Journal of Applied Analysis and Computation
Volume 13, Number 2, April 2023, 1014–1026
Website:http://www.jaac-online.com
DOI:10.11948/20220305

GLOBAL EXISTENCE AND BLOW-UP FOR ONE-DIMENSIONAL WAVE EQUATION WITH WEIGHTED EXPONENTIAL NONLINEARITY

Thanaa Alarfaj[1,2], Lulwah Al-Essa[1,2], Fatimah Alkathiri[1,2] and Mohamed Majdoub[1,2,†]

Abstract We consider the initial value problem for a one-dimensional wave equation with weighted exponential nonlinearity. We show global existence for small amplitude initial data. We also prove that blow-up in finite time occurs if the initial data are localized and the initial velocity being on the average positive.

Keywords Nonlinear wave equation, blow-up, differential inequalities, Hammerstein-type Volterra integral equations.

MSC(2010) 35K58, 35A01, 35B40, 46E30.

1. Introduction

We are concerned in this paper with global existence and blow-up for the following Cauchy problem:

$$\begin{cases} u_{tt} - u_{xx} = G(x, u(x,t)) & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x, t=0) = \phi(x), & x \in \mathbb{R}, \\ u_t(x, t=0) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\phi \in C^2(\mathbb{R}), \psi \in C^1(\mathbb{R})$ and the nonlinearity is given by

$$G(x,u) = \frac{g(u)}{(1+x^2)^{\frac{a+1}{2}}}, \tag{1.2}$$

for some continuous function $g : \mathbb{R} \to \mathbb{R}$.

The problem of global existence, decay and blow-up of solutions of nonlinear hyperbolic equations of type

$$u_{tt} - \Delta u = f(t,x,u), \quad x \in \mathbb{R}^n, \tag{1.3}$$

has attracted considerable attention in the mathematical community and the literature is very extensive. See e.g. [2, 4, 5, 7, 8, 12–14, 17, 20, 21, 23] and references

†The corresponding author. Email: mmajdoub@iau.edu.sa(M. Majdoub)
[1]Department of Mathematics, College of Science, Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam, Saudi Arabia
[2]Basic and Applied Scientific Research Center, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, 31441, Dammam, Saudi Arabia

therein. In a pioneering paper [7], F. John considers the case $f(t,x,u) \geq b|u|^p$, where $b > 0$. He showed that solutions of (1.3) in three space dimensions blow up if $1 < p < 1+\sqrt{2}$ but exist for all time if the data are small and $p > 1+\sqrt{2}$. Later on, T. Kato [8] gave a very simple proof of an analogue of John's result in n space dimensions, with the bound $1+\sqrt{2}$ replaced by $\frac{n+1}{n-1}$ under some additional assumptions on the initial data. According to Strauss's conjecture [17], the bound $\frac{n+1}{n-1}$ is not optimal. Indeed, Strauss conjectured that when the space dimension $n \geq 2$ the value $p_0(n)$, which separates global existence and blow-up, is given by the positive root of the quadratic equation

$$-(n-1)p^2 + (n+1)p + 2 = 0. \tag{1.4}$$

Clearly $p_0(3) = 1+\sqrt{2}$. This conjecture was verified by Glassey [5] in two space dimension, by Zhou [23] for $n = 4$, by Lindblad and Sogge [13] for $n \leq 8$. See also [4] for $n \geq 4$, $p \leq \frac{n+3}{n-1}$ and [14] for the critical case $p = p_0(n)$ in dimensions $n = 2, 3$. Note that a much more succinct proof using Gronwall's type inequality of John's result was given in [21].

For weighted nonlinear terms, we quote [12,18] where problem (1.1) is considered with $g(u) = |u|^{p-1}u$, $p > 1$, $a \geq -1$. It was shown that global solutions exist for small initial data. Moreover, an upper bound of the lifespan was obtained. See also [1] for other weighted nonlinear terms.

Note that the damped case was considered by many authors. See [6] and references therein.

Our aim here is to generalize earlier existing works and reveal new aspects by using other tools especially for blow-up.

