{R}$, $\mu(t) > 0$ and $h_2(t) \geq 0$, constants $p \in \mathbb{N}^+, m \geq 0, M \geq 0, \vartheta_1 \in \mathbb{R}, \vartheta_2 \geq 0$, such that for all $t \geq 0$ and $x = (x_1, x_2) \in \mathbb{R}^n$, the following conditions hold:*

(I) $\alpha(t)^m \|x_1\|^p \leq G(t)V(x,t)$ *and* $\lim\limits_{t\to+\infty}\sup \frac{\ln G(t)}{\ln \alpha(t)} = a$, $a \in \mathbb{R}$;

(II) $LV(x,t) \leq h_1(t)V(x,t) + \mu(t)$ *and* $\lim\limits_{t\to+\infty}\sup \frac{\int_0^t h_1(s)ds}{\ln \alpha(t)} \leq \vartheta_1$;

(III) $\lim\limits_{t\to+\infty}\sup \frac{t}{\ln \alpha(t)} = M$ *and* $\lim\limits_{t\to+\infty} \frac{\mu(t)}{\alpha(t)^m} = \sigma > 0$;

(IV) $\|V_x(x,t)g(x,t)\|^2 \geq h_2(t)V^2(x,t)$ *and* $\lim\limits_{t\to+\infty}\inf \frac{\int_0^t h_2(s)ds}{\ln \alpha(t)} \geq \vartheta_2$.

Let $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$ and $E\|x_0\|^2 < \infty$, such that the corresponding solution $x(t,t_0,x_0) = (x_1(t,t_0,x_0), x_2(t,t_0,x_0))$ with the initial value $x(t_0) = x_0 \in \mathbb{R}^n$ satisfies:

(1) $x_2(t,t_0,x_0)$ is globally uniformly bounded with probability one.

Then there is $\sigma_0 \geq \sigma > 0$, with $\|x_1(t,t_0,x_0)\| > \sigma_0$, such that the following formula holds

$$\lim_{t\to+\infty}\sup \frac{\ln(\|x_1(t)\| - (\sigma_0)^{\frac{1}{p}})}{\ln \alpha(t)} \leq -\gamma^*, a.s. \tag{3.1}$$

where $\gamma^* = \begin{cases} m - \vartheta_1 - \frac{3}{2}M - a, & M > \vartheta_2 \\ m - \vartheta_1 - a + \frac{1}{2}\vartheta_2 - M - \sqrt{M\vartheta_2}, & M \leq \vartheta_2 \end{cases}.$

Further, if $\gamma^* > 0$, the solution $x_1(t)$ of equation (2.1) is deemed to converge to the ball $\mathcal{B}_r := \{x \in \mathbb{R}^n : \|x\| \leq (\sigma_0)^{\frac{1}{p}}\}$ with respect to $x_1(t)$ with attenuation function $\alpha(t)$ and order at least γ^*. So the solution $x(t) = (x_1(t), x_2(t))$ of system (2.1) is almost sure partial practical stable with respect to x_1 with decay function $\alpha(t)$.

Proof. Without loss of generality, we assume that the initial moment is $t_0 = 0$, so for any standard initial value of $x_0 \in \mathbb{R}^n, x_0 \neq 0$, there exists a unique solution $x(t,0,x_0) = (x_1(t,0,x_0), x_2(t,0,x_0))$ of system (2.1), and lemma 2.1 states that the sample paths of solution $x(t,0,x_0)$ cannot return to the origin at any time, *i.e.*, $x(t) = (x_1(t), x_2(t)) \neq 0$, $\forall t \geq 0$ almost surely. In the following, we will simply set

$x(t, 0, x_0)$ as $x(t)$. We divide the proof into the following two cases:
case 1 Let $\mathfrak{D}_1 = \{t \in [0, +\infty) : \|x_1(t, 0, x_0)\|^p \leq \sigma_0\}$, then

$$\|x_1(t, 0, x_0)\| \leq (\sigma_0)^{\frac{1}{p}}, \quad t \in \mathfrak{D}_1.$$

case 2 Except for **case 1** above, for any $t \in \mathfrak{D}_2 := [0, +\infty) \setminus \mathfrak{D}_1$, such that $\|x_1(t, 0, x_0)\|^p > \sigma_0$. And then we know from condition (**III**),

$$\lim_{t\to+\infty} \frac{\mu(t)}{\alpha(t)^m} = \sigma > 0.$$

By $\sigma_0 \geq \sigma > 0$, there exists T_1 such that for any $t \geq T_1$, $\frac{\mu(t)}{\alpha(t)^m} \leq \sigma_0$, and thus we can obtain

$$\|x_1(t, t_0, x_0)\| > \frac{\mu(t)}{\alpha(t)^m}, \quad t \in [T_1, +\infty) \cap \mathfrak{D}_2.$$

Form condition (**I**) and $\mu(t) > 0$, we obtain

$$\alpha(t)^m \|x_1(t)\|^p - \mu(t) \leq \alpha(t)^m \|x_1(t)\|^p \leq G(t)V(x(t), t). \tag{3.2}$$

Let's study the formula $\alpha(t)^m \|x_1(t)\|^p - \mu(t)$, obviously

$$\begin{aligned}\alpha(t)^m \|x_1(t)\|^p - \mu(t) &= \alpha(t)^m \left(\|x_1(t)\|^p - \frac{\mu(t)}{\alpha(t)^m}\right) \\ &= \alpha^m(t) \left(\|x_1(t)\|^p - \left(\left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}}\right)^p\right),\end{aligned} \tag{3.3}$$

thanks to $p \in \mathbb{N}^*$

$$\begin{aligned}&\left(\|x_1(t)\|^p - \left(\left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}}\right)^p\right) \\ &= \left(\|x_1(t)\| - \left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}}\right) \sum_{n=1}^{p} \|x_1(t)\|^{p-n} \left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{n-1}{p}},\end{aligned} \tag{3.4}$$

by condition (**IV**)

$$\lim_{t\to+\infty} \frac{\mu(t)}{\alpha(t)^m} = \sigma.$$

By limit, we get: $\forall \theta \in (0, \sigma)$, $\exists T_2 \geq 0$

$$\frac{\mu(t)}{\alpha(t)^m} \geq \theta, \quad \forall t \geq T_2. \tag{3.5}$$

Again by condition (**I**)

$$\|x_1(t)\| > \left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}}, \tag{3.6}$$

we get

$$\|x_1(t)\|^{p-n} \geq \left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{p-n}{p}} \geq \theta^{\frac{p-n}{n}}. \tag{3.7}$$

