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SOLVING FUZZY FRACTIONAL EVOLUTION EQUATIONS WITH DELAY AND NONLOCAL CONDITIONS*

Xuping Zhang[1,†] and Donal O'Regan[2]

Abstract In this paper, we prove existence and uniqueness of two kinds of fuzzy mild solutions for fuzzy fractional evolution equations with delay and nonlocal conditions under Caputo gH differentiability. In particular, the strong restriction on the constants in the condition of Lipschitzian is completely removed when the nonlocal term $g \equiv 0$. An example is provided to illustrate our results.

Keywords Fuzzy fractional evolution equations, delay, nonlocal delay initial condition, fuzzy mild solution, fuzzy strongly continuous semigroup.

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1. Introduction

Fuzzy differential equations were proposed to handle uncertainty due to incomplete information that appears in many computer or mathematical models of some deterministic real world phenomena. There are several methods to researching fuzzy differential equations, see [1–4, 10, 18, 19, 26, 29]. The first approach uses the Hukuhara derivative (H derivative) of a fuzzy-valued function, but for some fuzzy differential equations within that framework, the diameter of the solution $u(t)$ is unbounded as the time t increases, which is quite different from the crisp cases. In order to overcome the above problem to some degree, Bede etc [4, 5] introduced a new concept of fuzzy derivatives called the generalized Hukuhara differentiability (gH differentiability) of fuzzy-valued functions. This concept has been studied and applied in the context of fuzzy differential equations by Alikhani and Bahrami [3], Khastan ect [18] and Mosleh and Otadi [22].

Fractional differential equations is an emerging field, which are used to characterize hereditary and memory properties of different processes and materials in physics, electrodynamics, chemistry, engineering, aerodynamics of complex medium and polymer rheology, see [8, 9, 30, 31]. Recently, fuzzy fractional differential equations have emerged as a significant topic, see [6, 13, 27]. Many complex processes in

†The corresponding author. Email: lanyu9986@126.com(X. Zhang)
1Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
2School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
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technology and nature are described by functional differential equations and they receive attention nowadays. Fuzzy differential equations with delay is one of the important type of functional differential equations, in which the response of the system depends not only on the current state of system, but also on the past history of the system. For more details on this topic, see for example, [14, 15, 17, 20, 24, 25, 32, 33].

Evolution equations can be interpreted as the differential law of the development in time of a system. Obtaining explicit solutions of an evolution equation in Banach space is based mainly on local compactness or interior estimations of normal linear spaces and on techniques of semigroups, see [11, 12]. Unfortunately, these properties do not hold in the case of fuzzy number spaces, in the context of which lack of separability and linearity are the main difficulties for mathematicians to obtain similar results as in classical cases [21]. Results on evolution equations in the fuzzy metric spaces are still at the initial stage of development.

The first paper dealing formally with semigroups of operators on spaces of fuzzy-valued functions was by Gal and Gal [16]. In 2018, Son [28] proved the existence of two kinds of fuzzy mild solutions of the Cauchy problem for fuzzy fractional evolution equations under Caputo gH differentiability by using the Schauder fixed point theorem. However so far we have not seen relevant papers that study fuzzy fractional evolution equations with *delay*.

Inspired by the above-mentioned aspects, in this paper, we study the existence of two kinds of fuzzy mild solutions for a class of fuzzy fractional evolution equations with *delay* and *nonlocal conditions* in the space of triangular fuzzy numbers $\mathcal{T}$:

$$\begin{aligned} &{}_{gH}^{C}\mathcal{D}_{0+}^{q}x(t) = Ax(t) \oplus f(t, x(t), x_t), \quad t \in [0, a], \\ &x(t) = g(x)(t) \oplus \varphi(t), \quad t \in [-h, 0], \end{aligned} \tag{1.1}$$

where the state function $x(\cdot)$ takes values in the space of triangular fuzzy numbers $\mathcal{T}$, a and h are two constants such that $0 < h; a < +\infty$, $A : D(A) \subset \mathcal{T} \to \mathcal{T}$ is a closed linear operator and generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{T}$, ${}_{gH}^{C}\mathcal{D}_{0+}^{q}$ is the fuzzy Caputo fractional gH derivative of order $q \in (0, 1)$, $f : [0, a] \times \mathcal{T} \times C([-h, 0], \mathcal{T}) \to \mathcal{T}$ is a continuous nonlinear function, $\varphi \in C([-h, 0], \mathcal{T})$ is a priori given history, while the function $g : C([-h, a], \mathcal{T}) \to C([-h, 0], \mathcal{T})$ implicitly defines a complementary history, chosen by the system itself, x_t denotes the function in $C([-h, 0], \mathcal{T})$ defined as $x_t(\tau) = x(t + \tau)$ for $\tau \in [-h, 0]$, and $x_t(\cdot)$ represent the time history of the state from the time $t - h$ up to the present time t.

The rest of this paper is organized as follows. In Section 2, for the convenience of the reader, we introduce some notations and preliminaries which are used throughout this paper. In Section 3, we prove existence and uniqueness of two kinds of fuzzy mild solutions fuzzy fractional evolution equations with *delay* and *nonlocal conditions*. In addition, the strong restriction on the constants in the condition of Lipschitzian is completely removed when the nonlocal term $g \equiv 0$ in this section. An example is given in the final section to illustrate our theory.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of the real axis $x : \mathbb{R} \to [0, 1]$, satisfying the following properties:

(i) x is normal, i.e., there exists $b_0 \in \mathbb{R}$ such that $x(b_0) = 1$;

(ii) x is fuzzy convex, i.e., $x(\lambda b_1 + (1-\lambda)b_2) \geq \min\{x(b_1), x(b_2)\}$, for $b_1, b_2 \in \mathbb{R}$ and $\lambda \in (0,1]$;

(iii) x is upper semicontinuous;

(iv) $[x]^0 = \overline{\{n \in \mathbb{R} : x(n) > 0\}}$ is compact.