Following [12], we define the weighted space $X(a)$ by

$$X(a) := \left\{ v \in C([0,\infty]^2) : \sup_{(x,t)\in[0,\infty]^2} \frac{\langle x+t\rangle\langle x-t\rangle^a}{\langle x\rangle}|v(x,t)| < \infty \right\},$$

where

$$\|v\|_{X(a)} = \sup_{(x,t)\in[0,\infty]^2} \frac{\langle x+t\rangle\langle x-t\rangle^a}{\langle x\rangle}|v(x,t)| \tag{1.5}$$

and $\langle x\rangle = 1+|x|$. We also define $Y(a)$ as in [12] by

$$\begin{aligned} Y(a) = \Big\{ (\phi,\psi) \in C^1(\mathbb{R}) \times C(\mathbb{R}) :\ & \sup_{x\geq 0} \Big\{ \langle x\rangle^a|\phi(x)| + \langle x\rangle^{a+1}|\phi'(x)| \\ & + \langle x\rangle^{a+1}|\psi(x)| \Big\} < \infty \Big\}, \end{aligned}$$

where

$$\|(\phi,\psi)\|_{Y(a)} = \sup_{x\geq 0} \left\{ \langle x\rangle^a|\phi(x)| + \langle x\rangle^{a+1}|\phi'(x)| + \langle x\rangle^{a+1}|\psi(x)| \right\}. \tag{1.6}$$

Our first main result can be formulated as follows.

Theorem 1.1. *Suppose that $g(u) = u(e^{u^2}-1)$ and $a > \frac{1}{3}$. Assume that $(\phi,\psi) \in (C^2(\mathbb{R}) \times (C^1(\mathbb{R})) \cap Y(a)$ and ϕ,ψ are odd functions. Then there are constants $\epsilon_0 > 0$ and $C > 0$ such that if $||(\phi,\psi)||_Y(a) \leq \epsilon$ for $0 < \epsilon < \epsilon_0$ there exists a unique global C^2 solution u of (1.1)-(1.2) satisfying $||u||_X(a) < C\epsilon$.*

Remark 1.1.

- If the given function G and the initial data ϕ, ψ are so smooth then the Cauchy problem (1.1) has a classical local (in time) solution. This follows from Duhamel's formula via the usual fixed point argument in suitable complete metric space. See Appendix A for more details.
- As it will be clear in the proof, Theorem 1.1 holds true for more general nonlinearities $g(u)$ satisfying $g(0) = 0$ and

$$|g(u) - g(v)| \leq c|u - v| \left(|u|^m e^{\lambda |u|^q} + |v|^m e^{\lambda |v|^q} \right) \tag{1.7}$$

  for some positive constants c, m, q, λ with $a > \frac{1}{m+1}$.
- It is an interesting open problem to find sharp upper and lower bounds for the lifespan as in the power case [10, 11, 19]. This will be studied elsewhere.

The following theorem shows that blow-up in finite time happen under suitable assumptions on the nonlinearity g and the the initial data (ϕ, ψ). More precisely, we suppose that the nonlinearity satisfies

$$g(0) = 0, \quad g \quad \text{is increasing,} \tag{1.8}$$

and

$$g(u) \geq C_0 \, u^p \quad \forall \quad u \geq 0, \tag{1.9}$$

for some constants $C_0 > 0$, $p > 1$.

Theorem 1.2. *Suppose that* (1.8)-(1.9) *are fulfilled and* $a \leq 1$. *Let* $(\phi, \psi) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$ *be two nonnegative functions such that, for some constant* $\rho > 0$, *we have*

$$\phi(x) = \psi(x) = 0 \;\; \textit{for} \;\; |x| \geq \rho \quad \textit{and} \quad \int_{-\rho}^{\rho} \psi(x) dx > 0. \tag{1.10}$$

Then, the solution of (1.1)-(1.2) *blows-up in a finite time.*

Remark 1.2. The main novelty in this Theorem lies in its proof which uses some ideas related to Hammerstein-type Volterra equations. See Corollary 2.1 below.

The rest of this paper is organized as follows. In the next section, we recall some basic facts and useful tools. Section 3 is devoted to the proof of Theorem 1.1. The forth section deals with blowing up solutions, that is, Theorem 1.2. Finally, for the sake of completeness, we give in the Appendix the proof of local existence of smooth solutions for more general nonlinear wave equations as well as the finite speed of propagation.

2. Background material

For future convenience, we recall some known and useful tools which will play an important role in the proof of our main results.

