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ON A SYSTEM OF COUPLED NONLOCAL SINGULAR FRACTIONAL BOUNDARY VALUE PROBLEMS WITH δ-LAPLACIAN OPERATORS

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Abstract We investigate the existence of at least one or two positive solutions of a system of two Riemann-Liouville fractional differential equations with δ-Laplacian operators and singular nonlinearities, supplemented with coupled nonlocal boundary conditions which contain Riemann-Stieltjes integrals and several fractional derivatives of different orders. We apply the Guo-Krasnosel'skii fixed point theorem in the proof of our main existence results.

Keywords Riemann-Liouville fractional differential equations, singular nonlinearities, nonlocal boundary conditions, existence, multiplicity.

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1. Introduction

In this paper, we consider the system of two nonlinear ordinary fractional differential equations with δ_1-Laplacian and δ_2-Laplacian operators

$$\begin{cases} D_{0+}^{\gamma_1}\left(\varphi_{\delta_1}\left(D_{0+}^{\zeta_1}\phi(\eta)\right)\right)+\Lambda(\eta,\phi(\eta),\psi(\eta))=0, \ \ \eta\in(0,1), \\ D_{0+}^{\gamma_2}\left(\varphi_{\delta_2}\left(D_{0+}^{\zeta_2}\psi(\eta)\right)\right)+\Upsilon(\eta,\phi(\eta),\psi(\eta))=0, \ \ \eta\in(0,1), \end{cases} \tag{1.1}$$

supplemented with the coupled nonlocal boundary conditions

$$\begin{cases} \phi^{(k)}(0)=0, \ k=0,\dots,p-2; \ D_{0+}^{\zeta_1}\phi(0)=0, \ D_{0+}^{\alpha_0}\phi(1)=\sum_{j=1}^{n}\int_0^1 D_{0+}^{\alpha_j}\psi(\eta)\,d\mathfrak{H}_j(\eta), \\ \psi^{(k)}(0)=0, \ k=0,\dots,q-2; \ D_{0+}^{\zeta_2}\psi(0)=0, \ D_{0+}^{\beta_0}\psi(1)=\sum_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}\phi(\eta)\,d\mathfrak{K}_j(\eta), \end{cases} \tag{1.2}$$

where $\gamma_1,\ \gamma_2\in(0,1]$, $\zeta_1\in(p-1,p]$, $\zeta_2\in(q-1,q]$, $p,\ q\in\mathbb{N}$, $p,\ q\geq 3$, $n,m\in\mathbb{N}$, $\alpha_j\in\mathbb{R}$ for all $j=0,1,\dots,n$, $0\leq\alpha_1<\alpha_2<\cdots<\alpha_n\leq\beta_0<\zeta_2-1$, $\beta_0\geq 1$, $\beta_j\in\mathbb{R}$ for all $j=0,1,\dots,m$, $0\leq\beta_1<\beta_2<\cdots<\beta_m\leq\alpha_0<\zeta_1-1$, $\alpha_0\geq 1$, $\delta_1,\ \delta_2>1$,

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$\varphi_{\delta_i}(\eta) = |\eta|^{\delta_i-2}\eta$, $\varphi_{\delta_i}^{-1} = \varphi_{\omega_i}$, $\omega_i = \frac{\delta_i}{\delta_i-1}$, $i = 1, 2$, the functions Λ and Υ are nonnegative and they may be singular at $\eta = 0$ and/or $\eta = 1$, the integrals from the boundary conditions (2) are Riemann-Stieltjes integrals with $\mathfrak{H}_i$, $i = 1, \ldots, n$ and $\mathfrak{K}_j$, $j = 1, \ldots, m$ functions of bounded variation, and D_{0+}^{ρ} denotes the Riemann-Liouville derivative of order ρ (for $\rho = \gamma_1, \zeta_1, \gamma_2, \zeta_2, \alpha_i$ for $i = 0, 1, \ldots, n$, β_j for $j = 0, 1, \ldots, m$).

We will present various assumptions on the singular functions Λ and Υ such that our problem (1.1), (1.2) has at least one or two positive solutions. A positive solution of problem (1.1), (1.2) is a pair of functions $(\phi, \psi) \in (C([0, 1], [0, \infty))^2)$ which satisfy the system (1.1) and the boundary conditions (1.2), with $\phi(\eta) > 0$ for all $\eta \in (0, 1]$, or $\psi(\eta) > 0$ for all $\eta \in (0, 1]$. In the main existence results we apply the Guo-Krasnosel'skii fixed point theorem, (see [11]). The system (1.1) with two positive parameters λ and μ, and nonsingular and nonnegative nonlinearities, subject to the boundary conditions (1.2) was studied in [26], by using the Guo-Krasnosel'skii fixed point theorem. In [26], the authors give various intervals for the parameters λ and μ, and some conditions for the nonlinearities of the system such that positive solutions exist or not. The existence of positive solutions for the system (1.1) subject to the uncoupled boundary conditions

$$\begin{cases} \phi^{(k)}(0)=0,\ k=0,\ldots,p-2;\ D_{0+}^{\zeta_1}\phi(0) = 0,\ D_{0+}^{\alpha_0}\phi(1) = \sum_{j=1}^{n} \int_0^1 D_{0+}^{\alpha_j}\phi(\eta)\, d\mathfrak{H}_j(\eta), \\ \psi^{(k)}(0)=0,\ k=0,\ldots,q-2;\ D_{0+}^{\zeta_2}\psi(0) = 0,\ D_{0+}^{\beta_0}\psi(1) = \sum_{j=1}^{m} \int_0^1 D_{0+}^{\beta_j}\psi(\eta)\, d\mathfrak{K}_j(\eta), \end{cases}$$

where $n, m \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ for all $i = 0, \ldots, n$, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le \alpha_0 < \zeta_1 - 1$, $\alpha_0 \ge 1$, $\beta_i \in \mathbb{R}$ for all $i = 0, \ldots, m$, $0 \le \beta_1 < \beta_2 < \cdots < \beta_m \le \beta_0 < \zeta_2 - 1$, $\beta_0 \ge 1$, has been investigated in the paper [8]. The system of fractional differential equations without δ-Laplacian operators

$$\begin{cases} D_{0+}^{\gamma}\phi(\eta) + \Lambda(\eta, \phi(\eta), \psi(\eta)) = 0, & \eta \in (0, 1), \\ D_{0+}^{\zeta}\psi(\eta) + \Upsilon(\eta, \phi(\eta), \psi(\eta)) = 0, & \eta \in (0, 1), \end{cases}$$

with the uncoupled nonlocal boundary conditions

$$\begin{cases} \phi(0) = \phi'(0) = \cdots = \phi^{(p-2)}(0) = 0, & D_{0+}^{\alpha_0}\phi(1) = \sum_{j=1}^{n} \int_0^1 D_{0+}^{\alpha_j}\phi(\eta)\, d\mathfrak{H}_j(\eta), \\ \psi(0) = \psi'(0) = \cdots = \psi^{(q-2)}(0) = 0, & D_{0+}^{\beta_0}\psi(1) = \sum_{j=1}^{m} \int_0^1 D_{0+}^{\beta_j}\psi(\eta)\, d\mathfrak{K}_j(\eta), \end{cases}$$

where $\gamma, \zeta \in \mathbb{R}$, $\gamma \in (p-1, p]$, $\zeta \in (q-1, q]$, $p, q \in \mathbb{N}$, $p \ge 3$, $q \ge 3$, $n, m \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ for all $i = 0, \ldots, n$, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le \alpha_0 < \gamma - 1$, $\alpha_0 \ge 1$, $\beta_i \in \mathbb{R}$ for all $i = 0, \ldots, m$, $0 \le \beta_1 < \beta_2 < \cdots < \beta_m \le \beta_0 < \zeta - 1$, $\beta_0 \ge 1$, and the functions Λ and Υ are nonnegative and singular at $\eta = 0$ and/or $\eta = 1$, was studied in [27].

Fractional differential equations and systems of fractional differential equations with or without δ-Laplacian operators, with parameters or without parameters, supplemented with various Riemann-Stieltjes integral boundary conditions were recently studied in [1–7], [12–18], [21, 25, 28–31]. We also mention the books [9, 10, 19, 20, 22–24], and their references, where their authors present various applications of the fractional differential equations in many scientific and engineering domains.

In Section 2, we study a nonlocal linear boundary value problem for fractional differential equations with δ-Laplacian operators, and we give some properties of the associated Green functions. Section 3 is concerned with the main existence theorems for the positive solutions of problem (1),(2), and in Section 4, we present two examples which illustrate our results.

2. Preliminary results

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{\gamma_1}\left(\varphi_{\delta_1}\left(D_{0+}^{\zeta_1}\phi(\eta)\right)\right)+\mathfrak{h}(\eta)=0, \quad \eta\in(0,1),\\ D_{0+}^{\gamma_2}\left(\varphi_{\delta_2}\left(D_{0+}^{\zeta_2}\psi(\eta)\right)\right)+\mathfrak{k}(\eta)=0, \quad \eta\in(0,1),\end{cases} \tag{2.1}$$

with the coupled boundary conditions (1.2), where $\mathfrak{h}$, $\mathfrak{k}\in C(0,1)\cap L^1(0,1)$.

We denote by

$$\begin{aligned}\mathfrak{a}=&\frac{\Gamma(\zeta_1)\Gamma(\zeta_2)}{\Gamma(\zeta_1-\alpha_0)\Gamma(\zeta_2-\beta_0)}-\left(\sum_{j=1}^{n}\frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_j)}\int_0^1\nu^{\zeta_2-\alpha_j-1}\,d\mathfrak{H}_j(\nu)\right)\\ &\times\left(\sum_{j=1}^{m}\frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1-\beta_j)}\int_0^1\nu^{\zeta_1-\beta_j-1}\,d\mathfrak{K}_j(\nu)\right).\end{aligned} \tag{2.2}$$

Lemma 2.1 ([26]). *If $\mathfrak{a}\neq 0$, then the unique solution $(\phi,\psi)\in C[0,1]\times C[0,1]$ of problem (2.1),(1.2) is given by*

$$\begin{cases}\phi(\eta)=\int_0^1\mathfrak{G}_1(\eta,\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\mathfrak{h}(\nu))\,d\nu+\int_0^1\mathfrak{G}_2(\eta,\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\mathfrak{k}(\nu))\,d\nu, \quad \forall\,\eta\in[0,1],\\ \psi(\eta)=\int_0^1\mathfrak{G}_3(\eta,\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\mathfrak{h}(\nu))\,d\nu+\int_0^1\mathfrak{G}_4(\eta,\nu)\varphi_{\omega_2}(I_{0+}^{\alpha_2}\mathfrak{k}(\nu))\,d\nu, \quad \forall\,\eta\in[0,1],\end{cases} \tag{2.3}$$

where the Green functions $\mathfrak{G}_i$, $i=1,\dots,4$ are

$$\begin{aligned}\mathfrak{G}_1(\eta,\nu)&=\mathfrak{g}_1(\eta,\nu)+\frac{\eta^{\zeta_1-1}}{\mathfrak{a}}\left(\sum_{j=1}^{n}\frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_j)}\int_0^1\varsigma^{\zeta_2-\alpha_j-1}\,d\mathfrak{H}_j(\varsigma)\right)\\ &\quad\times\left(\sum_{j=1}^{m}\int_0^1\mathfrak{g}_{1j}(\varsigma,\nu)\,d\mathfrak{K}_j(\varsigma)\right),\\ \mathfrak{G}_2(\eta,\nu)&=\frac{\eta^{\zeta_1-1}\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)}\sum_{j=1}^{n}\int_0^1\mathfrak{g}_{2j}(\varsigma,\nu)\,d\mathfrak{H}_j(\varsigma),\\ \mathfrak{G}_3(\eta,\nu)&=\frac{\eta^{\zeta_2-1}\Gamma(\zeta_1)}{\mathfrak{a}\Gamma(\zeta_1-\alpha_0)}\sum_{j=1}^{m}\int_0^1\mathfrak{g}_{1j}(\varsigma,\nu)\,d\mathfrak{K}_j(\varsigma),\\ \mathfrak{G}_4(\eta,\nu)&=\mathfrak{g}_2(\eta,\nu)+\frac{\eta^{\zeta_2-1}}{\mathfrak{a}}\left(\sum_{j=1}^{m}\frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1-\beta_j)}\int_0^1\varsigma^{\zeta_1-\beta_j-1}\,d\mathfrak{K}_j(\varsigma)\right)\\ &\quad\times\left(\sum_{j=1}^{n}\int_0^1\mathfrak{g}_{2j}(\varsigma,\nu)\,d\mathfrak{H}_j(\varsigma)\right),\end{aligned} \tag{2.4}$$

for all $(\eta,\nu)\in[0,1]\times[0,1]$, *and*

$$\begin{aligned}
\mathfrak{g}_1(\eta,\nu) &= \frac{1}{\Gamma(\zeta_1)}\begin{cases}\eta^{\zeta_1-1}(1-\nu)^{\zeta_1-\alpha_0-1}-(\eta-\nu)^{\zeta_1-1}, & 0\le\nu\le\eta\le1,\\ \eta^{\zeta_1-1}(1-\nu)^{\zeta_1-\alpha_0-1}, & 0\le\eta\le\nu\le1,\end{cases}\\
\mathfrak{g}_{1i}(\eta,\nu) &= \frac{1}{\Gamma(\zeta_1-\beta_i)}\begin{cases}\eta^{\zeta_1-\beta_i-1}(1-\nu)^{\zeta_1-\alpha_0-1}-(\eta-\nu)^{\zeta_1-\beta_i-1}, & 0\le\nu\le\eta\le1,\\ \eta^{\zeta_1-\beta_i-1}(1-\nu)^{\zeta_1-\alpha_0-1}, & 0\le\eta\le\nu\le1,\end{cases}\\
\mathfrak{g}_2(\eta,\nu) &= \frac{1}{\Gamma(\zeta_2)}\begin{cases}\eta^{\zeta_2-1}(1-\nu)^{\zeta_2-\beta_0-1}-(\eta-\nu)^{\zeta_2-1}, & 0\le\nu\le\eta\le1,\\ \eta^{\zeta_2-1}(1-\nu)^{\zeta_2-\beta_0-1}, & 0\le\eta\le\nu\le1,\end{cases}\\
\mathfrak{g}_{2j}(\eta,\nu) &= \frac{1}{\Gamma(\zeta_2-\alpha_j)}\begin{cases}\eta^{\zeta_2-\alpha_j-1}(1-\nu)^{\zeta_2-\beta_0-1}-(\eta-\nu)^{\zeta_2-\alpha_j-1}, & 0\le\nu\le\eta\le1,\\ \eta^{\zeta_2-\alpha_j-1}(1-\nu)^{\zeta_2-\beta_0-1}, & 0\le\eta\le\nu\le1,\end{cases}
\end{aligned} \tag{2.5}$$

for all $i=1,\dots,m$ *and* $j=1,\dots,n$.