It follows from (3.5), (3.6) and (3.7) that

$$\begin{aligned}\sum_{n=1}^{p}\|x_1(t)\|^{p-n}\left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{n-1}{p}} =&\|x_1(t)\|^{p-1}+\|x_1(t)\|^{p-2}\left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}}\\ &+\|x_1(t)\|^{p-3}\left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{2}{p}}+\ldots+\left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{p-1}{p}}\\ \geq& p\theta^{\frac{p-1}{p}},\quad \forall t\geq T. \end{aligned}\tag{3.8}$$

Let $\tilde{\theta}=p\theta^{\frac{p-1}{p}}$, by (3.3) and (3.8),

$$\alpha(t)^m\|x_1(t)\|^p-\mu(t)\geq\alpha(t)^m\left(\|x_1(t)\|-\left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}}\right)\tilde{\theta}. \tag{3.9}$$

Form (3.2), we obtain

$$\alpha(t)^m\left(\|x_1(t)\|-\left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}}\right)\tilde{\theta}\leq\alpha(t)^m\|x_1(t)\|^p-\mu(t)\leq G(t)V(x(t),t). \tag{3.10}$$

Because of the basic form of $LV(x)$ and the Itô formula (see [26]), for the solution $x(t)$ of system (2.1), we have

$$V(x(t),t)=V(x(0),0)+M_t+\int_0^t LV(x(s),s)\mathrm{d}s, \tag{3.11}$$

where

$$\begin{aligned}M_t=&\int_0^t V_x(x(s),s)g(x(s),s)\mathrm{d}W(s)\\ &+\int_0^t\int_{\|z\|<c}\left[V(x(s)+H(x(s-),z,s),s)-V(x(s),s)\right]\tilde{N}(\mathrm{d}s,\mathrm{d}z),\end{aligned}$$

and

$$\begin{aligned}LV(x,t)=&V_t(x,t)+V_x(x,t)f(x,t)+\int_{\|z\|<c}V(x+H(x,z,t),t)-V(x,t)\\ &-H(x,z,t)V_x(x,t)\nu(\mathrm{d}z)+\frac{1}{2}\mathrm{trace}\left[g^T(x,t)V_{xx}(x,t)g(x,t)\right].\end{aligned}\tag{3.12}$$

Next, let use the Itô formula to study $\ln(V(x(t),t))$,

$$\begin{aligned}&\ln(V(x(t),t))\\ =&\ln(V(x(0),0))+\int_0^t(\frac{V_t(x(s),s)}{V(x(s),s)}+\frac{V_x(x(s),s)f(x(s),s)}{V(x(s),s)})\mathrm{d}s\\ &+\int_0^t\int_{\|z\|<c}\ln(V(x(s)+H(x(s-),z,s),s)-\ln V(x(s),s))\\ &-\frac{H(x(s-),z,s)V_x(x(s),s)}{V(x(s),s)}\nu(\mathrm{d}z)\mathrm{d}s-\frac{1}{2}\int_0^t\frac{\|V_x(x(s),s)g(x(s),s)\|^2}{V^2(x(s),s)}\mathrm{d}s\end{aligned}$$

$$+ \frac{1}{2V(x(s),s)} \text{trace}\left[g^T(x(s),s)V_{xx}(x(s),s)g(x(s),s)\right] \mathrm{d}s$$
$$+ \int_0^t \int_{\|z\|<c} \left[\ln(V(x(s)+H(x(s-),z,s),s)) - \ln(V(x(s),s))\right] \tilde{N}(\mathrm{d}s,\mathrm{d}z)$$
$$+ \int_0^t \frac{V_x(x(s),s)g(x(s),s)}{V(x(s),s)} \mathrm{d}W(s). \tag{3.13}$$

Now, using the L defined in (3.12), we obtain

$$\begin{aligned}\ln(V(x(t),t)) =& D + \int_0^t \frac{LV(x(s),s)}{V(x(s),s)} \mathrm{d}s + \tilde{M}_t - \frac{1}{2}\int_0^t \frac{\|V_x(x(s),s)g(x(s),s)\|^2}{V^2(x(s),s)} \mathrm{d}s \\ &- \int_0^t \int_{\|z\|<c} \frac{V(x(s)+H(x(s-),z,s),s) - V(x(s),s)}{V(x(s),s)} \nu(\mathrm{d}z)\mathrm{d}s \\ &+ \int_0^t \int_{\|z\|<c} \ln \frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)} \nu(\mathrm{d}z)\mathrm{d}s,\end{aligned} \tag{3.14}$$

where $D = \ln(V(x(0),0))$ is a constant value and

$$\begin{aligned}\tilde{M}_t =& \int_0^t \frac{V_x(x(s),s)g(x(s),s)}{V(x(s),s)} dW(s) \\ &+ \int_0^t \int_{\|z\|<c} [\ln(V(x(s)+H(x(s-),z,s),s)) \\ &- \ln((V(x(s),s))]\tilde{N}(\mathrm{d}s,\mathrm{d}z).\end{aligned} \tag{3.15}$$

The main task of the following is to study $\tilde{M}_t$. We choose the standard initial value $x(0) = x_0$ and guarantee that $E\|x_0\|^2 < \infty$. We can use the Exponential Martingale inequality, letting $T = k$, $\epsilon = \varepsilon$, $\eta = \frac{k-1}{2\varepsilon}$ and $\varepsilon \in (0,1)$, $k \in \mathbb{N}^+$ and $k > 1$. Then we can get naturally that for any $k > 1$,

$$P\left[\sup\left\{\tilde{M}_t - Y(t,\varepsilon)|0 \le t \le k\right\} > \frac{k-1}{2\varepsilon}\right] \le e^{-\frac{1}{2}(k-1)},$$

where

$$\begin{aligned}Y(t,\varepsilon) =& \frac{\varepsilon}{2}\int_0^t \frac{\|V_x(x(s),s)g(x(s),s)\|^2}{V^2(x(s),s)} \mathrm{d}s + \frac{1}{\varepsilon}\int_0^t \int_{\|z\|<c} \left(\frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)}\right)^\varepsilon \\ &- 1 - \varepsilon \ln \frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)} \nu(\mathrm{d}z)\mathrm{d}s.\end{aligned}$$