Consider the Hammerstein-type Volterra equation

$$u(t) = A + \int_0^t k(t - s) g(u(s)) ds, \tag{2.1}$$

where $A > 0$, the kernel $k : (0, \infty) \to [0, \infty)$ is locally integrable and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (1.8)-(1.9). Define the quantities

$$\mathbf{I}(t) = \int_0^t k(\zeta)d\zeta, \quad F(t,u) = \mathbf{I}(t)g(u) - u, \quad F_{min}(t) = \min_{u \geqslant 0} F(t,u). \tag{2.2}$$

In our context, [3, Theorem 2.12, p. 494] translate to

Theorem 2.1. *Suppose that assumptions* (1.8)-(1.9) *are fulfilled and*

$$\int_0^\infty k(\zeta)\, d\zeta = \infty. \tag{2.3}$$

If

$$k(\zeta) = \zeta^{\beta-1} k_1(\zeta), \quad \inf_{\zeta \in [0,\delta]} k_1(\zeta) > 0, \tag{2.4}$$

where $\beta, \delta > 0$ *and* k_1 *is a nonnegative function, then the solution of* (2.1) *blows up in finite time.*

Proof. Using (1.9), we see that

$$F_{min}(t) \geq \frac{1-p}{p}(C_0\, p\mathbf{I}(t))^{\frac{-1}{1-p}}.$$

It follows by (2.3) that a time $t^* > 0$ exists such that $A + F_{min}(t^*) > 0$. Again by (1.9) we have

$$\int_U^\infty \left(\frac{u}{g(u)}\right)^{1/\beta} \frac{du}{u} < \infty \quad \text{for all} \quad U > 0.$$

Therefore, all assumptions in [3, Theorem 2.12, p. 494] are satisfied. This completes the proof of Theorem 2.1. □

As a consequence, we have

Corollary 2.1. *Suppose that assumptions* (1.8), (1.9), (2.3) *and* (2.4) *are fulfilled. If* v *is a continuous nonnegative solution of*

$$v(t) \geq A + \int_0^t k(t-s)g(v(s))ds, \ t \geq 0, \tag{2.5}$$

then v *blows up in finite time.*

Proof. It suffices to show that $v(t) \geq u(t)$ where u is the solution of (2.1). Define, for $\varepsilon > 0$, the function u_ε by

$$u_\varepsilon(t) = \begin{cases} A & \text{if} \quad 0 \leq t \leq \varepsilon, \\ u(t-\varepsilon) & \text{if} \quad \varepsilon < t. \end{cases} \tag{2.6}$$

Then

$$u_\varepsilon(t) \leq A + \int_0^t k(t-s)g(u_\varepsilon(s))ds. \tag{2.7}$$

From (2.5) we see that $v(t) > A$ for all $t > 0$. Hence $v(t) > u_\varepsilon(t)$ for all $0 < t \leq \varepsilon$. Arguing as in [3, Lemma 2.4, p. 489], we find that $v(t) > u_\varepsilon(t)$ for all $t > 0$. In particular,

$$v(t) > u(t-\varepsilon), \quad \varepsilon < t. \tag{2.8}$$

Letting $\varepsilon \to 0$ we deduce that $v(t) \geq u(t)$ for all $t \geq 0$. This finishes the proof. □

As an application of the above result, we have

Lemma 2.1. *Let $C_1 > 0, C_2 > 0, p > 1$ and $b > -1$. If v is a continuous nonnegative solution of*

$$v(t) \geq C_1 + C_2 \int_0^t \{1 - e^{-(t-s)}\}^b v(s)^p ds, \quad t \geq 0, \tag{2.9}$$

then v blows up in a finite time.

Proof. Clearly v satisfies (2.5) with $A = C_1$, $g(u) = C_2\, u^p$ and $k(\zeta) = (1 - e^{-\zeta})^b$. By writing

$$k(\zeta) = \zeta^b \left(\frac{1 - e^{-\zeta}}{\zeta} \right)^b := \zeta^b\, k_1(\zeta)$$

we see that (2.4) is satisfied with $\beta = b + 1 > 0$. This shows that all assumptions (1.8), (1.9), (2.3) and (2.4) are fulfilled. The proof is then easily concluded. □

Next we recall two lemmas that we will use later. See [12] for the proofs.

Lemma 2.2. *Let $a > 0$ and $(x, t) \in [0, \infty)^2$. Then we have*

$$\int_{|x-t|}^{x+t} \frac{1}{\langle y \rangle^{a+1}} dy \leq \frac{2a^{-1} \max\{1, a\} \min\{x, t\}}{\langle x + t \rangle \langle x - t \rangle^a}. \tag{2.10}$$

Lemma 2.3. *Let $p > 1$ and $pa > 1$. Then, there exists a positive constant C such that*

$$\iint_{D(x,t)} \frac{\langle y \rangle^{p-a-1}}{\langle y + s \rangle^p \langle y - s \rangle^{pa}} dyds \leq \frac{C \langle x \rangle}{\langle x + t \rangle \langle x - t \rangle^a}, (x, t) \in [0, \infty)^2, \tag{2.11}$$

where $D(x, t) = \{(y, s) \in [0, \infty) \times [0, t) : |x - (t - s)| \leq y \leq x + t - s\}$.