Based on the properties of functions $\mathfrak{g}_1,\ \mathfrak{g}_{1i},\ \mathfrak{g}_2,\ \mathfrak{g}_{2j},\ i=1,\dots,m,\ j=1,\dots,n$ from (2.5) (see [13]), we obtain for the Green functions $\mathfrak{G}_i,\ i=1,\dots,4$ the following properties.

Lemma 2.2 ([26]). *Assume that* $\mathfrak{a}>0$, $\mathfrak{H}_i,\ i=1,\dots,n$, $\mathfrak{K}_j,\ j=1,\dots,m$ *are nondecreasing functions. Then the functions* $\mathfrak{G}_i,\ i=1,\dots,4$ *given by (2.4) have the properties:*

a) $\mathfrak{G}_i:[0,1]\times[0,1]\to[0,\infty)$, $i=1,\dots,4$ *are continuous functions;*

b) $\mathfrak{G}_1(\eta,\nu)\le\mathfrak{J}_1(\nu)$ *for all* $(\eta,\nu)\in[0,1]\times[0,1]$, *where*

$$\mathfrak{J}_1(\nu)=\mathfrak{h}_1(\nu)+\frac{1}{\mathfrak{a}}\left(\sum_{j=1}^{n}\frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_j)}\int_0^1\varsigma^{\zeta_2-\alpha_j-1}\,d\mathfrak{H}_j(\varsigma)\right)\left(\sum_{j=1}^{m}\int_0^1\mathfrak{g}_{1j}(\varsigma,\nu)\,d\mathfrak{K}_j(\varsigma)\right),$$

with $\mathfrak{h}_1(\nu)=\frac{1}{\Gamma(\zeta_1)}(1-\nu)^{\zeta_1-\alpha_0-1}(1-(1-\nu)^{\alpha_0})$, *for all* $\nu\in[0,1]$;

c) $\mathfrak{G}_1(\eta,\nu)\ge\eta^{\zeta_1-1}\mathfrak{J}_1(\nu)$ *for all* $(\eta,\nu)\in[0,1]\times[0,1]$;

d) $\mathfrak{G}_2(\eta,\nu)\le\mathfrak{J}_2(\nu)$, *for all* $(\eta,\nu)\in[0,1]\times[0,1]$, *where*

$$\mathfrak{J}_2(\nu)=\frac{\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)}\sum_{j=1}^{n}\int_0^1\mathfrak{g}_{2j}(\varsigma,\nu)\,d\mathfrak{H}_j(\varsigma),\quad\forall\,\nu\in[0,1];$$

e) $\mathfrak{G}_2(\eta,\nu)=\eta^{\zeta_1-1}\mathfrak{J}_2(\nu)$ *for all* $(\eta,\nu)\in[0,1]\times[0,1]$;

f) $\mathfrak{G}_3(\eta,\nu)\le\mathfrak{J}_3(\nu)$ *for all* $(\eta,\nu)\in[0,1]\times[0,1]$, *where*

$$\mathfrak{J}_3(\nu)=\frac{\Gamma(\zeta_1)}{\mathfrak{a}\Gamma(\zeta_1-\alpha_0)}\sum_{j=1}^{m}\int_0^1\mathfrak{g}_{1j}(\varsigma,\nu)\,d\mathfrak{K}_j(\varsigma),\quad\forall\,\nu\in[0,1];$$

g) $\mathfrak{G}_3(\eta,\nu)=\eta^{\zeta_2-1}\mathfrak{J}_3(\nu)$ *for all* $(\eta,\nu)\in[0,1]\times[0,1]$;

h) $\mathfrak{G}_4(\eta,\nu)\le\mathfrak{J}_4(\nu)$ *for all* $(\eta,\nu)\in[0,1]\times[0,1]$, *where*

$$\mathfrak{J}_4(\nu)=\mathfrak{h}_2(\nu)+\frac{1}{\mathfrak{a}}\left(\sum_{j=1}^{m}\frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1-\beta_j)}\int_0^1\varsigma^{\zeta_1-\beta_j-1}\,d\mathfrak{K}_j(\varsigma)\right)\left(\sum_{j=1}^{n}\int_0^1\mathfrak{g}_{2j}(\varsigma,\nu)\,d\mathfrak{H}_j(\varsigma)\right),$$

with $\mathfrak{h}_2(\nu)=\frac{1}{\Gamma(\zeta_2)}(1-\nu)^{\zeta_2-\beta_0-1}(1-(1-\nu)^{\beta_0})$, *for all* $\nu\in[0,1]$;

i) $\mathfrak{G}_4(\eta,\nu)\ge\eta^{\zeta_2-1}\mathfrak{J}_4(\nu)$, *for all* $(\eta,\nu)\in[0,1]\times[0,1]$.

From the definitions of functions $\mathfrak{J}_i,\ i=1,\dots,4$, we see that $\mathfrak{J}_i(\nu)\geq 0$, for all $\nu\in[0,1]$, $i=1,\dots,4$, $\mathfrak{J}_1\not\equiv 0$ and $\mathfrak{J}_4\not\equiv 0$; the function $\mathfrak{J}_2$ is the null function if all functions $\mathfrak{H}_i,\ i=1,\dots,n$ are constant, and the function $\mathfrak{J}_3$ is the null function if all functions $\mathfrak{K}_j,\ j=1,\dots,m$ are constant.

Lemma 2.3. *Assume that* $\mathfrak{a}>0$, $\mathfrak{H}_i,\ i=1,\dots,n$, $\mathfrak{K}_j,\ j=1,\dots,m$ *are nondecreasing functions, and* $\mathfrak{h},\mathfrak{k}\in C(0,1)\cap L^1(0,1)$ *with* $\mathfrak{h}(\eta)\geq 0$, $\mathfrak{k}(\eta)\geq 0$ *for all* $\eta\in(0,1)$. *Then the solution* (ϕ,ψ) *of problem (2.1),(1.2) given by (2.3) satisfies the inequalities* $\phi(\eta)\geq 0$, $\psi(\eta)\geq 0$ *for all* $\eta\in[0,1]$. *In addition, we have the inequalities* $\phi(\eta)\geq\eta^{\zeta_1-1}\phi(\varsigma)$ *and* $\psi(\eta)\geq\eta^{\zeta_2-1}\psi(\varsigma)$ *for all* $\eta,\varsigma\in[0,1]$.

Proof. Based on the assumptions of this lemma, by using relations (2.3) and Lemma 2.2, we deduce that $\phi(\eta)\geq 0$ and $\psi(\eta)\geq 0$ for all $\eta\in[0,1]$. Besides, for all $\eta,\varsigma\in[0,1]$, we obtain the following inequalities

$$\begin{aligned}
\phi(\eta)&\geq\eta^{\zeta_1-1}\left(\int_0^1\mathfrak{J}_1(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\mathfrak{h}(\nu))\,d\nu+\int_0^1\mathfrak{J}_2(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\mathfrak{k}(\nu))\,d\nu\right)\\
&\geq\eta^{\zeta_1-1}\left(\int_0^1\mathfrak{G}_1(\varsigma,\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\mathfrak{h}(\nu))\,d\nu+\int_0^1\mathfrak{G}_2(\varsigma,\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\mathfrak{k}(\nu))\,d\nu\right)\\
&=\eta^{\zeta_1-1}\phi(\varsigma),\\
\psi(\eta)&\geq\eta^{\zeta_2-1}\left(\int_0^1\mathfrak{J}_3(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\mathfrak{h}(\nu))\,d\nu+\int_0^1\mathfrak{J}_4(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\mathfrak{k}(\nu))\,d\nu\right)\\
&\geq\eta^{\zeta_2-1}\left(\int_0^1\mathfrak{G}_3(\varsigma,\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\mathfrak{h}(\nu))\,d\nu+\int_0^1\mathfrak{G}_4(\varsigma,\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\mathfrak{k}(\nu))\,d\nu\right)\\
&=\eta^{\zeta_2-1}\psi(\varsigma).
\end{aligned}$$

□

3. Existence and multiplicity of positive solutions

We investigate in this section the existence of at least one or two positive solutions for problem (1.1),(1.2) under various assumptions on the functions Λ and Υ which may be singular at $\eta=0$ and/or $\eta=1$. We give now the basic assumptions that we will use in our main results.

$(A1)$ $\gamma_1,\gamma_2\in(0,1]$, $\zeta_1\in(p-1,p]$, $\zeta_2\in(q-1,q]$, $p,q\in\mathbb{N}$, $p,q\geq 3$, $n,m\in\mathbb{N}$, $\alpha_j\in\mathbb{R}$ for all $j=0,1,\dots,n$, $0\leq\alpha_1<\alpha_2<\dots<\alpha_n\leq\beta_0<\zeta_2-1$, $\beta_0\geq 1$, $\beta_j\in\mathbb{R}$ for all $j=0,1,\dots,m$, $0\leq\beta_1<\beta_2<\dots<\beta_m\leq\alpha_0<\zeta_1-1$, $\alpha_0\geq 1$, $\mathfrak{H}_j:[0,1]\to\mathbb{R}$, $j=1,\dots,n$ and $\mathfrak{K}_k:[0,1]\to\mathbb{R}$, $k=1,\dots,m$ are nondecreasing functions, $\mathfrak{a}>0$ ($\mathfrak{a}$ is given by (2.2)), $\delta_i>1$, $\varphi_{\delta_i}(\nu)=|\nu|^{\delta_i-2}\nu$, $\varphi_{\delta_i}^{-1}=\varphi_{\omega_i}$, $\omega_i=\frac{\delta_i}{\delta_i-1}$, $i=1,2$.

$(A2)$ The functions $\Lambda,\Upsilon\in C((0,1)\times[0,\infty)\times[0,\infty),[0,\infty))$ and there exist the functions $\theta_i\in C((0,1),[0,\infty))$ and $\mu_i\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$, $i=1,2$, with $L_1,L_2\in(0,\infty)$ such that

$$\Lambda(\eta,z,w)\leq\theta_1(\eta)\mu_1(\eta,z,w),\quad\Upsilon(\eta,z,w)\leq\theta_2(\eta)\mu_2(\eta,z,w),\tag{3.1}$$

for all $\eta\in(0,1)$ and $z,w\in[0,\infty)$, where $L_1=\int_0^1(1-\nu)^{\zeta_1-\alpha_0-1}\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu$, and $L_2=\int_0^1(1-\nu)^{\zeta_2-\beta_0-1}\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\,d\nu$.

By using Lemma 2.1 (the relations (2.3)), the pair (ϕ, ψ) is a solution of problem (1.1),(1.2) if and only if (ϕ, ψ) is a solution of the nonlinear system of integral equations

$$\begin{cases} \phi(\eta) = \displaystyle\int_0^1 \mathfrak{G}_1(\eta, \nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu, \phi(\nu), \psi(\nu)))\, d\nu \\ \qquad\quad + \displaystyle\int_0^1 \mathfrak{G}_2(\eta, \nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\Upsilon(\nu, \phi(\nu), \psi(\nu)))\, d\nu, \ \ \eta \in [0, 1], \\ \psi(\eta) = \displaystyle\int_0^1 \mathfrak{G}_3(\eta, \nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu, \phi(\nu), \psi(\nu)))\, d\nu \\ \qquad\quad + \displaystyle\int_0^1 \mathfrak{G}_4(\eta, \nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\Upsilon(\nu, \phi(\nu), \psi(\nu)))\, d\nu, \ \ \eta \in [0, 1]. \end{cases}$$

We introduce the Banach space $\mathfrak{X} = C[0, 1]$ with the supremum norm $\|\phi\| = \sup_{\eta\in[0,1]} |\phi(\eta)|$, and the Banach space $\mathfrak{Y} = \mathfrak{X} \times \mathfrak{X}$ with the norm $\|(\phi, \psi)\|_{\mathfrak{Y}} = \|\phi\| + \|\psi\|$. We define the cone $\mathfrak{P} \subset \mathfrak{Y}$ by

$$\mathfrak{P} = \{(\phi, \psi) \in \mathfrak{Y}, \ \ \phi(\eta) \geq 0, \ \ \psi(\eta) \geq 0, \ \ \forall\, \eta \in [0, 1]\}.$$

We also define the operators $\mathfrak{T}_1$, $\mathfrak{T}_2 : \mathfrak{Y} \to \mathfrak{X}$ and $\mathfrak{T} : \mathfrak{Y} \to \mathfrak{Y}$ by

$$\begin{aligned} \mathfrak{T}_1(\phi, \psi)(\eta) &= \int_0^1 \mathfrak{G}_1(\eta, \nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu, \phi(\nu), \psi(\nu)))\, d\nu \\ &\quad + \int_0^1 \mathfrak{G}_2(\eta, \nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\Upsilon(\nu, \phi(\nu), \psi(\nu)))\, d\nu, \ \ \eta \in [0, 1], \\ \mathfrak{T}_2(\phi, \psi)(\eta) &= \int_0^1 \mathfrak{G}_3(\eta, \nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu, \phi(\nu), \psi(\nu)))\, d\nu \\ &\quad + \int_0^1 \mathfrak{G}_4(\eta, \nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\Upsilon(\nu, \phi(\nu), \psi(\nu)))\, d\nu, \ \ \eta \in [0, 1], \end{aligned}$$

and $\mathfrak{T}(\phi, \psi) = (\mathfrak{T}_1(\phi, \psi), \mathfrak{T}_2(\phi, \psi))$, $(\phi, \psi) \in \mathfrak{Y}$. We see that (ϕ, ψ) is a solution of problem (1.1),(1.2) if and only if (ϕ, ψ) is a fixed point of operator $\mathfrak{T}$.