Since

$$\sum_{k=2}^{\infty} e^{-\frac{1}{2}(k-1)} < \infty,$$

by Borel-Cantelli Lemma, we see that for almost all $\omega \in \Omega, \exists \tilde{k} > 0$, where $\tilde{k}$ only related to $\omega \in \Omega$. We have

$$P\left[\lim_{k\to+\infty} \inf(\sup\left\{\tilde{M}_t - Y(t,\varepsilon)|0 \le t \le k\right\}) \le \frac{k-1}{2\varepsilon}\right] = 1,$$

i.e., for all $k-1 \le t \le k, k \ge \tilde{k}$,

$$\begin{aligned}\tilde{M}_t \le & \frac{\varepsilon}{2}\int_0^t \frac{\|V_x(x(s),s)g(x(s),s)\|^2}{V^2(x(s),s)}\mathrm{d}s \\ & + \frac{1}{\varepsilon}\int_0^t \int_{\|z\|<c} (\frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)})^{\varepsilon} \\ & - 1 - \varepsilon \ln \frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)}\nu(\mathrm{d}z)\mathrm{d}s + \frac{k-1}{2\varepsilon}, \quad a.s.\end{aligned} \tag{3.16}$$

We know that the following inequality holds for all $n \in (0,1)$, $m > 0$,

$$m^n < 1 + n(m-1). \tag{3.17}$$

Let

$$\begin{aligned}P(\varepsilon,x,z,t) = & \frac{1}{\varepsilon}[(\frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)})^{\varepsilon} - 1 \\ & - \varepsilon \ln \frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)}],\end{aligned} \tag{3.18}$$

and let $m = \frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)}$, $n = \varepsilon$. Then by formula (3.17) and the previous assumption, we get

$$(\frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)})^{\varepsilon} \le 1 + \varepsilon(\frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)} - 1),$$

we easily deduce that

$$\begin{aligned}P(\varepsilon,x,z,t) \le & \frac{1}{\varepsilon}[1 + \varepsilon(\frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)} - 1) - 1 \\ & - \varepsilon \ln \frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)}] \\ = & \frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)} - 1 \\ & - \ln \frac{V(x(s)+H(x(s-),z,s),s)}{V(x(s),s)}.\end{aligned} \tag{3.19}$$

Substituting (3.15), (3.16), (3.18) and (3.19) into (3.14) gives

$$\begin{aligned}\ln(V(x(t),t)) \le & D + \int_0^t \frac{LV(x(s),s)}{V(x(s),s)}\mathrm{d}s - \frac{1}{2}\int_0^t \frac{\|V_x(x(s),s)g(x(s),s)\|^2}{V^2(x(s),s)}\mathrm{d}s \\ & + \frac{\varepsilon}{2}\int_0^t \frac{\|V_x(x(s),s)g(x(s),s)\|^2}{V^2(x(s),s)}\mathrm{d}s + \frac{k-1}{2\varepsilon},\end{aligned} \tag{3.20}$$

by conditions (**II**) and (**IV**)

$$\ln(V(x(t),t)) \le D + \int_0^t h_1(s)\mathrm{d}s - \frac{1-\varepsilon}{2}\int_0^t h_2(s)\mathrm{d}s + \frac{1+2\varepsilon}{2\varepsilon}t;$$

further, for all $k-1 \le t \le k, k \ge \tilde{k}$,

$$\lim_{t\to+\infty} \sup \frac{\ln(V(x(t),t))}{\ln\alpha(t)} \le \vartheta_1 - \frac{(1-\varepsilon)}{2}\vartheta_2 + \frac{1+2\varepsilon}{2\varepsilon}M. \tag{3.21}$$

By (3.10), it is not difficult to get

$$m\ln\alpha(t) + \ln(\|x_1(t)\| - \left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}}) + \ln\tilde{\theta} \le \ln G(t) + \ln(V(x(t),t)),$$

further

$$\frac{\ln(\|x_1(t)\| - \left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}})}{\ln\alpha(t)} \le -m - \frac{\ln\tilde{\theta}}{\ln\alpha(t)} + \frac{\ln(V(x(t),t))}{\ln\alpha(t)} + \frac{\ln G(t)}{\ln\alpha(t)}.$$

By (3.21) and the property of $\alpha(t)$, letting $k \to +\infty$ yields

$$\lim_{t\to+\infty} \sup \frac{\ln(\|x_1(t)\| - \left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}})}{\ln\alpha(t)} \le -(m-\vartheta_1+\frac{(1-\varepsilon)}{2}\vartheta_2-\frac{1+2\varepsilon}{2\varepsilon}M-a). \tag{3.22}$$

Let $\gamma(\varepsilon) = m - \vartheta_1 + \frac{(1-\varepsilon)}{2}\vartheta_2 - \frac{1+2\varepsilon}{2\varepsilon}M - a$, which shows that the decaying order depends on the parameter ε. The next main task is to find the optimal valued $\gamma^* = \sup_{\varepsilon\in(0,1)} \gamma(\varepsilon)$. Note

$$\frac{d\gamma(\varepsilon)}{d\varepsilon} = \frac{1}{2\varepsilon^2}M - \frac{1}{2}\vartheta_2, \tag{3.23}$$

which implies that

$$\gamma^* = m - \vartheta_1 - \frac{3}{2}M - a, M > \vartheta_2, \tag{3.24}$$

$$\gamma^* = m - \vartheta_1 - a + \frac{1}{2}\vartheta_2 - M - \sqrt{M\vartheta_2}, M \le \vartheta_2. \tag{3.25}$$

Finally, we use the condition (**III**) to obtain

$$\lim_{t\to+\infty} \frac{\mu(t)}{\alpha(t)^m} = \sigma \le \sigma_0.$$

Taking limit, we get that when $t \ge T_1$,

$$\frac{\mu(t)}{\alpha(t)^m} \le \sigma_0,$$

hence

$$\lim_{t\to+\infty} \sup \frac{\ln(\|x_1(t)\| - (\sigma_0)^{\frac{1}{p}})}{\ln\alpha(t)} \le \lim_{t\to+\infty} \sup \frac{\ln(\|x_1(t)\| - \left(\frac{\mu(t)}{\alpha(t)^m}\right)^{\frac{1}{p}})}{\ln\alpha(t)} \le -\gamma^*.$$

