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EXISTENCE OF PERIODIC SOLUTIONS FOR TWO CLASSES OF SECOND ORDER *P*-LAPLACIAN DIFFERENTIAL EQUATIONS*

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Abstract In this paper, by using the Manásevich-Mawhin theorem on continuity of the topological degree, we prove the existence of periodic solutions for two classes of second order p-Laplacian polynomial differential equations. Finally, some examples are given to show applications of the conclusions.

Keywords Periodic solution, p-Laplacian, Manásevich-Mawhin theorem.

MSC(2010) 34B16, 34C25.

1. Introduction

In the past few years, the periodic solutions of second order polynomial differential equations have attracted the attention of many researchers, because this class of equations can well describe some mathematical models that appear in biology and physics. Such as the equations

$$x''(t) + cx'(t) = r(t)x(t)^{\alpha} - s(t)x(t)^{\beta}, \tag{1.1}$$

where $c \geqslant 0$, $0 < \alpha < \beta < 1$ and r, s are continuous T-periodic functions on $\mathbb{R}$. Equation (1.1) describes the Liebau phenomenon [18, 23], which refers to the preferential flow direction obtained due to asymmetric periodic oscillations in a mechanical system without valves. In [5–8, 17, 27], the authors use different classical theories, such as Krasnosel'skiĭ-Guo fixed point theorem, upper and lower solution method and fixed point theorem, to study the existence of positive periodic solutions to equation (1.1).

In equation (1.1), if $c = 0$, $\alpha = 0$ and $\beta < 0$, then equation (1.1) becomes Lazer-Solimini equation

$$x''(t) + \frac{a(t)}{x(t)^{\alpha}} = h(t), \tag{1.2}$$

where $\alpha > 0$ and a, h are continuous T-periodic functions on $\mathbb{R}$. Equation (1.2) can describe the motion of a charged charge and linear motions in a periodically forced Kepler problem. Many scholars have studied the existence, uniqueness and

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stability of periodic solutions of singular second order differential equations (1.2), see for instance the papers [4, 10, 16, 19, 24–26, 29]. In equation (1.1), if $\alpha < 0$ and $\beta < 0$, some scholars also studied the existence of periodic solutions of this type of equation in [9, 14].

At the same time, some scholars study the existence of periodic solutions of the following second order cubic differential equation

$$x'' + a(t)x - b(t)x^2 - c(t)x^3 = 0, \tag{1.3}$$

where $a(t), b(t), c(t)$ are are continuous T-periodic functions on $\mathbb{R}$. Equation (1.3) can describe a biomathematics model related to circle of willis aneurysm [11], where x is the velocity of blood flow in the aneurysm, $a(t), b(t), c(t)$ are coefficient functions related to aneurysm. For the study of periodic solutions of equation (1.3), see [1–3, 12, 13, 15].

Motivated by the above mentioned work, in this paper, our purpose is use the Manásevich-Mawhin theorem on continuity of the topological degree to establish the existence of positive periodic solutions of the second order p-Laplacian polynomial differential equations

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + r(t)x^\gamma - s(t)x^\beta = 0, \tag{1.4}$$

and

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + a(t)x^\alpha - b(t)x^{\alpha+1} - c(t)x^{\alpha+2} = 0, \tag{1.5}$$

where $p > 1$, $\varphi_p(x) = |x|^{p-2}x$ for $x \neq 0$ and $\varphi_p(0) = 0$, $f : (0, +\infty) \longrightarrow \mathbb{R}$ is a continuous function, equation (1.4) satisfies $\beta, \gamma \in \mathbb{R}$, $s(t), r(t)$ are continuous T-periodic functions on $\mathbb{R}$, equation (1.5) satisfies $\alpha \in \mathbb{R}$, $a(t), b(t), c(t)$ are continuous T-periodic functions on $\mathbb{R}$. Obviously, equation (1.1) and (1.2) are special forms of equation (1.4), and equation (1.3) is special forms of equation (1.5). For the study of the existence of periodic solutions of second order p-Laplacian differential equations, we can refer to [20, 21, 28].

The paper is organized as follows: after this Introduction, in Section 2 for the convenience of the readers we collected some general results in order to prove our main theorems. In Section 3 we apply the Manásevich-Mawhin theorem on continuity of the topological degree to obtain the existence of T-periodic solutions of the equation (1.4) and (1.5). In section 4, we apply the previous results to some examples.

2. Preliminaries

In this section, we given some preliminary results which will paly important roles in the prove of our main results.

