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A NEW NUMERICAL TECHNIQUE FOR SOLVING ψ-FRACTIONAL RICCATI DIFFERENTIAL EQUATIONS

Amjid Ali[1,†] and Teruya Minamoto[1]

Abstract This paper proposes a new numerical technique for solving a specific class of fractional differential equations, which includes the ψ-Caputo fractional derivative. The class under consideration is nonlinear ψ-fractional Riccati differential equations (ψ-FRDEs). Our approach relies on the ψ-Haar wavelet (ψ-HW) operational matrix, which is a novel type of operational matrix of fractional integration. We derive an explicit formula for the ψ-fractional integral of the HW. This operational matrix has been used successfully to solve nonlinear ψ-FRDEs.The Quasi-linearization technique is employed to linearize the non-linear ψ-FRDEs. This technique reduces the problem to an algebraic equation that can be easily solved. The technique is a useful and straightforward mathematical tool for solving nonlinear ψ-FRDEs. The computational complexity of the operational matrix technique is minimal. The error analysis of the proposed method is thoroughly investigated. To justify the method's accuracy and efficiency, numerical results are given.

Keywords ψ-HW Operational matrices, ψ-Caputo fractional integral and derivative, Riccati Fractional differential equations, quasi linearization, collocation points, convergence.

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1. Introduction

Fractional calculus is a branch of mathematics that deals with arbitrary order derivatives. Fractional calculus is a field of pure mathematics that is gradually being studied in a variety of areas. Nowadays, fractional calculus and fractional differential equations have been implemented in mathematics, chemistry, physics engineering and biology [11,24,25,32]. Many engineering and mathematical physics models have been modeled via distributed order fractional differential equations [10,21,27,28,30,31]. Caputo introduced a fractional derivative of distributed-order and later developed by him in 1995 [15]. The fractional order differential equations and their implementations in engineering and other fields have received a lot of attention. For instance, the general result of linear fractional order differential equation was discuss scientifically in Bagley et al. [13,14], in the constitutive equation of dielectric media, fractional order equations were established [15] and Sokolov et al. explored distributed order fractional kinetics [33].

†The corresponding author. Email address:amjidmaths85@gmail.com (A. Ali)
[1]Faculty of Science and Engineering, Saga University, 1 Honjomachi, Saga, 840-8502, Japan

Some of the most recently established methods for solving fractional differential equations are given in [1–3, 20].

The Riccati differential equation is regarded as an essential type of non-linear differential equation because of its capacity to describe a wide range of engineering and physical phenomena, including stochastic control, river flow, transmission line phenomena, dynamic games, financial mathematics, and others [4, 17, 18, 26]. The general form of Riccati equation is given by

$$u'(\varkappa) + a(\varkappa)u(\varkappa) + b(\varkappa)u^2(\varkappa) = g(\varkappa), \tag{1.1}$$

where $a(\varkappa), b(\varkappa)$ and $g(\varkappa)$ are continuous functions of $\varkappa$. The fractional Riccati differential equation is a broad view of the traditional Riccati differential equation achieved when substituting a fractional derivative of order γ for the first order derivative in Eq. (1.1). Fractional Riccati equation has the following form

$$\mathbb{D}^{\gamma}u(\varkappa) + a(\varkappa)u^2(\varkappa) + b(\varkappa)u(\varkappa) = g(\varkappa), \quad 0 < \varkappa \leq 1, \quad 0 < \gamma \leq 1. \tag{1.2}$$

The fractional derivative in Eq. (1.2) is defined in the Caputo perspective. Due to its various implementations in engineering and science, many numerical and analytical methods were investigated for solving fractional Riccati differential equations. Ozturk et al. [29] applied the collocation technique with the help of Taylor expansion for converting the fractional Riccati differential equations into a scheme of non-linear algebraic equation. Khan et al. [22] used the homotopy perturbation technique for solving fractional Riccati differential equation. In addition, other numerical approaches for solving fractional Riccati differential equations were investigated, see [19, 23, 34].

In this article, we are concerned with numerical approximation of the Riccati fractional differential equation in which the fractional derivative is given in the ψ-Caputo sense, called the ψ-FRDE.

The ψ-FRDE is given by

$${}^{C}\mathbb{D}^{\gamma,\psi}u(\varkappa) + a(\varkappa)u^2(\varkappa) + b(\varkappa)u(\varkappa) = g(\varkappa), \quad 0 < \varkappa \leq 1, \quad 0 < \gamma \leq 1. \tag{1.3}$$

We propose the ψ-HW Operational-matrix approach for solving nonlinear ψ-FRDEs. Operational matrices approach has been applied for the first time to ψ-FRDEs. To the best of our knowledge there is no related work in literature in this direction. However the method reduces exactly to the classical Haar wavelet operational matrix method when $\psi(\varkappa)$ is chosen to be $\varkappa$. To assure the convergence of the suggested approach, we constructed an inequality in the perspective of error-analysis. We put some problems to the test to see how effective the proposed method is. The outcomes of these examples are presented graphically and in tables.

Organization of the paper: The layout of this paper is as follows

Section 2 reviews some fundamental concepts of ψ-fractional calculus theory. HW and function approximation using HW were also explored, which is an important aspect of this paper. We construct the generalized fractional integration of HW in section 3. In section 4, we got a precise upper-bound on the error-estimate for the suggested technique. In the section 5, some numerical examples are presented to demonstrate the correctness and efficacy of the proposed method. In the last section of the proposed study, the conclusions are given.

2. Preliminaries

In this section, we shall define some ψ-fractional integral and differential operators. Let γ be a positive real number, n be a natural number and $g : [a_1, a_2] \to \mathbb{R}$ and $\psi \in C^1([a_1, a_2])$ be functions such that g is integrable and ψ is increasing with $\psi'(\varkappa) \neq 0 \ \forall \ \varkappa \in [a_1, a_2]$.

Definition 2.1 ([6,9,24]). The ψ-Riemann-Liouvile (ψ-RL) fractional integration of order γ is given by:

$$\mathcal{J}_{a_1}^{\gamma,\psi} g(\varkappa) = \frac{1}{\Gamma(\gamma)} \int_{a_1}^{\varkappa} \psi'(\wp)\big(\psi(\varkappa) - \psi(\wp)\big)^{\gamma-1} g(\wp) d\wp. \tag{2.1}$$

The ψ-RL fractional derivative of order γ is given as:

$$\begin{aligned}\mathbb{D}_{a_1}^{\gamma,\psi} g(\varkappa) &= \left(\frac{1}{\psi'(\varkappa)} \frac{d}{d\varkappa}\right)^n \mathcal{J}_{a_1}^{n-\gamma,\psi} g(\varkappa) \\ &= \frac{1}{\Gamma(n-\gamma)} \quad \left(\frac{1}{\psi'(\varkappa)} \frac{d}{d\varkappa}\right)^n \int_{a_1}^{\varkappa} \psi'(\wp)\big(\psi(\varkappa) - \psi(\wp)\big)^{n-\gamma-1} g(\wp) d\wp,\end{aligned}$$

where $n = \lfloor \gamma \rfloor + 1$.