Thus if $\gamma^* > 0$, letting $r = (\sigma_0)^{\frac{1}{p}}$, we get

$$\lim_{t\to+\infty} \sup \frac{\ln(\|x_1(t)\| - r)}{\ln\alpha(t)} \le -\gamma^*, a.s., \tag{3.26}$$

where

$$\gamma^* = \begin{cases} m - \vartheta_1 - \frac{3}{2}M - a, & M > \vartheta_2, \\ m - \vartheta_1 - a + \frac{1}{2}\vartheta_2 - M - \sqrt{M\vartheta_2}, & M \leq \vartheta_2. \end{cases}$$

From (3.26) and **case 1**, it is not difficult to obtain that there exists time $\widetilde{T} \geq \max\{T_1, T_2\}$, such that when $t \geq \widetilde{T}$, then the ball $\mathcal{B}_r = \{x \in \mathbb{R}^n : \|x\| \leq (\sigma_0)^{\frac{1}{p}}\}$ is said to be surely globally practically uniformly stable with respect to the decay function $\alpha(t)$, *i.e.* the solution $x(t)$ of system (2.1) converges to the ball $\mathcal{B}_r := \{x \in \mathbb{R}^n : \|x\| \leq (\sigma_0)^{\frac{1}{p}}\}$ with respect to $x_1(t)$ with attenuation function $\alpha(t)$ and order at least γ^*. Finally, from condition (1), according to Definition 2.4, the solution $x(t)$ of system (2.1) almost sure partial practical stable with respect to x_1 with decay function $\alpha(t)$. □

Remark 3.1. To sum up, by establishing a suitable Lyapunov function and using Exponential martingale inequality and Borel-Cantelli theorem, we have sufficient conditions that can guarantee the almost suer partial practical stability of stochastic differential equations driven by Lévy noise. There have remained two important problems: one is the selection of the attenuation function, the other is whether the Lyapunov function used in the theorem exists. The first problem is generally easy to solve. For example, we can choose $\alpha(t) = O(e^t)$ or $\alpha(t) = O(\ln(t+1))$, so we can get partial practical exponential stability or partial practical logarithmic stability.

Remark 3.2. When we study the almost sure partial practical stability for system (2.1), in general, it is difficult to find suitable Lyapunov functions. In order to find a more appropriate Lyapunov function, further thinking, let $Q : \mathbb{R}^1_+ \to \mathbb{R}^{n\times n}$ be a C^1 positive-definite function with $Q(t) = Q(t)^T$. We take the Lyapunov function as $V(x,t) = x^T Q(t)x$, where $x \in \mathbb{R}^n$. Let $C^1(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}^n)$ denote the family of all nonnegative functions $f(x,t)$ on $\mathbb{R}^n \times \mathbb{R}_+$ which are once differentiable in x.

Corollary 3.1. *Suppose besides (2.7) and $f \in C^1(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}^n)$ that there are continuous functions $G(t) \geq 0$, $\varphi_{1,2}(t) \in \mathbb{R}$, $\mu(t) > 0$ and $h_2(t) \geq 0$, constants $p = 2$, $m \geq 0$, $M \geq 0$, $\vartheta_1 \in \mathbb{R}$, $\vartheta_2 \geq 0$, such that for all $t \geq 0$ and $x = (x_1, x_2) \in \mathbb{R}^n$, the following conditions hold:*

(1) $\dot{Q}(t) + \frac{\partial f^T(x,t)}{\partial x} Q(t) + Q(t)\frac{\partial f(x,t)}{\partial x} \leq \varphi_1(t)Q(t);$

(2) $trace\left[g^T(x,t)Q(t)g(x,t)\right] + \int_{\|z\|<c} H^T(x,z,t)Q(t)H(x,z,t)\nu(dz)$
$+ 2f^T(0,t)Q(t)x(t) \leq \varphi_2(t)x^T Q(t)x + \mu(t);$

(3) $\left\|x^T Q(t)g(x,t)\right\|^2 \geq h_2(t)\left\|x^T Q(t)x\right\|^2$ *and* $\lim_{t\to+\infty}\inf \frac{\int_0^t h_2(s)ds}{\ln\alpha(t)} \geq \vartheta_2;$

(4) $\lim_{t\to+\infty}\sup \frac{\int_0^t(\varphi_1(s)+\varphi_2(s))ds}{\ln\alpha(t)} \leq \vartheta_1$ *and* $\lim_{t\to+\infty}\inf \frac{\int_0^t Q(s)ds}{\ln\alpha(t)} \geq b$, $b \in \mathbb{R}$,

where $\|Q(t)\|$ is the determinant of the matrix $Q(t)$ at time t, and assumptions (III), (1) of Theorem 3.1 are satisfied. Then the conclusion in Theorem 3.1 holds.

Proof. Let $V(x,t) = x^T Q(t)x$. It is obvious that $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, and by Definition 2.1, we obtain

$$V_t(x,t) = x^T \dot{Q}(t)x,$$

$$V_x(x,t)f(x,t) = x^T Q(t) f(x,t) + f^T(x,t)Q(t)x,$$
$$V_{xx}(x,t) = 2Q(t),$$

and

$$\begin{aligned}
LV(x,t) =& V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}\mathrm{trace}\left[g^T(x,t)V_{xx}(x,t)g(x,t)\right] \\
&+ \int_{\|z\|<c} (V(x+H(x,z,t),t) - V(x,t) - H(x,z,t)V_x(x,t))\nu(\mathrm{d}z) \\
=& x^T\dot{Q}(t)x + x^T Q(t)f(x,t) + f^T(x,t)Q(t)x \\
&+ \mathrm{trace}\left[g^T(x,t)Q(t)g(x,t)\right] \\
&+ \int_{\|z\|<c} H^T(x,z,t)Q(t)H(x,z,t)\nu(\mathrm{d}z).
\end{aligned} \tag{3.27}$$

Now let's analyze equation (3.27):

$$\begin{aligned}
LV(x(t),t) =& x^T(t)\dot{Q}(t)x(t) + x^T(t)Q(t)\left[f(x(t),t) - f(0,t)\right] \\
&+ \left[f(x(t),t) - f(0,t)\right]^T Q(t)x(t) + \mathrm{trace}\left[g^T(x(t),t)Q(t)g(x(t),t)\right] \\
&+ \int_{\|z\|<c} H^T(x,z,t)Q(t)H(x,z,t)\nu(\mathrm{d}z) + 2f^T(0,t)Q(t)x(t).
\end{aligned}$$