Then, $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. The α-cuts or level sets of $x \in \mathbb{R}_{\mathcal{F}}$ are defined by $[x]^\alpha = \{n \in \mathbb{R} : x(n) \geq \alpha\}$ for $\alpha \in (0,1]$.

The mapping $x : \mathbb{R} \to [0,1]$ takes a particularly nice form since the level sets of x become intervals with parametric form $x = (x^-, x^+)$ such that $[x(n)]^\alpha = [x^-_\alpha(n), x^+_\alpha(n)]$, $\alpha \in [0,1]$, $n \in \mathbb{R}$. Denote the zero fuzzy number $\hat{0}$ by $[\hat{0}]^\alpha = [0,0]$ for all $\alpha \in [0,1]$. The operations of addition and scalar multiplication of fuzzy numbers on $\mathbb{R}_{\mathcal{F}}$ have the form

$$[x \oplus y]^\alpha = [x]^\alpha + [y]^\alpha, \quad [\lambda \odot x]^\alpha = \lambda [x]^\alpha, \quad \forall \alpha \in [0,1],$$

where $\lambda \in \mathbb{R}$, $[x]^\alpha + [y]^\alpha$ means the usual addition of two intervals of $\mathbb{R}$ and $\lambda[x]^\alpha$ means the usual product between a scalar and a real interval number.

Definition 2.1 ([5]). The generalized Hukuhara differential of two fuzzy numbers $u, w \in \mathbb{R}_{\mathcal{F}}$ (gH difference for short) is defined as follows

$$x \ominus_{gH} y = z \Leftrightarrow \begin{cases} \text{(i)} \;\; x = y \oplus z, \quad \text{or} \\ \text{(ii)} \; y = x \oplus (-1) \odot z. \end{cases}$$

Denote $d : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^+ \cup \{0\}$ by

$$d(x,y) = \sup_{0 \leq \alpha \leq 1} d_H([x]^\alpha, [y]^\alpha), \;\text{ for all } x, y \in \mathbb{R}_{\mathcal{F}},$$

where d_H is the Hausdorff metric in the set consisting of all nonempty, compact and convex subsets of $\mathbb{R}$. Now $d(x,y)$ is called the sup-metric between fuzzy numbers x and y. For more properties of the sup-metric d, see [5, 6]. Denote $\mathcal{T}$ by the set of all triangular fuzzy number in $\mathbb{R}_{\mathcal{F}}$. Now $(\mathcal{T}, d)$ is a subspace of the metric space $(\mathbb{R}_{\mathcal{F}}, d)$ and it is a complete metric space. We denote by $C([0,a], \mathcal{T})$ the set of all continuous mappings $f : [0,a] \to \mathcal{T}$, and for any $t \in [0,a]$, we denote by $C^{\mathcal{T}}_t := C([-h,t], \mathcal{T})$ the complete metric space of all the continuous functions from $[-h,t]$ into $\mathcal{T}$ with the sup-metric $D_t(f,g) = \sup\limits_{-h \leq \tau \leq t} d(f(\tau), g(\tau))$. Let $L([0,a], \mathcal{T})$ be the set of Lebesgue integrable fuzzy-valued functions from $[0,a]$ into $\mathcal{T}$ (for the definition of Lebesgue integrable fuzzy-valued functions, see [7]).

Definition 2.2 ([3, 5]). Let $f : [0,a] \to \mathbb{R}_{\mathcal{F}}$ and $t_0 \in (0,a)$. We say f is gH differentiable at t_0, if there exists an element $\mathcal{D}_{gH} f(t_0) \in \mathbb{R}_{\mathcal{F}}$, such that

(i) for all $r > 0$ sufficiently small, there exist $f(t_0 + r) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - r)$ and the limits

$$\mathcal{D}_{gH} f(t_0) = \lim_{r \to 0} \frac{f(t_0 + r) \ominus f(t_0)}{r} = \lim_{r \to 0} \frac{f(t_0) \ominus f(t_0 - r)}{r},$$

or

(ii) for all $r > 0$ sufficiently small, there exist $f(t_0) \ominus f(t_0 + r)$, $f(t_0 - r) \ominus f(t_0)$ and the limits

$$\mathcal{D}_{gH} f(t_0) = \lim_{r\to 0} \frac{f(t_0) \ominus f(t_0 + r)}{-r} = \lim_{r\to 0} \frac{f(t_0 - r) \ominus f(t_0)}{-r},$$

(r and $-r$ at denominators means $\frac{1}{r}\cdot$ and $-\frac{1}{r}\cdot$, respectively).

A fuzzy-valued function will be referred to as (i)-gH differentiable on $t_0 \in (0, a)$ if it is gH differentiable on $t_0 \in (0, a)$ in the case (i) etc. The fuzzy-valued function f is called right (i)-gH differentiable at t_0, if in the case (i), there exists an element $\mathcal{D}^+_{gH} f(t_0) \in \mathbb{R}_{\mathcal{F}}$, such that for all $r > 0$ sufficiently small, there exist only $f(t_0 + r) \ominus f(t_0)$ and the limit

$$\mathcal{D}^+_{gH} f(t_0) = \lim_{r\to 0} \frac{f(t_0 + r) \ominus f(t_0)}{r}.$$

Also f is called left (i)-gH differentiable at t_0, if in the case (i), there exists an element $\mathcal{D}^-_{gH} f(t_0) \in \mathbb{R}_{\mathcal{F}}$, such that for all $r > 0$ sufficiently small, there exist only $f(t_0) \ominus f(t_0 - r)$ and the limit

$$\mathcal{D}^-_{gH} f(t_0) = \lim_{r\to 0} \frac{f(t_0) \ominus f(t_0 - r)}{r}.$$

Analogous concepts can be defined for right and left (ii)-gH differentiability at t_0.