Consider the following periodic boundary value problem

$$\begin{cases} (\varphi_p(u'))' = \tilde{f}(t, u, u'), \\ u(0) = u(T), u'(0) = u'(T), \end{cases} \tag{2.1}$$

where $p > 1$, $\tilde{f} : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be Carathéodory functions.

Lemma 2.1 ([22, Manásevich-Mawhin theorem]). *Assume that Ω is an open bounded set in C_T^1 such that the following conditions hold:*

(a1) *For each $\lambda \in (0,1)$ the problem*

$$\begin{cases} (\varphi_p(u'))' = \lambda \tilde{f}(t,u,u'), \\ u(0) = u(T), u'(0) = u'(T), \end{cases}$$

has no solution on $\partial\Omega$.

(a2) *The equation*

$$F(e) = \frac{1}{T}\int_0^T \tilde{f}(t,e,0)\mathrm{d}t = 0,$$

has no solution on $\partial\Omega \cap \mathbb{R}^N$.

(a3) *The Brouwer degree $\deg\{F, \Omega \cap \mathbb{R}^N, 0\} \neq 0$.*

Then the problem (2.1) has at least one solution in $\overline{\Omega}$.

By applications of Lemma 2.1, we obtain the following results.

Lemma 2.2. *Assume that there exist positive constants M_1, M_2, N_1 and $M_1 < M_2$ such that the following conditions hold:*

(g1) *For $\lambda \in (0,1]$, each possible positive periodic solution of equation*

$$(\varphi_p(x'(t)))' + \lambda f(x(t))x'(t) + \lambda r(t)x(t)^\gamma - \lambda s(t)x(t)^\beta = 0 \tag{2.2}$$

satisfies $M_1 < x(t) < M_2$ and $|x'|_\infty < N_1$ for all $t \in [0,T]$, where $|x'|_\infty = \max_{t\in[0,T]}|x'(t)|$.

(g2) *Each possible positive solution e_1 to the equation*

$$r(t)e_1^\gamma - s(t)e_1^\beta = 0$$

satisfies $M_1 < e_1 < M_2$.

(g3) *For all $t \in [0,T]$, we have*

$$(r(t)M_1^\gamma - s(t)M_1^\beta)(r(t)M_2^\gamma - s(t)M_2^\beta) < 0.$$

Then the equation (1.4) has at least one positive T-periodic solution $x(t)$ satisfied $M_1 < x(t) < M_2$ for all $t \in [0,T]$.

Lemma 2.3. *Assume that there exist positive constants M_3, M_4, N_2 and $M_3 < M_4$ such that the following conditions hold:*

(k1) *For $\lambda \in (0,1]$, each possible positive periodic solution of equation*

$$(\varphi_p(x'(t)))' + \lambda f(x(t))x'(t) + \lambda a(t)x(t)^\alpha - \lambda b(t)x(t)^{\alpha+1} - \lambda c(t)x(t)^{\alpha+2} = 0 \tag{2.3}$$

satisfies $M_3 < x(t) < M_4$ and $|x'|_\infty < N_2$ for all $t \in [0,T]$, where $|x'|_\infty = \max_{t\in[0,T]}|x'(t)|$.

(k2) Each possible positive solution e_2 to the equation

$$a(t)e_2^{\alpha} - b(t)e_2^{\alpha+1} - c(t)e_2^{\alpha+2} = 0$$

satisfied $M_3 < e_2 < M_4$.

(k3) For all $t \in [0, T]$, we get

$$(a(t)M_3^{\alpha} - b(t)M_3^{\alpha+1} - c(t)M_3^{\alpha+2})(a(t)M_4^{\alpha} - b(t)M_4^{\alpha+1} - c(t)M_4^{\alpha+2}) < 0.$$

Then the equations (1.5) *has at least one positive T-periodic solution $x(t)$ satisfied $M_3 < x(t) < M_4$ for all $t \in [0, T]$.*

Throughout this paper, let Banach spaces $X = C_T^1 = \{x \in C^1(\mathbb{R}, \mathbb{R}) : x(t + T) = x(t)$ for all $t \in \mathbb{R}\}$ with the norm $\|x\|_{C_T} = \max\{|x|_\infty, |x'|_\infty\}$, where $|x|_\infty = \max\limits_{t\in[0,T]} |x(t)|$, $|x'|_\infty = \max\limits_{t\in[0,T]} |x'(t)|$. For a given continuous function $w : [0, T] \to \mathbb{R}$, we denote

$$w^+ = \max\{w(t) : t \in [0, T]\},\ w^- = \min\{w(t) : t \in [0, T]\}.$$

3. The main results

In this section, we are ready to state and prove our main results.