3. Proof of Theorem 1.1

The solution of (1.1) can be seen as a fixed point in the appropriate function space to the following nonlinear integral operator

$$\begin{aligned} \Phi[u](z) &= u_0(z) + L[u](z) \qquad (3.1) \\ &:= u_0(z) + \frac{1}{2} \iint_{D(z)} G(y, u(y, s)) dyds, \end{aligned}$$

where $z = (x, t)$ and u_0 is the solution to the homogeneous wave equation given by

$$u_0(z) = \frac{\varphi(x + t) + \varphi(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \psi(y)\, dy. \tag{3.2}$$

As φ, ψ are odd functions, u_0, u are odd in x and therefore we only consider $(x, t) \in [0, \infty)^2$. To conclude the proof, we will show that Φ is a contraction on some closed ball in $X(a)$.

A first step consists in obtaining useful estimates of $L[v]$ whenever $v \in X(a)$. More precisely, we have

Proposition 3.1. *Let $a > \frac{1}{3}$. If $v \in X(a)$, then $L[v] \in X(a)$. Furthermore, there exists a constant $C_0 > 0$ such that*

$$||L[v]||_{X(a)} \leq C_0 ||v||_{X(a)} \left(e^{||v||^2_{X(a)}} - 1 \right), \tag{3.3}$$

$$||L[v] - L[w]||_{X(a)} \leq C_0 ||v - w||_{X(a)} \left[(e^{2||v||^2_{X(a)}} - 1) + (e^{2||w||^2_{X(a)}} - 1) \right], \tag{3.4}$$

hold for all $v, w \in X(a)$.

Proof. Let $v \in X(a)$. Using the Taylor expansion

$$g(u) = u(e^{u^2} - 1) = \sum_{k=1}^{\infty} \frac{u^{2k+1}}{k!},$$

we get

$$\begin{aligned} |L[v](z)| &\leq \frac{1}{2} \iint_{D(z)} |G(y, v(y, s))| dyds \\ &\leq \frac{1}{2} \iint_{D(z)} \frac{|v(e^{v^2} - 1)|}{(1 + y^2)^{\frac{1+a}{2}}} dyds \\ &\leq \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k!} \iint_{D(z)} \frac{|v^{2k+1}(y, s)|}{(1 + y^2)^{\frac{1+a}{2}}} dyds. \end{aligned}$$

Owing to (1.5) we see that

$$|v(y, s)| \leq ||v||_{X(a)} \left(\frac{< y >}{< y + s >< y - s >^a} \right).$$

Hence

$$|L[v](z)| \leq \frac{1}{2} \sum_{k=1}^{\infty} \frac{||v||^{2k+1}_{X(a)}}{k!} \mathbf{I}(a, k)(z),$$

where

$$\mathbf{I}(a, k)(z) = \iint_{D(z)} \frac{< y >^{p_k - a - 1}}{< y + s >^{p_k} < y - s >^{ap_k}} dyds, \tag{3.5}$$

and $p_k = 2k + 1$. By invoking Lemma 2.3 and observing that $ap_k > 1$ for all $k \geq 1$, we arrive at

$$|L[v](z)| \leq \frac{C}{2} \left(\frac{< x >}{< x + t >< x - t >^a} \right) \sum_{k=1}^{\infty} \frac{||v||^{2k+1}_{X(a)}}{k!}.$$

Therefore

$$\|L[v]\|_{X(a)} \leq \frac{C}{2} \sum_{k=1}^{\infty} \frac{||v||^{2k+1}_{X(a)}}{k!} = \frac{C}{2} \|v\|_{X(a)} \left(e^{\|v\|^2_{X(a)}} - 1 \right).$$

This finishes the proof of (3.3). Next, we turn to (3.4). Observe that

$$|g(v) - g(w)| \leq C|v - w| \left[(e^{2v^2} - 1) + (e^{2w^2} - 1) \right],$$

where g is as in (1.2). Arguing as in the proof of (3.3), we find that

$$|L[v](z)-L[w](z)|\leq\frac{C}{2}||v-w||_{X(a)}\sum_{k=1}^{\infty}2^k\frac{||v||_{X(a)}^{2k}+||w||_{X(a)}^{2k}}{k!}\mathbf{I}(a,k)(z),$$

where $\mathbf{I}(a,k)(z)$ is given by (3.5).