Since $f \in C^1(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}^n)$, by Lagrange's mean value theorem, $\exists \varepsilon_t \in (0, x(t))$, such that

$$f(x(t),t) - f(0,t) = f_x(\varepsilon_t x(t), t)x(t),$$

where $f_x(x,t,i) = \frac{\partial f(x,t,i)}{\partial x}$, we obtain

$$\begin{aligned}
LV(x(t),t) =& x^T(t)\dot{Q}(t)x(t) + x^T(t)Q(t)f_x(\varepsilon_t x(t),t)x(t) \\
&+ x^T(t)Q_i(t)f_x^T(\varepsilon_t x(t),t)x(t) + \mathrm{trace}\left[g^T(x(t),t)Q(t)g(x(t),t)\right] \\
&+ \int_{\|z\|<c} H^T(x,z,t)Q(t)H(x,z,t)\nu(\mathrm{d}z) + 2f^T(0,t)Q(t)x(t).
\end{aligned}$$

It suffices to show conditions (**1**) and (**2**). We have

$$LV(x(t),t) \le (\varphi_1(t) + \varphi_2(t))V(x(t),t) + \mu(t).$$

We can make $G(t) = \frac{\alpha(t)^m}{Q(t)}$. To sum up, Corollary 3.1 satisfies all conditions of Theorem 3.1, which allows us to conclude that

$$\lim_{t\to+\infty} \sup \frac{\ln(\|x_1(t)\| - r)}{\ln \alpha(t)} \le -\gamma^*, \quad a.s.,$$

where

$$\gamma^* = \begin{cases} b - \vartheta_1 - \frac{3}{2}M, & M > \vartheta_2, \\ b - \vartheta_1 + \frac{1}{2}\vartheta_2 - M - \sqrt{M\vartheta_2}, & M \le \vartheta_2. \end{cases}$$

Thus if $\gamma^* > 0$, the solution $x(t)$ of system (2.1) almost sure partial practical stable with respect to x_1 with decay function $\alpha(t)$. □

Remark 3.3. In Theorem 3.1, when $\mu(t) = 0$, we can get the almost sure asymptotic stability of stochastic differential equations driven by Lévy noise with a general decay rate. The specific content is as follows.

Theorem 3.2. *Suppose besides **(H1)** that there are continuous functions $V(x,t) \in C^{2,1}(\mathbb{R}^n;\mathbb{R}_+)$, $G(t) \geq 0$, $h_1(t) \in \mathbb{R}$ and $h_2(t) \geq 0$, constants $p \in \mathbb{N}^+$, $m \geq 0$, $M \geq 0$, $\vartheta_1 \in \mathbb{R}$, $\vartheta_2 \geq 0$, such that for all $t \geq t_0 \geq 0$ and $x \in \mathbb{R}^n$, the following conditions hold:*

(I) $\alpha(t)^m \|x\|^2 \leq G(t)V(x,t)$ *and* $\lim_{t\to+\infty} \sup \frac{\ln G(t)}{\ln \alpha(t)} = a$, $a \in \mathbb{R}$;

(II) $LV(x,t) \leq h_1(t)V(x,t)$ *and* $\lim_{t\to+\infty} \sup \frac{\int_0^t h_1(s)ds}{\ln \alpha(t)} \leq \vartheta_1$;

(III) $\lim_{t\to+\infty} \sup \frac{t}{\ln \alpha(t)} = M$;

(IV) $\|V_x(x,t)g(x,t)\|^2 \geq h_2(t)V^2(x,t)$ *and* $\lim_{t\to+\infty} \inf \frac{\int_0^t h_2(s)ds}{\ln \alpha(t)} \geq \vartheta_2$.

Let $x_0 \in \mathbb{R}^n, x_0 \neq 0$. Then

$$\lim_{t\to+\infty} \sup \frac{\ln(\|x(t)\|)}{\ln \alpha(t)} < -\beta^*, a.s., \tag{3.28}$$

where

$$\beta^* = \begin{cases} \dfrac{1}{p}(m - \vartheta_1 - \frac{3}{2}M - a), \quad M > \vartheta_2, \\ \dfrac{1}{p}(m - \vartheta_1 - a + \frac{1}{2}\vartheta_2 - M - \sqrt{M\vartheta_2}), \; M \leq \vartheta_2. \end{cases}$$

Further, if $\beta^* > 0$, the solution $x(t)$ of (2.1) is deemed to converge to zero with attenuation function $\alpha(t)$ and order at least β^*.

Proof. For any $x_0 \neq 0$, if the system meets (2.7), we have a unique solution interval $x(t,t_0,x_0)$ and we can guarantee that $\mathbb{P}(x(t,t_0,x_0) \neq 0, \forall t \geq t_0) = 1$ by Lemma 2.1 By (**I**) of Theorem 3.2

$$\alpha(t)^m \|x(t)\|^p \leq G(t)V(x(t),t).$$

Apply Itô formula $\ln(V(x(t)))$, then, for each $t \geq t_0$,

$$\begin{aligned}
&\ln(V(x(t),t)) \\
=&\ln(V(x(t_0),t_0)) + \int_{t_0}^t (\frac{V_t(x(s),s)}{V(x(s),s)} + \frac{V_x(x(s),s)f(x(s),s)}{V(x(s),s)})\mathrm{d}s \\
&+ \int_{t_0}^t \int_{\|z\|<c} \ln(V(x(s) + H(x(s-),z,s),s)) - \ln(V(x(s),s)) \\
&- \frac{H(x(s-),z,s)V_x(x(s),s)}{V(x(s),s)}\nu(\mathrm{d}z)\mathrm{d}s - \frac{1}{2}\int_{t_0}^t \frac{\|V_x(x(s),s)g(x(s),s)\|^2}{V^2(x(s),s)}\mathrm{d}s \\
&+ \frac{1}{2V(x(s),s)}\mathrm{trace}\left[g^T(x(s),s)V_{xx}(x(s),s)g(x(s),s)\right]\mathrm{d}s \\
&+ \int_{t_0}^t \int_{\|z\|<c} \left[\ln(V(x(s) + H(x(s-),z,s),s)) - \ln(V(x(s),s))\right] \tilde{N}(\mathrm{d}s,\mathrm{d}z) \\
&+ \int_{t_0}^t \frac{V_x(x(s),s)g(x(s),s)}{V(x(s),s)}\mathrm{d}W(s).
\end{aligned}$$

Similar to (3.13)-(3.20), we have that

$$V(x(t),t) \leq D + \int_0^t h_1(s)\mathrm{d}s - \frac{1-\varepsilon}{2}\int_0^t h_2(s)\mathrm{d}s + \frac{1+2\varepsilon}{2\varepsilon}t.$$