Theorem 3.1. *Assume $\beta > \gamma$, and $r(t), s(t)$ are positive continuous T-periodic functions. Then equations* (1.4) *has at least one positive T-periodic solution.*

Proof. We will verify that all the conditions of Lemma 2.2 are true.

Since $r(t), s(t)$ are positive continuous T-periodic functions, hence we have $0 < r^- \leqslant r(t) \leqslant r^+$, $0 < s^- \leqslant s(t) \leqslant s^+$. Let $0 < M_1 < (\frac{r^-}{s^+})^{\frac{1}{\beta-\gamma}}$ and $M_2 > (\frac{r^+}{s^-})^{\frac{1}{\beta-\gamma}}$ are constants. By $\beta > \gamma$, we obtain

$$0 < M_1 < (\frac{r^-}{s^+})^{\frac{1}{\beta-\gamma}} \leqslant (\frac{r(t)}{s(t)})^{\frac{1}{\beta-\gamma}} \leqslant (\frac{r^+}{s^-})^{\frac{1}{\beta-\gamma}} < M_2$$

uniformly in t. Furthermore, we get

$$r(t) - s(t)M_1^{\beta-\gamma} \geqslant r^- - s^+ M_1^{\beta-\gamma} > 0$$

and

$$r(t) - s(t)M_2^{\beta-\gamma} \leqslant r^+ - s^- M_2^{\beta-\gamma} < 0$$

uniformly in $t \in [0, T]$. For $\forall t \in [0, T]$, when $x(t) \in (0, M_1]$, we have

$$r(t) - s(t)x(t)^{\beta-\gamma} > 0 \tag{3.1}$$

uniformly in $t \in [0, T]$. For $\forall t \in [0, T]$, when $x(t) \in [M_2, +\infty)$, we obtain

$$r(t) - s(t)x(t)^{\beta-\gamma} < 0 \tag{3.2}$$

uniformly in $t \in [0, T]$.

Let $\underline{t}$ and $\bar{t}$ respectively stand for the global minimum and maximum points $x(t)$ on $t \in [0,T]$, that is

$$x(\bar{t}) = \max_{t\in[0,T]} x(t), \quad x(\underline{t}) = \min_{t\in[0,T]} x(t).$$

Obviously, we have

$$x'(\bar{t}) = 0, \quad x'(\underline{t}) = 0.$$

We claim that

$$(\varphi_p(x'(\bar{t})))' \leqslant 0. \tag{3.3}$$

In fact, if (3.3) does not hold, then $(\varphi_p(x'(\bar{t})))' > 0$, and there exists $\varepsilon > 0$ such that $(\varphi_p(x'(t)))' > 0$ for $t \in (\bar{t}-\varepsilon, \bar{t}+\varepsilon)$, hence $\varphi_p(x'(t))$ is strictly increasing for $t \in (\bar{t}-\varepsilon, \bar{t}+\varepsilon)$. then we get that $x'(t)$ is strictly increasing for $t \in (\bar{t}-\varepsilon, \bar{t}+\varepsilon)$. This contradicts the definition of $\bar{t}$. Therefore, (3.3) is true. Similarly, we get

$$(\varphi_p(x'(\underline{t})))' \geqslant 0. \tag{3.4}$$

Suppose $x(t)$ is an arbitrary T-periodic positive solution of equation (2.2), we claim that

$$M_1 < x(\underline{t}) \leqslant x(t) \leqslant x(\bar{t}) < M_2. \tag{3.5}$$

In fact, if (3.5) fails, then $x(\bar{t}) \geqslant M_2$ or $0 < x(\underline{t}) \leqslant M_1$ at least one hold. When $x(\bar{t}) \geqslant M_2$, by (3.2) and (3.3), we have

$$\begin{aligned}
0 =&(\varphi_p(x'(\bar{t})))' + \lambda f(x(\bar{t}))x'(\bar{t}) + \lambda r(\bar{t})x(\bar{t})^\gamma - \lambda s(\bar{t})x^\beta\\
=&(\varphi_p(x'(\bar{t})))' + \lambda x(\bar{t})^\gamma(r(\bar{t}) - s(\bar{t})x(t)^{\beta-\gamma})\\
<&0.
\end{aligned}$$

When $0 < x(\underline{t}) \leqslant M_1$, by (3.1) and (3.4), we get

$$\begin{aligned}
0 =&(\varphi_p(x'(\underline{t})))' + \lambda f(x(\underline{t}))x'(\underline{t}) + \lambda x(\underline{t})^\gamma r(\underline{t}) - \lambda s(\underline{t})x(\underline{t})^\beta\\
=&(\varphi_p(x'(\underline{t})))' + \lambda x(\underline{t})^\gamma(r(\underline{t}) - s(\underline{t})x^{\beta-\gamma})\\
>&0.
\end{aligned}$$