Definition 2.2 ([5,7,8]). Let γ be a positive real number, n a natural number and $g, \psi \in C^n([a_1, a_2])$ where ψ is an increasing function in such a way that $\psi'(\varkappa) \neq 0$ $\forall \ \varkappa \in [a_1, a_2]$. The ψ-Caputo fractional derivative of order γ is given by:

$${}^{C}\mathbb{D}_{a_1}^{\gamma,\psi} g(\varkappa) = \frac{1}{\Gamma(n-\gamma)} \int_{a_1}^{\varkappa} \psi'(\wp)\big(\psi(\varkappa) - \psi(\wp)\big)^{n-\gamma-1} \mathbb{D}^{n,\psi} g(\wp) d\wp,$$

where $\mathbb{D}^{n,\psi} g(\varkappa) = \left(\frac{1}{\psi'(\varkappa)} \frac{d}{d\varkappa}\right)^n g(\varkappa)$, $n = \lfloor \gamma \rfloor + 1$ for $\gamma \notin \mathbb{N}$ and $n = \gamma$ when $\gamma \in \mathbb{N}$.

For specific choices of the function $\psi(\varkappa)$ the ψ-fractional operators result in the following classical fractional operators .

(1) $\psi(\varkappa) = \varkappa$ gives RL and Caputo fractional operators.

(2) $\psi(\varkappa) = \ln(\varkappa)$ gives the Hadamard and Caputo-Hadamard fractional operators.

2.1. Function Approximation by Haar Wavelet

Let's start with a quick review of the Haar functions. The Haar functions are an orthogonal family of switched rectangular waveform with amplitudes that vary from one to the next.

The ith Haar function $h_i(\varkappa)$, $\varkappa \in [a_1, a_2]$ is given by:

$$h_i(\varkappa) = \begin{cases} 1, & \text{when } \ \varkappa \in [a_1 + (a_2 - a_1)\dfrac{k}{m}, \ \ a_1 + (a_2 - a_1)\dfrac{2k+1}{2m}); \\ -1, & \text{when } \ \varkappa \in [a_1 + (a_2 - a_1)\dfrac{2k+1}{2m}, \ \ a_1 + (a_2 - a_1)\dfrac{k+1}{m}); \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

where $m = 2^j, k = 0, 1, 2, 3, \cdots, m-1$, and $j = 0, 1, 2, \cdots, J$. The parameters i, j, k are related by the equation $i = 2^j + k + 1$, i is called the wavelet number. Equation (2.2) is valid $\forall\ i \geq 3$.

Scaling functions for the HW family for $i = 1$ and $i = 2$ are defined as follows:

$$h_1(\varkappa) = \begin{cases} 1, & \text{when } \varkappa \in [a_1, a_2); \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h_2(\varkappa) = \begin{cases} 1, & \text{when } \varkappa \in [a_1, \dfrac{a_1 + a_2}{2}); \\ -1, & \text{when } \varkappa \in [\dfrac{a_1 + a_2}{2}, a_2); \\ 0, & \text{elsewhere.} \end{cases} \tag{2.3}$$

HW can approximate a square integrable function $u(\varkappa)$ on (a_1, a_2) as follows:

$$u(\varkappa) = \sum_{i=0}^{\infty} c_i h_i(\varkappa), \tag{2.4}$$

where $c_i = \langle u(\varkappa), h_i(\varkappa) \rangle$, $\langle . \rangle$ denotes the inner-product. Approximation of functions is done with the first m terms, that is

$$u(\varkappa) \cong u_m(\varkappa) = \sum_{i=0}^{m-1} c_i h_i(\varkappa),$$

in matrix form, this can be represented as:

$$u(\varkappa) \cong u_m(\varkappa) = C_m^T H_m(\varkappa), \tag{2.5}$$

where $C = [c_0, c_1, c_2, \cdots, c_{m-1}]^T$, generated by $c_i = \langle u(\varkappa), h_i(\varkappa) \rangle$ is the coefficient matrix and $H = [h_0(\varkappa), h_1(\varkappa), h_2(\varkappa), \cdots, h_{m-1}(\varkappa)]^T$ is the vector of Haar functions.

3. ψ-Haar wavelts Operational-matrix

The ψ-RL integration of fractional order γ of the HW is given by:

$$\mathcal{J}^{\gamma,\psi} h_1(\varkappa) = \frac{1}{\Gamma(\gamma+1)} \big[\psi(\varkappa) - \psi(a_1)\big]^{\gamma}, \tag{3.1}$$

$$P_l^{\gamma,\psi}(\varkappa) = \mathcal{J}^{\gamma,\psi} h_l(\varkappa) = \frac{1}{\Gamma(\gamma)} \int_{a_1}^{\varkappa} \psi'(\wp)\big(\psi(\varkappa) - \psi(\wp)\big)^{\gamma-1} h_i(\wp) d\wp$$

$$= \begin{cases} 0, & \text{if } \varkappa < \zeta_1(l); \\ \dfrac{1}{\Gamma(\gamma+1)} \big[\psi(\varkappa) - \psi(\zeta_1(l))\big]^{\gamma}, & \text{if } \varkappa \in [\zeta_1(l), \zeta_2(l)); \\ \dfrac{1}{\Gamma(\gamma+1)} \Big[\big(\psi(\varkappa) - \psi(\zeta_1(l))\big)^{\gamma} - 2\big(\psi(\varkappa) - \psi(\zeta_2(l))\big)^{\gamma}\Big], & \text{if } \varkappa \in (\zeta_2(l), \zeta_3(l)]; \\ \dfrac{1}{\Gamma(\gamma+1)} \Big[\big(\psi(\varkappa) - \psi(\zeta_1(l))\big)^{\gamma} - 2\big(\psi(\varkappa) - \psi(\zeta_2(l))\big)^{\gamma} & \\ +\big(\psi(\varkappa) - \psi(\zeta_3(l))\big)^{\gamma}\Big], & \text{if } \varkappa > \zeta_3(l), \end{cases} \tag{3.2}$$

where $\zeta_1(l) = a_1+(a_2-a_1)\frac{k}{m}$, $\zeta_2(l) = a_1+(a_2-a_1)\frac{2k+1}{2m}$, $\zeta_3(l) = a_1+(a_2-a_1)\frac{k+1}{m}$. The ψ-HW operational matrix $P^{\gamma,\psi}$ is computed for $\psi(\varkappa) = \sin(\varkappa)$ and $\gamma = 0.8$.