Lemma 3.1. *Assume that (A1) and (A2) hold. Then $\mathfrak{T} : \mathfrak{P} \to \mathfrak{P}$ is a completely continuous operator.*

Proof. We denote by

$$\begin{aligned} \Psi_1 &= \int_0^1 \mathfrak{J}_1(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\, d\nu, \ \ \Psi_2 = \int_0^1 \mathfrak{J}_2(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\, d\nu, \\ \Psi_3 &= \int_0^1 \mathfrak{J}_3(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\, d\nu, \ \ \Psi_4 = \int_0^1 \mathfrak{J}_4(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\, d\nu. \end{aligned}$$

By using $(A2)$ and Lemma 2.2, we deduce that $\Psi_1 > 0$, $\Psi_2 \geq 0$, $\Psi_3 \geq 0$, $\Psi_4 > 0$, and

$$\begin{aligned} \Psi_1 = \int_0^1 \Bigg[\mathfrak{h}_1(\nu) + \frac{1}{\mathfrak{a}} \left(\sum_{i=1}^{n} \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2 - \alpha_i)} \int_0^1 \varsigma^{\zeta_2-\alpha_i-1} d\mathfrak{H}_i(\varsigma) \right) \\ \times \left(\sum_{i=1}^{m} \int_0^1 \mathfrak{g}_{1i}(\varsigma, \nu)\, d\mathfrak{K}_i(\varsigma) \right) \Bigg] \varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\, d\nu \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \left[\frac{1}{\Gamma(\zeta_1)} (1-\nu)^{\zeta_1-\alpha_0-1} + \frac{1}{\mathfrak{a}} \left(\sum_{i=1}^{n} \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_i)} \int_0^1 \varsigma^{\zeta_2-\alpha_i-1}\, d\mathfrak{H}_i(\varsigma) \right) \right. \\
&\quad \left. \times \left(\sum_{i=1}^{m} \int_0^1 \frac{1}{\Gamma(\zeta_1-\beta_i)} \varsigma^{\zeta_1-\beta_i-1} (1-\nu)^{\zeta_1-\alpha_0-1}\, d\mathfrak{K}_i(\varsigma) \right) \right] \varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\, d\nu \\
&= L_1 \left[\frac{1}{\Gamma(\zeta_1)} + \frac{1}{\mathfrak{a}} \left(\sum_{i=1}^{n} \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_i)} \int_0^1 \varsigma^{\zeta_2-\alpha_i-1}\, d\mathfrak{H}_i(\varsigma) \right) \right. \\
&\quad \left. \times \left(\sum_{i=1}^{m} \int_0^1 \frac{1}{\Gamma(\zeta_1-\beta_i)} \varsigma^{\zeta_1-\beta_i-1}\, d\mathfrak{K}_i(\varsigma) \right) \right] < \infty,
\end{aligned}
$$

$$
\begin{aligned}
\Psi_2 &= \int_0^1 \frac{\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)} \left(\sum_{i=1}^{n} \int_0^1 \mathfrak{g}_{2i}(\varsigma,\nu)\, d\mathfrak{H}_i(\varsigma) \right) \varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\, d\nu \\
&\leq \int_0^1 \frac{\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)} \left(\sum_{i=1}^{n} \int_0^1 \frac{1}{\Gamma(\zeta_2-\alpha_i)} \varsigma^{\zeta_2-\alpha_i-1} (1-\nu)^{\zeta_2-\beta_0-1}\, d\mathfrak{H}_i(\varsigma) \right) \varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\, d\nu \\
&= L_2 \frac{\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)} \left(\sum_{i=1}^{n} \int_0^1 \frac{1}{\Gamma(\zeta_2-\alpha_i)} \varsigma^{\zeta_2-\alpha_i-1}\, d\mathfrak{H}_i(\varsigma) \right) < \infty,
\end{aligned}
$$

$$
\begin{aligned}
\Psi_3 &= \int_0^1 \frac{\Gamma(\zeta_1)}{\mathfrak{a}\Gamma(\zeta_1-\alpha_0)} \left(\sum_{i=1}^{m} \int_0^1 \mathfrak{g}_{1i}(\varsigma,\nu)\, d\mathfrak{K}_i(\varsigma) \right) \varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\, d\nu \\
&\leq \int_0^1 \frac{\Gamma(\zeta_1)}{\mathfrak{a}\Gamma(\zeta_1-\alpha_0)} \left(\sum_{i=1}^{m} \int_0^1 \frac{1}{\Gamma(\zeta_1-\beta_i)} \varsigma^{\zeta_1-\beta_i-1} (1-\nu)^{\zeta_1-\alpha_0-1}\, d\mathfrak{K}_i(\varsigma) \right) \varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\, d\nu \\
&= L_1 \frac{\Gamma(\zeta_1)}{\mathfrak{a}\Gamma(\zeta_1-\alpha_0)} \left(\sum_{i=1}^{m} \frac{1}{\Gamma(\zeta_1-\beta_i)} \varsigma^{\zeta_1-\beta_i-1}\, d\mathfrak{K}_i(\varsigma) \right) < \infty,
\end{aligned}
$$

$$
\begin{aligned}
\Psi_4 &= \int_0^1 \left[\mathfrak{h}_2(\nu) + \frac{1}{\mathfrak{a}} \left(\sum_{i=1}^{m} \frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1-\beta_i)} \int_0^1 \varsigma^{\zeta_1-\beta_i-1}\, d\mathfrak{K}_i(\varsigma) \right) \right. \\
&\quad \left. \times \left(\sum_{i=1}^{n} \int_0^1 \mathfrak{g}_{2i}(\varsigma,\nu)\, d\mathfrak{H}_i(\varsigma) \right) \right] \varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\, d\nu \\
&\leq \int_0^1 \left[\frac{1}{\Gamma(\zeta_2)} (1-\nu)^{\zeta_2-\beta_0-1} + \frac{1}{\mathfrak{a}} \left(\sum_{i=1}^{m} \frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1-\beta_i)} \int_0^1 \varsigma^{\zeta_1-\beta_i-1}\, d\mathfrak{K}_i(\varsigma) \right) \right. \\
&\quad \left. \times \left(\sum_{i=1}^{n} \int_0^1 \frac{1}{\Gamma(\zeta_2-\alpha_i)} \varsigma^{\zeta_2-\alpha_i-1} (1-\nu)^{\zeta_2-\beta_0-1}\, d\mathfrak{H}_i(\varsigma) \right) \right] \varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\, d\nu \\
&= L_2 \left[\frac{1}{\Gamma(\zeta_2)} + \frac{1}{\mathfrak{a}} \left(\sum_{i=1}^{m} \frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1-\beta_i)} \int_0^1 \varsigma^{\zeta_1-\beta_i-1}\, d\mathfrak{K}_i(\varsigma) \right) \right. \\
&\quad \left. \times \left(\sum_{i=1}^{n} \int_0^1 \frac{1}{\Gamma(\zeta_2-\alpha_i)} \varsigma^{\zeta_2-\alpha_i-1}\, d\mathfrak{H}_i(\varsigma) \right) \right] < \infty.
\end{aligned}
$$

By Lemma 2.3, we also find that $\mathfrak{T}$ maps $\mathfrak{P}$ into $\mathfrak{P}$.

We show now that $\mathfrak{T}$ maps bounded sets into relatively compact sets. We assume $\mathfrak{D} \subset \mathfrak{P}$ is an arbitrary bounded set. Then there exists $\Xi_1 > 0$ such that $\|(\phi,\psi)\|_{\mathfrak{Y}} \leq$

Ξ_1 for all $(\phi,\psi)\in\mathfrak{D}$. By using the continuity of μ_1 and μ_2, we deduce that there exists $\Xi_2>0$ such that

$$\Xi_2=\max\left\{\sup_{\eta\in[0,1],\ z,w\in[0,\Xi_1]}\mu_1(\eta,z,w),\ \sup_{\eta\in[0,1],\ z,w\in[0,\Xi_1]}\mu_2(\eta,z,w)\right\}<\infty.$$

By Lemma 2.2, for any $(\phi,\psi)\in\mathfrak{D}$ and $\eta\in[0,1]$, we obtain

$$\begin{aligned}
&\mathfrak{T}_1(\phi,\psi)(\eta)\\
&\le\int_0^1\mathfrak{J}_1(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu,\phi(\nu),\psi(\nu)))\,d\nu+\int_0^1\mathfrak{J}_2(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\Upsilon(\nu,\phi(\nu),\psi(\nu)))\,d\nu\\
&\le\int_0^1\mathfrak{J}_1(\nu)\varphi_{\omega_1}\left(\frac{1}{\Gamma(\gamma_1)}\int_0^\nu(\nu-\varsigma)^{\gamma_1-1}\theta_1(\varsigma)\mu_1(\varsigma,\phi(\varsigma),\psi(\varsigma))\,d\varsigma\right)d\nu\\
&\quad+\int_0^1\mathfrak{J}_2(\nu)\varphi_{\omega_2}\left(\frac{1}{\Gamma(\gamma_2)}\int_0^\nu(\nu-\varsigma)^{\gamma_2-1}\theta_2(\varsigma)\mu_2(\varsigma,\phi(\varsigma),\psi(\varsigma))\,d\varsigma\right)d\nu\\
&\le\Xi_2^{\omega_1-1}\int_0^1\mathfrak{J}_1(\nu)\varphi_{\omega_1}\left(\frac{1}{\Gamma(\gamma_1)}\int_0^\nu(\nu-\varsigma)^{\gamma_1-1}\theta_1(\varsigma)\,d\varsigma\right)d\nu\\
&\quad+\Xi_2^{\omega_2-1}\int_0^1\mathfrak{J}_2(\nu)\varphi_{\omega_2}\left(\frac{1}{\Gamma(\gamma_2)}\int_0^\nu(\nu-\varsigma)^{\gamma_2-1}\theta_2(\varsigma)\,d\varsigma\right)d\nu\\
&=\Xi_2^{\omega_1-1}\int_0^1\mathfrak{J}_1(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu+\Xi_2^{\omega_2-1}\int_0^1\mathfrak{J}_2(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\,d\nu\\
&=\Psi_1\Xi_2^{\omega_1-1}+\Psi_2\Xi_2^{\omega_2-1},\\
&\mathfrak{T}_2(\phi,\psi)(\eta)\\
&\le\int_0^1\mathfrak{J}_3(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu,\phi(\nu),\psi(\nu)))\,d\nu+\int_0^1\mathfrak{J}_4(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\Upsilon(\nu,\phi(\nu),\psi(\nu)))\,d\nu\\
&\le\int_0^1\mathfrak{J}_3(\nu)\varphi_{\omega_1}\left(\frac{1}{\Gamma(\gamma_1)}\int_0^\nu(\nu-\varsigma)^{\gamma_1-1}\theta_1(\varsigma)\mu_1(\varsigma,\phi(\varsigma),\psi(\varsigma))\,d\varsigma\right)d\nu\\
&\quad+\int_0^1\mathfrak{J}_4(\nu)\varphi_{\omega_2}\left(\frac{1}{\Gamma(\gamma_2)}\int_0^\nu(\nu-\varsigma)^{\gamma_2-1}\theta_2(\varsigma)\mu_2(\varsigma,\phi(\varsigma),\psi(\varsigma))\,d\varsigma\right)d\nu\\
&\le\Xi_2^{\omega_1-1}\int_0^1\mathfrak{J}_3(\nu)\varphi_{\omega_1}\left(\frac{1}{\Gamma(\gamma_1)}\int_0^\nu(\nu-\varsigma)^{\gamma_1-1}\theta_1(\varsigma)\,d\varsigma\right)d\nu\\
&\quad+\Xi_2^{\omega_2-1}\int_0^1\mathfrak{J}_4(\nu)\varphi_{\omega_2}\left(\frac{1}{\Gamma(\gamma_2)}\int_0^\nu(\nu-\varsigma)^{\gamma_2-1}\theta_2(\varsigma)\,d\varsigma\right)d\nu\\
&=\Xi_2^{\omega_1-1}\int_0^1\mathfrak{J}_3(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu+\Xi_2^{\omega_2-1}\int_0^1\mathfrak{J}_4(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\,d\nu\\
&=\Psi_3\Xi_2^{\omega_1-1}+\Psi_4\Xi_2^{\omega_2-1}.
\end{aligned}$$

Then $\|\mathfrak{T}_1(\phi,\psi)\|\le\Psi_1\Xi_2^{\omega_1-1}+\Psi_2\Xi_2^{\omega_2-1}$, $\|\mathfrak{T}_2(\phi,\psi)\|\le\Psi_3\Xi_2^{\omega_1-1}+\Psi_4\Xi_2^{\omega_2-1}$ for all $(\phi,\psi)\in\mathfrak{D}$, and so $\mathfrak{T}_1(\mathfrak{D})$, $\mathfrak{T}_2(\mathfrak{D})$ and $\mathfrak{T}(\mathfrak{D})$ are bounded.

We will show next that $\mathfrak{T}(\mathfrak{D})$ is equicontinuous. By using Lemma 2.1, for $(\phi,\psi)\in\mathfrak{D}$ and $\eta\in[0,1]$, we obtain

$$\begin{aligned}
&\mathfrak{T}_1(\phi,\psi)(\eta)\\
&=\int_0^1\left[\mathfrak{g}_1(\eta,\nu)+\frac{\eta^{\zeta_1-1}}{\mathfrak{a}}\left(\sum_{i=1}^n\frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_i)}\int_0^1\varsigma^{\zeta_2-\alpha_i-1}\,d\mathfrak{H}_i(\varsigma)\right)\right.
\end{aligned}$$

$$\times\left(\sum_{i=1}^{m}\int_0^1 \mathfrak{g}_{1i}(\varsigma,\nu)\,d\mathfrak{K}_i(\varsigma)\right)\Bigg]\,\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu,\phi(\nu),\psi(\nu)))\,d\nu$$
$$+\int_0^1 \frac{\eta^{\zeta_1-1}\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)}\left(\sum_{i=1}^{n}\int_0^1 \mathfrak{g}_{2i}(\varsigma,\nu)\,d\mathfrak{H}_i(\varsigma)\right)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\Upsilon(\nu,\phi(\nu),\psi(\nu)))\,d\nu$$
$$=\int_0^\eta \frac{1}{\Gamma(\zeta_1)}\left[\eta^{\zeta_1-1}(1-\nu)^{\zeta_1-\alpha_0-1}-(\eta-\nu)^{\zeta_1-1}\right]\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu,\phi(\nu),\psi(\nu)))\,d\nu$$
$$+\int_\eta^1 \frac{1}{\Gamma(\zeta_1)}\eta^{\zeta_1-1}(1-\nu)^{\zeta_1-\alpha_0-1}\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu,\phi(\nu),\psi(\nu)))\,d\nu$$
$$+\frac{\eta^{\zeta_1-1}}{\mathfrak{a}}\left(\sum_{i=1}^{n}\frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_i)}\int_0^1 \varsigma^{\zeta_2-\alpha_i-1}\,d\mathfrak{H}_i(\varsigma)\right)\int_0^1\left(\sum_{i=1}^{m}\int_0^1 \mathfrak{g}_{1i}(\varsigma,\nu)\,d\mathfrak{K}_i(\varsigma)\right)$$
$$\times\,\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu,\phi(\nu),\psi(\nu)))\,d\nu$$
$$+\frac{\eta^{\zeta_1-1}\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)}\int_0^1\left(\sum_{i=1}^{n}\int_0^1 \mathfrak{g}_{2i}(\varsigma,\nu)\,d\mathfrak{H}_i(\varsigma)\right)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\Upsilon(\nu,\phi(\nu),\psi(\nu)))\,d\nu.$$

Then for any $\eta\in(0,1)$ we have

$$(\mathfrak{T}_1(\phi,\psi))'(\eta)$$
$$=\int_0^\eta \frac{1}{\Gamma(\zeta_1)}\left[(\zeta_1-1)\eta^{\zeta_1-2}(1-\nu)^{\zeta_1-\alpha_0-1}-(\zeta_1-1)(\eta-\nu)^{\zeta_1-2}\right]$$
$$\times\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu,\phi(\nu),\psi(\nu)))\,d\nu$$
$$+\int_\eta^1 \frac{1}{\Gamma(\zeta_1)}(\zeta_1-1)\eta^{\zeta_1-2}(1-\nu)^{\zeta_1-\alpha_0-1}\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu,\phi(\nu),\psi(\nu)))\,d\nu$$
$$+\frac{(\zeta_1-1)\eta^{\zeta_1-2}}{\mathfrak{a}}\left(\sum_{i=1}^{n}\frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_i)}\int_0^1 \varsigma^{\zeta_2-\alpha_i-1}\,d\mathfrak{H}_i(\varsigma)\right)\int_0^1\left(\sum_{i=1}^{m}\int_0^1 \mathfrak{g}_{1i}(\varsigma,\nu)\,d\mathfrak{K}_i(\varsigma)\right)$$
$$\times\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu,\phi(\nu),\psi(\nu)))\,d\nu$$
$$+\frac{(\zeta_1-1)\eta^{\zeta_1-2}\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)}\int_0^1\left(\sum_{i=1}^{n}\int_0^1 \mathfrak{g}_{2i}(\varsigma,\nu)\,d\mathfrak{H}_i(\varsigma)\right)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\Upsilon(\nu,\phi(\nu),\psi(\nu)))\,d\nu.$$