But these are contradiction with equation (2.2). Hence, we have

$$M_1 < x(\underline{t}) \leqslant x(t) \leqslant x(\bar{t}) < M_2.$$

Multiplying both sides of equation (2.2) by $x(t)$ and integrating over the interval $[0,T]$, we get

$$\int_0^T (\varphi_p(x'(t)))'x(t)\mathrm{d}t + \lambda\int_0^T f(x(t))x'(t)x(t)\mathrm{d}t + \lambda\int_0^T [r(t)x^\gamma - s(t)x^\beta]x(t)\mathrm{d}t = 0.$$

Since $\int_0^T (\varphi_p(x'(t)))'x(t)\mathrm{d}t = -\int_0^T |x'(t)|^p\mathrm{d}t$ and $\int_0^T f(x(t))x'(t)x(t)\mathrm{d}t = 0$. Then we obtain

$$\int_0^T |x'(t)|^p\mathrm{d}t =\lambda\int_0^T f(x(t))x'(t)x(t)\mathrm{d}t + \lambda\int_0^T [r(t)x^\gamma - s(t)x^\beta]x(t)\mathrm{d}t$$

$$< \int_0^T |r(t)x^{\gamma+1} - s(t)x^{\beta+1}|\mathrm{d}t.$$

We need to classify discuss the relationship of β, γ and -1, respectively. If $\gamma \geqslant -1$, we have

$$\begin{aligned}\int_0^T |x'(t)|^p \mathrm{d}t <& \int_0^T |r(t)x^{\gamma+1} - s(t)x^{\beta+1}|\mathrm{d}t \\ \leqslant& T(r^+ M_2^{\gamma+1} + s^+ M_2^{\beta+1}),\end{aligned}$$

if $\beta \leqslant -1$, we get

$$\begin{aligned}\int_0^T |x'(t)|^p \mathrm{d}t <& \int_0^T |r(t)x^{\gamma+1} - s(t)x^{\beta+1}|\mathrm{d}t \\ \leqslant& T(r^+ M_1^{\gamma+1} + s^+ M_1^{\beta+1}),\end{aligned}$$

if $\gamma < -1 < \beta$, we obtian

$$\begin{aligned}\int_0^T |x'(t)|^p \mathrm{d}t <& \int_0^T |r(t)x^{\gamma+1} - s(t)x^{\beta+1}|\mathrm{d}t \\ \leqslant& T(r^+ M_1^{\gamma+1} + s^+ M_2^{\beta+1}).\end{aligned}$$

Regardless of the above situation, it may easily be shown that there exists a positive constant $K = \max\{T(r^+M_2^{\gamma+1} + s^+M_2^{\beta+1}), T(r^+M_1^{\gamma+1} + s^+M_1^{\beta+1}), T(r^+M_1^{\gamma+1} + s^+M_2^{\beta+1})\}$ such that

$$\int_0^T |x'(t)|^p \mathrm{d}t \leqslant K.$$

Since $x(0) = x(T)$, hence there exists a point $t_0 \in [0, T]$ such that $x'(t_0) = 0$, then $\varphi_p(x'(t_0)) = 0$. Put $G := \max\{|f(x)|, x \in [M_1, M_2]\}$. By (2.2) and Hölder's inequality, we have

$$\begin{aligned}\big|\varphi_p(x'(t))\big| =& \big|\int_{t_0}^t (\varphi_p(x'(t)))'\mathrm{d}t\big| \\ \leqslant& \lambda\big(\int_0^T |f(x(t))x'(t)|\mathrm{d}t + \int_0^T |r(t)x^{\gamma+1} - s(t)x^{\beta+1}|\mathrm{d}t\big) \\ \leqslant& \int_0^T |f(x(t))x'(t)|\mathrm{d}t + \int_0^T |r(t)x^{\gamma+1} - s(t)x^{\beta+1}|\mathrm{d}t \\ \leqslant& G\int_0^T |x'(t)|\mathrm{d}t + K \\ \leqslant& GT^{\frac{1}{q}}\big(\int_0^T |x'(t)|^p\mathrm{d}t\big)^{\frac{1}{p}} + K \\ \leqslant& GT^{\frac{1}{q}}K^{\frac{1}{p}} + K \\ :=& H,\end{aligned} \tag{3.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Now, we claim that there exists a positive constant N_1, for all $t \in \mathbb{R}$ satisfies

$$|x'|_\infty \leqslant N_1. \tag{3.7}$$

In fact, if (3.7) fails, then there exists $N' > 0$ such that

$$|x'|_\infty > N'.$$

Then we have

$$|\varphi_p(x')|_\infty = |x'|_\infty^{p-1} > (N')^{p-1},$$

which is a contradiction. Hence (3.7) holds.