In this paper, we denote (i)-gH derivative and (ii)-gH derivative of the fuzzy-valued function f on $t_0 \in [0, a]$ by $D^i_{gH}(t_0)$ and $D^{ii}_{gH}(t_0)$, respectively. In addition, we say that the fuzzy-valued f does not have any switch at t_0. If f has a switch at t_0, then f may be not (i)-gH differentiable at t_0 and not (ii)-gH differentiable at t_0.

Denote $\mathcal{L}(\mathcal{T})$ by the space of all bounded linear operator $p : \mathcal{T} \to \mathcal{T}$, and specify that

$$\|p\|_{\mathcal{L}(\mathcal{T})} := \inf\{M : d(px, \hat{0}) \le M d(x, \hat{0}),\ \forall x \in \mathcal{T}\} \tag{2.1}$$

is the norm of p, and

$$(\alpha p_1 + \beta p_2)(x) = \alpha \odot (p_1 x) \oplus \beta \odot (p_2 x),\quad \forall x \in \mathcal{T},$$

is a linear operation on $\mathcal{L}(\mathcal{T})$, where $\alpha, \beta \in \mathbb{R}$, $p_1, p_2 \in \mathcal{L}(\mathcal{T})$. It is easy to see that

$$d(px, \hat{0}) \le \|p\|_{\mathcal{L}(\mathcal{T})} d(x, \hat{0}),\quad d(px, py) \le \|p\|_{\mathcal{L}(\mathcal{T})} d(x, y),\ \ x, y \in \mathcal{T}.$$

Definition 2.3 ([23]). Let $x \in L([0, a], \mathcal{T})$ and $[x(t)]^\alpha = [x^-_\alpha(t), x^+_\alpha(t)]$, for all $t \in [0, a]$, $\alpha \in [0, 1]$. The mixed Riemann-Liouville fractional integral of order $q \in (0, 1]$ with the lower limit zero for a fuzzy-valued function x is defined as

$${}^{RL}_{\mathcal{F}}\mathcal{I}^q_{0+} x(t) = \frac{1}{\Gamma(q)} \odot \int_0^t (t - s)^{q-1} \odot x(s) ds,$$

by levelsetwise

$$[{}^{RL}_{\mathcal{F}}\mathcal{I}^q_{0+} x(t)]^\alpha = [{}^{RL}I^q_{0+} x^-_\alpha(t),\ {}^{RL}I^q_{0+} x^+_\alpha(t)],\ \ \alpha \in [0, 1],\ t \in [0, a],$$

where ${}^{RL}I^q_{0+}$ is Riemann-Liouville fractional integral operator of order $q \in (0, 1]$ of a real-valued function.

Concretely, we say that x is Caputo (i)-gH differentiable if

$$ {}^{C}_{gH}\mathcal{D}^{q}_{0^+}x(t) = {}^{RL}_{\mathcal{F}}\mathcal{I}^{1-q}_{0^+}(\mathcal{D}^{i}_{gH}x(t)), $$

and we say that x is Caputo (ii)-gH differentiable if

$$ {}^{C}_{gH}\mathcal{D}^{q}_{0^+}x(t) = {}^{RL}_{\mathcal{F}}\mathcal{I}^{1-q}_{0^+}(\mathcal{D}^{ii}_{gH}x(t)). $$

Definition 2.4 ([28]). Suppose that $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(\mathcal{T})$ is a semigroup of operators on $\mathcal{T}$ satisfying $\lim\limits_{t\to 0^+} T(t)x = x$ for all $x \in \mathcal{T}$. Then we say that $\{T(t)\}_{t\geq 0}$ is a strong continuous semigroup (or C_0-semigroup) on $\mathcal{T}$.

Proposition 2.1 ([28]). *Suppose that $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(\mathcal{T})$ is a C_0-semigroup on $\mathcal{T}$. Then the mapping $T(\cdot)x : [0, +\infty) \to \mathcal{T}$ is continuous for all $x \in \mathcal{T}$.*

Definition 2.5 ([16]). Let $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(\mathcal{T})$ is a C_0-semigroup on $\mathcal{T}$. Then the infinitesimal generator $A : D(A) \subset \mathcal{T} \to \mathcal{T}$ is defined by

$$ \begin{aligned} Ax &= \lim_{r\to 0^+} \frac{1}{r} \odot (T(r)x \ominus_{gH} x), \\ D(A) &= \{x \in \mathcal{T} : \lim_{r\to 0^+} \frac{1}{r} \odot (T(r)x \ominus_{gH} x) \text{ exists in } \mathcal{T}\}. \end{aligned} $$

For more details about the properties of the operator semigroups, we refer to the paper by [28]. Throughout this paper, let $A : D(A) \subset \mathcal{T} \to \mathcal{T}$ be a closed linear operator and let A generate a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{T}$. Then from Lemma 5.1 in [28], we know that

$$ M := \sup_{t\in[0,a]} \|T(t)\|_{\mathcal{L}(\mathcal{T})} \geq 1. \tag{2.2} $$