Applying again Lemma 2.3, we deduce that

$$\begin{aligned}&|L[v](z)-L[w](z)|\\ \leq& C\left(\frac{<x>}{<x+t><x-t>^a}\right)||v-w||_{X(a)}\sum_{k=1}^{\infty}2^k\frac{||v||_{X(a)}^{2k}+||w||_{X(a)}^{2k}}{k!}\\ \leq& C\left(\frac{<x>}{<x+t><x-t>^a}\right)||v-w||_{X(a)}\left[(e^{2||v||_{X(a)}^2}-1)+(e^{2||w||_{X(a)}^2}-1)\right].\end{aligned}$$

This yields (3.4) as claimed. The proof of Proposition 3.1 is now complete. □

End of the proof of Theorem 1.1. Let $C_1>0$ be as in [12, Proposition 3.1], that is,

$$||u_0|_{X(a)}\leq C_1||(\phi,\psi)||_{Y(a)}. \tag{3.6}$$

Suppose that $||(\phi,\psi)||_{Y(a)}\leq\epsilon$ where $\epsilon>0$ to be chosen later. Let $\mathbf{B}_\epsilon$ be the closed ball in $X(a)$ centered at the origin and with radius $2C_1\epsilon$, that is,

$$\mathbf{B}_\epsilon=\left\{u\in X(a);\quad ||u||_{X(a)}\leq 2C_1\varepsilon\right\}.$$

From Proposition 3.1 and thanks to (3.6), one has

$$||\Phi[u]||_{X(a)}\leq C_1\epsilon+2C_0C_1\epsilon\left(e^{8C_1^2\epsilon^2}-1\right),$$

and

$$||\Phi[u]-\Phi[v]||_{X(a)}\leq 2C_0\left(e^{8C_1^2\epsilon^2}-1\right)||u-v||_{X(a)},$$

provided that $u,v\in\mathbf{B}_\epsilon$. Choosing $\epsilon>0$ small enough such that

$$2C_0\left(e^{8C_1^2\epsilon^2}-1\right)\leq\frac{1}{2}, \tag{3.7}$$

we conclude that Φ maps $\mathbf{B}_\epsilon$ into itself and it is a contraction. This achieves the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

We argue by contradiction assuming that the solution $u\in C^2(\mathbb{R}\times[0,\infty)$ exists globally in time. Since the initial data (ϕ,ψ) are nonnegative functions then $u(x,t)\geq 0$ for all $(x,t)\in\mathbb{R}\times[0,\infty)$. Also, from assumption (1.10) and Corollary B.1, we have that $u(x,t)=0$ for $|x|\geq\rho+t$. For $t\geq 0$, define the function

$$w(t)=\int_{-\infty}^{\infty}u(x,t)dx=\int_{-(\rho+t)}^{\rho+t}u(x,t)dx,$$

which satisfies

$$\begin{aligned} w''(t) &= \int_{-\infty}^{\infty} u_{tt}(x,t)dx, \\ &= \int_{-\infty}^{\infty} u_{xx}(x,t)dx + \int_{-\infty}^{\infty} \frac{g(u(x,t))}{(1+x^2)^{\frac{a+1}{2}}}dx \\ &= \int_{-\infty}^{\infty} \frac{g(u(x,t))}{(1+x^2)^{\frac{a+1}{2}}}dx. \end{aligned}$$

Now, using Hölder's inequality we infer

$$\begin{aligned} w(t) &= \int_{-(\rho+t)}^{\rho+t} (1+x^2)^{\frac{a+1}{2p}} \frac{u(x,t)}{(1+x^2)^{\frac{a+1}{2p}}}dx, \\ &\leq \left(2\int_0^{\rho+t} (1+x^2)^{\frac{a+1}{2(p-1)}}dx\right)^{1-\frac{1}{p}} \left(\int_{-\infty}^{\infty} \frac{u^p(x,t)}{(1+x^2)^{\frac{a+1}{2}}}\right)^{\frac{1}{p}}, \end{aligned}$$

where $p > 1$ is as in (1.9). By (1.9) and the estimate

$$\int_0^{\rho+t} (1+x^2)^{\frac{a+1}{2(p-1)}}\, dx \lesssim (1+t)^{\frac{a+p}{p-1}},$$

we deduce that

$$w''(t) \geq C\frac{w^p(t)}{(1+t)^{p+a}}, \tag{4.1}$$

for some positive constant C depending only on p, a and ρ.