Hence for any $\eta\in(0,1)$ we find

$$|(\mathfrak{T}_1(\phi,\psi))'(\eta)|$$
$$\le\frac{1}{\Gamma(\zeta_1-1)}\int_0^\eta\left[\eta^{\zeta_1-2}(1-\nu)^{\zeta_1-\alpha_0-1}+(\eta-\nu)^{\zeta_1-2}\right]$$
$$\times\varphi_{\omega_1}(I_{0+}^{\gamma_1}(\theta_1(\nu)\mu_1(\nu,\phi(\nu),\psi(\nu))))\,d\nu$$
$$+\frac{1}{\Gamma(\zeta_1-1)}\int_\eta^1 \eta^{\zeta_1-2}(1-\nu)^{\zeta_1-\alpha_0-1}\varphi_{\omega_1}(I_{0+}^{\gamma_1}(\theta_1(\nu)\mu_1(\nu,\phi(\nu),\psi(\nu))))\,d\nu$$
$$+\frac{(\zeta_1-1)\eta^{\zeta_1-2}}{\mathfrak{a}}\left(\sum_{i=1}^{n}\frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_i)}\int_0^1 \varsigma^{\zeta_2-\alpha_i-1}\,d\mathfrak{H}_i(\varsigma)\right)\int_0^1\left(\sum_{i=1}^{m}\int_0^1 \mathfrak{g}_{1i}(\varsigma,\nu)\,d\mathfrak{K}_i(\varsigma)\right)$$
$$\times\varphi_{\omega_1}(I_{0+}^{\gamma_1}(\theta_1(\nu)\mu_1(\nu,\phi(\nu),\psi(\nu))))\,d\nu$$
$$+\frac{(\zeta_1-1)\eta^{\zeta_1-2}\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)}\int_0^1\left(\sum_{i=1}^{n}\int_0^1 \mathfrak{g}_{2i}(\varsigma,\nu)d\mathfrak{H}_i(\varsigma)\right)\varphi_{\omega_2}(I_{0+}^{\gamma_2}(\theta_2(\nu)\mu_2(\nu,\phi(\nu),\psi(\nu))))d\nu.$$

Therefore for any $\eta \in (0,1)$ we deduce

$$\begin{aligned}
&|(\mathfrak{T}_1(\phi,\psi))'(\eta)| \\
&\leq \Xi_2^{\omega_1-1}\left[\frac{1}{\Gamma(\zeta_1-1)}\int_0^\eta \left[\eta^{\zeta_1-2}(1-\nu)^{\zeta_1-\alpha_0-1}+(\eta-\nu)^{\zeta_1-2}\right]\right. \\
&\quad \times \varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu + \frac{1}{\Gamma(\zeta_1-1)}\int_\eta^1 \eta^{\zeta_1-2}(1-\nu)^{\zeta_1-\alpha_0-1}\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu \\
&\quad + \frac{(\zeta_1-1)\eta^{\zeta_1-2}}{\mathfrak{a}}\left(\sum_{i=1}^n \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_i)}\int_0^1 \varsigma^{\zeta_2-\alpha_i-1}\,d\mathfrak{H}_i(\varsigma)\right) \\
&\quad \left.\times \int_0^1 \left(\sum_{i=1}^m \int_0^1 \mathfrak{g}_{1i}(\varsigma,\nu)\,d\mathfrak{K}_i(\varsigma)\right)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu\right] \\
&\quad + \Xi_2^{\omega_2-1}\frac{(\zeta_1-1)\eta^{\zeta_1-2}\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)}\int_0^1\left(\sum_{i=1}^n \int_0^1 \mathfrak{g}_{2i}(\varsigma,\nu)\,d\mathfrak{H}_i(\varsigma)\right)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\,d\nu \\
&=: \sigma(\eta).
\end{aligned} \tag{3.2}$$

We denote by

$$\begin{aligned}
\sigma_0(\eta) &= \frac{1}{\Gamma(\zeta_1-1)}\int_0^1 \left[\eta^{\zeta_1-2}(1-\nu)^{\zeta_1-\alpha_0-1}+(\eta-\nu)^{\zeta_1-2}\right]\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu \\
&\quad + \frac{1}{\Gamma(\zeta_1-1)}\int_\eta^1 \eta^{\zeta_1-2}(1-\nu)^{\zeta_1-\alpha_0-1}\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu.
\end{aligned}$$

For the integral of function σ_0, by exchanging the order of integration, we conclude that

$$\begin{aligned}
\int_0^1 \sigma_0(\eta)\,d\eta &= \frac{1}{\Gamma(\zeta_1)}\int_0^1 (1-\nu)^{\zeta_1-\alpha_0-1}(1+(1-\nu)^{\alpha_0})\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu \\
&\leq \frac{2}{\Gamma(\zeta_1)}\int_0^1 (1-\nu)^{\zeta_1-\alpha_0-1}\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu = \frac{2L_1}{\Gamma(\zeta_1)} < \infty.
\end{aligned}$$

The integral of function σ is

$$\begin{aligned}
&\int_0^1 \sigma(\eta)\,d\eta \\
&= \Xi_2^{\omega_1-1}\int_0^1 \sigma_0(\eta)\,d\eta + \Xi_2^{\omega_1-1}\frac{\zeta_1-1}{\mathfrak{a}}\left(\int_0^1 \eta^{\zeta_1-2}\,d\eta\right)\left(\sum_{i=1}^n \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_i)}\right. \\
&\quad \left.\times \int_0^1 \varsigma^{\zeta_2-\alpha_i-1}\,d\mathfrak{H}_i(\varsigma)\right)\int_0^1\left(\sum_{i=1}^m \int_0^1 \mathfrak{g}_{1i}(\varsigma,\nu)\,d\mathfrak{K}_i(\varsigma)\right)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu \\
&\quad + \Xi_2^{\omega_2-1}\frac{(\zeta_1-1)\Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2-\beta_0)}\left(\int_0^1 \eta^{\zeta_1-2}\,d\eta\right) \\
&\quad \times \int_0^1\left(\sum_{i=1}^n \int_0^1 \mathfrak{g}_{2i}(\varsigma,\nu)\,d\mathfrak{H}_i(\varsigma)\right)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\,d\nu \\
&\leq \frac{2L_1\Xi_2^{\omega_1-1}}{\Gamma(\zeta_1)} + \frac{L_1\Xi_2^{\omega_1-1}}{\mathfrak{a}}\left(\sum_{i=1}^n \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2-\alpha_i)}\int_0^1 \varsigma^{\zeta_2-\alpha_i-1}\,d\mathfrak{H}_i(\varsigma)\right)
\end{aligned}$$

$$\times \left(\sum_{i=1}^{m} \int_0^1 \frac{1}{\Gamma(\zeta_1 - \beta_i)} \varsigma^{\zeta_1 - \beta_i - 1} \, d\mathfrak{K}_i(\varsigma) \right)$$
$$+ \frac{L_2 \Xi_2^{\omega_2 - 1} \Gamma(\zeta_2)}{\mathfrak{a}\Gamma(\zeta_2 - \beta_0)} \left(\sum_{i=1}^{n} \int_0^1 \frac{1}{\Gamma(\zeta_2 - \beta_i)} \varsigma^{\zeta_2 - \alpha_i - 1} \, d\mathfrak{H}_i(\varsigma) \right) < \infty. \tag{3.3}$$

Then we obtain that $\sigma \in L^1(0,1)$. So for any $\tau_1, \tau_2 \in [0,1]$ with $\tau_1 \le \tau_2$ and $(\phi, \psi) \in \mathfrak{D}$, by relations (3.2) and (3.3) we find

$$|\mathfrak{T}_1(\phi,\psi)(\tau_1) - \mathfrak{T}_1(\phi,\psi)(\tau_2)| = \left| \int_{\tau_1}^{\tau_2} (\mathfrak{T}_1(\phi,\psi))'(\eta) \, d\eta \right| \le \int_{\tau_1}^{\tau_2} \sigma(\eta) \, d\eta. \tag{3.4}$$

By (3.3), (3.4) and the property of absolute continuity of the integral function, we obtain that $\mathfrak{T}_1(\mathfrak{D})$ is equicontinuous. By using similar arguments, we deduce that $\mathfrak{T}_2(\mathfrak{D})$ is also equicontinuous, and then $\mathfrak{T}(\mathfrak{D})$ is equicontinuous. We apply now the Arzela-Ascoli theorem, and we find that $\mathfrak{T}_1(\mathfrak{D})$ and $\mathfrak{T}_2(\mathfrak{D})$ are relatively compact sets, and so $\mathfrak{T}(\mathfrak{D})$ is a relatively compact set. Then $\mathfrak{T}$ is a compact operator. In addition, we can prove that $\mathfrak{T}_1$, $\mathfrak{T}_2$ and $\mathfrak{T}$ are continuous on $\mathfrak{P}$ (see Lemma 1.4.1 from [12]). Therefore $\mathfrak{T}$ is a completely continuous operator on $\mathfrak{P}$. □

Now we define the cone

$$\mathfrak{P}_0 = \left\{ (\phi,\psi) \in \mathfrak{P}, \ \min_{\eta \in [0,1]} \phi(\eta) \ge \eta^{\zeta_1 - 1} \|\phi\|, \ \min_{\eta \in [0,1]} \psi(\eta) \ge \eta^{\zeta_2 - 1} \|\psi\| \right\}.$$

Under the assumptions $(A1)$ and $(A2)$, by using Lemma 2.3, we find $\mathfrak{T}(\mathfrak{P}_0) \subset \mathfrak{P}_0$, and $\mathfrak{T}|_{\mathfrak{P}_0} : \mathfrak{P}_0 \to \mathfrak{P}_0$ (denoted again by $\mathfrak{T}$) is also a completely continuous operator. For $\xi > 0$, we denote by B_ξ the open ball centered at zero of radius ξ, by $\overline{B}_\xi$ its closure, and by ∂B_ξ its boundary.

Theorem 3.1. *We suppose that $(A1)$ and $(A2)$ hold. If the functions μ_1, μ_2, Λ and Υ also satisfy the conditions*

$(A3)$ *There exist $\rho_1 \ge 1$ and $\rho_2 \ge 1$ such that*

$$\mu_{10} = \lim_{\substack{z+w \to 0 \\ z,w \ge 0}} \sup_{\eta \in [0,1]} \frac{\mu_1(\eta,z,w)}{\varphi_{\delta_1}((z+w)^{\rho_1})} = 0, \ \textit{and} \ \mu_{20} = \lim_{\substack{z+w \to 0 \\ z,w \ge 0}} \sup_{\eta \in [0,1]} \frac{\mu_2(\eta,z,w)}{\varphi_{\delta_2}((z+w)^{\rho_2})} = 0;$$

$(A4)$ *There exists $[\sigma_1, \sigma_2] \subset [0,1]$, $0 < \sigma_1 < \sigma_2 < 1$ such that*

$$\Lambda_\infty^i = \lim_{\substack{z+w \to \infty \\ z,w \ge 0}} \inf_{\eta \in [\sigma_1,\sigma_2]} \frac{\Lambda(\eta,z,w)}{\varphi_{\delta_1}(z+w)} = \infty, \ \textit{or} \ \Upsilon_\infty^i = \lim_{\substack{z+w \to \infty \\ z,w \ge 0}} \inf_{\eta \in [\sigma_1,\sigma_2]} \frac{\Upsilon(\eta,z,w)}{\varphi_{\delta_2}(z+w)} = \infty,$$

then problem (1.1),(1.2) has at least one positive solution $(\phi(\eta), \psi(\eta))$, $\eta \in [0,1]$.

Proof. We consider the cone $\mathfrak{P}_0$ defined above. By assumption $(A3)$, if $\Psi_2 > 0$ and $\Psi_3 > 0$, we deduce that for $\epsilon_1 = \min\left\{ \frac{1}{(4\Psi_1)^{\delta_1 - 1}}, \frac{1}{(4\Psi_3)^{\delta_1 - 1}} \right\}$ and $\epsilon_2 = \min\left\{ \frac{1}{(4\Psi_2)^{\delta_2 - 1}}, \frac{1}{(4\Psi_4)^{\delta_2 - 1}} \right\}$, there exists $R_1 \in (0,1)$ such that

$$\mu_i(\eta,z,w) \le \epsilon_i (z+w)^{\rho_i(\delta_i - 1)}, \ \forall \eta \in [0,1], \ z,w \ge 0, \ z+w \le R_1, \ i = 1,2. \tag{3.5}$$