3. Main results

Lemma 3.1. *The fuzzy fractional evolution equations with delay and nonlocal conditions* (1.1) *is equation to one of the following fuzzy integral equations*

$$ x(t) = \begin{cases} g(x)(t) \oplus \varphi(t), \quad t \in [-h, 0]; \\ U_q(t) \odot [g(x)(0) \oplus \varphi(0)] \\ \oplus \displaystyle\int_0^t (t-s)^{q-1} \odot V_q(t-s) f(s, x(s), x_s) ds, \quad t \in [0, a], \end{cases} \tag{3.1} $$

if x is Caputo (i)-gH differentiable,

$$ x(t) = \begin{cases} g(x)(t) \oplus \varphi(t), \quad t \in [-h, 0]; \\ U_q(t) \odot [g(x)(0) \oplus \varphi(0)] \\ \ominus (-1) \odot \displaystyle\int_0^t (t-s)^{q-1} \odot V_q(t-s) f(s, x(s), x_s) ds, \quad t \in [0, a]. \end{cases} \tag{3.2} $$

if x is Caputo (ii)-gH differentiable, where

$$ U_q(t) = \int_0^\infty \Psi_q(s) \odot T(t^q s) ds, \quad V_q(t) = q \odot \int_0^\infty s\Psi_q(s) \odot T(t^q s) ds, \tag{3.3} $$

and Ψ_q is the function of Wright type defined on $(0,\infty)$ which satisfies

$$\Psi_q(s) \geq 0, \quad \int_0^\infty \Psi_q(s)ds = 1, \quad \int_0^\infty s^\gamma \Psi_q(s)ds = \frac{\Gamma(1+\gamma)}{\Gamma(1+q\gamma)}, \quad \gamma \in [0,1]. \tag{3.4}$$

Proof. It is direct result of lemma 4.1 in [28]. □

Lemma 3.2 ([34]). *For any $t \geq 0$, the operators $U_q(t)$ and $V_q(t)$ are bounded, that is*

$$D(U_q(t)x, \hat{0}) \leq MD(x, \hat{0}) \ \text{and} \ D(V_q(t)x, \hat{0}) \leq \frac{M}{\Gamma(q)} D(x, \hat{0}), \quad \forall x \in \mathcal{T}.$$

Lemma 3.3 ([34]). *For any fixed $t \geq 0$, $U_q(t)$ and $V_q(t)$ are strong continuous operators, that is, for any $x \in \mathcal{T}$ and $t'' > t' \geq 0$,*

$$D(U_q(t'')x, U_q(t')x) \to 0 \ \text{and} \ D(V_q(t'')x, V_q(t')x) \to 0, \quad t' \to t''.$$

Let $(\hat{Z}x)(t) := U_q(t) \odot [g(x)(0) \oplus \varphi(0)] \ominus (-1) \odot \int_0^t (t-s)^{q-1} \odot V_q(t-s) f(s, x(s), x_s)ds$, $t \in [0,a]$, and let $\hat{C}_a^{\mathcal{T}} := \hat{C}([-h,a],\mathcal{T}) = \{x \in C_a^{\mathcal{T}} : (\hat{Z}x)(t) \text{ exists for any } t \in [0,a]\}$.

Definition 3.1. By the fuzzy mild solution in type 1 of fuzzy fractional evolution equations with delay and nonlocal conditions (1.1) on the interval $[-h,a]$, we mean a function $x \in C_a^{\mathcal{T}}$ satisfying

$$x(t) = \begin{cases} g(x)(t) \oplus \varphi(t), & t \in [-h,0]; \\ U_q(t) \odot [g(x)(0) \oplus \varphi(0)] & \\ \oplus \displaystyle\int_0^t (t-s)^{q-1} \odot V_q(t-s) f(s, x(s), x_s)ds, & t \in [0,a]. \end{cases} \tag{3.5}$$

By the fuzzy mild solution in type 2 of fuzzy fractional evolution equations with delay and nonlocal conditions (1.1) on the interval $[-h,a]$, we mean a function $x \in \hat{C}_a^{\mathcal{T}}$ satisfying

$$x(t) = \begin{cases} g(x)(t) \oplus \varphi(t), & t \in [-h,0]; \\ U_q(t) \odot [g(x)(0) \oplus \varphi(0)] & \\ \ominus (-1) \odot \displaystyle\int_0^t (t-s)^{q-1} \odot V_q(t-s) f(s, x(s), x_s)ds, & t \in [0,a]. \end{cases} \tag{3.6}$$

Assumption 3.1. *The following supposed conditions will be need in the next results:*

(i) There exist positive constants L_1 and L_2 such that

$$d(f(t,u,v), f(t,\tilde{u},\tilde{v})) \leq L_1 d(u,\tilde{u}) + L_2 D_0(v,\tilde{v})$$

for all $u,\tilde{u} \in \mathcal{T}$, $v,\tilde{v} \in C_0^{\mathcal{T}}$ and a.e. $t \in [0,a]$;

(ii) There exists positive constant L such that

$$D_0(g(w), g(\tilde{w})) \leq L D_a(w,\tilde{w}), \quad \forall\, w,\tilde{w} \in C_a^{\mathcal{T}}.$$

Theorem 3.1. *If the Assumption 3.1 hold, then for any $\varphi \in C_0^{\mathcal{T}}$, the fuzzy fractional evolution equations with delay and nonlocal conditions (1.1) has a unique fuzzy mild solution in type 1 in the space $C_a^{\mathcal{T}}$ provided that*

$$L + \frac{a^q(L_1+L_2)}{\Gamma(1+q)} < \frac{1}{M}. \tag{3.7}$$

Proof. Consider the operator $Q : C_a^{\mathcal{T}} \to C_a^{\mathcal{T}}$ defined by

$$(Qx)(t) = \begin{cases} g(x)(t) \oplus \varphi(t), \quad t \in [-h, 0]; \\ U_q(t) \odot [g(x)(0) \oplus \varphi(0)] \\ \oplus \displaystyle\int_0^t (t-s)^{q-1} \odot V_q(t-s) f(s, x(s), x_s) ds, \quad t \in [0, a]. \end{cases} \tag{3.8}$$

By direct calculation, it is easy to see that the operator Q is well defined. From Definition 3.1, we can know that the fuzzy mild solution of the problem (1.1) is equivalent to the fixed point of the operator Q defined by (3.8). In the following, we will prove the operator Q has a unique fixed point in the space $C_a^{\mathcal{T}}$.