From Condition (II), (III), (IV) and (3.21)-(3.25), we obtain

$$\lim_{t\to+\infty}\sup \frac{\ln(\|x(t)\|)}{\ln \alpha(t)} \leq -\beta^*,$$

where

$$\beta^* = \begin{cases} \dfrac{1}{p}(m-\vartheta_1-\frac{3}{2}M-a), & M > \vartheta_2, \\ \dfrac{1}{p}(m-\vartheta_1-a+\frac{1}{2}\vartheta_2-M-\sqrt{M\vartheta_2}), & M \leq \vartheta_2. \end{cases}$$

In the case where (2.7) fails to hold, we may assume without loss of generality that $H \neq 0$ and that $x(t,t_0,x_0)$ can reach zero. So to avoid some unnecessary trouble, we need to make some technical corrections to the solution $x(t)$ without affecting the final result. We may assume that the process $x(t)$ has N(N can be infinite) times to reach the far point (with probability one). We define the following stopping times:

$$\begin{aligned} \tau_1 &= \inf\left\{t \geq t_0 : x(t) = 0\right\}, \\ \tau_2 &= \inf\left\{t > \tau_1 : x(t) = 0\right\}, \\ &\cdots\cdots \\ \tau_n &= \inf\left\{t > \tau_{n-1} : x(t) = 0\right\}. \end{aligned}$$

Let

$$\tilde{x}(t) = \chi_{[t_0,\tau_1)}(t)x(t) + \sum_{n=1}^{N} \chi_{(\tau_{n-1},\tau_n)}(t)x(t),$$

which yields that

$$\mathbb{P}(\tilde{x}(t) \neq 0, \forall t \geq t_0) = 1,$$

and then finally we obtain

$$\lim_{t\to+\infty}\sup \frac{\ln(\|\tilde{x}(t)\|)}{\ln \alpha(t)} \leq -\beta^*.$$

□

4. Examples

In order to illustrate the validity of our main results, two examples are provided.
Example 4.1. Consider the following stochastic differential equation driven by

Lévy noise noise:

$$\begin{cases} \mathrm{d}x_1 = f_1(x_1(t), x_2(t), t)\mathrm{d}t + g_1(x_1(t), x_2(t), t)\mathrm{d}W(t) \\ \qquad + \int_{\|z\|<1} H_1(x_1(t-), x_2(t-), z, t)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \\ \mathrm{d}x_2 = f_2(x_1(t), x_2(t), t)\mathrm{d}t + g_2(x_1(t), x_2(t), t)\mathrm{d}W(t) \\ \qquad + \int_{\|z\|<1} H_2(x_1(t-), x_2(t-), z, t)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \end{cases} \tag{4.1}$$

with the initial condition $x(t_0) = x_0 = (x_{10}, x_{20})$, $f = (f_1, f_2)$, $g = (g_1, g_2)$, $H = (H_1, H_2)$ and $W(t)$ is a one dimensional Brownian motion, N is a Poisson random measure defined on $\mathbb{R}_+ \times (\mathbb{R} - \{0\})$ with compensator $\tilde{N}$ and intensity measure $\nu(\cdot)$. It is always assumed that N is independent of W. Let $\nu(\mathrm{d}z) = \frac{1}{1+z^2}\mathrm{d}z$ and

$$\begin{aligned} &f_1(x_1, x_2, t) = k_1 t x_1 + e^{-x_2}, \quad g_1(x_1, x_2, t) = k_2 t^{\frac{1}{2}} x_1, \quad H_1(x_1, x_2, z, t) = \|x_1\| z^2, \\ &f_2(x_1, x_2, t) = k_3 x_2 - \cos^2(x_1) x_2, \quad g_2(x_1, x_2, t) = \sqrt{2}\cos(x_1) x_2, \\ &H_2(x_1, x_2, z, t) = \|x_2\| z^2, \end{aligned}$$

where $k_{1,2,3} \in \mathbb{R}$. From the above definition, it can be easily deduced that system (4.1) satisfies the equation (2.7). Let $V := \alpha(t)^m x_1^2$, *i.e.*, $Q(t) = \alpha(t)^m$. And Taking $\alpha(t) = e^{t^2}$, $p = 2$ and $m = 0$, we have
Step1: by (**I**) of Theorem 3.1, $G(t) = 1$ and $a = 0$.
Step2:

$$\begin{aligned} LV(x,t) =& V_t(x,t) + V_x(x,t) f_1(x,t) + \frac{1}{2}\mathrm{trace}\left[g_1^T(x,t) V_{xx}(x,t) g_1(x,t)\right] \\ &+ \int_{\|z\|<c} (V(x + H_1(x,z,t), t) - V(x,t) - H_1(x,z,t) V_x(x,t))\nu(\mathrm{d}z) \\ =& 2x_1(k_1 t x_1 + e^{-x_2}) + k_2^2 t x_1^2 + \int_{\|z\|<1} \frac{x_1^2 z^4}{1+z^2}\mathrm{d}z \\ \le& (2k_1 t + k_2^2 t + \frac{3\pi - 8}{6}) x_1^2 + 2\|x_1\| \\ \le& (2k_1 t + k_2^2 t + \frac{3\pi - 2}{6}) x_1^2 + 1, \end{aligned}$$

which implies that $h_1(t) = 2k_1 t + k_2^2 t + \frac{3\pi-2}{6}$ and $\mu(t) = 1$, so $\sigma = 1$.
Step3: $V_x(x,t) g_1(x,t) = 2k_2 t^{\frac{1}{2}} x^2$, by (**III**) of Theorem 3.1, we have $h_2(t) = 4k_2^2 t$.