$$P^{\gamma,\psi} = \begin{bmatrix} 0.5606 & -0.2219 & -0.1403 & -0.0847 & -0.08164 & -0.0611 & -0.0484 & -0.0362 \\ 0.2769 & 0.0617 & -0.1403 & 0.1369 & -0.08164 & -0.0611 & 0.0896 & 0.0521 \\ 0.0656 & 0.0826 & 0.05017 & -0.0093 & -0.08164 & 0.0935 & -0.0092 & -0.0020 \\ 0.0689 & -0.0689 & 0 & 0.03492 & 0 & 0 & -0.0690 & 0.0696 \\ 0.0150 & 0.0188 & 0.04881 & -0.0011 & 0.03404 & -0.0054 & -0.0008 & -0.0003 \\ 0.0175 & 0.0231 & -0.04066 & -0.0044 & 0 & 0.0318 & -0.0048 & -0.0007 \\ 0.0178 & -0.0178 & 0 & 0.0401 & 0 & 0 & 0.0279 & -0.0041 \\ 0.0166 & -0.0166 & 0 & -0.0332 & 0 & 0 & 0 & 0.0224 \end{bmatrix}.$$

4. Error analysis

In [16], FDEs of the Caputo type were studied in the context of error-analysis. Furthermore, [12] provides a convergence analysis of HW's solution of non-linear Fredholm integral equations. In this section, we calculated the maximum absolute-error using the ψ-Caputo fractional differential operator, confirming the effectiveness of the ψ-HW technique for ψ-FDEs.

Theorem 4.1. *Assume $\mathbb{D}^{\ell} y$ is a continuous function on the interval $[a_1, a_2]$, and $M > 0$ such that $|\mathbb{D}^{\ell,\psi} u(\varkappa)| \leq M \ \forall \ \varkappa \in [a_1, a_2]$, where, $a_1, a_2 \in \mathbb{R}^+$, $\mathbb{D}^{\ell,\psi} u(\varkappa) = \left(\frac{1}{\psi'(\varkappa)}\frac{d}{d\varkappa}\right)^{\ell} u(\varkappa)$. Let ${}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u_\ell(\varkappa)$ is the approximation of ${}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa)$, then we have*

$$\left\|{}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa) - {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u_\ell(\varkappa)\right\|_E \leq \frac{(a_2-a_1)M\left(\psi'(a_2)\right)^{\ell-\gamma}}{\Gamma(\ell-\gamma+1)} \frac{1}{k^{(\ell-\gamma)}} \frac{1}{[1-2^{2(\gamma-\ell)}]^{\frac{1}{2}}}.$$

Proof. ${}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} y$ can be approximated by HW as:

$${}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa) = \sum_{i=a}^{\infty} c_i h_i(\varkappa),$$

here c_i is given by

$$c_i = \langle {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa), h_i(\varkappa)\rangle = \int_{a_1}^{a_2} \left({}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa)\right) h_i(\varkappa) d\varkappa. \tag{4.1}$$

Let the approximation of ${}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u$is ${}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u_\ell$ which is defined by

$${}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u_\ell(\varkappa) = \sum_{i=0}^{\ell-1} c_i h_i(\varkappa), \tag{4.2}$$

in which $\ell = 2^{\beta+1}, \quad \beta = 1, 2, 3, \cdots$. Therefore,

$$ {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa) - {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u_\ell(\varkappa) = \sum_{i=m}^{\infty} c_i h_i(\varkappa) = \sum_{i=2^{\beta+1}}^{\infty} c_i h_i(\varkappa), \tag{4.3} $$

this gives

$$ \begin{aligned} \left\| {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa) - {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u_\ell(\varkappa) \right\|_E^2 &= \int_{a_1}^{\varkappa} \left({}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa) - {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u_\ell(\varkappa) \right)^2 d\varkappa \\ &= \sum_{i=2^{\beta+1}}^{\infty} \sum_{i'=2^{\beta+1}}^{\infty} c_i c_{i'} \int_{a_1}^{\varkappa} h_i(\varkappa) h_{i'}(\varkappa) d\varkappa, \end{aligned} \tag{4.4} $$

the sequence $\{h_m(\varkappa)\}$ being orthogonal, we get $\int_{a_1}^{a_2} h_m(\varkappa) h_m(\varkappa) d\varkappa = I_m$, the symbol I_m denotes the mth order identity matrix.

Thus, equation (4.4) yields,

$$ \left\| {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa) - {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u_\ell(\varkappa) \right\|_E^2 = \sum_{i'=2^{\beta+1}}^{\infty} c_i^2. \tag{4.5} $$

Equation (4.1) gives:

$$ \begin{aligned} c_i =& \int_{a_1}^{a_2} \left({}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa) \right) h_i(\varkappa) d\varkappa \\ =& 2^{\frac{j}{2}} \left\{ \int_{a_1+(a_2-a_1)k2^{-j}}^{a_1+(a_2-a_1)(k+\frac{1}{2})2^{-j}} {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa) d\varkappa - \int_{a_1+(a_2-a_1)(k+\frac{1}{2})2^{-j}}^{a_1+(a_2-a_1)(k+1)2^{-j}} {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa) d\varkappa \right\}. \end{aligned} \tag{4.6} $$

Employing mean value theorem of integration: $\exists \ \ \varkappa_1, \varkappa_2 \in (a_1, a_2)$ where

$$ \begin{aligned} & a_1 + (a_2 - a_1)k2^{-j} < \varkappa_1 < a_1 + (a_2 - a_1)\big(k + \frac{1}{2}\big)2^{-j}, \\ & a_1 + (a_2 - a_1)\big(k + \frac{1}{2}\big)2^{-j} < \varkappa_2 < a_1 + (a_2 - a_1)(k + 1)2^{-j}, \end{aligned} $$

eq.(4.6) implies

$$ \begin{aligned} c_i =& 2^{\frac{j}{2}}(a_2 - a_1) \left\{ \left(a_1 + (k + \frac{1}{2})2^{-j} \quad - \quad (a_1 + k2^{-j}) \right) {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa_1) \right. \\ & \left. - \left(a_1 + (k+1)2^{-j} - (a_1 + (k + \frac{1}{2})2^{-j} \right) {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa_2) \right\} \\ =& 2^{\frac{j}{2}}(a_2 - a_1) \left\{ 2^{-j-1} \left({}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa_1) - {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa_2) \right) \right\}. \end{aligned} \tag{4.7} $$