For any $x, y \in C_a^{\mathcal{T}}$, we get from Assumption 3.1 and Lemma 3.3 that

$$\begin{aligned} & d((Qx)(t), (Qy)(t)) \\ =& d(g(x)(t) \oplus \varphi(t), g(y)(t) \oplus \varphi(t)) \\ =& d(g(x)(t), g(y)(t)) \le L(x(t), y(t)) \le L D_a(x, y), \quad t \in [-h, 0], \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} & d((Qx)(t), (Qy)(t)) \\ \le& M d(g(x)(0), g(y)(0)) + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} d(f(s, x(s), x_s), f(s, y(s), y_s)) ds \\ \le& M D_0(g(x), g(y)) + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} [L_1 d(x(s), y(s)) + L_2 d(x_s, y_s)] ds \\ \le& M L D_a(x, y) + \frac{M(L_1 + L_2) D_a(x, y)}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \\ \le& M \left[L + \frac{a^q (L_1 + L_2)}{\Gamma(1+q)} \right] D_a(x, y), \quad t \in [0, a]. \end{aligned} \tag{3.10}$$

It follows from (2.2), (3.9) and (3.10) that for any $x, y \in C_a^{\mathcal{T}}$,

$$\begin{aligned} D_a(Qx, Qy) =& \sup_{-h \le t \le a} d((Qx)(t), (Qy)(t)) \\ \le& M \left[L + \frac{a^q (L_1 + L_2)}{\Gamma(1+q)} \right] D_a(x, y). \end{aligned} \tag{3.11}$$

Combining (3.7) and (3.11), we have that

$$D_a(Qx, Qy) \le K D_a(x, y),$$

where $0 < K := M\left[L + \frac{a^q(L_1+L_2)}{\Gamma(1+q)}\right] < 1$. Therefore, it follows from the Banach fixed point theorem that the operator Q defined by (3.8) has a unique fixed point $x^* \in C_a^{\mathcal{T}}$, which is a unique fuzzy mild solution of the problem (1.1) in type 1. □

Theorem 3.2. *Suppose that $A : D(A) \subset \mathcal{T} \to \mathcal{T}$ is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t \ge 0} \subset \mathcal{L}(\mathcal{T})$, $\hat{C}_a^{\mathcal{T}} := \hat{C}([-h, a], \mathcal{T}) \neq \emptyset$ and for any $x \in \hat{C}_a^{\mathcal{T}}$, $(\hat{Z}x)(t)$ exists for all $t \in [0, a]$. If Assumption 3.1 are fulfilled, then the fuzzy fractional evolution equations with delay and nonlocal conditions* (1.1) *has a unique fuzzy mild solution in type 2 in space $\hat{C}_a^{\mathcal{T}}$.*

Proof. Consider the operator $\mathcal{F}:\hat{C}_a^{\mathcal{T}}\to\hat{C}_a^{\mathcal{T}}$ defined by

$$(\Phi x)(t)=\begin{cases} g(x)(t)\oplus\varphi(t), \quad t\in[-h,0];\\ U_q(t)\odot[g(x)(0)\oplus\varphi(0)]\\ \ominus(-1)\odot\displaystyle\int_0^t(t-s)^{q-1}\odot V_q(t-s)f(s,x(s),x_s)ds,\ t\in[0,a].\end{cases}\tag{3.12}$$

By direct calculation, it is easy to see that the operator Φ is well defined. From Definition 3.1, we can know that the fuzzy mild solution of problem (1.1) is equivalent to the fixed point of the operator Φ defined by (3.12). Similar to the proof of Theorem 3.2, we can prove that $\Phi:\hat{C}_a^{\mathcal{T}}\to\hat{C}_a^{\mathcal{T}}$ is a contraction. It follows from the Banach fixed point theorem that Φ has a unique fixed point $x^*\in\hat{C}_a^{\mathcal{T}}$, which is a unique fuzzy mild solution of the problem (1.1) in type 2. □

In order to obtain the existence of a fuzzy mild solution for fuzzy fractional evolution equations with delay and nonlocal conditions (1.1) in Theorems 3.1 and 3.2 we assume the constants in the Lipschitzian condition on the nonlinear term f and the nonlocal term g satisfy a very strong inequality (3.7). If the nonlocal term $g\equiv 0$, then we can remove the strong restriction condition (3.7) by using a new method of argument.

Theorem 3.3. *Assume that $A:D(A)\subset\mathcal{T}\to\mathcal{T}$ is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t\geq 0}\subset\mathcal{L}(\mathcal{T})$ and the nonlocal term $g\equiv 0$. If Assumption 3.1(i) holds, then the problem* (1.1) *has a unique fuzzy mild solution in type 1 in space $C_a^{\mathcal{T}}$.*

Proof. Let $\mathcal{Q}:C_a^{\mathcal{T}}\to C_a^{\mathcal{T}}$ be the operator defined by

$$(\mathcal{Q}x)(t)=\begin{cases}\varphi(t), \quad t\in[-h,0];\\ U_q(t)\odot\varphi(0)\\ \oplus\displaystyle\int_0^t(t-s)^{q-1}\odot V_q(t-s)f(s,x(s),x_s)ds, \quad t\in[0,a].\end{cases}\tag{3.13}$$

From direct calculation, it is easy to see that $\mathcal{Q}$ is well defined and from Definition 3.1, we can see that the fuzzy mild solution of problem (1.1) is equivalent to the fixed point of the operator $\mathcal{Q}$ defined by (3.13). Now, we prove the operator $\mathcal{Q}$ has a unique fixed point in the space $C_a^{\mathcal{T}}$.