Let

$$\Omega_1 := \{x \in X \mid M_1 < x(t) < M_2, |x'|_\infty < N_1\},$$

which is an open set in X. Obviously, condition (g1) of Lemma 2.2 is satisfied.

For a possible solution e_1 of the equation

$$r(t)e_1^\gamma - s(t)e_1^\beta = 0,$$

satisfied $M_1 < e_1 < M_2$, otherwise, it contradicts (3.1) and (3.2). Therefore, condition (g2) of Lemma 2.2 hold.

Finally, we verify that condition (g3) of Lemma 2.2 is true. By (3.1) and (3.2) we get

$$r(t)M_1^\gamma - s(t)M_1^\beta < 0$$

and

$$r(t)M_2^\gamma - s(t)M_2^\beta > 0$$

uniformly in $t \in [0, T]$. So condition (g3) of Lemma 2.2 is also satisfied.

In view of all the discussion above, from Lemma 2.2, we can conclude equation (1.4) has at least one positive T-periodic solution $x(t)$ satisfying $M_1 < x(t) < M_2$ for all $t \in [0, T]$. The proof is complete. $\square$

Below we introduce the result of the existence of the positive T-periodic solution of equation (1.5) and its proof.

Theorem 3.2. *Let $a(t), b(t)$ and $c(t)$ are positive continuous T-periodic functions with ${b^+}^2 - {b^-}^2 < 2a^-c^-$. Then equation (1.5) has at least one positive T-periodic solution.*

Proof. We will verify that all the conditions of Lemma 2.3 are true.

Let

$$0 < M_3 < \frac{\sqrt{{b^-}^2 + 2a^-c^-} - b^+}{2c^+} \text{ and } M_4 > \frac{\sqrt{{b^+}^2 + 4a^+c^+} - b^-}{2c^-} > 0 \tag{3.8}$$

are constants. From ${b^+}^2 - {b^-}^2 < 2a^-c^-$ we know that $\sqrt{{b^-}^2 + 2a^-c^-} - b^+ > 0$, that is, M_3 and M_4 are well defined.

By simple calculation we can get

$$0 < M_3 < \frac{\sqrt{{b^-}^2 + 2a^-c^-} - b^+}{2c^+} \leqslant \frac{\sqrt{b(t)^2 + 2a(t)c(t)} - b(t)}{2c(t)}$$

$$< \frac{\sqrt{b(t)^2+4a(t)c(t)}-b(t)}{2c(t)} \leqslant \frac{\sqrt{b^{+2}+4a^{+}c^{+}}-b^{-}}{2c^{-}} < M_4$$

uniformly in t. By (3.8), we have

$$\begin{aligned}
a(t)-b(t)M_4-c(t)M_4^2 =& -c(t)\big(M_4+\frac{\sqrt{b(t)^2+4a(t)c(t)}+b(t)}{2c(t)}\big)\\
&\times\big(M_4-\frac{\sqrt{b(t)^2+4a(t)c(t)}-b(t)}{2c(t)}\big)\\
&<0.\\
a(t)-b(t)M_3-c(t)M_3^2 >& \frac{a(t)}{2}-b(t)M_3-c(t)M_3^2\\
=& -c(t)\big(M_3+\frac{\sqrt{b(t)^2+2a(t)c(t)}+b(t)}{2c(t)}\big)\\
&\times\big(M_3-\frac{\sqrt{b(t)^2+2a(t)c(t)}-b(t)}{2c(t)}\big)\\
&\geqslant 0.
\end{aligned}$$

That is

$$a(t)-b(t)M_4-c(t)M_4^2<0$$

and

$$a(t)-b(t)M_3-c(t)M_3^2>0$$

uniformly in $t\in[0,T]$. Furthermore, for $\forall t\in[0,T]$, when $x(t)\in(0,M_3]$, we obtain

$$a(t)-b(t)x(t)-c(t)x(t)^2>0 \tag{3.9}$$

uniformly in $t\in[0,T]$. For $\forall t\in[0,T]$, when $x(t)\in[M_4,+\infty)$, we have

$$a(t)-b(t)x(t)-c(t)x(t)^2<0 \tag{3.10}$$

uniformly in $t\in[0,T]$.