Therefore, it follows from (4.1) that for $t \geq 1$,

$$\begin{aligned} w(t) &= w(0) + w'(0)t + \int_0^t (t-s)w''(s)ds, \\ &\geq C_1 t + C\int_0^t (t-s)\frac{w^p(s)}{(1+s)^{p+a}}ds, \\ &\geq C_1 t + C\int_1^t (t-s)\frac{w^p(s)}{s^{p+a}}ds, \end{aligned}$$

where $C_1 = \int_{\mathbb{R}} \psi(x)dx > 0$. By putting $f(t) = \frac{w(t)}{t}$, the above inequality reduces to

$$f(t) \geq C_1 + C\int_1^t (1-\frac{s}{t})\frac{f^p(s)}{s^a}ds. \tag{4.2}$$

By performing some elementary computations, we conclude from (4.2) that

$$v(t) \geq C_1 + C\int_0^t \left(1-e^{-(t-s)}\right) e^{(1-a)s}\, v^p(s)\, ds, \tag{4.3}$$

where $v(t) = f(e^t)$. Applying Lemma 2.1 to (4.3) after using $a \leq 1$, we deduce that v blows up in finite time. This finishes the proof of Theorem 1.2.

Acknowledgment

The authors thank the reviewer for the careful reading of the manuscript and helpful comments.

A. Appendix

In this section we explain how to obtain local smooth solutions for (1.1) for general nonlinearities. We also give a continuation argument. See [15, 16] for more details and complements.

Theorem A.1. *Suppose* $G \in C^2(\mathbb{R} \times \mathbb{R})$ *and* $\phi, \psi \in C_0^\infty(\mathbb{R})$, *then there exists* $T > 0$ *such that* (1.1) *admits a unique local solution* $u \in C^2(\mathbb{R} \times [0, T))$. *Moreover, if* $T < \infty$, *then* $\sup\limits_{t<T,\, x\in\mathbb{R}} |u(t, x)| = \infty$.

Although the proof is classical, we give it here for the sake of completeness.
Proof. For $T > 0$, let

$$X_T = \left\{ u \in C(\mathbb{R} \times [0, T)); \quad u(x, 0) = \phi(x) \quad \text{and} \quad \|u - u_0\|_{L^\infty} \le 1 \right\}, \tag{A.1}$$

where u_0 is the solution of the homogeneous wave equation given by (3.2). Recall the definition of the operator Φ given by (3.1)

$$\Phi[u](z) = u_0(z) + \frac{1}{2} \iint\limits_{D(z)} G(y, u(y, s))dyds,$$

where $z = (x, t)$ and $D(z) = \{(s, y) \in [0, t] \times \mathbb{R}; \quad |y - x| \le t - s\}$. It is clear that

$$u \in X_T \Longrightarrow |u(x, t)| \le 1 + \|u_0\|_{L^\infty} := \mathbf{K}, \quad (x, t) \in \mathbb{R} \times [0, T]. \tag{A.2}$$

Define

$$\mathbf{M}(y) = \sup_{|u|\le\mathbf{K}} |G(y, u)|, \tag{A.3}$$

$$\mathbf{N}(y) = \sup_{|u|\le\mathbf{K}} |\partial_2 G(y, u)|. \tag{A.4}$$

Note that $\mathbf{M}$ and $\mathbf{N}$ are continuous functions.

To draw the conclusion that (1.1) has a unique local solution, we will show that Φ is a contraction map on X_T for appropriately chosen $T > 0$ small enough. First, let us show that Φ maps X_T into itself. Let $u \in X_T$. Then, for $z \in \mathbb{R} \times [0, T]$, we have

$$\begin{aligned}
|\Phi[u](z) - u_0(z)| &\le \frac{1}{2} \iint\limits_{D(z)} \mathbf{M}(y)dyds \\
&\le \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t-s} M(y)dyds \\
&\le \frac{1}{2} \int_0^t \int_{-t+s}^{t-s} M(\xi + x)d\xi ds \\
&\le T^2 \sup_{|y|\le T} \mathbf{M}(y).
\end{aligned}$$