Therefore,

$$ c_i^2 = 2^{-j-2}(a_2 - a_1)^2 ({}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa_1) - {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa_2))^2. \tag{4.8} $$

Applying ψ-Caputo fractional derivative where, ψ is an increasing function and $|\mathbb{D}^{\ell,\psi} u(\varkappa)| \le M$, we get:

$$ \left| {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa_1) - {}^{C}\mathbb{D}^{\gamma,\psi}_{a_1} u(\varkappa_2) \right| $$

$$
\begin{aligned}
=&\frac{1}{\Gamma(\ell-\gamma)}\bigg|\int_{a_1}^{\varkappa_1} \psi^{'}(\varkappa)\Big(\psi(\varkappa_1)-\psi(\varkappa)\Big)^{\ell-\gamma-1}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\\
&-\int_{a_1}^{\varkappa_2} \psi^{'}(\varkappa)\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\bigg|\\
=&\frac{1}{\Gamma(\ell-\gamma)}\bigg|\int_{a_1}^{\varkappa_1} \psi^{'}(\varkappa)\Big(\psi(\varkappa_1)-\psi(\varkappa)\Big)^{\ell-\gamma-1}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\\
&-\int_{a_1}^{\varkappa_1} \psi^{'}(\varkappa)\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\\
&-\int_{x_1}^{x_2} \psi^{'}(x)\Big(\psi(x_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\bigg|\\
\leq&\frac{1}{\Gamma(\ell-\gamma)}\bigg|\int_{a_1}^{\varkappa_1} \psi^{'}(\varkappa)\Big(\psi(\varkappa_1)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\\
&-\int_{a_1}^{\varkappa_1} \psi^{'}(\varkappa)\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\bigg|\\
&+\bigg|\int_{\varkappa_1}^{\varkappa_2} \psi^{'}(\varkappa)\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\bigg|\\
=&\frac{1}{\Gamma(\ell-\gamma)}\Bigg(\bigg|\int_{a_1}^{\varkappa_1} \psi^{'}(\varkappa)\bigg[\Big(\psi(\varkappa_1)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\\
&-\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\bigg]\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\bigg|\\
&+\bigg|\int_{\varkappa_1}^{\varkappa_2} \psi^{'}(\varkappa)\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\bigg|\Bigg)\\
\leq&\frac{1}{\Gamma(\ell-\gamma)}\Bigg(\int_{a_1}^{\varkappa_1} \bigg|\psi^{'}(\varkappa)\bigg[\Big(\psi(\varkappa_1)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\\
&-\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\bigg]\mathbb{D}^{\ell,\psi}u(\varkappa)\bigg|d\varkappa\\
&+\int_{\varkappa_1}^{\varkappa_2} \bigg|\psi^{'}(\varkappa)\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\mathbb{D}^{\ell,\psi}u(\varkappa)\bigg|d\varkappa\Bigg)\\
\leq&\frac{1}{\Gamma(\ell-\gamma)}\Bigg(\int_{a_1}^{\varkappa_1} \psi^{'}(\varkappa)\bigg[\Big(\psi(\varkappa_1)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\\
&-\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\bigg]\bigg|\mathbb{D}^{\ell,\psi}u(\varkappa)\bigg|d\varkappa\\
&+\int_{\varkappa_1}^{\varkappa_2} \psi^{'}(\varkappa)\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\bigg|\mathbb{D}^{\ell,\psi}u(\varkappa)\bigg|d\varkappa\Bigg), \text{where } \ell>1+\gamma\\
\leq&\frac{M}{\Gamma(\ell-\gamma)}\Bigg(\int_{a_1}^{\varkappa_1} \psi^{'}(\varkappa)\bigg[\Big(\psi(\varkappa_1)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}-\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}\bigg]d\varkappa
\end{aligned}
$$

$$+\int_{\varkappa_1}^{\varkappa_2}\psi'(\varkappa)\Big(\psi(\varkappa_2)-\psi(\varkappa)\Big)^{\ell-(\gamma+1)}d\varkappa\Bigg)$$

$$=\frac{M}{\Gamma(\ell-\gamma)}\frac{1}{(\ell-\gamma)}\Bigg(\Big(\psi(\varkappa_1)-\psi(a_1)\Big)^{\ell-\gamma}+\Big(\psi(\varkappa_2)-\psi(\varkappa_1)\Big)^{\ell-\gamma}-\Big(\psi(\varkappa_2)-\psi(a_1)\Big)^{\ell-\gamma}$$

$$+\Big(\psi(\varkappa_2)-\psi(\varkappa_1)\Big)^{\ell-\gamma}\Bigg)$$

$$=\frac{M}{\Gamma(\ell-\gamma+1)}\Bigg(\Big(\psi(\varkappa_1)-\psi(a_1)\Big)^{\ell-\gamma}-\Big(\psi(\varkappa_2)-\psi(a_1)\Big)^{\ell-\gamma}+2\Big(\psi(\varkappa_2)-\psi(\varkappa_1)\Big)^{\ell-\gamma}\Bigg).$$

Since $\varkappa_1 > a_1,\ \varkappa_2 > a_1$ and $\varkappa_2 > \varkappa_1$ and $\psi(\varkappa)$ is an increasing function, so

$$\Big(\psi(\varkappa_1)-\psi(a_1)\Big)^{\ell-\gamma}-\Big(\psi(\varkappa_2)-\psi(a_1)\Big)^{\ell-\gamma}<0.$$

Therefore,

$$|{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}y(\varkappa_1)-{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}y(\varkappa_2)|\leq\frac{2M}{\Gamma(\ell-\gamma+1)}\Big(\psi(\varkappa_2)-\psi(\varkappa_1)\Big)^{\ell-\gamma}.$$

By mean value theorem, there exists $\zeta\in[\varkappa_1,\varkappa_2]\subseteq[a_1,a_2]$ such that $\psi(\varkappa_2)-\psi(\varkappa_1)\leq(\varkappa_2-\varkappa_1)\psi'(\zeta)$, we get:

$$\begin{aligned}|{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}y(\varkappa_1)-{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}y(\varkappa_2)| &\leq\frac{2M}{\Gamma(\ell-\gamma+1)}\Big((\varkappa_2-\varkappa_1)\psi'(\zeta)\Big)^{\ell-\gamma}\\ &\leq\frac{2M}{\Gamma(\ell-\gamma+1)2^{j(\ell-\gamma)}}\Big(\psi'(a_2)\Big)^{\ell-\gamma},\end{aligned}$$

which implies that,

$$\Big({}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}y(\varkappa_1)-{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}y(\varkappa_2)\Big)^2\leq\frac{4M^2}{\Gamma^2(\ell-\gamma+1)2^{2j(\ell-\gamma)}}\Big(\psi'(a_2)\Big)^{2(\ell-\gamma)}. \tag{4.9}$$