Let $\underline{t}$ and $\bar{t}$ respectively represents the global minimum and maximum points $x(t)$ on $t\in[0,T]$, then

$$x(\bar{t})=\max_{t\in[0,T]}x(t),\quad x(\underline{t})=\min_{t\in[0,T]}x(t).$$

Similar to the derivation of (3.3) and (3.4), we get

$$(\varphi_p(x'(\underline{t})))'\geqslant 0 \text{ and } (\varphi_p(x'(\bar{t})))'\leqslant 0.$$

Suppose $x(t)$ is an arbitrary T-periodic positive solution of equation (2.3). Similar to the proof of Theorem 3.1, we get

$$M_3<x(\underline{t})\leqslant x(t)\leqslant x(\bar{t})<M_4.$$

Multiplying both sides of equation (2.3) by $x(t)$ and integrating over the interval $[0,T]$, we get

$$\begin{aligned}&\int_0^T(\varphi_p(x'(t)))'x(t)\mathrm{d}t+\lambda\int_0^T f(x(t))x'(t)x(t)\mathrm{d}t\\&+\lambda\int_0^T[a(t)x^\alpha-b(t)x^{\alpha+1}-c(t)x^{\alpha+2}]x(t)\mathrm{d}t=0.\end{aligned}$$

Note that $\int_0^T(\varphi_p(x'(t)))'x(t)\mathrm{d}t=-\int_0^T|x'(t)|^p\mathrm{d}t$ and $\int_0^T f(x(t))x'(t)x(t)\mathrm{d}t=0$. Put $G_1:=\max\{|f(x)|,x\in[M_3,M_4]\}$. Then we get

$$\begin{aligned}\int_0^T|x'(t)|^p\mathrm{d}t=&\lambda\int_0^T[a(t)x^\alpha-b(t)x^{\alpha+1}-c(t)x^{\alpha+2}]x(t)\mathrm{d}t\\&+\lambda\int_0^T f(x(t))x'(t)x(t)\mathrm{d}t\\\leqslant&\int_0^T|a(t)x^{\alpha+1}-b(t)x^{\alpha+2}-c(t)x^{\alpha+3}|\mathrm{d}t,\end{aligned}$$

if $\alpha>-1$, we get

$$\begin{aligned}\int_0^T|x'(t)|^p\mathrm{d}t\leqslant&\int_0^T|a(t)x^{\alpha+1}-b(t)x^{\alpha+2}-c(t)x^{\alpha+3}|\mathrm{d}t\\<&T(a^+M_4^{\alpha+1}+b^+M_4^{\alpha+2}+c^+M_4^{\alpha+3}),\end{aligned}$$

if $\alpha=-1$, we have

$$\begin{aligned}\int_0^T|x'(t)|^p\mathrm{d}t\leqslant&\int_0^T|a(t)x^{\alpha+1}-b(t)x^{\alpha+2}-c(t)x^{\alpha+3}|\mathrm{d}t\\<&T(a^++b^+M_4+c^+M_4^2),\end{aligned}$$

if $-2<\alpha<-1$, we obtain

$$\begin{aligned}\int_0^T|x'(t)|^p\mathrm{d}t\leqslant&\int_0^T|a(t)x^{\alpha+1}-b(t)x^{\alpha+2}-c(t)x^{\alpha+3}|\mathrm{d}t\\<&T(a^+M_3^{\alpha+1}+b^+M_4^{\alpha+2}+c^+M_4^{\alpha+3}),\end{aligned}$$

if $\alpha=-2$, we get

$$\begin{aligned}\int_0^T|x'(t)|^p\mathrm{d}t\leqslant&\int_0^T|a(t)x^{\alpha+1}-b(t)x^{\alpha+2}-c(t)x^{\alpha+3}|\mathrm{d}t\\<&T(a^+\frac{1}{M_3}+b^++c^+M_4),\end{aligned}$$

if $-3<\alpha<-2$, we have

$$\begin{aligned}\int_0^T|x'(t)|^p\mathrm{d}t\leqslant&\int_0^T|a(t)x^{\alpha+1}-b(t)x^{\alpha+2}-c(t)x^{\alpha+3}|\mathrm{d}t\\<&T(a^+M_3^{\alpha+1}+b^+M_3^{\alpha+2}+c^+M_4^{\alpha+3}),\end{aligned}$$

if $\alpha = -3$, we get

$$\begin{aligned}\int_0^T |x'(t)|^p \mathrm{d}t \leqslant& \int_0^T |a(t)x^{\alpha+1} - b(t)x^{\alpha+2} - c(t)x^{\alpha+3}|\mathrm{d}t \\ <& T(a^+ \frac{1}{M_3^2} + b^+ \frac{1}{M_4} + c^+),\end{aligned}$$

if $\alpha < -3$, we obtain

$$\begin{aligned}\int_0^T |x'(t)|^p \mathrm{d}t \leqslant& \int_0^T |a(t)x^{\alpha+1} - b(t)x^{\alpha+2} - c(t)x^{\alpha+3}|\mathrm{d}t \\ <& T(a^+ M_3^{\alpha+1} + b^+ M_3^{\alpha+2} + c^+ M_3^{\alpha+3}).\end{aligned}$$