Since $T^2 \sup\limits_{|y|\le T} \mathbf{M}(y) \to 0$ as $T \to 0$, we deduce that $\Phi[u] \in X_T$ for $T > 0$ small enough. Next, let $u, v \in X_T$ and write

$$|\Phi[u](z) - \Phi[v](z)| \le \frac{1}{2} \iint\limits_{D(z)} |G(y, u(y, s)) - G(y, v(y, s))|dyds$$

$$\leq \frac{1}{2}\iint\limits_{D(z)} \mathbf{N}(y)\,|u(y,s)-v(y,s)|dyds$$
$$\leq \frac{1}{2}\|u-v\|_{L^\infty}\iint\limits_{D(z)} \mathbf{N}(y)\,dyds$$
$$\leq T^2\sup_{|y|\leq T}\mathbf{N}(y)\,\|u-v\|_{L^\infty}.$$

Choosing $T>0$ small enough such that $\sup_{|y|\leq T}\mathbf{N}(y)\leq \frac{1}{2}$, we end up with

$$\|\Phi[u]-\Phi[v]\|_{L^\infty}\leq \frac{1}{2}\,\|u-v\|_{L^\infty}.$$

Plugging all estimates above together, we deduce that $\Phi: X_T \longrightarrow X_T$ is a contraction for $T>0$ small enough. Existence of a local solution as claimed in Theorem A.1 is now a simple consequence of the Banach fixed point theorem. $\square$

B. Appendix

Here we recall an uniqueness result (or finite speed of propagation) in higher dimension $N\geq 1$ although we used it in dimension one. Consider the nonlinear wave equation in dimension $N\geq 1$

$$u_{tt}-\Delta\,u = G(x,u), \tag{B.1}$$

where $G:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a C^1-function. For any fixed $(x_0,T)\in\mathbb{R}^N\times(0,\infty)$, we introduce

$$\Omega=\Big\{\,(x,t);\quad 0\leq t\leq T,\ \ |x-x_0|\leq T-t\,\Big\} \tag{B.2}$$

which is called the backward light cone trough (x_0,T). The base B_0 of Ω is defined as

$$B_0=\{\,x;\quad |x-x_0|\leq T\,\}=B(x_0,T).$$

Theorem B.1. *If $u\in C^2(\Omega)$ solves* (B.1) *in Ω. Then u is uniquely determined by its data u and u_t on B_0. In other words, if $u,v\in C^2(\Omega)$ are two solutions of* (B.1) *in Ω, with the same data in B_0, then $u=v$ in Ω.*

The proof of this theorem uses energy method and can be found in many references. See for example [15, 16]. For completeness, we give a detailed proof here. Let us first prove the following result which yields easily Theorem 3.1. Consider the linear wave equation

$$\mathbf{w}_{tt}-\Delta\mathbf{w}=a(x,t)\mathbf{w}, \tag{B.3}$$

where a is a continuous function in (x,t).

Proposition B.1. *Let $\mathbf{w}$ be a C^2 solution of* (B.3) *in Ω. If $\mathbf{w}=\mathbf{w}_t=0$ in B_0, then $\mathbf{w}=0$ in Ω.*

Proof. [Proof of Proposition B.1] Consider for $0\leq t\leq T$ the function

$$E(t)=\int_{B(x_0,T-t)}\left(|\mathbf{w}(x,t)|^2+|\mathbf{w}_t(x,t)|^2+|\nabla\mathbf{w}(x,t)|^2\right)\,dx$$

$$= \int_0^{T-t} \int_{\partial B(x_0,\tau)} \left(|\mathbf{w}|^2 + |\mathbf{w}_t|^2 + |\nabla \mathbf{w}|^2\right) d\sigma\, d\tau.$$

We have

$$\begin{aligned}
\frac{dE}{dt}(t) &= 2\int_{B(x_0,T-t)} (\mathbf{w}\mathbf{w}_t + \mathbf{w}_t\mathbf{w}_{tt} + \nabla\mathbf{w}\cdot\nabla\mathbf{w}_t)\, dx \\
&\quad - \int_{\partial B(x_0,T-t)} \left(|\mathbf{w}|^2 + |\mathbf{w}_t|^2 + |\nabla \mathbf{w}|^2\right) d\sigma \\
&= 2\int_{B(x_0,T-t)} \mathbf{w}_t\,(\mathbf{w} + \mathbf{w}_{tt} - \Delta\mathbf{w})\, dx + 2\int_{B(x_0,T-t)} \operatorname{div}(\mathbf{w}_t\nabla\mathbf{w})\, dx \\
&\quad - \int_{\partial B(x_0,T-t)} \left(|\mathbf{w}|^2 + |\mathbf{w}_t|^2 + |\nabla \mathbf{w}|^2\right) d\sigma,
\end{aligned}$$

where we have used the fact that

$$\operatorname{div}(\mathbf{w}_t\nabla\mathbf{w}) = \nabla\mathbf{w}_t\cdot\nabla\mathbf{w} + \mathbf{w}_t\Delta\mathbf{w}.$$