Putting (4.9) in (4.8), we get:

$$c_i^2\leq 2^{-j-2}(a_2-a_1)^2\frac{4M^2}{\Gamma^2(\ell-\gamma+1)2^{2j(\ell-\gamma)}}\Big(\psi'(a_2)\Big)^{2(\ell-\gamma)}. \tag{4.10}$$

Putting together equations (4.5) and (4.10), we have

$$\begin{aligned}&\big\|{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}u(\varkappa)-{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}u_\ell(\varkappa)\big\|_E^2\\ &=\sum_{i=2^{\beta+1}}^{\infty}c_i^2=\sum_{j=\beta+1}^{\infty}\left(\sum_{i=2^j}^{2^{j+1}-1}c_i^2\right)\\ &\leq\sum_{j=\beta+1}^{\infty}(a_2-a_1)^2\frac{M^2}{\Gamma^2(\ell-\gamma+1)2^{2j(\ell-\gamma)+j}}\Big(\psi'(a_2)\Big)^{2(\ell-\gamma)}(2^{j+1}-1-2^j+1)\\ &=\frac{(a_2-a_1)^2M^2\Big(\psi'(a_2)\Big)^{2(\ell-\gamma)}}{\Gamma^2(\ell-\gamma+1)}\sum_{j=\beta+1}^{\infty}\frac{1}{2^{2j(\ell-\gamma)}}\end{aligned}$$

$$=\frac{(a_2-a_1)^2M^2\left(\psi^{'}(a_2)\right)^{2(\ell-\gamma)}}{\Gamma^2(\ell-\gamma+1)}\frac{1}{2^{2(\beta+1)(\ell-\gamma)}}\frac{1}{1-2^{2(\gamma-\ell)}}, \tag{4.11}$$

which implies that

$$\left\|{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}u(\varkappa)-{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}u_\ell(\varkappa)\right\|_E\leq\frac{(a_2-a_1)M\left(\psi^{'}(a_2)\right)^{\ell-\gamma}}{\Gamma(\ell-\gamma+1)}\frac{1}{2^{(\beta+1)(\ell-\gamma)}}\frac{1}{[1-2^{2(\gamma-\ell)}]^{\frac{1}{2}}}. \tag{4.12}$$

Using $k=2^{\beta+1}$, (4.12) becomes:

$$\left\|{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}u(\varkappa)-{}^{C}\mathbb{D}_{a_1}^{\gamma,\psi}u_\ell(\varkappa)\right\|_E\leq\frac{(a_2-a_1)M\left(\psi^{'}(a_2)\right)^{\ell-\gamma}}{\Gamma(\ell-\gamma+1)}\frac{1}{k^{(\ell-\gamma)}}\frac{1}{[1-2^{2(\gamma-\ell)}]^{\frac{1}{2}}}. \tag{4.13}$$

We need the value of M to compute the error bound, therefore we will determine M firstly. As $\mathbb{D}^\ell u(\varkappa)$ is bounded and continuous on the interval $[a_1,a_2]$, therefore, so is $\mathbb{D}^{\ell,\psi}u(\varkappa)$ and is approximated by:

$$\mathbb{D}^{\ell,\psi}u(\varkappa)\cong\sum_{i=0}^{r-1}c_ih_i(\varkappa)=C_r^TH_r(\varkappa). \tag{4.14}$$

Integrating(4.14), yields:

$$\mathbb{D}^{\ell-1,\psi}u(\varkappa)=\int_{a_1}^{\varkappa}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa+\mathbb{D}^{\ell-1,\psi}u(a_1)=\int_{a_1}^{\varkappa}\mathbb{D}^{\ell,\psi}u(\varkappa)d\varkappa\cong C_r^TP^{1,\psi}H_\ell(\varkappa). \tag{4.15}$$

Similarly,

$$\mathbb{D}^{\ell-2,\psi}u(\varkappa)=\int_{a_1}^{\varkappa}\mathbb{D}^{\ell-1,\psi}u(\varkappa)d\varkappa+\mathbb{D}^{\ell-2,\psi}y(a_1)=\int_{a_1}^{\varkappa}\mathbb{D}^{\ell-1,\psi}u(\varkappa)d\varkappa\cong C_\ell^TP^{2,\psi}H_\ell(\varkappa). \tag{4.16}$$

Proceeding in the same way we get:

$$\mathbb{D}^{\psi}u(\varkappa)\cong C_\ell^TP^{\ell,\psi}H_\ell(\varkappa). \tag{4.17}$$

Taking the points $\varkappa_j=\frac{j-1/2}{\ell}$, where $j=0,1,2,\cdots,m$, and utilizing them in (4.17), gives:

$$\mathbb{D}^{\psi}u(\varkappa_j)\cong C_\ell^TP^{\ell,\psi}H_\ell(\varkappa_j). \tag{4.18}$$

The matrix form of (4.18) is as:

$$\begin{aligned}&\mathbb{D}^{\psi}U^T\cong C_\ell^TP^{\ell,\psi}H_\ell(\varkappa_j)\\&\text{where } \mathbb{D}^{\psi}U^T=[\mathbb{D}^{\psi}u(\varkappa_1),\mathbb{D}^{\psi}u(\varkappa_2),\mathbb{D}^{\psi}u(\varkappa_3),\cdots,\mathbb{D}^{\psi}u(\varkappa_\ell)]^T.\end{aligned} \tag{4.19}$$

Eq. (4.19), yields the value of C_ℓ^T, which can then be used to calculate $\mathbb{D}^{\ell,\psi}(\varkappa)$ $\forall\ \varkappa\in[a_1,a_2]$ using (4.14).

Let $\tau_i\in[a_1,a_2]$ then for the equally spaced points $i=1,2,3,\ldots,\ell$, $\mathbb{D}^{\ell,\psi}u(t_i)$ can be calculated. Then M is approximated by $\varepsilon+\max|\mathbb{D}^\ell u(t_i)|$. □

Theorem 4.2. *Assume that* ${}^{C}\mathbb{D}^{\gamma,\psi}_{a_1}u_\ell$*, computed from* ψ*-HW is estimated by* ${}^{C}\mathbb{D}^{\gamma,\psi}_{a_1}u$*, then we have*

$$\|u(\varkappa) - u_\ell(\varkappa)\|_E \leq \frac{MN}{\Gamma(\gamma+1)\Gamma(\ell-\gamma+1)} \frac{1}{k^{(\ell-\gamma)}} \frac{1}{[1-2^{2(\gamma-\ell)}]^{\frac{1}{2}}}. \tag{4.20}$$

where $N = \max|(a_2-a_1)(\psi(a_2))^{\ell-\gamma}(\psi(\varkappa)-\psi(0))^{\gamma}|$.