Regardless of the above situation, it may easily be shown that there exists a positive constant K' such that

$$\int_0^T |x'(t)|^p \mathrm{d}t \leqslant K'.$$

Note that $x(0) = x(T)$, hence there exists a point $t_0 \in [0, T]$ such that $x'(t_0) = 0$, then $\varphi_p(x'(t_0)) = 0$. Put $G_1 := \max\{|f(x)|, x \in [M_3, M_4]\}$ By (2.3) and Hölder's inequality, we get

$$\begin{aligned}\big|\varphi_p(x'(t))\big| =& \big|\int_{t_0}^t (\varphi_p(x'(t)))'\mathrm{d}t\big| \\ \leqslant& \lambda\big(\int_0^T |f(x(t))x'(t)|\mathrm{d}t + \int_0^T |a(t)x^{\alpha+1} - b(t)x^{\alpha+2} - c(t)x^{\alpha+3}|\mathrm{d}t\big) \\ \leqslant& \int_0^T |f(x(t))x'(t)|\mathrm{d}t + \int_0^T |a(t)x^{\alpha+1} - b(t)x^{\alpha+2} - c(t)x^{\alpha+3}|\mathrm{d}t \\ \leqslant& G_1 \int_0^T |x'(t)|\mathrm{d}t + K' \\ \leqslant& G_1 T^{\frac{1}{q}} \big(\int_0^T |x'(t)|^p \mathrm{d}t\big)^{\frac{1}{p}} + K' \\ \leqslant& G_1 T^{\frac{1}{q}} (K')^{\frac{1}{p}} + K' \\ :=& H'.\end{aligned} \tag{3.11}$$

Now, we claim that there exists a positive constant N_2, for all $t \in \mathbb{R}$ such that

$$|x'|_\infty \leqslant N_2. \tag{3.12}$$

In fact, if (3.12) fails, then there exists $N' > 0$ such that

$$|x'|_\infty > N'.$$

Then we have

$$|\varphi_p(x')|_\infty = |x'|_\infty^{p-1} > (N')^{p-1},$$

which is a contradiction. Hence (3.12) holds.

Let

$$\Omega_2 := \{x \in X \mid M_3 < x(t) < M_4, |x'|_\infty < N_2\},$$

which is an open set in X. Obviously, condition (k1) of Lemma 2.3 is satisfied.

For a possible solution e_2 of the equation

$$-a(t)e_2^\alpha + b(t)e_2^{\alpha+1} + c(t)e_2^{\alpha+2} = 0,$$

satisfied $M_3 < e_2 < M_4$, Otherwise, it contradicts (3.9) and (3.10). Therefore, condition (k2) of Lemma 2.3 is satisfied.

Finally, we verify that condition (k3) of Lemma 2.3 is also satisfied. By (3.9) and (3.10) we obtain that

$$-a(t)M_3^\alpha + b(t)M_3^{\alpha+1} + c(t)M_3^{\alpha+2} < 0$$

and

$$-a(t)M_4^\alpha + b(t)M_4^{\alpha+1} + c(t)M_4^{\alpha+2} > 0$$

uniformly in $t \in [0, T]$. So condition (k3) of Lemma 2.3 is also satisfied.

In view of all the discussion above, applying Lemma 2.3, equation (1.5) has at least one T-periodic solution x satisfied $M_3 < x(t) < M_4$ for all $t \in [0, T]$. Theorem 3.2 is proved. □

4. Examples

In this section, we will give some specific examples and apply the theorems obtained in the previous section to prove the existence of positive periodic solutions for these examples, to illustrating the applicability of the conclusions obtained in this paper.

Example 4.1. Consider the following Liebau type differential equation

$$x''(t) + 9x' + (\cos t + 2)x^{\frac{1}{4}} - (\sin t + 3)x^{\frac{1}{2}} = 0. \tag{4.1}$$

Obviously, equation (4.1) is the case of equations (1.4) when $p = 2$, $f(x) = 9$, $r(t) = \cos t + 2$, $s(t) = \sin t + 3$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{4}$. Equation (4.1) also is the case of equations (1.1). We have $\beta > \gamma$, $r(t) > 0$, $s(t) > 0$ and $r^- = 1$, $r^+ = 3$, $s^- = 2$, $s^+ = 4$.