Using $\mathbf{w}_{tt} = \Delta\mathbf{w} + a\mathbf{w}$ and the divergence theorem, we obtain

$$\begin{aligned}
\frac{dE}{dt}(t) &= 2\int_{B(x_0,T-t)} \left(\mathbf{w}\mathbf{w}_t + a\mathbf{w}^2\right) dx \\
&\quad +2\int_{\partial B(x_0,T-t)} \mathbf{w}_t\nabla\mathbf{w}\cdot\nu\, d\sigma - \int_{\partial B(x_0,T-t)} \left(|\mathbf{w}|^2 + |\mathbf{w}_t|^2 + |\nabla \mathbf{w}|^2\right) d\sigma,
\end{aligned}$$

where ν denotes the outward unit normal to $\partial B(x_0, T-t)$. Since

$$2|\mathbf{w}_t\nabla\mathbf{w}\cdot\nu| \le |\mathbf{w}_t|^2 + |\nabla\mathbf{w}|^2,$$

we deduce that

$$\frac{dE}{dt}(t) \le 2\int_{B(x_0,T-t)} \left(\mathbf{w}\mathbf{w}_t + a\mathbf{w}^2\right) dx.$$

Recalling that a is continuous, there exists a constant $M > 0$ such that $|a(x,t)| \le M$ for all $(x,t) \in \Omega$. Hence

$$2\left(\mathbf{w}\mathbf{w}_t + a\mathbf{w}_t^2\right) \le 2\mathbf{w}\mathbf{w}_t + 2M\mathbf{w}^2 \le \mathbf{w}^2 + \mathbf{w}_t^2 + 2M\mathbf{w}^2 \le (2M+1)(\mathbf{w}^2 + \mathbf{w}_t^2).$$

It follows that

$$\frac{dE}{dt}(t) \le (2M+1)E(t).$$

This implies that $\frac{d}{dt}\left[E(t)\mathrm{e}^{-(2M+1)t}\right] \le 0$. Therefore $E(t) \le E(0)\mathrm{e}^{(2M+1)t}$ for $0 \le t \le T$. Since $E(0) = 0$, we conclude that $\mathbf{w} = 0$ in Ω. □

A straightforward consequence of Proposition B.1 is Theorem 3.1.

Proof of Theorem 3.1. Suppose that $u, v \in C^2(\Omega)$ both solve (B.1), and with identical data in $B(x_0, T)$. Let $\mathbf{w} = u - v$. Then

$$\mathbf{w}_{tt} - \Delta\mathbf{w} = G(x,u) - G(x,v), \quad \mathbf{w} = \mathbf{w}_t = 0 \quad \text{in} \quad B(x_0,T).$$

Let us write

$$G(x,u) - G(x,v) = \int_0^1 \frac{d}{d\tau}\Big[G(x,(1-\tau)v + \tau u)\Big]\, d\tau$$

$$= \left(\int_0^1 \partial_2 G(x, (1-\tau)v + \tau u)\, d\tau \right) (u - v)$$
$$= a(x,t)\mathbf{w},$$

where

$$a(x,t) = \int_0^1 \partial_2 G(x, (1-\tau)v + \tau u)\, d\tau.$$

Recalling that G is C^1 and $u, v \in C^2$, we deduce that a is a continuous function in (x,t). Applying Proposition B.1 we conclude the proof of Theorem 3.1. $\square$

As an application, we have

Corollary B.1. *Let $(\phi, \psi) \in C^2(\mathbb{R}) \times C^1(\mathbb{R}), G \in C^1(\mathbb{R}^2)$ such that $G(x,0) = 0$ for all $x \in \mathbb{R}$. Also, suppose that there exists a constant $r > 0$ such that $\phi(x) = \psi(x) = 0$ for $|x| \geq r$. Then, the solution $u \in C^2(\mathbb{R} \times [0,T))$ of (1.1) satisfies*

$$u(x,t) = 0 \quad \text{for} \quad |x| \geq r + t.$$

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