Then by (3.5) and Lemma 2.2, for any $(\phi,\psi)\in\partial B_{R_1}\cap\mathfrak{P}_0$ and $\eta\in[0,1]$, we find

$$\begin{aligned}
&\mathfrak{T}_1(\phi,\psi)(\eta)\\
&\le \int_0^1 \mathfrak{J}_1(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}(\theta_1(\nu)\mu_1(\nu,\phi(\nu),\psi(\nu))))\,d\nu\\
&\quad+\int_0^1 \mathfrak{J}_2(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}(\theta_2(\nu)\mu_2(\nu,\phi(\nu),\psi(\nu))))\,d\nu\\
&\le \int_0^1 \mathfrak{J}_1(\nu)\varphi_{\omega_1}\left(I_{0+}^{\gamma_1}\left(\theta_1(\nu)\epsilon_1(\phi(\nu)+\psi(\nu))^{\rho_1(\delta_1-1)}\right)\right)\,d\nu\\
&\quad+\int_0^1 \mathfrak{J}_2(\nu)\varphi_{\omega_2}\left(I_{0+}^{\gamma_2}\left(\theta_2(\nu)\epsilon_2(\phi(\nu)+\psi(\nu))^{\rho_2(\delta_2-1)}\right)\right)\,d\nu\\
&\le \epsilon_1^{\omega_1-1}\int_0^1 \mathfrak{J}_1(\nu)\varphi_{\omega_1}\left(I_{0+}^{\gamma_1}\theta_1(\nu)(\|\phi\|+\|\psi\|)^{\rho_1(\delta_1-1)}\right)\,d\nu\\
&\quad+\epsilon_2^{\omega_2-1}\int_0^1 \mathfrak{J}_2(\nu)\varphi_{\omega_2}\left(I_{0+}^{\gamma_2}\theta_2(\nu)(\|\phi\|+\|\psi\|)^{\rho_2(\delta_2-1)}\right)\,d\nu\\
&= \epsilon_1^{\omega_1-1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\rho_1}\int_0^1 \mathfrak{J}_1(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu\\
&\quad+\epsilon_2^{\omega_2-1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\rho_2}\int_0^1 \mathfrak{J}_2(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\,d\nu\\
&= \Psi_1\epsilon_1^{\omega_1-1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\rho_1}+\Psi_2\epsilon_2^{\omega_2-1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\rho_2}\\
&\le (\Psi_1\epsilon_1^{\omega_1-1}+\Psi_2\epsilon_2^{\omega_2-1})\|(\phi,\psi)\|_{\mathfrak{Y}}\le\left(\tfrac14+\tfrac14\right)\|(\phi,\psi)\|_{\mathfrak{Y}}=\tfrac12\|(\phi,\psi)\|_{\mathfrak{Y}},
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{T}_2(\phi,\psi)(\tau)\\
&\le \int_0^1 \mathfrak{J}_3(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}(\theta_1(\nu)\mu_1(\nu,\phi(\nu),\psi(\nu))))\,d\nu\\
&\quad+\int_0^1 \mathfrak{J}_4(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}(\theta_2(\nu)\mu_2(\nu,\phi(\nu),\psi(\nu))))\,d\nu\\
&\le \int_0^1 \mathfrak{J}_3(\nu)\varphi_{\omega_1}\left(I_{0+}^{\gamma_1}\left(\theta_1(\nu)\epsilon_1(\phi(\nu)+\psi(\nu))^{\rho_1(\delta_1-1)}\right)\right)\,d\nu\\
&\quad+\int_0^1 \mathfrak{J}_4(\nu)\varphi_{\omega_2}\left(I_{0+}^{\gamma_2}\left(\theta_2(\nu)\epsilon_2(\phi(\nu)+\psi(\nu))^{\rho_2(\delta_2-1)}\right)\right)\,d\nu\\
&\le \epsilon_1^{\omega_1-1}\int_0^1 \mathfrak{J}_3(\nu)\varphi_{\omega_1}\left(I_{0+}^{\gamma_1}\theta_1(\nu)(\|\phi\|+\|\psi\|)^{\rho_1(\delta_1-1)}\right)\,d\nu\\
&\quad+\epsilon_2^{\omega_2-1}\int_0^1 \mathfrak{J}_4(\nu)\varphi_{\omega_2}\left(I_{0+}^{\gamma_2}\theta_2(\nu)(\|\phi\|+\|\psi\|)^{\rho_2(\delta_2-1)}\right)\,d\nu\\
&= \epsilon_1^{\omega_1-1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\rho_1}\int_0^1 \mathfrak{J}_3(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu\\
&\quad+\epsilon_2^{\omega_2-1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\rho_2}\int_0^1 \mathfrak{J}_4(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\,d\nu\\
&= \Psi_3\epsilon_1^{\omega_1-1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\rho_1}+\Psi_4\epsilon_2^{\omega_2-1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\rho_2}\\
&\le (\Psi_3\epsilon_1^{\omega_1-1}+\Psi_4\epsilon_2^{\omega_2-1})\|(\phi,\psi)\|_{\mathfrak{Y}}\le\left(\tfrac14+\tfrac14\right)\|(\phi,\psi)\|_{\mathfrak{Y}}=\tfrac12\|(\phi,\psi)\|_{\mathfrak{Y}}.
\end{aligned}$$

So we conclude that $\|\mathfrak{T}_1(\phi,\psi)\|\le\frac12\|(\phi,\psi)\|_{\mathfrak{Y}}$, $\|\mathfrak{T}_2(\phi,\psi)\|\le\frac12\|(\phi,\psi)\|_{\mathfrak{Y}}$ for all $(\phi,\psi)\in\partial B_{R_1}\cap\mathfrak{P}_0$, and then

$$\|\mathfrak{T}(\phi,\psi)\|_{\mathfrak{Y}}\le\|(\phi,\psi)\|_{\mathfrak{Y}},\ \ \forall\,(\phi,\psi)\in\partial B_{R_1}\cap\mathfrak{P}_0. \tag{3.6}$$

If $\Psi_2 = 0$ and $\Psi_3 \neq 0$, then we choose $\epsilon_1 = \min\left\{\frac{1}{(4\Psi_1)^{\delta_1-1}}, \frac{1}{(4\Psi_3)^{\delta_1-1}}\right\}$ and $\epsilon_2 = \frac{1}{(2\Psi_4)^{\delta_2-1}}$; if $\Psi_2 \neq 0$ and $\Psi_3 = 0$, then we choose $\epsilon_1 = \frac{1}{(2\Psi_1)^{\delta_1-1}}$ and $\epsilon_2 = \min\left\{\frac{1}{(4\Psi_2)^{\delta_2-1}}, \frac{1}{(4\Psi_4)^{\delta_2-1}}\right\}$; if $\Psi_2 = \Psi_3 = 0$, then we choose $\epsilon_1 = \frac{1}{(2\Psi_1)^{\delta_1-1}}$ and $\epsilon_2 = \frac{1}{(2\Psi_4)^{\delta_2-1}}$. In all these cases we obtain as above the inequality (3.6).

Next, in assumption $(A4)$, we suppose that $\Upsilon^i_\infty = \infty$ (in a similar manner we treat the case $\Lambda^i_\infty = \infty$). Then for $\epsilon_3 = 2(\mathfrak{E}\sigma_0)^{1-\delta_2}$, where

$$\mathfrak{E} = \frac{\sigma_1^{\zeta_1-1}}{(\Gamma(\gamma_2+1))^{\omega_2-1}} \int_{\sigma_1}^{\sigma_2} \mathfrak{J}_2(\nu)(\nu-\sigma_1)^{\gamma_2(\omega_2-1)}\, d\nu$$

and $\sigma_0 = \min\{\sigma_1^{\zeta_1-1}, \sigma_1^{\zeta_2-1}\}$, there exists $M_1 > 0$ such that

$$\Upsilon(\eta, z, w) \geq \epsilon_3(z+w)^{\delta_2-1} - M_1, \ \ \forall\, \eta \in [\sigma_1, \sigma_2], \ \ z,\, w \geq 0. \tag{3.7}$$

Then by (3.7), for any $(\phi, \psi) \in \mathfrak{P}_0$ and $\eta \in [\sigma_1, \sigma_2]$, we find

$$\begin{aligned}
&\mathfrak{T}_1(\phi,\psi)(\eta)\\
&\geq \int_0^1 \mathfrak{G}_2(\eta,\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\Upsilon(\nu,\phi(\nu),\psi(\nu)))\, d\nu\\
&\geq \int_{\sigma_1}^{\sigma_2} \mathfrak{G}_2(\eta,\nu)\left(\frac{1}{\Gamma(\gamma_2)}\int_{\sigma_1}^{\nu}(\nu-\varsigma)^{\gamma_2-1}\Upsilon(\varsigma,\phi(\varsigma),\psi(\varsigma))\, d\varsigma\right)^{\omega_2-1} d\nu\\
&\geq \sigma_1^{\zeta_1-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_2(\nu)\frac{1}{(\Gamma(\gamma_2))^{\omega_2-1}}\\
&\quad\times\left(\int_{\sigma_1}^{\nu}(\nu-\varsigma)^{\gamma_2-1}\left(\epsilon_3(\phi(\varsigma)+\psi(\varsigma))^{\delta_2-1}-M_1\right)\, d\varsigma\right)^{\omega_2-1} d\nu\\
&\geq \sigma_1^{\zeta_1-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_2(\nu)\frac{1}{(\Gamma(\gamma_2))^{\omega_2-1}}\left(\int_{\sigma_1}^{\nu}(\nu-\varsigma)^{\gamma_2-1}\right.\\
&\quad\left.\times\left(\epsilon_3\left(\sigma_1^{\zeta_1-1}\|\phi\|+\sigma_1^{\zeta_2-1}\|\psi\|\right)^{\delta_2-1}-M_1\right)\, d\varsigma\right)^{\omega_2-1} d\nu\\
&= \frac{\sigma_1^{\zeta_1-1}}{(\Gamma(\gamma_2))^{\omega_2-1}}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_2(\nu)\left(\epsilon_3\left(\sigma_1^{\zeta_1-1}\|\phi\|+\sigma_1^{\zeta_2-1}\|\psi\|\right)^{\delta_2-1}-M_1\right)^{\omega_2-1}\\
&\quad\times\frac{(\nu-\sigma_1)^{\gamma_2(\omega_2-1)}}{\gamma_2^{\omega_2-1}}\, d\nu\\
&\geq \frac{\sigma_1^{\zeta_1-1}}{(\Gamma(\gamma_2+1))^{\omega_2-1}}\left(\epsilon_3\sigma_0^{\delta_2-1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1}-M_1\right)^{\omega_2-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_2(\nu)(\nu-\sigma_1)^{\gamma_2(\omega_2-1)}\, d\nu\\
&= \left(\mathfrak{E}^{\frac{1}{\omega_2-1}}\epsilon_3\sigma_0^{\delta_2-1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1}-\mathfrak{E}^{\frac{1}{\omega_2-1}}M_1\right)^{\omega_2-1}\\
&= \left(2\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1}-M_2\right)^{\omega_2-1}, \ \ M_2 = \mathfrak{E}^{\delta_2-1}M_1.
\end{aligned}$$

Then we deduce

$$\|\mathfrak{T}_1(\phi,\psi)\| \geq \left(2\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1}-M_2\right)^{\omega_2-1}, \ \ \forall\,(\phi,\psi)\in\mathfrak{P}_0.$$

We choose $R_2 \geq \max\left\{1, M_2^{\omega_2-1}\right\}$ and we obtain

$$\|\mathfrak{T}(\phi,\psi)\|_{\mathfrak{Y}} \geq \|\mathfrak{T}_1(\phi,\psi)\| \geq \|(\phi,\psi)\|_{\mathfrak{Y}}, \ \ \forall\,(\phi,\psi)\in\partial B_{R_2}\cap\mathfrak{P}_0. \tag{3.8}$$

Now by Lemma 3.1, the relations (3.6), (3.8), and the Guo-Krasnosel'skii fixed point theorem, we conclude that operator $\mathfrak{T}$ has a fixed point $(\phi,\psi) \in (\overline{B}_{R_2} \setminus B_{R_1}) \cap \mathfrak{P}_0$, that is $R_1 \leq \|(\phi,\psi)\|_{\mathfrak{Y}} \leq R_2$, and $\phi(\eta) \geq \eta^{\zeta_1-1}\|\phi\|$ and $\psi(\eta) \geq \eta^{\zeta_2-1}\|\psi\|$ for all $\eta \in [0,1]$. Then $\|\phi\| > 0$ or $\|\psi\| > 0$, and so $\phi(\eta) > 0$ for all $\eta \in (0,1]$ or $\psi(\eta) > 0$ for all $\eta \in (0,1]$. Hence, $(\phi(\eta),\psi(\eta))$, $\eta \in [0,1]$ is a positive solution of problem (1.1),(1.2). □

Theorem 3.2. *We suppose that* $(A1)$ *and* $(A2)$ *hold. If the functions* μ_1, μ_2, Λ *and* Υ *also satisfy the conditions*

$(A5)$ $\mu_{1\infty} = \lim\limits_{\substack{z+w\to\infty \\ z,w\geq 0}} \sup\limits_{\eta\in[0,1]} \dfrac{\mu_1(\eta,z,w)}{\varphi_{\delta_1}(z+w)} = 0$, *and* $\mu_{2\infty} = \lim\limits_{\substack{z+w\to\infty \\ z,w\geq 0}} \sup\limits_{\eta\in[0,1]} \dfrac{\mu_2(\eta,z,w)}{\varphi_{\delta_2}(z+w)} = 0$;

$(A6)$ *There exist* $[\sigma_1,\sigma_2] \subset [0,1]$, $0 < \sigma_1 < \sigma_2 < 1$, $\varrho_1 \in (0,1]$ *and* $\varrho_2 \in (0,1]$ *such that*

$$\Lambda_0^i = \lim_{\substack{z+w\to 0 \\ z,w\geq 0}} \inf_{\eta\in[\sigma_1,\sigma_2]} \frac{\Lambda(\eta,z,w)}{\varphi_{\delta_1}((z+w)^{\varrho_1})} = \infty, \ \textit{or}$$

$$\Upsilon_0^i = \lim_{\substack{z+w\to \infty \\ z,w\geq 0}} \inf_{\eta\in[\sigma_1,\sigma_2]} \frac{\Upsilon(\eta,z,w)}{\varphi_{\delta_2}((z+w)^{\varrho_2})} = \infty,$$

then problem (1.1),(1.2) has at least one positive solution $(\phi(\eta),\psi(\eta))$, $\eta \in [0,1]$.

Proof. We consider again the cone $\mathfrak{P}_0$. By assumption $(A5)$, we find that for $\epsilon_4 \in \left(0, \min\left\{\frac{1}{\varphi_{\delta_1}(2\Psi_1+2\Psi_3)}, \frac{1}{2\varphi_{\delta_1}(\Psi_1+\Psi_3)}\right\}\right)$ and $\epsilon_5 \in \left(0, \min\left\{\frac{1}{\varphi_{\delta_2}(2\Psi_2+2\Psi_4)}, \frac{1}{2\varphi_{\delta_2}(\Psi_2+\Psi_4)}\right\}\right)$, there exist $M_3 > 0$ and $M_4 > 0$ such that

$$\mu_1(\eta,z,w) \leq \epsilon_4(z+w)^{\delta_1-1} + M_3,\ \mu_2(\eta,z,w) \leq \epsilon_5(z+w)^{\delta_2-1} + M_4,\ \forall \eta\in[0,1],\ z,w\geq 0. \tag{3.9}$$