We can easily obtain that $\vartheta_1 = k_1 + \frac{1}{2}k_2^2$, $\vartheta_2 = 2k_2^2$ and $M = 0$ by the conditions of Theorem 3.1 and the proof procedure, hence we have $\gamma^* = \frac{1}{2}k_2^2 - k_1$.
Step4: The following task is to investigate whether the following system is globally uniformly bounded with probability:

$$\mathrm{d}x_2 = [k_3 x_2 - \cos^2(x_1) x_2]\mathrm{d}t + \sqrt{2}\cos(x_1) x_2 \mathrm{d}W(t) + \int_{\|z\|<1} \|x_2\| z^2 \tilde{N}(\mathrm{d}t, \mathrm{d}z).$$

Let $V(x_2, t) = x_2^2$, and we can get

$$LV(x_2,t) = V_t(x,t) + V_x(x,t) f_1(x,t) + \frac{1}{2}\mathrm{trace}\left[g_1^T(x,t) V_{xx}(x,t) g_1(x,t)\right]$$

$$+\int_{\|z\|<c}(V(x+H_1(x,z,t),t)-V(x,t)-H_1(x,z,t)V_x(x,t))\nu(\mathrm{d}z)$$
$$=2k_3x_2^2-2\cos^2(x_1)x_2^2+2\cos^2(x_1)x_2^2+\frac{3\pi-8}{6}x_2^2=(2k_3+\frac{3\pi-8}{6})x_2^2,$$

and

$$\int_{\|z\|<1}\|H_2(x,z,t)\|^2\nu(\mathrm{d}z)=\int_{\|z\|<1}\frac{x_2^2z^4}{1+z^2}\mathrm{d}z=(\frac{3\pi-8}{6})x_2^2.$$

We can chose $k_3 \le \frac{8-3\pi}{12}$ such that $LV(x_2,t)\le 0$. So $x_2(t)$ is globally uniformly stable in probability by Theorem 2.3 in [3], which in turn implies $x_2(t)$ is globally uniformly bounded with probability one.

To sum up, we choose $k_1=\frac{3}{2}$, $k_2=3$, $k_3=-2$, which we can obtain $\gamma^*=3>0$, hence we obtain that the solution $x(t)$ of equation (4.1) is deemed to converge to the ball $\mathcal{B}_r$ with $r=1$ almost surely with respect to $x_1(t)$ with attenuation function e^{t^2} and order at least 3, *i.e.*, the solution $x(t)$ of equation (4.1) is almost sure partial practical stable with respect to $x_1(t)$ with attenuation function e^{t^2} by Theorem 3.1.

Example 4.2. Consider the following nonlinear stochastic differential equation driven by Lévy noise:

$$\mathrm{d}x(t)=f(x(t),t)\mathrm{d}t+g(x(t),t)\mathrm{d}W(t)+\int_{\|z\|<c}H(x(t-),z,t)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \tag{4.2}$$

where

$$f(x(t),t)=d_1(t)\sin x(t),\qquad g(x(t),t)=d_2(t)x(t),$$
$$H(x(t),z,t)=r(t)x(t)U(z,t),$$

with the initial condition $x(t_0)=x_0$ and $W(t)$ is a one dimensional Brownian motion, N is a Poisson random measure defined on $\mathbb{R}_+\times(\mathbb{R}-\{0\})$ with compensator $\tilde{N}$ and intensity measure $\nu(\cdot)$. We need to make the following assumptions:

(1) $\int_{\|z\|<c}\|U(z,t)\|^2\nu(\mathrm{d}z)=L_1(t)$ and $\int_{\|z\|<c}\|U(z,t)\|\nu(\mathrm{d}z)=L_2(t)$;

(2) $\lim\limits_{t\to+\infty}\sup\frac{\int_0^t 2d_1(s)+d_2^2(s)+r^2(s)L_1(s)\mathrm{d}s}{\ln\alpha(t)}\le\widetilde{\vartheta}_1$ and $\lim\limits_{t\to+\infty}\inf\frac{\int_0^t d_2^2(s)\mathrm{d}s}{\ln\alpha(t)}\ge\widetilde{\vartheta}_2$;

where $d_1(t)$, $d_2(t)$, $r(t)$, $L_1(t)$, $L_2(t)$ are continuous functions and $\widetilde{\vartheta}_1\in\mathbb{R}$, $\widetilde{\vartheta}_1\ge 0$.

Based on the above assumptions, we can obtain the following corollary.

Corollary 4.1. *Let $x_0\in\mathbb{R}^n$, $x_0\ne 0$ and $E\|x_0\|^2<\infty$. The solution with initial value $x(t_0)=x_0(x_0\ne 0)$ of system (4.2) is deemed to converge to zero with attenuation function e^{t^2} and order at least $\beta^*=\widetilde{\vartheta}_2-\frac{1}{2}\widetilde{\vartheta}_1$.*

Proof. Using the above assumptions, it is easy to prove that system (4.2) satisfies conditions (H1). Let $V(x,t)=x^TQx$, where $x\in\mathbb{R}^n$ and Q is an n dimensional symmetric positive-definite matrix.

By Definition 2.1, we get

$$\begin{aligned}LV(x,t)=&2d_1x^TQ\sin x+d_2^2x^TQx\\&+\int_{\|z\|<c}[x+rxU(z,t)]^TQ[x+rxU(z,t)]\\&-x^TQx-2rU(z,t)x^TQx\nu(\mathrm{d}z)\\\le&(2d_1(t)+d_2^2(t)+r^2(t)L_1(t))x^TQx,\end{aligned} \tag{4.3}$$

on the other hand, $\|V_x(x,t)g(x,t)\|^2 = \left\|x^T Q_i g(x,t)\right\|^2 = 4d_2^2(t)\left\|x^T Q_i x\right\|^2$. Let $\alpha(t) = e^{t^2}$, so $G(t) = \frac{\alpha(t)^m}{\lambda_{min}(Q)}$ by (**I**) of Theorem 3.2. To sum up, we obtain $a = m$, $h_1(t) = 2d_1(t) + d_2^2(t) + r^2(t)L_1(t)$, $h_2(t) = 4d_2^2(t)$, $M = 0$, $\vartheta_1 = \widetilde{\vartheta}_1$ and $\vartheta_2 = 4\widetilde{\vartheta}_2$. Therefore, we have by Theorem 3.2,

$$\lim_{t\to+\infty} \sup \frac{\ln(\|x(t)\|)}{\ln \alpha(t)} < -\beta^*, a.s.,$$

where $\beta^* = \widetilde{\vartheta}_2 - \frac{1}{2}\widetilde{\vartheta}_1$. We can take the coefficient $\widetilde{\vartheta}_2$ to be sufficiently large or $\widetilde{\vartheta}_1$ to be sufficiently small so as to ensure that $\beta^* > 0$. This shows that the solution $x(t)$ of (2.1) is deemed to converge to zero with attenuation function e^{t^2} and order at least $\widetilde{\vartheta}_2 - \frac{1}{2}\widetilde{\vartheta}_1$. □

Acknowledgements

The authors sincerely thank Professor Yong Li for instructions and many invaluable suggestions. The authors are grateful to all those who give our helpful comments and suggestions for this thesis.

References

[1] D. Applebaum, *Lévy processes and stochastic calculus*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2004.