For any $t\in[0,a]$, from the formulation of the operator $\mathcal{Q}$ defined by (3.5), using the Lemma 3.2 and Assumption 3.1(i), we get that for any $x,y\in C_a^{\mathcal{T}}$ and $t\in[0,a]$,

$$\begin{aligned} d((\mathcal{Q}x)(t),(\mathcal{Q}y)(t)) &\leq\frac{M}{\Gamma(q)}\int_0^t(t-s)^{q-1}d(f(s,x(s),x_s),f(s,y(s),y_s))ds\\ &\leq\frac{M}{\Gamma(q)}\int_0^t(t-s)^{q-1}[L_1d(x(s),y(s))+L_2D_0(x_s,y_s)]ds\\ &\leq\frac{M(L_1+L_2)D_a(x,y)}{\Gamma(q)}\int_0^t(t-s)^{q-1}ds\\ &\leq\frac{M(L_1+L_2)t^q}{\Gamma(1+q)}D_a(x,y). \end{aligned}\tag{3.14}$$

Again by Lemma 3.2, Assumption 3.1(i), (3.5) and (3.14), we get that for any $t\in[0,a]$,

$$d((\mathcal{Q}^2u)(t),(\mathcal{Q}^2v)(t))$$

$$
\begin{aligned}
&= d(U_q(t)\odot(\mathcal{Q}x)(0), U_q(t)\odot(\mathcal{Q}y)(0)) \\
&\quad + d\Big(\int_0^t (t-s)^{q-1}\odot V_q(t-s)f(s,(\mathcal{Q}x)(s),(\mathcal{Q}x)_s)ds, \\
&\quad \int_0^t (t-s)^{q-1}\odot V_q(t-s)f(s,(\mathcal{Q}y)(s),(\mathcal{Q}y)_s)ds\Big) \\
&= d(U_q(t)\odot\varphi(0), U_q(t)\odot\varphi(0)) \\
&\quad + \int_0^t (t-s)^{q-1} d\Big(V_q(t-s)f(s,(\mathcal{Q}x)(s),(\mathcal{Q}x)_s)ds, \\
&\quad V_q(t-s)f(s,(\mathcal{Q}y)(s),(\mathcal{Q}y)_s)\Big)ds \\
&\le \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1} d\Big(f(s,(\mathcal{Q}x)(s),(\mathcal{Q}x)_s), f(s,(\mathcal{Q}y)(s),(\mathcal{Q}y)_s)\Big)ds \\
&\le \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}[L_1 d((\mathcal{Q}x)(s),(\mathcal{Q}y)(s)) + L_2 D_0((\mathcal{Q}x)_s,(\mathcal{Q}y)_s)]ds \\
&\le \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}[L_1 d((\mathcal{Q}x)(s),(\mathcal{Q}y)(s)) \\
&\quad + L_2 \sup_{-h\le\tau\le 0} d((\mathcal{Q}x)(s+\tau),(\mathcal{Q}y)(s+\tau)]ds \\
&= \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}[L_1 d((\mathcal{Q}x)(s),(\mathcal{Q}y)(s)) \\
&\quad + L_f^2 \sup_{0\le s+\tau\le s} d((\mathcal{Q}x)(s+\tau),(\mathcal{Q}y)(s+\tau)]ds \\
&\le \frac{M}{\Gamma(q)}\Big\{\frac{M(L_1+L_2)D_a(x,y)}{\Gamma(1+q)}\Big[L_1\int_0^t (t-s)^{q-1}s^q ds \\
&\quad + L_2\int_0^t (t-s)^{q-1}\sup_{0\le s+\tau\le s}(s+\tau)^q ds\Big]\Big\} \\
&\le \frac{M}{\Gamma(q)}\Big\{\frac{M(L_1+L_2)D_a(x,y)}{\Gamma(1+q)}\Big[L_1\int_0^t (t-s)^{q-1}s^q ds + L_2\int_0^t (t-s)^{q-1}s^q ds\Big]\Big\} \\
&\le \frac{[M(L_1+L_2)]^2}{\Gamma(q)\Gamma(1+q)}\int_0^t (t-s)^{q-1}s^q ds D_a(x,y) \\
&\le \frac{[M(L_1+L_2)t^q]^2}{\Gamma(1+2q)\mathcal{H}(1+q,q)}\int_0^1 (1-s)^{q-1}s^q ds D_a(x,y) \\
&= \frac{[M(L_1+L_2)t^q]^2}{\Gamma(1+2q)} D_a(x,y),
\end{aligned} \tag{3.15}
$$

where $x, y \in C_a^{\mathcal{T}}$ and $\mathcal{H}(1+q,q) = \int_0^1 (1-s)^{q-1}s^q ds$ is the Beta function .