By using $(A2)$ and (3.9), for any $(\phi,\psi) \in \mathfrak{P}_0$, we have

$$\begin{aligned}
&\mathfrak{T}_1(\phi,\psi)(\eta) \\
&\leq \int_0^1 \mathfrak{J}_1(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}(\theta_1(\nu)\mu_1(\nu,\phi(\nu),\psi(\nu)))\,d\nu \\
&\quad + \int_0^1 \mathfrak{J}_2(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}(\theta_2(\nu)\mu_2(\nu,\phi(\nu),\psi(\nu)))\,d\nu \\
&\leq \int_0^1 \mathfrak{J}_1(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}(\theta_1(\nu)(\epsilon_4(\phi(\nu)+\psi(\nu))^{\delta_1-1} + M_3)))\,d\nu \\
&\quad + \int_0^1 \mathfrak{J}_2(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}(\theta_2(\nu)(\epsilon_5(\phi(\nu)+\psi(\nu))^{\delta_2-1} + M_4)))\,d\nu \\
&\leq \int_0^1 \mathfrak{J}_1(\nu)\frac{1}{(\Gamma(\gamma_1))^{\omega_1-1}}\left(\epsilon_4\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_1-1} + M_3\right)^{\omega_1-1}\left(\int_0^\nu (\nu-\varsigma)^{\gamma_1-1}\theta_1(\varsigma)\,d\varsigma\right)^{\omega_1-1} d\nu \\
&\quad + \int_0^1 \mathfrak{J}_2(\nu)\frac{1}{(\Gamma(\gamma_2))^{\omega_2-1}}\left(\epsilon_5\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1} + M_4\right)^{\omega_2-1}\left(\int_0^\nu (\nu-\varsigma)^{\gamma_2-1}\theta_2(\varsigma)\,d\varsigma\right)^{\omega_2-1} d\nu \\
&\leq \Psi_1\left(\epsilon_4\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_1-1} + M_3\right)^{\omega_1-1} + \Psi_2\left(\epsilon_5\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1} + M_4\right)^{\omega_2-1},\ \ \forall \eta\in[0,1], \\
&\mathfrak{T}_2(\phi,\psi)(\eta) \\
&\leq \int_0^1 \mathfrak{J}_3(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}(\theta_1(\nu)\mu_1(\nu,\phi(\nu),\psi(\nu)))\,d\nu
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \mathfrak{J}_4(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}(\theta_2(\nu)\mu_2(\nu,\phi(\nu),\psi(\nu)))\,d\nu\\
\le&\int_0^1 \mathfrak{J}_3(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}(\theta_1(\nu)(\epsilon_4(\phi(\nu)+\psi(\nu))^{\delta_1-1}+M_3)))\,d\nu\\
&+\int_0^1 \mathfrak{J}_4(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}(\theta_2(\nu)(\epsilon_5(\phi(\nu)+\psi(\nu))^{\delta_2-1}+M_4)))\,d\nu\\
\le&\int_0^1 \mathfrak{J}_3(\nu)\frac{1}{(\Gamma(\gamma_1))^{\omega_1-1}}\left(\epsilon_4\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_1-1}+M_3\right)^{\omega_1-1}\left(\int_0^\nu(\nu-\varsigma)^{\gamma_1-1}\theta_1(\varsigma)\,d\varsigma\right)^{\omega_1-1}d\nu\\
&+\int_0^1 \mathfrak{J}_4(\nu)\frac{1}{(\Gamma(\gamma_2))^{\omega_2-1}}\left(\epsilon_5\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1}+M_4\right)^{\omega_2-1}\left(\int_0^\nu(\nu-\varsigma)^{\gamma_2-1}\theta_2(\varsigma)\,d\varsigma\right)^{\omega_2-1}d\nu\\
\le&\Psi_3\left(\epsilon_4\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_1-1}+M_3\right)^{\omega_1-1}+\Psi_4\left(\epsilon_5\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1}+M_4\right)^{\omega_2-1},\ \ \forall\eta\in[0,1].
\end{aligned}
$$

Then we deduce

$$
\begin{aligned}
\|\mathfrak{T}_1(\phi,\psi)\| &\le \Psi_1\left(\epsilon_4\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_1-1}+M_3\right)^{\omega_1-1}+\Psi_2\left(\epsilon_5\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1}+M_4\right)^{\omega_2-1},\\
\|\mathfrak{T}_2(\phi,\psi)\| &\le \Psi_3\left(\epsilon_4\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_1-1}+M_3\right)^{\omega_1-1}+\Psi_4\left(\epsilon_5\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1}+M_4\right)^{\omega_2-1},
\end{aligned}
$$

and so

$$
\begin{aligned}
\|\mathfrak{T}(\phi,\psi)\|_{\mathfrak{Y}} \le{}& (\Psi_1+\Psi_3)\left(\epsilon_4\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_1-1}+M_3\right)^{\omega_1-1}\\
&+(\Psi_2+\Psi_4)\left(\epsilon_5\|(\phi,\psi)\|_{\mathfrak{Y}}^{\delta_2-1}+M_4\right)^{\omega_2-1},\ \forall(\phi,\psi)\in\mathfrak{P}_0.
\end{aligned}
$$

We choose

$$
\begin{aligned}
R_3>\max\Bigg\{&1,\frac{(\Psi_1+\Psi_3)M_3^{\omega_1-1}+(\Psi_2+\Psi_4)M_4^{\omega_2-1}}{1-\left[(\Psi_1+\Psi_3)\epsilon_4^{\omega_1-1}+(\Psi_2+\Psi_4)\epsilon_5^{\omega_2-1}\right]},\\
&\frac{(\Psi_1+\Psi_3)2^{\omega_1-2}M_3^{\omega_1-1}+(\Psi_2+\Psi_4)2^{\omega_2-2}M_4^{\omega_2-1}}{1-\left[(\Psi_1+\Psi_3)2^{\omega_1-2}\epsilon_4^{\omega_1-1}+(\Psi_2+\Psi_4)2^{\omega_2-2}\epsilon_5^{\omega_2-1}\right]},\\
&\frac{(\Psi_1+\Psi_3)M_3^{\omega_1-1}+(\Psi_2+\Psi_4)2^{\omega_2-2}M_4^{\omega_2-1}}{1-\left[(\Psi_1+\Psi_3)\epsilon_4^{\omega_1-1}+(\Psi_2+\Psi_4)2^{\omega_2-2}\epsilon_5^{\omega_2-1}\right]},\\
&\frac{(\Psi_1+\Psi_3)2^{\omega_1-2}M_3^{\omega_1-1}+(\Psi_2+\Psi_4)M_4^{\omega_2-1}}{1-\left[(\Psi_1+\Psi_3)2^{\omega_1-2}\epsilon_4^{\omega_1-1}+(\Psi_2+\Psi_4)\epsilon_5^{\omega_2-1}\right]}\Bigg\},
\end{aligned}
\tag{3.10}
$$

and then we conclude that

$$
\|\mathfrak{T}(\phi,\psi)\|_{\mathfrak{Y}}\le\|(\phi,\psi)\|_{\mathfrak{Y}},\ \ \forall\,(\phi,\psi)\in\partial B_{R_3}\cap\mathfrak{P}_0. \tag{3.11}
$$

In the definition of R_3, we relied on the well-known inequalities $(c+d)^\theta\le 2^{\theta-1}(c^\theta+d^\theta)$ for $\theta\ge1$ and $c,d\ge0$, and $(c+d)^\theta\le c^\theta+d^\theta$ for $\theta\in(0,1]$ and $c,d\ge0$. Here $\theta=\omega_1-1$ or ω_2-1. We prove inequality (3.11) in one case, namely $\omega_1\in(1,2]$ and $\omega_2\ge2$. In this case, by using (3.10) and the inequalities $(\Psi_1+\Psi_3)\epsilon_4^{\omega_1-1}<\frac12$, $(\Psi_2+\Psi_4)2^{\omega_2-2}\epsilon_5^{\omega_2-1}<\frac12$ (from the definition of ϵ_4 and ϵ_5) we have the inequalities

$$
(\Psi_1+\Psi_3)(\epsilon_4R_3^{\delta_1-1}+M_3)^{\omega_1-1}+(\Psi_2+\Psi_4)(\epsilon_5R_3^{\delta_2-1}+M_4)^{\omega_2-1}
$$

$$\begin{aligned}&\le(\Psi_1+\Psi_3)(\epsilon_4^{\omega_1-1}R_3+M_3^{\omega_1-1})+(\Psi_2+\Psi_4)2^{\omega_2-2}(\epsilon_5^{\omega_2-1}R_3+M_4^{\omega_2-1})\\&=\left[(\Psi_1+\Psi_3)\epsilon_4^{\omega_1-1}+(\Psi_2+\Psi_4)2^{\omega_2-2}\epsilon_5^{\omega_2-1}\right]R_3\\&\quad+(\Psi_1+\Psi_3)M_3^{\omega_1-1}+(\Psi_2+\Psi_4)2^{\omega_2-2}M_4^{\omega_2-1}<R_3.\end{aligned}$$

In a similar manner we consider the cases $\omega_1 \in (1,2]$ and $\omega_2 \in (1,2]$; $\omega_1 \ge 2$ and $\omega_2 \in (1,2]$; $\omega_1 \ge 2$ and $\omega_2 \ge 2$.

Next, in assumption $(A6)$, we consider $\Lambda_0^i = \infty$ (in a similar manner we can study the case $\Upsilon_0^i = \infty$). For $\epsilon_6 = \sigma_0^{\varrho_1(1-\delta_1)}\widetilde{\mathfrak{E}}^{1-\delta_1}$, where $\sigma_0 = \min\left\{\sigma_1^{\zeta_1-1}, \sigma_1^{\zeta_2-1}\right\}$ and $\widetilde{\mathfrak{E}} = \frac{\sigma_1^{\zeta_1-1}}{(\Gamma(\gamma_1+1))^{\omega_1-1}}\int_{\sigma_1}^{\sigma_2}(\nu-\sigma_1)^{\gamma_1(\omega_1-1)}\mathfrak{J}_1(\nu)\,d\nu$, there exists $R_4 \in (0,1]$ such that

$$\Lambda(\eta,z,w) \ge \epsilon_6(z+w)^{\varrho_1(\delta_1-1)},\ \ \forall\,\eta\in[\sigma_1,\sigma_2],\ z,w\ge 0,\ z+w\le R_4. \tag{3.12}$$

Then by using (3.12), for any $(\phi,\psi)\in\partial B_{R_4}\cap\mathfrak{P}_0$ and $\eta\in[\sigma_1,\sigma_2]$ we obtain

$$\begin{aligned}&\mathfrak{T}_1(\phi,\psi)(\eta)\\&\ge\int_0^1\mathfrak{G}_1(\eta,\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\Lambda(\nu,\phi(\nu),\psi(\nu)))\,d\nu\\&\ge\int_{\sigma_1}^{\sigma_2}\mathfrak{G}_1(\eta,\nu)\left(\frac{1}{\Gamma(\gamma_1)}\int_{\sigma_1}^{\nu}(\nu-\varsigma)^{\gamma_1-1}\Lambda(\varsigma,\phi(\varsigma),\psi(\varsigma))\,d\varsigma\right)^{\omega_1-1}d\nu\\&\ge\sigma_1^{\zeta_1-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_1(\nu)\frac{1}{(\Gamma(\gamma_1))^{\omega_1-1}}\left(\int_{\sigma_1}^{\nu}(\nu-\varsigma)^{\gamma_1-1}\epsilon_6(\phi(\varsigma)+\psi(\varsigma))^{\varrho_1(\delta_1-1)}\,d\varsigma\right)^{\omega_1-1}d\nu\\&\ge\sigma_1^{\zeta_1-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_1(\nu)\frac{1}{(\Gamma(\gamma_1))^{\omega_1-1}}\\&\quad\times\left(\int_{\sigma_1}^{\nu}(\nu-\varsigma)^{\gamma_1-1}\epsilon_6\left(\sigma_1^{\zeta_1-1}\|\phi\|+\sigma_1^{\zeta_2-1}\|\psi\|\right)^{\varrho_1(\delta_1-1)}d\varsigma\right)^{\omega_1-1}d\nu\\&\ge\sigma_1^{\zeta_1-1}\epsilon_6^{\omega_1-1}\sigma_0^{\varrho_1}\|(\phi,\psi)\|_{\mathfrak{Y}}^{\varrho_1}\frac{1}{(\Gamma(\gamma_1+1))^{\omega_1-1}}\int_{\sigma_1}^{\sigma_2}(\nu-\sigma_1)^{\gamma_1(\omega_1-1)}\mathfrak{J}_1(\nu)\,d\nu\\&=\|(\phi,\psi)\|_{\mathfrak{Y}}^{\varrho_1}\ge\|(\phi,\psi)\|_{\mathfrak{Y}}.\end{aligned}$$

Therefore $\|\mathfrak{T}_1(\phi,\psi)\| \ge \|(\phi,\psi)\|_{\mathfrak{Y}}$ for all $(\phi,\psi)\in\partial B_{R_4}\cap\mathfrak{P}_0$, and then

$$\|\mathfrak{T}(\phi,\psi)\|_{\mathfrak{Y}} \ge \|\mathfrak{T}_1(\phi,\psi)\| \ge \|(\phi,\psi)\|_{\mathfrak{Y}},\ \ \forall\,(\phi,\psi)\in\partial B_{R_4}\cap\mathfrak{P}_0. \tag{3.13}$$

So, by using Lemma 3.1, the relations (3.11), (3.13), and the Guo-Krasnosel'skii fixed point theorem, we deduce that operator $\mathfrak{T}$ has at least one fixed point $(\phi,\psi)\in(\overline{B}_{R_3}\setminus B_{R_4})\cap\mathfrak{P}_0$, that is $R_3 \le \|(\phi,\psi)\|_{\mathfrak{Y}} \le R_4$, which is a positive solution of problem (1.1),(1.2). □

Theorem 3.3. *We suppose that* $(A1)$, $(A2)$, $(A4)$ *and* $(A6)$ *hold. If the functions* μ_1 *and* μ_2 *also satisfy the assumption*

$(A7)$ $\mathcal{D}_0^{\omega_1-1}\Psi_1 < \frac{1}{4},\ \mathcal{D}_0^{\omega_2-1}\Psi_2 < \frac{1}{4},\ \mathcal{D}_0^{\omega_1-1}\Psi_3 < \frac{1}{4},\ \mathcal{D}_0^{\omega_2-1}\Psi_4 < \frac{1}{4},$

where $\mathcal{D}_0 = \max\{\max_{\eta,z,w\in[0,1]}\mu_1(\eta,z,w), \max_{\eta,z,w\in[0,1]}\mu_2(\eta,z,w)\}$*, then problem (1.1),(1.2) has at least two positive solutions* $(\phi_1(\eta),\psi_1(\eta))$, $(\phi_2(\eta),\psi_2(\eta))$, $\eta\in[0,1]$.