[2] D. Applebaum, *Lévy processes and stochastic calculus*, Second edition, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2009.

[3] D. Applebaum and M. Siakalli, *Asymptotic stability of stochastic differential equations driven by Lévy noise*, J. Appl. Probab., 2009, 46(4), 1116–1129.

[4] L. Arnold and B. Schmalfuss, *Lyapunov's second method for random dynamical systems*, J. Differential Equations, 2001, 177(1), 235–265.

[5] J. Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1996.

[6] S. P. Bhat and D. S. Bernstein, *Finite-time stability of continuous autonomous systems*, SIAM J. Control Optim., 2000, 38(3), 751–766.

[7] T. Caraballo, F. Ezzine and M. A. Hammami, *Partial stability analysis of stochastic differential equations with a general decay rate*, J. Engrg. Math., 2021, 130(4), 17.

[8] T. Caraballo, F. Ezzine, M. A. Hammami and L. Mchiri, *Practical stability with respect to a part of variables of stochastic differential equations*, Stochastics, 2021, 93(5), 647–664.

[9] T. Caraballo, M. A. Hammami and L. Mchiri, *Practical asymptotic stability of nonlinear stochastic evolution equations*, Stoch. Anal. Appl., 2014, 32(1), 77–87.

[10] T. Caraballo, M. A. Hammami and L. Mchiri, *On the practical global uniform asymptotic stability of stochastic differential equations*, Stochastics, 2016, 88(1), 45–56.

[11] R. Denk, M. Kupper and M. Nendel, *A semigroup approach to nonlinear Lévy processes*, Stochastic Process. Appl., 2020, 130(3), 1616–1642.

[12] C. Fei, W. Fei and X. Mao, *A note on sufficient conditions of asymptotic stability in distribution of stochastic differential equations with G-Brownian motion*, Appl. Math. Lett., 2023, 136, 108448.

[13] O. Ignatyev, *Partial asymptotic stability in probability of stochastic differential equations*, Statist. Probab. Lett., 2009, 79(5), 597–601.

[14] A. E. Kyprianou, *Introductory lectures on fluctuations of Lévy processes with applications*, Universitext., Springer-Verlag, Berlin, 2006.

[15] C. Li, J. Sun and R. Sun, *Stability analysis of a class of stochastic differential delay equations with nonlinear impulsive effects*, J. Franklin Inst., 2010, 347(7), 1186–1198.

[16] M. Li and F. Deng, *Almost sure stability with general decay rate of neutral stochastic delayed hybrid systems with Lévy noise*, Nonlinear Anal. Hybrid Syst., 2017, 24, 171–185.

[17] Y. Li, Z. Liu and W. Wang, *Almost periodic solutions and stable solutions for stochastic differential equations*, Discrete Contin. Dyn. Syst. Ser. B, 2019, 24(11), 5927–5944.

[18] Z. Li and L. Xu, *Almost automorphic solutions for stochastic differential equations driven by Lévy noise*, Phys. A, 2020, 545, 19.

[19] M. Liao, *Lévy processes in Lie groups*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2004, 162.

[20] H. Lin and P. J. Antsaklis, *Stability and stabilizability of switched linear systems: a survey of recent results*, IEEE Trans. Automat. Control, 2009, 54(2), 308–322.

[21] K. Y. Lum, D. S. Bernstein and V. T. Coppola, *Global stabilization of the spinning top with mass imbalance*, Dynam. Stability Systems, 1995, 10(4), 339–365.

[22] X. Mao, *Almost sure polynomial stability for a class of stochastic differential equations*, Quart. J. Math. Oxford Ser., 1992, 43(171), 339–348.

[23] X. Mao, *Stability of stochastic differential equations with Markovian switching*, Stochastic Process. Appl., 1999, 79(1), 45–67.

[24] X. Mao, *Stochastic differential equations and applications*, Second edition, Horwood Publishing Limited, Chichester, 2008.

[25] K. Peiffer and N. Rouche, *Liapunov's second method applied to partial stability*, J. Mécanique, 1969, 8, 323–334.

[26] A. V. Skorokhod, *Asymptotic methods in the theory of stochastic differential equations*, American Mathematical Society, Providence, RI, 1989.

[27] A. V. Skorokhod, *On stability of solution in inertial navigation problem*, Certain problems on dynamics of mechanical systems, Moscow, 1991, 46–50.

[28] L. Tan, W. Jin and Y. Suo, *Stability in distribution of neutral stochastic functional differential equations*, Statist. Probab. Lett., 2015, 107, 27–36.

[29] V. I. Vorotnikov, *Partial stability and control*, Translated from the Russian by Igor E. Merinov. Birkhäuser Boston, Inc., Boston, MA, 1998.

[30] B. Wang and Q. Zhu, *Stability analysis of Markov switched stochastic differential equations with both stable and unstable subsystems*, Systems Control Lett., 2017, 105, 55–61.

[31] H. Wu and J. Sun, *p-moment stability of stochastic differential equations with impulsive jump and Markovian switching*, Automatica J. IFAC, 2006, 42(10), 1753–1759.

[32] S. Wu, D. Han and X. Meng, *p-moment stability of stochastic differential equations with jumps*, Appl. Math. Comput., 2004, 152(2), 505–519.

[33] Y. Xu, J. Duan and W. Xu, *An averaging principle for stochastic dynamical systems with Lévy noise*, Phys. D, 2011, 240(17), 1395–1401.

[34] Y. Xu, B. Pei and G. Guo, *Existence and stability of solutions to non-Lipschitz stochastic differential equations driven by Lévy noise*, Appl. Math. Comput., 2015, 263, 398–409.

[35] Y. Xu, X. Wang, H. Zhang and W. Xu, *Stochastic stability for nonlinear systems driven by Lévy noise*, Nonlinear Dynam., 2012, 68(1–2), 7–15.

[36] C. Yuan and X. Mao, *Robust stability and controllability of stochastic differential delay equations with Markovian switching*, Automatica J. IFAC, 2004, 40(3), 343–354.

[37] Q. Zhu, *Asymptotic stability in the pth moment for stochastic differential equations with Lévy noise*, J. Math. Anal. Appl., 2014, 416(1), 126–142.

[38] Q. Zhu, *Razumikhin-type theorem for stochastic functional differential equations with Lévy noise and Markov switching*, Internat. J. Control, 2017, 90(8), 1703–1712.

[39] Q. Zhu, *Stability analysis of stochastic delay differential equations with Lévy noise*, Systems Control Lett., 2018, 118, 62–68.