From Theorem 3.1 we get that the equations (4.1) has at least one positive 2π-periodic solution $x(t)$ satisfied $M_1 < x(t) < M_2$ for all $t \in [0, 2\pi]$, where $0 < M_1 < \frac{1}{256}$, $M_2 > \frac{81}{16}$ are constants.

Example 4.2. Consider the following second order p-Laplacian differential equation

$$(\varphi_p(x'(t)))' + 2x' + \big(\cos\big(\frac{2\pi t}{T}\big) + 7\big)x^{-1} - \big(\sin\big(\frac{2\pi t}{T}\big) + 2\big)x = 0. \tag{4.2}$$

It is clear that equation (4.2) is the case of equations (1.4) when $f(x) = 2$, $r(t) = \cos(\frac{2\pi t}{T}) + 7$, $s(t) = \sin(\frac{2\pi t}{T}) + 2$, $\beta = 1$, $\gamma = -1$. We have $\beta > \gamma$, $\frac{1}{\beta-\gamma} = \frac{1}{2}$ and $r^- = 6$, $r^+ = 8$, $s^- = 1$, $s^+ = 3$.

Then Theorem 3.1 guarantees that the equations (4.2) has at least one positive T-periodic solution $x(t)$ satisfied $M_1' < x(t) < M_2'$ for all $t \in [0, T]$, where $0 < M_1' < \sqrt{2}$, $M_2' > 2\sqrt{2}$ are constants.

As can be seen from the above two examples, equation (1.4) is a direct generalization of equation (1.1) and (1.2).

Example 4.3. Consider the following second order cubic differential equation

$$x''(t) + \big(\cos\big(\frac{2\pi t}{T}\big) + 5\big)x - \big(\sin\big(\frac{2\pi t}{T}\big) + 6\big)x^2 - \big(\sin\big(\frac{2\pi t}{T}\big) + 5\big)x^3 = 0. \tag{4.3}$$

In equations (1.5), when $p = 2,\ f(x) = 0,\ a(t) = \cos(\frac{2\pi t}{T}) + 5, b(t) = \sin(\frac{2\pi t}{T}) + 6$ and $c(t) = \sin(\frac{2\pi t}{T}) + 5,\ \alpha = 1$, equations (1.5) reduce to equations (4.3). Clearly, equation (4.3) is also a special case of equation (1.3). We have $a(t) > 0,\ b(t) > 0,\ c(t) > 0$ and $a^- = 4, a^+ = 6, b^- = 5, b^+ = 7, c^- = 4, c^+ = 6$. Direct calculation we can get ${b^+}^2 - {b^-}^2 = 49 - 25 = 24 < 32 = 2a^-c^-$.

Then for each $T > 0$, we from Theorem 3.2 conclude that the equations (4.3) has at least one positive T-periodic solution $x(t)$ satisfied $M_3 < x(t) < M_4$ for all $t \in [0, T]$, where $0 < M_3 < \frac{\sqrt{59}}{12} - \frac{7}{12},\ M_4 > \frac{\sqrt{193}}{10} - \frac{1}{2}$ are constants.

Example 4.4. Consider the following second order p-Laplacian differential equation

$$(\varphi_p(x'(t)))' + 2x^2x' + (\cos t + 3)x^{-\frac{3}{2}} - (\sin t + 2)x^{-\frac{1}{2}} - (\cos t + 4)x^{\frac{1}{2}} = 0. \tag{4.4}$$

In equations (1.5), when $f(x) = 2x^2,\ a(t) = \cos t + 3, b(t) = \sin t + 2$ and $c(t) = \cos t + 4,\ \alpha = -\frac{3}{2}$, equations (1.5) reduce to equations (4.4). Clearly, we have $a(t) > 0,\ b(t) > 0,\ c(t) > 0$ and $a^- = 2, a^+ = 4, b^- = 1, b^+ = 3, c^- = 3, c^+ = 5$. Direct calculation we can get ${b^+}^2 - {b^-}^2 = 9 - 1 = 8 < 12 = 2a^-c^-$.

Then Theorem 3.2 guarantees that the equations (4.4) has at least one positive 2π-periodic solution $x(t)$ satisfied $M_3' < x(t) < M_4'$ for all $t \in [0, 2\pi]$, where $0 < M_3' < \frac{\sqrt{13}-3}{10},\ M_4' > \frac{\sqrt{89}-1}{6}$ are constants.

From examples 4.3 and 4.4, it can be seen that equations (1.5) is not only a generalization of equations (1.3) but also equations (1.5) can deal with some singular polynomial differential equations.

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