Proof. If assumptions $(A1)$, $(A2)$ and $(A4)$ are satisfied, then by the proof of Theorem 3.1, we find that there exists $R_2 > 1$ such that

$$\|\mathfrak{T}(\phi,\psi)\|_{\mathfrak{Y}} \geq \|(\phi,\psi)\|_{\mathfrak{Y}}, \ \forall\,(\phi,\psi) \in \partial B_{R_2} \cap \mathfrak{P}_0. \tag{3.14}$$

If assumptions $(A1)$, $(A2)$ and $(A6)$ are satisfied, then by the proof of Theorem 3.2, we deduce that there exists $R_4 < 1$ (we can choose $R_4 < 1$) such that

$$\|\mathfrak{T}(\phi,\psi)\|_{\mathfrak{Y}} \geq \|(\phi,\psi)\|_{\mathfrak{Y}}, \ \forall\,(\phi,\psi) \in \partial B_{R_4} \cap \mathfrak{P}_0. \tag{3.15}$$

We consider now the set $B_1 = \{(\phi,\psi) \in \mathfrak{Y}, \ \|(\phi,\psi)\|_{\mathfrak{Y}} < 1\}$. By $(A7)$, for any $(\phi,\psi) \in \partial B_1 \cap \mathfrak{P}_0$ and $\eta \in [0,1]$, we obtain

$$\begin{aligned}
&\mathfrak{T}_1(\phi,\psi)(\eta)\\
&\leq \int_0^1 \mathfrak{J}_1(\nu)\left(\frac{1}{\Gamma(\gamma_1)}\int_0^\nu (\nu-\varsigma)^{\gamma_1-1}\theta_1(\varsigma)\mu_1(\varsigma,\phi(\varsigma),\psi(\varsigma))\,d\varsigma\right)^{\omega_1-1} d\nu\\
&\quad + \int_0^1 \mathfrak{J}_2(\nu)\left(\frac{1}{\Gamma(\gamma_2)}\int_0^\nu (\nu-\varsigma)^{\gamma_2-1}\theta_2(\varsigma)\mu_2(\varsigma,\phi(\varsigma),\psi(\varsigma))\,d\varsigma\right)^{\omega_2-1} d\nu\\
&\leq \mathcal{D}_0^{\omega_1-1}\int_0^1 \mathfrak{J}_1(\nu)\left(\frac{1}{\Gamma(\gamma_1)}\int_0^\nu (\nu-\varsigma)^{\gamma_1-1}\theta_1(\varsigma)\,d\varsigma\right)^{\omega_1-1} d\nu\\
&\quad + \mathcal{D}_0^{\omega_2-1}\int_0^1 \mathfrak{J}_2(\nu)\left(\frac{1}{\Gamma(\gamma_2)}\int_0^\nu (\nu-\varsigma)^{\gamma_2-1}\theta_2(\varsigma)\,d\varsigma\right)^{\omega_2-1} d\nu\\
&= \mathcal{D}_0^{\omega_1-1}\int_0^1 \mathfrak{J}_1(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu + \mathcal{D}_0^{\omega_2-1}\int_0^1 \mathfrak{J}_2(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\,d\nu\\
&= \mathcal{D}_0^{\omega_1-1}\Psi_1 + \mathcal{D}_0^{\omega_2-1}\Psi_2 < \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2},\\
&\mathfrak{T}_2(\phi,\psi)(\eta) \leq \int_0^1 \mathfrak{J}_3(\nu)\left(\frac{1}{\Gamma(\gamma_1)}\int_0^\nu (\nu-\varsigma)^{\gamma_1-1}\theta_1(\varsigma)\mu_1(\varsigma,\phi(\varsigma),\psi(\varsigma))\,d\varsigma\right)^{\omega_1-1} d\nu\\
&\quad + \int_0^1 \mathfrak{J}_4(\nu)\left(\frac{1}{\Gamma(\gamma_2)}\int_0^\nu (\nu-\varsigma)^{\gamma_2-1}\theta_2(\varsigma)\mu_2(\varsigma,\phi(\varsigma),\psi(\varsigma))\,d\varsigma\right)^{\omega_2-1} d\nu\\
&\leq \mathcal{D}_0^{\omega_1-1}\int_0^1 \mathfrak{J}_3(\nu)\left(\frac{1}{\Gamma(\gamma_1)}\int_0^\nu (\nu-\varsigma)^{\gamma_1-1}\theta_1(\varsigma)\,d\varsigma\right)^{\omega_1-1} d\nu\\
&\quad + \mathcal{D}_0^{\omega_2-1}\int_0^1 \mathfrak{J}_4(\nu)\left(\frac{1}{\Gamma(\gamma_2)}\int_0^\nu (\nu-\varsigma)^{\gamma_2-1}\theta_2(\varsigma)\,d\varsigma\right)^{\omega_2-1} d\nu\\
&= \mathcal{D}_0^{\omega_1-1}\int_0^1 \mathfrak{J}_3(\nu)\varphi_{\omega_1}(I_{0+}^{\gamma_1}\theta_1(\nu))\,d\nu + \mathcal{D}_0^{\omega_2-1}\int_0^1 \mathfrak{J}_4(\nu)\varphi_{\omega_2}(I_{0+}^{\gamma_2}\theta_2(\nu))\,d\nu\\
&= \mathcal{D}_0^{\omega_1-1}\Psi_3 + \mathcal{D}_0^{\omega_2-1}\Psi_4 < \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.
\end{aligned}$$

Then we conclude $\|\mathfrak{T}_1(\phi,\psi)\| < \frac{1}{2}$ and $\|\mathfrak{T}_2(\phi,\psi)\| < \frac{1}{2}$ for all $(\phi,\psi) \in \partial B_1 \cap \mathfrak{P}_0$. Therefore we find

$$\|\mathfrak{T}(\phi,\psi)\|_{\mathfrak{Y}} = \|\mathfrak{T}_1(\phi,\psi)\| + \|\mathfrak{T}_2(\phi,\psi)\| < 1 = \|(\phi,\psi)\|_{\mathfrak{Y}}, \ \forall\,(\phi,\psi) \in \partial B_1 \cap \mathfrak{P}_0. \tag{3.16}$$

Hence, by (3.14), (3.16) and the Guo-Krasnosel'skii fixed point theorem, we deduce that problem (1.1),(1.2) has one positive solution $(\phi_1,\psi_1) \in \mathfrak{P}_0$ with $1 < \|(\phi_1,\psi_1)\|_{\mathfrak{Y}} \leq R_2$. By (3.15), (3.16) and the Guo-Krasnosel'skii fixed point theorem, we conclude that problem (1.1),(1.2) has another positive solution $(\phi_2,\psi_2) \in \mathfrak{P}_0$ with $R_4 \leq \|(\phi_2,\psi_2)\|_{\mathfrak{Y}} < 1$. So problem (1.1),(1.2) has at least two positive solutions $(\phi_1(\eta),\psi_1(\eta))$, $(\phi_2(\eta),\psi_2(\eta))$, $\eta \in [0,1]$. □

Remark 3.1. Theorem 3.1 remains valid if the functions $\mu_1,\ \mu_2$ and Υ satisfy the inequalities (3.5) and (3.7), instead of $(A3)$ and $(A4)$. Theorem 3.2 remains valid if the functions $\mu_1,\ \mu_2$ and Λ satisfy the inequalities (3.9) and (3.12), instead of $(A5)$ and $(A6)$. Theorem 3.3 remains valid if the functions Λ and Υ satisfy the inequalities (3.12) and (3.7), instead of $(A6)$ and $(A4)$.

4. Examples

Let $\gamma_1 = 1/4,\ \gamma_2 = 1/5,\ \zeta_1 = 7/2\ (p = 4),\ \zeta_2 = 8/3\ (q = 3),\ \delta_1 = 3,\ \delta_2 = 4,$ $\omega_1 = 3/2,\ \omega_2 = 4/3,\ n = 1,\ m = 2,\ \alpha_0 = 9/4,\ \beta_0 = 3/2,\ \alpha_1 = 1/6,\ \beta_1 = 3/4,$ $\beta_2 = 6/5$, $\mathfrak{H}_1(\eta) = \{1,\ \eta \in [0, 1/3);\ 3/2,\ \eta \in [1/3, 1]\}$, $\mathfrak{K}_1(\eta) = \eta/4$ for all $\eta \in [0, 1]$, and $\mathfrak{K}_2(\eta) = \{1/2,\ \eta \in [0, 1/2);\ 7/2,\ \eta \in [1/2, 1]\}$.

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{1/4}\left(\varphi_3\left(D_{0+}^{7/2}\phi(\eta)\right)\right) + \Lambda(\eta, \phi(\eta), \psi(\eta)) = 0,\ \ \eta \in (0, 1), \\ D_{0+}^{1/5}\left(\varphi_4\left(D_{0+}^{8/3}\psi(\eta)\right)\right) + \Upsilon(\eta, \phi(\eta), \psi(\eta)) = 0,\ \ \eta \in (0, 1), \end{cases} \tag{4.1}$$

with the coupled nonlocal boundary conditions

$$\begin{cases} \phi(0) = \phi'(0) = \phi''(0) = 0,\ \ D_{0+}^{7/2}\phi(0) = 0,\ \ D_{0+}^{9/4}\phi(1) = \frac{1}{2}D_{0+}^{1/6}\psi\left(\frac{1}{3}\right), \\ \psi(0) = \psi'(0) = 0,\ \ D_{0+}^{8/3}\psi(0) = 0,\ \ D_{0+}^{3/2}\psi(1) = \frac{1}{4}\int_0^1 D_{0+}^{3/4}\phi(\eta)\, d\eta + 3D_{0+}^{6/5}\phi\left(\frac{1}{2}\right). \end{cases} \tag{4.2}$$

We obtain here $\mathfrak{a} \approx 5.54794669 > 0$, so assumption $(A1)$ is satisfied. In addition, we find

$$\mathfrak{g}_1(\eta, \nu) = \frac{1}{\Gamma(7/2)} \begin{cases} \eta^{5/2}(1-\nu)^{1/4} - (\eta-\nu)^{5/2},\ \ 0 \le \nu \le \eta \le 1, \\ \eta^{5/2}(1-\nu)^{1/4},\ \ 0 \le \eta \le \nu \le 1, \end{cases}$$

$$\mathfrak{g}_{11}(\varsigma, \nu) = \frac{1}{\Gamma(11/4)} \begin{cases} \varsigma^{7/4}(1-\nu)^{1/4} - (\varsigma-\nu)^{7/4},\ \ 0 \le \nu \le \varsigma \le 1, \\ \varsigma^{7/4}(1-\nu)^{1/4},\ \ 0 \le \varsigma \le \nu \le 1, \end{cases}$$

$$\mathfrak{g}_{12}(\varsigma, \nu) = \frac{1}{\Gamma(23/10)} \begin{cases} \varsigma^{13/10}(1-\nu)^{1/4} - (\varsigma-\nu)^{13/10},\ \ 0 \le \nu \le \varsigma \le 1, \\ \varsigma^{13/10}(1-\nu)^{1/4},\ \ 0 \le \varsigma \le \nu \le 1, \end{cases}$$

$$\mathfrak{g}_2(\eta, \nu) = \frac{1}{\Gamma(8/3)} \begin{cases} \eta^{5/3}(1-\nu)^{1/6} - (\eta-\nu)^{5/3},\ \ 0 \le \nu \le \eta \le 1, \\ \eta^{5/3}(1-\nu)^{1/6},\ \ 0 \le \eta \le \nu \le 1, \end{cases}$$

$$\mathfrak{g}_{21}(\varsigma, \nu) = \frac{1}{\Gamma(5/2)} \begin{cases} \varsigma^{3/2}(1-\nu)^{1/6} - (\varsigma-\nu)^{3/2},\ \ 0 \le \nu \le \varsigma \le 1, \\ \varsigma^{3/2}(1-\nu)^{1/6},\ \ 0 \le \varsigma \le \nu \le 1, \end{cases}$$

$$\mathfrak{h}_1(\nu) = \frac{1}{\Gamma(7/2)}\left[(1-\nu)^{1/4} - (1-\nu)^{5/2}\right],\ \ \nu \in [0, 1],$$

$$\mathfrak{h}_2(\nu) = \frac{1}{\Gamma(8/3)}\left[(1-\nu)^{1/6} - (1-\nu)^{5/3}\right],\ \ \nu \in [0, 1],$$

$$\mathfrak{G}_1(\eta,\nu)=\mathfrak{g}_1(\eta,\nu)+\frac{\eta^{5/2}}{\mathfrak{a}}\left(\frac{\Gamma(8/3)}{2\Gamma(5/2)}\left(\frac{1}{3}\right)^{3/2}\right)\left[\frac{1}{4}\int_0^1\mathfrak{g}_{11}(\varsigma,\nu)\,d\varsigma+3\mathfrak{g}_{12}\left(\frac{1}{2},\nu\right)\right],$$
$$\mathfrak{G}_2(\eta,\nu)=\frac{\eta^{5/2}\Gamma(8/3)}{2\mathfrak{a}\Gamma(7/6)}\mathfrak{g}_{21}\left(\frac{1}{3},\nu\right),$$
$$\mathfrak{G}_3(\eta,\nu)=\frac{\eta^{5/3}\Gamma(7/2)}{\mathfrak{a}\Gamma(5/4)}\left[\frac{1}{4}\int_0^1\mathfrak{g}_{11}(\varsigma,\nu)\,d\varsigma+3\mathfrak{g}_{12}\left(\frac{1}{2},\nu\right)\right],$$
$$\mathfrak{G}_4(\eta,\nu)=\mathfrak{g}_2(\eta,\nu)+\frac{\eta^{5/3}}{2\mathfrak{a}}\left[\frac{\Gamma(7/2)}{11\Gamma(11/4)}+\frac{3\Gamma(7/2)}{\Gamma(23/10)}\left(\frac{1}{2}\right)^{13/10}\right]\mathfrak{g}_{21}\left(\frac{1}{3},\nu\right),$$

for all $(\eta,\nu)\in[0,1]\times[0,1]$.

Besides we deduce

$$\mathfrak{J}_1(\nu)=\begin{cases}\mathfrak{h}_1(\nu)+\frac{1}{\mathfrak{a}}\left(\frac{\Gamma(8/3)}{2\Gamma(5/2)}\left(\frac{1}{3}\right)^{3/2}\right)\left\{\frac{1}{11\Gamma(11/4)}\left[(1-\nu)^{1/4}-(1-\nu)^{11/4}\right]\right.\\ \quad\left.+\frac{3}{\Gamma(23/10)}\left[\left(\frac{1}{2}\right)^{13/10}(1-\nu)^{1/4}-\left(\frac{1}{2}-\nu\right)^{13/10}\right]\right\},\ 0\le\nu<\frac{1}{2},\\ \mathfrak{h}_1(\nu)+\frac{1}{\mathfrak{a}}\left(\frac{\Gamma(8/3)}{2\Gamma(5/2)}\left(\frac{1}{3}\right)^{3/2}\right)\left\{\frac{1}{11\Gamma(11/4)}\left[(1-\nu)^{1/4}-(1-\nu)^{11/4}\right]\right.\\ \quad\left.+\frac{3}{\Gamma(23/10)}\left(\frac{1}{2}\right)^{13/10}(1-\nu)^{1/4}\right\},\ \frac{1}{2}\le\nu\le1,\end{cases}$$
$$\mathfrak{J}_2(\nu)=\begin{cases}\frac{\Gamma(8/3)}{2\mathfrak{a}\Gamma(7/6)\Gamma(5/2)}\left[\left(\frac{1}{3}\right)^{3/2}(1-\nu)^{1/6}-\left(\frac{1}{3}-\nu\right)^{3/2}\right],\ 0\le\nu<\frac{1}{3},\\ \frac{\Gamma(8/3)}{2\mathfrak{a}\Gamma(7/6)\Gamma(5/2)}\left(\frac{1}{3}\right)^{3/2}(1-\nu)^{1/6},\ \frac{1}{3}\le\nu\le1,\end{cases}$$
$$\mathfrak{J}_3(\nu)=\begin{cases}\frac{\Gamma(7/2)}{\mathfrak{a}\Gamma(5/4)}\left\{\frac{1}{11\Gamma(11/4)}(1-\nu)^{1/4}-\frac{1}{11\Gamma(11/4)}(1-\nu)^{11/4}+\frac{3}{\Gamma(23/10)}\right.\\ \quad\left.\times\left[\left(\frac{1}{2}\right)^{13/10}(1-\nu)^{1/4}-\left(\frac{1}{2}-\nu\right)^{13/10}\right]\right\},\ 0\le\nu<\frac{1}{2},\\ \frac{\Gamma(7/2)}{\mathfrak{a}\Gamma(5/4)}\left[\frac{1}{11\Gamma(11/4)}(1-\nu)^{1/4}-\frac{1}{11\Gamma(11/4)}(1-\nu)^{11/4}+\frac{3}{\Gamma(23/10)}\right.\\ \quad\left.\times\left(\frac{1}{2}\right)^{13/10}(1-\nu)^{1/4}\right],\ \frac{1}{2}\le\nu\le1,\end{cases}$$
$$\mathfrak{J}_4(\nu)=\begin{cases}\mathfrak{h}_2(\nu)+\frac{1}{2\mathfrak{a}}\left[\frac{\Gamma(7/2)}{11\Gamma(11/4)}+\frac{3\Gamma(7/2)}{\Gamma(23/10)}\left(\frac{1}{2}\right)^{13/10}\right]\frac{1}{\Gamma(5/2)}\\ \quad\times\left[\left(\frac{1}{3}\right)^{3/2}(1-\nu)^{1/6}-\left(\frac{1}{3}-\nu\right)^{3/2}\right],\ 0\le\nu<\frac{1}{3},\\ \mathfrak{h}_2(\nu)+\frac{1}{2\mathfrak{a}}\left[\frac{\Gamma(7/2)}{11\Gamma(11/4)}+\frac{3\Gamma(7/2)}{\Gamma(23/10)}\left(\frac{1}{2}\right)^{13/10}\right]\frac{1}{\Gamma(5/2)}\left(\frac{1}{3}\right)^{3/2}(1-\nu)^{1/6},\\ \frac{1}{3}\le\nu\le1.\end{cases}$$