For any $x, y \in C_a^{\mathcal{T}}$ and $t \in [0,a]$, we assume that

$$
d((\mathcal{Q}^k x)(t), (\mathcal{Q}^k y)(t)) \le \frac{[M(L_1+L_2)t^q]^k}{\Gamma(1+kq)} D_a(x,y), \tag{3.16}
$$

where $k \ge 3$ is a integer. From (3.5), (3.14) and (3.16), using Lemma 3.2 and Assumption 3.1(i), we get that for any $t \in [0,a]$ and $x, y \in C_a^{\mathcal{T}}$,

$$
d((\mathcal{Q}^{k+1}x)(t), (\mathcal{Q}^{k+1}y)(t))
$$

$$
\begin{aligned}
=&d(U_q(t)\odot(\mathcal{Q}^k x)(0),U_q(t)\odot(\mathcal{Q}^k y)(0))\\
&+d\Big(\int_0^t (t-s)^{q-1}\odot V_q(t-s)f(s,(\mathcal{Q}^k x)(s),(\mathcal{Q}^k y)_s)ds,\\
&\int_0^t (t-s)^{q-1}\odot V_q(t-s)f(s,(\mathcal{Q}^k y)(s),(\mathcal{Q}^k y)_s)ds\Big)\\
=&d(U_q(t)\odot\varphi(0),U_q(t)\odot\varphi(0))\\
&+\int_0^t (t-s)^{q-1}d\Big(V_q(t-s)f(s,(\mathcal{Q}^k x)(s),(\mathcal{Q}^k x)_s)ds,\\
&V_q(t-s)f(s,(\mathcal{Q}^k y)(s),(\mathcal{Q}^k y)_s)\Big)ds\\
\le&\frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}d\Big(f(s,(\mathcal{Q}^k x)(s),(\mathcal{Q}^k x)_s),f(s,(\mathcal{Q}^k y)(s),(\mathcal{Q}^k y)_s)\Big)ds\\
\le&\frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}[L_1 d((\mathcal{Q}^k x)(s),(\mathcal{Q}^k y)(s))+L_2 D_0((\mathcal{Q}^k x)_s,(\mathcal{Q}^k y)_s)]ds\\
\le&\frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}[L_1 d((\mathcal{Q}^k x)(s),(\mathcal{Q}^k y)(s))\\
&+L_2\sup_{-h\le\tau\le 0} d((\mathcal{Q}^k x)(s+\tau),(\mathcal{Q}^k x)(s+\tau)]ds\\
=&\frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}[L_1 d((\mathcal{Q}^k x)(s),(\mathcal{Q}^k x)(s))\\
&+L_f^2\sup_{0\le s+\tau\le s} d((Q^k x)(s+\tau),(Q^k x)(s+\tau)]ds\\
\le&\frac{M}{\Gamma(q)}\Big\{\frac{[M(L_f^1+L_f^2)]^k D_a(u,v)}{\Gamma(1+kq)}\Big[L_1\int_0^t (t-s)^{q-1}s^q ds\\
&+L_2\int_0^t (t-s)^{q-1}\sup_{0\le s+\tau\le s}(s+\tau)^q ds\Big]\Big\}\\
\le&\frac{M}{\Gamma(q)}\Big\{\frac{[M(L_1+L_2)]^k D_a(x,y)}{\Gamma(1+kq)}\Big[L_1\int_0^t (t-s)^{q-1}s^q ds+L_2\int_0^t (t-s)^{q-1}s^q ds\Big]\Big\}\\
\le&\frac{[M(L_1+L_2)]^{k+1}}{\Gamma(q)\Gamma(1+kq)\mathcal{H}(1+q,q)}\int_0^t (t-s)^{q-1}s^q ds D_a(x,y)\\
\le&\frac{[M(L_1+L_2)t^q]^{k+1}}{\Gamma(1+(k+1)q)}D_a(x,y). \qquad (3.17)
\end{aligned}
$$

Therefore, by the method of mathematical induction, for any positive integer n, $t\in[0,a]$ and $x,y\in C_a^{\mathcal{T}}$, we obtain

$$d((\mathcal{Q}^n x)(t),(\mathcal{Q}^n y)(t))\le\frac{[M(L_1+L_2)t^q]^n}{\Gamma(1+nq)}D_a(x,y). \qquad (3.18)$$

In addition, by the formulation of the operator $\mathcal{Q}$ defined by (3.5), we know that for any positive integer n, $t\in[-h,0]$ and $x,y\in C_a^{\mathcal{T}}$,

$$d((\mathcal{Q}^n x)(t),(\mathcal{Q}^n y)(t))=0. \qquad (3.19)$$

Consequently, by (3.18) and (3.19), we see that for any positive integer n and

$x, y \in C_a^{\mathcal{T}}$,

$$D_a(\mathcal{Q}^n x, \mathcal{Q}^n y) = \sup_{-h \le t \le a} d((\mathcal{Q}^n x)(t), (\mathcal{Q}^n y)(t)) \\ \le \frac{[M(L_1 + L_2)a^q]^n}{\Gamma(1+nq)} D_a(x, y). \tag{3.20}$$

By the well-known Stirling's formula, we know that for any positive integer n,

$$\Gamma(1+nq) = \sqrt{2\pi nq}\Big(\frac{nq}{e}\Big)^{nq} e^{\frac{\sigma}{12nq}}, \quad 0 < \sigma < 1. \tag{3.21}$$

Hence, from the fact that

$$\frac{[M(L_1+L_2)a^q]^n}{\sqrt{2\pi nq}\Big(\frac{nq}{e}\Big)^{nq} e^{\frac{\sigma}{12nq}}} \to 0 \quad \text{as} \quad n \to \infty,$$

we get that there exists a large enough positive integer n^* such that

$$\frac{[M(L_1+L_2)a^q]^{n^*}}{\sqrt{2\pi n^* q}\Big(\frac{n^* q}{e}\Big)^{n^* q} e^{\frac{\sigma}{12n^* q}}} < 1.$$

It follows that the operator $\mathcal{Q}^{n^*}$ is a contraction, and then has a unique $x^* \in C_a^{\mathcal{T}}$ such that $\mathcal{Q}^{n^*} u^* = u^*$.