Theorem 4.2. can be proved easily by following the procedure of Theorem 4.1. From Eq. (4.20) we noted that $\|u(\varkappa) - u_\ell(\varkappa)\|_E$ tends to 0 as ℓ tends to ∞. As a result, the ψ-HW technique is inferred to be convergent.

5. Numerical Examples

Here we provide numerical solution of some non-linear ψ-RFDEs using the ψ-HW operational matrix approach.

Example 5.1. Consider the non-linear ψ-RFDE:

$${}^{C}\mathbb{D}^{\gamma,\psi}u(\varkappa) + u(\varkappa) + u^2(\varkappa) = g(\varkappa), \quad 0 < \gamma \leq 1, \quad 0 < \varkappa \leq 1, \tag{5.1}$$

with the initial condition $u(0) = 0$.

At $g(x) = \frac{2}{\Gamma(3-\gamma)}(\psi(\varkappa))^{2-\gamma} + (\psi(\varkappa))^2 + (\psi(\varkappa))^4$, the actual solution of the problem (5.1) is $u(\varkappa) = (\psi(\varkappa))^2$. First we apply Quasilinearization technique to Eq.(5.1) to linearize the non-linear terms in it.

$$\begin{aligned}
{}^{C}\mathbb{D}^{\gamma,\psi}u(\varkappa) &= g(\varkappa) - u(\varkappa) - u^2(\varkappa),\\
{}^{C}\mathbb{D}^{\gamma,\psi}u_{r+1}(\varkappa) &= g(\varkappa) - u_r(\varkappa) - u_r^2(\varkappa) + (u_{r+1}(\varkappa) - u_r(\varkappa))(-1 - 2u_r(\varkappa))\\
&= g(\varkappa) - u_r^2(\varkappa) - (1 + 2u_r(\varkappa))u_{r+1}(\varkappa) + 2u_r(\varkappa),
\end{aligned}$$

which implies that

$${}^{C}\mathbb{D}^{\gamma,\psi}u_{r+1}(\varkappa) + (1 + 2u_r(\varkappa))u_{r+1}(\varkappa) = g(\varkappa) + u_r^2(\varkappa), \quad \text{with } u_{r+1}(0) = 0. \tag{5.2}$$

Let

$${}^{C}\mathbb{D}^{\gamma,\psi}u_{r+1}(\varkappa) = C_m^T H_m(\varkappa). \tag{5.3}$$

Operating ${}^{C}\mathcal{J}^{\gamma,\psi}_{a_1}$ on Eq. (5.3) and using the initial conditions, we have

$$u_{r+1}(\varkappa) = {}^{C}\mathcal{J}^{\gamma,\psi}C_m^T H_m(x) = C_m^T P^{\gamma,\psi}_{m\times m}H_m(\varkappa). \tag{5.4}$$

Substituting (5.3) and (5.4) in (5.2) we have,

$$C_m^T\big(H_m(\varkappa) + P^{\gamma,\psi}_{m\times m}H_m(\varkappa)\big) = g(\varkappa) - u_r(\varkappa) - u_r^2(\varkappa). \tag{5.5}$$

Eq. (5.5) can be expressed in matrix notation as:

$$C_m^T\big(H_m(\varkappa) + 2u_r(\varkappa)P^{\gamma,\psi}_{m\times m}H_m(\varkappa)\big) = G, \tag{5.6}$$

where $G = g(\varkappa) - u_r(\varkappa) - u_r^2(\varkappa)$. We can determine the value of C form the linear system (5.6), and then putting the value of C in Eq. (5.4) gives the approximate results. Table 1 shows the max absolute error for $\psi(\varkappa) = \varkappa^3$. In Fig.1, Approximate results for different choices of ψ are also given in graphs.

Table 1. Results of the max absolute error for problem (5.1) using different γ and J.

γ	$J=5$	$J=6$	$J=7$	$J=8$	$J=9$
0.6	4.3854×10^{-4}	1.4252×10^{-4}	4.6203×10^{-5}	1.4996×10^{-5}	4.8789×10^{-6}
0.7	3.3031×10^{-4}	1.0001×10^{-4}	3.0183×10^{-5}	9.1122×10^{-6}	2.7562×10^{-6}
0.8	2.4252×10^{-4}	6.8593×10^{-5}	1.9314×10^{-5}	5.4339×10^{-6}	1.5301×10^{-6}
0.9	1.7673×10^{-4}	4.6930×10^{-5}	1.2396×10^{-5}	3.2674×10^{-6}	8.6081×10^{-7}
1.0	1.3575×10^{-4}	3.4133×10^{-5}	8.5580×10^{-6}	2.1426×10^{-6}	5.3605×10^{-7}

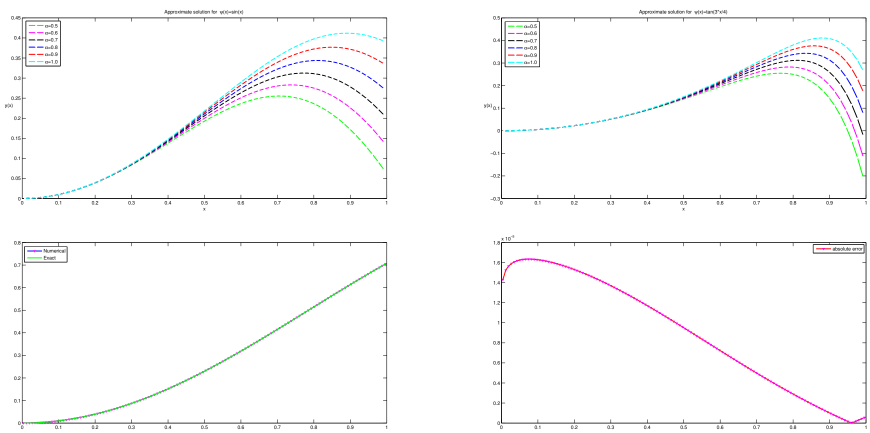

Figure 1. Actual and approximate results of the problem (5.1) for $J=6$ and various values of γ.