Example 4.1. We consider the functions

$$\Lambda(\eta,z,w)=\frac{(z+w)^{2a}}{\sqrt[4]{\eta}\,\sqrt[5]{1-\eta}},\quad \Upsilon(\eta,z,w)=\frac{(z+w)^{3b}}{\sqrt[3]{\eta}\,\sqrt{1-\eta}},\quad \eta\in(0,1),\quad z,w\ge0,\tag{4.3}$$

where $a>1$, $b>1$. Here $\Lambda(\eta,z,w)=\theta_1(\eta)\mu_1(\eta,z,w)$, $\Upsilon(\eta,z,w)=\theta_2(\eta)\mu_2(\eta,z,w)$, $\theta_1(\eta)=\frac{1}{\sqrt[4]{\eta}\,\sqrt[5]{1-\eta}}$, $\theta_2(\eta)=\frac{1}{\sqrt[3]{\eta}\,\sqrt{1-\eta}}$ for all $\eta\in(0,1)$, $\mu_1(\eta,z,w)=(z+w)^{2a}$, $\mu_2(\eta,z,w)=(z+w)^{3b}$ for all $\eta\in[0,1]$, $z,w\ge0$.

We have

$$L_1=\int_0^1(1-\nu)^{1/4}\left(I_{0+}^{1/4}\theta_1(\nu)\right)^{1/2}d\nu$$

$$=\frac{1}{(\Gamma(1/4))^{1/2}}\int_0^1 (1-\nu)^{1/4}\left(\int_0^\nu (\nu-\varsigma)^{-3/4}\frac{1}{\sqrt[4]{\varsigma}\,\sqrt[5]{1-\varsigma}}\,d\varsigma\right)^{1/2} d\nu$$

$$\overset{\varsigma=\nu x}{=}\frac{1}{(\Gamma(1/4))^{1/2}}\int_0^1 (1-\nu)^{1/4}\left(\int_0^1 (\nu-\nu x)^{-3/4}(\nu x)^{-1/4}(1-\nu x)^{-1/5}\nu\,dx\right)^{1/2} d\nu$$

$$=\frac{1}{(\Gamma(1/4))^{1/2}}\int_0^1 (1-\nu)^{1/4}\left(\int_0^1 (1-x)^{-3/4}x^{-1/4}(1-\nu x)^{-1/5}\,dx\right)^{1/2} d\nu$$

$$=(\Gamma(3/4))^{1/2}\int_0^1 (1-\nu)^{1/4}\left({}_2F_1\left[\frac{1}{5},\frac{3}{4},1,\nu\right]\right)^{1/2} d\nu \approx 0.93696479,$$

$$L_2=\int_0^1 (1-\nu)^{1/6}\left(I_{0+}^{1/5}\theta_2(\nu)\right)^{1/3} d\nu$$

$$=\frac{1}{(\Gamma(1/5))^{1/3}}\int_0^1 (1-\nu)^{1/6}\left(\int_0^\nu (\nu-\varsigma)^{-4/5}\frac{1}{\sqrt[3]{\varsigma}\,\sqrt{1-\varsigma}}\,d\varsigma\right)^{1/3} d\nu$$

$$\overset{\varsigma=\nu x}{=}\frac{1}{(\Gamma(1/5))^{1/3}}\int_0^1 (1-\nu)^{1/6}\left(\int_0^1 (\nu-\nu x)^{-4/5}(\nu x)^{-1/3}(1-\nu x)^{-1/2}\nu\,dx\right)^{1/3} d\nu$$

$$=\frac{1}{(\Gamma(1/5))^{1/3}}\int_0^1 (1-\nu)^{1/6}\left(\nu^{-2/15}\int_0^1 (1-x)^{-4/5}x^{-1/3}(1-\nu x)^{-1/2}\,dx\right)^{1/3} d\nu$$

$$=(\Gamma(2/3))^{1/3}\int_0^1 (1-\nu)^{1/6}\left(\nu^{-2/15}\,{}_2F_1\left[\frac{1}{2},\frac{2}{3},\frac{13}{15},\nu\right]\right)^{1/3} d\nu \approx 1.07711433,$$

where ${}_2F_1[\widetilde{a},\widetilde{b},\widetilde{c},z]=\frac{1}{\Gamma(\widetilde{b})\Gamma(\widetilde{c}-\widetilde{b})}\int_0^1 \nu^{\widetilde{b}-1}(1-\nu)^{\widetilde{c}-\widetilde{b}-1}(1-\nu z)^{-\widetilde{a}}\,d\nu$ is the regularized hypergeometric function. So $L_i\in(0,\infty),\ i=1,2$, and then assumptions $(A1)$ and $(A2)$ are satisfied.

In addition, in $(A3)$, for $\rho_1=\rho_2=1$, we obtain $\mu_{10}=0$, $\mu_{20}=0$, and in $(A4)$ for $[\sigma_1,\sigma_2]\subset[0,1]$, $0<\sigma_1<\sigma_2<1$, we have $\Lambda_\infty^i=\infty$ (and $\Upsilon_\infty^i=\infty$). Then by Theorem 3.1, we deduce that problem (4.1),(4.2) with the nonlinearities (4.3) has at least one positive solution $(\phi(\eta),\psi(\eta)),\ \eta\in[0,1]$.

Example 4.2. We consider the functions

$$\begin{aligned}\Lambda(\eta,z,w)&=\frac{a_0(\eta+2)}{(\eta^2+9)\sqrt[3]{1-\eta}}\left[(z+w)^{a_1}+(z+w)^{a_2}\right],\ \ \eta\in[0,1),\ \ z,w\ge 0,\\ \Upsilon(\eta,z,w)&=\frac{b_0(3+\cos\eta)}{(\eta+1)^3\sqrt[4]{\eta}}\left(z^{a_3}+w^{a_4}\right),\ \ \eta\in(0,1],\ \ z,w\ge 0,\end{aligned}\tag{4.4}$$

where $a_0>0$, $b_0>0$, $a_1>2$, $a_2\in(0,2)$, $a_3>0$, $a_4>0$.

Here we have $\theta_1(\eta)=\frac{1}{\sqrt[3]{1-\eta}},\ \eta\in[0,1)$, $\theta_2(\eta)=\frac{1}{\sqrt[4]{\eta}},\ \eta\in(0,1]$, and

$$\begin{aligned}\mu_1(\eta,z,w)&=\frac{a_0(\eta+2)}{\eta^2+9}\left[(z+w)^{a_1}+(z+w)^{a_2}\right],\\ \mu_2(\eta,z,w)&=\frac{b_0(3+\cos\eta)}{(\eta+1)^3}\left(z^{a_3}+w^{a_4}\right),\ \eta\in[0,1],\ z,w\ge 0.\end{aligned}$$

Further, we obtain

$$L_1=\int_0^1 (1-\nu)^{1/4}\left(I_{0+}^{1/4}\theta_1(\nu)\right)^{1/2} d\nu$$

$$
\begin{aligned}
&= \frac{1}{(\Gamma(1/4))^{1/2}} \int_0^1 (1-\nu)^{1/4} \left(\int_0^\nu (\nu-\varsigma)^{-3/4} \frac{1}{\sqrt[3]{1-\varsigma}}\, d\varsigma \right)^{1/2} d\nu \\
&\overset{\varsigma=\nu x}{=} \frac{1}{(\Gamma(1/4))^{1/2}} \int_0^1 (1-\nu)^{1/4} \left(\int_0^\nu (\nu-\nu x)^{-3/4} (1-\nu x)^{-1/3} \nu\, dx \right)^{1/2} d\nu \\
&= \frac{1}{(\Gamma(1/4))^{1/2}} \int_0^1 (1-\nu)^{1/4} \left(\nu^{1/4} \int_0^1 (1-x)^{-3/4} (1-\nu x)^{-1/3} dx \right)^{1/2} d\nu \\
&= \int_0^1 (1-\nu)^{1/4} \left(\nu^{1/4}\, {}_2F_1 \left[\frac{1}{3}, 1, \frac{5}{4}, \nu \right] \right)^{1/2} d\nu \approx 0.82086172, \\
L_2 &= \int_0^1 (1-\nu)^{1/6} \left(I_{0+}^{1/5} \theta_2(\nu) \right)^{1/3} d\nu \\
&= \frac{1}{(\Gamma(1/5))^{1/3}} \int_0^1 (1-\nu)^{1/6} \left(\int_0^\nu (\nu-\varsigma)^{-4/5} \frac{1}{\sqrt[4]{\varsigma}} d\varsigma \right)^{1/3} d\nu \\
&\overset{\varsigma=\nu x}{=} \frac{1}{(\Gamma(1/5))^{1/3}} \int_0^1 (1-\nu)^{1/6} \left(\int_0^1 (\nu-\nu x)^{-4/5} (\nu x)^{-1/4} \nu\, dx \right)^{1/3} d\nu \\
&= \frac{1}{(\Gamma(1/5))^{1/3}} \int_0^1 (1-\nu)^{1/6} \left(\nu^{-1/20} \int_0^1 (1-x)^{-4/5} x^{-1/4} dx \right)^{1/3} d\nu \\
&= \frac{1}{(\Gamma(1/5))^{1/3}} \int_0^1 (1-\nu)^{1/6} \left(\nu^{-1/20} B\left(\frac{3}{4}, \frac{1}{5} \right) \right)^{1/3} d\nu \\
&\approx 0.9247852,
\end{aligned}
$$

where $B(\widetilde{p},\widetilde{q}) = \int_0^1 \nu^{\widetilde{p}-1}(1-\nu)^{\widetilde{q}-1}\, d\nu,\ \widetilde{p},\widetilde{q} > 0$ is the first Euler function (the beta function). Hence assumptions $(A1)$ and $(A2)$ are satisfied.

For $[\sigma_1,\sigma_2] \subset (0,1)$ we obtain $\Lambda_\infty^i = \infty$, and if we consider $0 < \varrho_1 \le 1$, $2\varrho_1 > a_2$, we find $\Lambda_0^i = \infty$. Then assumptions $(A4)$ and $(A6)$ are also satisfied. In addition, after some computations, we deduce

$$
\begin{aligned}
\Psi_1 &= \int_0^1 \mathfrak{J}_1(\nu) \left(I_{0+}^{1/4} \theta_1(\nu) \right)^{1/2} d\nu \\
&= \int_0^1 \mathfrak{J}_1(\nu) \left(\nu^{1/4}\, {}_2F_1 \left[\frac{1}{3}, 1, \frac{5}{4}, \nu \right] \right)^{1/2} d\nu \approx 0.18645088, \\
\Psi_2 &= \int_0^1 \mathfrak{J}_2(\nu) \left(I_{0+}^{1/5} \theta_2(\nu) \right)^{1/3} d\nu \\
&= \frac{1}{(\Gamma(1/5))^{1/3}} \int_0^1 \mathfrak{J}_2(\nu) \left(\nu^{-1/20} B\left(\frac{3}{4}, \frac{1}{5} \right) \right)^{1/3} d\nu \approx 0.01643815, \\
\Psi_3 &= \int_0^1 \mathcal{J}_3(\nu) \left(I_{0+}^{1/4} \theta_1(\nu) \right)^{1/2} d\nu \\
&= \int_0^1 \mathcal{J}_3(\nu) \left(\nu^{1/4}\, {}_2F_1 \left[\frac{1}{3}, 1, \frac{5}{4}, \nu \right] \right)^{1/2} d\nu \approx 0.46636539, \\
\Psi_4 &= \int_0^1 \mathfrak{J}_4(\nu) \left(I_{0+}^{1/5} \theta_2(\nu) \right)^{1/3} d\nu \\
&= \frac{1}{(\Gamma(1/5))^{1/3}} \int_0^1 \mathfrak{J}_4(\nu) \left(\nu^{-1/20} B\left(\frac{3}{4}, \frac{1}{5} \right) \right)^{1/3} d\nu \approx 0.37996349.
\end{aligned}
$$

Besides, we find $\mathcal{D}_0 = \max\left\{\frac{3a_0}{10}(2^{a_1}+2^{a_2}), 8b_0\right\}$. If

$$a_0 < \frac{10}{3(2^{a_1}+2^{a_2})}\min\left\{\frac{1}{16\Psi_1^2}, \frac{1}{64\Psi_2^3}, \frac{1}{16\Psi_3^2}, \ \frac{1}{64\Psi_4^3}\right\},$$

and

$$b_0 < \frac{1}{8}\min\left\{\frac{1}{16\Psi_1^2}, \frac{1}{64\Psi_2^3}, \frac{1}{16\Psi_3^2}, \frac{1}{64\Psi_4^3}\right\},$$

then the inequalities $\mathcal{D}_0^{1/2}\Psi_1 < \frac{1}{4}$, $\mathcal{D}_0^{1/3}\Psi_2 < \frac{1}{4}$, $\mathcal{D}_0^{1/2}\Psi_3 < \frac{1}{4}$, $\mathcal{D}_0^{1/3}\Psi_4 < \frac{1}{4}$ are satisfied (that is, assumption $(A7)$ is satisfied). For example, if $a_1 = 3$, $a_2 = 1$, and $a_0 \leq 0.094$ and $b_0 \leq 0.035$, then the above inequalities are satisfied. By Theorem 3.3 we conclude that problem (4.1),(4.2) with the nonlinearities (4.4) has at least two positive solutions $(\phi_1(\eta), \psi_1(\eta))$, $(\phi_2(\eta), \psi_2(\eta))$, $\eta \in [0,1]$.

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