Last, we will prove that x^* is also the fixed point of the operator $\mathcal{Q}$. Now $\mathcal{Q}^{n^*} x^* = x^*$ implies that $\mathcal{Q}^{(n^*+1)} x^* = \mathcal{Q} x^*$. In addition, $\mathcal{Q}^{(n^*+1)} x^* = \mathcal{Q} x^*$ and we can write $\mathcal{Q}^{n^*}(\mathcal{Q} x^*) = \mathcal{Q} x^*$, which means that $\mathcal{Q} x^*$ is also a fixed point of the operator $\mathcal{Q}^{n^*}$. Since the fixed point the operator Q^{n^*} is unique, we get $\mathcal{Q} x^* = x^*$. Hence, $\mathcal{Q}$ has a unique fixed point $x^* \in C_a^{\mathcal{T}}$, which is a unique fuzzy mild solution of the problem (1.1) in type 1. □

Theorem 3.4. *Assume that $A : D(A) \subset \mathcal{T} \to \mathcal{T}$ is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t \ge 0} \subset \mathcal{L}(\mathcal{T})$, the nonlocal term $g \equiv 0$, $\hat{C}_a^{\mathcal{T}} := \hat{C}([-h, a], \mathcal{T}) \ne \emptyset$ and for any $x \in \hat{C}_a^{\mathcal{T}}$, $(\hat{Z}x)(t)$ exists for all $t \in [0, a]$. If Assumption 3.1(i) holds, then the problem* (1.1) *has a unique fuzzy mild solution in type 2 in space $\hat{C}_a^{\mathcal{T}}$.*

Proof. Consider the operator $F : \hat{C}_a^{\mathcal{T}} \to \hat{C}_a^{\mathcal{T}}$ defined by

$$(\Psi u)(t) = \begin{cases} \varphi(t), \quad t \in [-h, 0]; \\ U_q(t) \odot \varphi(0) \\ \ominus (-1) \odot \displaystyle\int_0^t (t-s)^{q-1} \odot V_q(t-s) f(s, x(s), x_s) ds, \ t \in [0, a]. \end{cases} \tag{3.22}$$

By direct calculation, it is easy to see that the operator Ψ is well defined. From Definition 3.1, we see that the fuzzy mild solution of problem (1.1) is equivalent to the fixed point of the operator Ψ defined by (3.12). Similar to the proof of Theorem 3.3, we can prove that there exists a large enough positive integer p such that Ψ^p has a unique fixed point $\tilde{x} \in \hat{C}_a^{\mathcal{T}}$ and then $\tilde{x}$ is also the fixed point of the operator Ψ. It follows that $\tilde{x} \in \hat{C}_a^{\mathcal{T}}$ is unique fuzzy mild solution of the problem (1.1) in type 2. □

4. An example

In this section, we give an example to illustrate the feasibility of our results. We consider the fuzzy fractional partial differential equations with delay of the form

$$\begin{aligned}
{}_{gH}^{C}\mathcal{D}_{0^+}^{q}x(t) &= Ax(t) \oplus \frac{\cos t}{\Gamma(0.5)} \odot x(t), \\
&\qquad \oplus \int_{-h}^{0} \rho(s) \odot x_t(s)ds \oplus te^{-6t}\mathcal{A}, \quad t \in [0,a],, \qquad (4.1)\\
x(t) &= \varphi(t), \quad -h \le t \le 0.
\end{aligned}$$

where the state function $x \in C_a^{\mathcal{T}}$, ${}_{gH}^{C}\mathcal{D}_{0^+}^{q}$ is the Caputo gH fractional derivative of order $q \in (0,1]$, $A : D(A) \subset \mathcal{T} \to \mathcal{T}$ is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t\ge 0}$ in $\mathcal{T}$, a, h are two constants such that $0 < h, a < +\infty$, the map $\rho : [-h, 0] \to \mathbb{R}^+$ is integrable, $\mathcal{A} = (0,1,2) \in \mathcal{T}$ is a symmetric triangular fuzzy number, $\varphi \in C_0^{\mathcal{T}}$.

Suppose that

$$f(t, x(t), x_t) = \frac{\cos t}{\Gamma(0.5)} \odot x(t) \oplus \int_{-h}^{0} \rho(s) \odot x_t(s)ds \oplus te^{-6t}\mathcal{A}. \qquad (4.2)$$

Then the problem (4.1) can be abstracted into

$$\begin{aligned}
&{}_{gH}^{C}\mathcal{D}_{0^+}^{q}x(t) = Ax(t) \oplus f(t, x(t), x_t), \quad t \in [0,a], \\
&x(t) = \varphi(t), \quad -h \le t \le 0.
\end{aligned} \qquad (4.3)$$

For any $t \in [0,a]$, by (4.2), it is easy to see that the function $f : [0,a] \times \mathcal{T} \times C_0^{\mathcal{T}} \to \mathcal{T}$ is continuous, and for any $u, v \in \mathcal{T}$ and $\tilde{u}, \tilde{v} \in C_0^{\mathcal{T}}$,

$$d(f(t,u,v), f(t,\tilde{u},\tilde{v}) \le \frac{1}{\sqrt{\pi}} d(u,\tilde{u}) + \int_{-h}^{0} \rho(s)dsD_0(v,\tilde{v}),$$

which means that the nonlinearity f satisfies (Hf) with $L_1 = \frac{1}{\sqrt{\pi}}$ and $L_2 = \int_{-h}^{0} \rho(s)ds$. Hence for every $\varphi \in C_0^{\mathcal{T}}$ problem (4.1) has a unique fuzzy mild solution in type 1 in space $C_a^{\mathcal{T}}$ by Theorem 3.3.

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