Example 5.2. Here we take the following non-linear ψ-RFDE

$$ {}^{C}\mathbb{D}^{\gamma,\psi}u(\varkappa)+u^2(\varkappa)=g(\varkappa),\quad 0<\varkappa\le 1,\quad 0<\gamma\le 1. \tag{5.7}$$

$$ y(0)=0. \tag{5.8}$$

For $g(\varkappa)=\frac{2}{\Gamma(3-\gamma)}(\psi(\varkappa))^{2-\gamma}+\frac{\gamma}{\Gamma(2-\gamma)}(\psi(\varkappa))^{1-\gamma}+(\psi(\varkappa))^4+\gamma^2(\psi(\varkappa))^2+2\gamma(\psi(\varkappa))^3$. One may verify that $u(\varkappa)=(\psi(\varkappa))^2+\gamma\psi(\varkappa)$ is the actual result for the problem (5.7). The nonlinear terms of (5.7) are linearized by Quasilinearization techniques.The linearized form of (5.7) is

$$ {}^{C}\mathbb{D}^{\gamma,\psi}u_{r+1}(\varkappa)+2u_{r+1}(\varkappa)u_r(\varkappa)=g(\varkappa)+u_r^2(\varkappa),\quad \text{with}\quad u_{r+1}(0)=0. \tag{5.9}$$

For numerical results we employ the ψ-HW procedure.

Let

$$ {}^{C}\mathbb{D}^{\gamma,\psi}u_{r+1}(\varkappa)=C_m^T H_m(\varkappa). \tag{5.10}$$

Applying the ψ-Caputo integral operator, ${}^{C}\mathcal{J}_{a_1}^{\gamma,\psi}$, on both sides of (5.10) and utilizing the initial condition, we have:

$$ u_{r+1}(\varkappa)={}^{C}\mathcal{J}^{\gamma,\psi}C_m^T H_m(x)=C_m^T P_{m\times m}^{\gamma,\psi}H_m(\varkappa). \tag{5.11}$$

Substituting Equations (5.10) and (5.11) in equation (5.7), we get

$$ C_m^T\big(H_m(\varkappa)+2u_r(\varkappa)P_{m\times m}^{\gamma,\psi}H_m(\varkappa)\big)=g(\varkappa)+u_r^2(\varkappa). \tag{5.12}$$

Equation (5.12) in matrix form is given as:

$$C_m^T\big(H_m(\varkappa)+2u_r(\varkappa)P_{m\times m}^{\gamma,\psi}H_m(\varkappa)\big)=g(\varkappa)+u_r^2(\varkappa). \tag{5.13}$$

C can be determined from the linear system (5.13), using C in Eq. (5.11) gives the approximate results. Table 2 contains the max absolute error for distinct values of γ and J. In Fig.2, approximate results for different choices ψ are also given in graphs.

Table 2. Values of the max absolute error for problem (5.8) using different γ and J.

γ	$J=5$	$J=6$	$J=7$	$J=8$	$J=9$
0.6	1.2733×10^{-3}	6.2471×10^{-4}	3.0934×10^{-4}	1.5391×10^{-4}	7.6766×10^{-5}
0.7	1.2910×10^{-3}	6.3216×10^{-4}	3.1273×10^{-4}	1.5552×10^{-4}	7.7551×10^{-5}
0.8	1.1161×10^{-3}	5.4369×10^{-4}	2.6824×10^{-4}	1.3321×10^{-4}	6.6382×10^{-5}
0.9	7.1349×10^{-4}	3.4173×10^{-4}	1.6710×10^{-4}	8.2612×10^{-5}	4.1071×10^{-5}
1.0	6.1030×10^{-5}	1.5258×10^{-5}	3.8146×10^{-6}	9.5367×10^{-7}	2.3841×10^{-7}

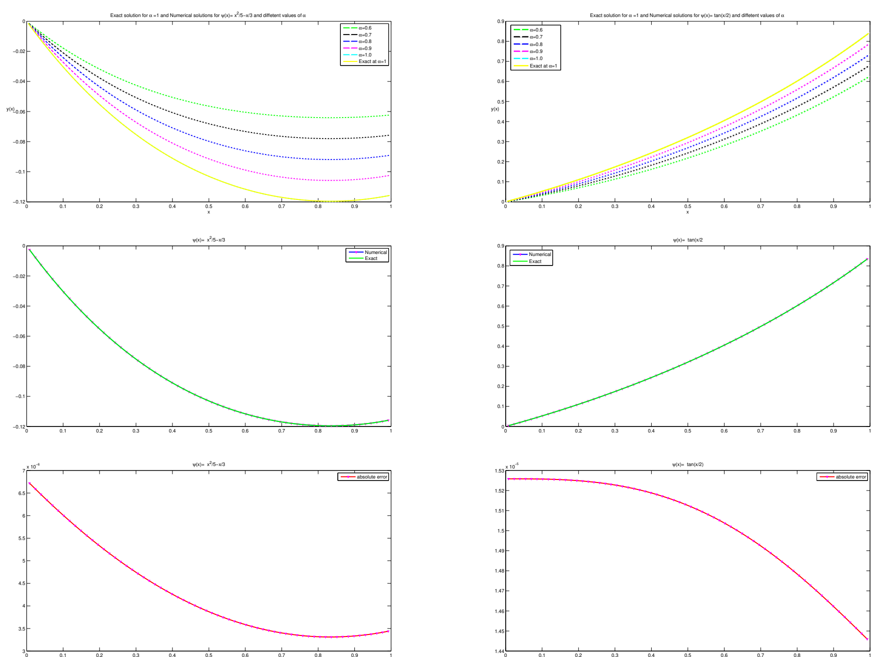

Figure 2. Actual and Approximate results of problem (5.7) for different fractional orders, γ, and their max absolute error.

Example 5.3. Consider the generalized non-linear ψ-RFDE:

$${}^{C}\mathbb{D}^{\gamma,\psi}u(\varkappa)+a(x)u^2(x)+b(x)u(\varkappa)=g(\varkappa),\quad 0<\gamma\le 1,\quad x\in[0,1]\ \text{ and }\ y(0)=u_0. \tag{5.14}$$

For $g(\varkappa) = \frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+1)}(\psi(\varkappa))^{\gamma} + a(\varkappa)(\psi(\varkappa))^{4\gamma} + b(x)(\psi(\varkappa))^{2\gamma}$,the actual solution of Eq. (5.14). is $u(x) = (\psi(\varkappa))^{2\gamma}$ To make it simple we consider $a(\varkappa) = b(\varkappa) = 1$ First we apply Quasilinearization technique to (5.14). The linearized form of (5.14) is

$$^{C}\mathbb{D}^{\gamma,\psi}u_{r+1}(\varkappa) + (1 + 2u_r(\varkappa))u_{r+1}(\varkappa) = u_r^2(\varkappa) + g(\varkappa), \text{ with } u_{r+1}(0) = (\psi(0))^{2\gamma}. \tag{5.15}$$

Let

$$^{C}\mathbb{D}^{\gamma,\psi}u_{r+1}(\varkappa) = C_m^T H_m(\varkappa). \tag{5.16}$$

Operating $^{C}\mathcal{J}_{a_1}^{\gamma,\psi}$ onto Eq. (5.16) and using the initial conditions, we have

$$\begin{aligned} u_{r+1}(\varkappa) &= {}^{C}\mathcal{J}^{\gamma,\psi}C_m^T H_m(x) + c_0 \\ &= C_m^T P_{m\times m}^{\gamma,\psi} H_m(\varkappa) + \frac{1}{2}. \end{aligned} \tag{5.17}$$

Substituting (5.16) and (5.17) in (5.15) we have,

$$C_m^T\big(H_m(\varkappa) + (1 + 2u_r(\varkappa))P_{m\times m}^{\gamma,\psi}H_m(\varkappa)\big) = g(\varkappa) + u_r^2(\varkappa). \tag{5.18}$$

Eq.(5.18) is expressed in the form of a matrix as:

$$C_m^T\big(H_m(\varkappa) + (1 - 2u_r(\varkappa))P_{m\times m}^{\gamma,\psi}H_m(\varkappa)\big) = G, \tag{5.19}$$

where $G = g(\varkappa) + u_r^2(\varkappa)$. C can be determined from the linear system (5.19), substituting C in Eq. (5.17) gives the approximate results. The max absolute error for several values of γ and J is shown in Table 3. It demonstrates that when the value of J increases, the error decreases. In Fig. 3, the approximate results for various functions ψ are also given in pictorial form.

Table 3. Results of max absolute error for the problem (5.14) using $\psi(\varkappa) = \varkappa^2$ and various J and γ

γ	$J = 5$	$J = 6$	$J = 7$	$J = 8$	$J = 9$
0.6	2.2790×10^{-4}	7.4702×10^{-5}	2.4498×10^{-5}	8.0425×10^{-6}	2.6428×10^{-6}
0.7	2.1513×10^{-4}	6.5313×10^{-5}	1.9862×10^{-5}	6.0507×10^{-6}	1.8462×10^{-6}
0.8	2.0818×10^{-4}	5.8754×10^{-5}	1.6603×10^{-5}	4.6985×10^{-6}	1.3317×10^{-6}
0.9	2.0718×10^{-4}	5.4730×10^{-5}	1.4458×10^{-5}	3.8213×10^{-6}	1.0106×10^{-6}
1.0	2.0363×10^{-4}	5.3432×10^{-5}	1.3359×10^{-5}	3.3400×10^{-6}	8.3502×10^{-7}

Example 5.4. Consider the non-linear ψ-RFDE:

$$^{C}\mathbb{D}^{\gamma,\psi}u(\varkappa) - u^2(\varkappa) - 1 = 0, \quad 0 < \gamma \leq 1, \quad x \in [0,1] \quad \text{and} \quad y(0) = 0. \tag{5.20}$$

The actual solution of Eq.(5.20) is $u(\varkappa) = \tan(\psi(\varkappa))$. First we apply the Quasilinearization procedure to (5.20). The linearized form of (5.20) is:

$$^{C}\mathbb{D}^{\gamma,\psi}u_{r+1}(\varkappa) - 2u_r(\varkappa)u_{r+1}(\varkappa) = 1 - u_r^2(\varkappa), \quad \text{with} \quad u_{r+1}(0) = 0. \tag{5.21}$$

Let

$$^{C}\mathbb{D}^{\gamma,\psi}u_{r+1}(\varkappa) = C_m^T H_m(\varkappa). \tag{5.22}$$

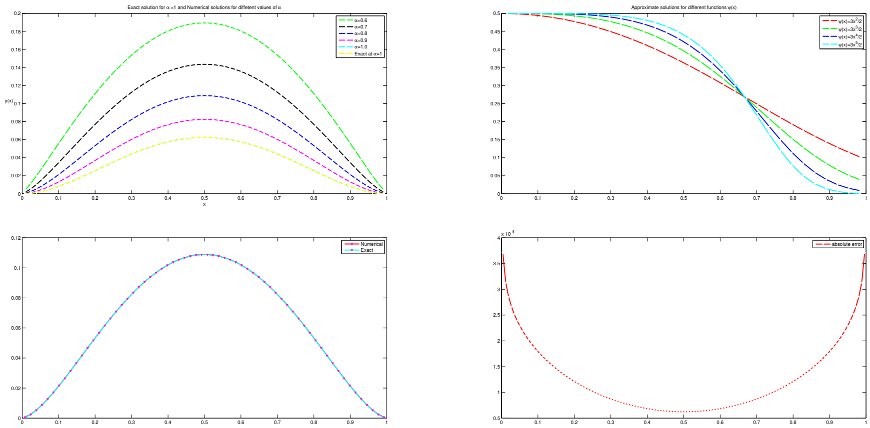

Figure 3. Approx results for various γ values and functions $\psi(x)$.

Operating ${}^{C}\mathcal{J}^{\gamma,\psi}_{a_1}$, on Eq. (5.22) and using the initial conditions, we have

$$u_{r+1}(\varkappa) = {}^{C}\mathcal{J}^{\gamma,\psi} C_m^T H_m(x) = C_m^T P^{\gamma,\psi}_{m\times m} H_m(\varkappa). \tag{5.23}$$

Substituting (5.22) and (5.23) in (5.21) we have,

$$C_m^T\big(H_m(\varkappa) - 2u_r(\varkappa) P^{\gamma,\psi}_{m\times m} H_m(\varkappa)\big) = 1 - u_r^2(\varkappa). \tag{5.24}$$

Eq.(5.24) can be expressed in matrix form as following:

$$C_m^T\big(H_m(\varkappa) - 2u_r(\varkappa) P^{\gamma,\psi}_{m\times m} H_m(\varkappa)\big) = G, \tag{5.25}$$

where $G = 1 - u_r^2(\varkappa)$. C can be determined from the linear system (5.19), substituting C in Eq. (5.23) gives the approximate results. Table 4 shows the max absolute error for various γ and J values. It shows that increasing the value of J lowers the error. In Fig. 4, the approximate results for various functions ψ are also represented graphically.

Table 4. max absolute error for problem (5.20) with $\psi(\varkappa) = \varkappa^3$, at various values of γ and J.

γ	$J=5$	$J=6$	$J=7$	$J=8$	$J=9$
0.6	2.2790×10^{-4}	7.4702×10^{-5}	2.4498×10^{-5}	8.0425×10^{-6}	2.6428×10^{-6}
0.7	2.1513×10^{-4}	6.5313×10^{-5}	1.9862×10^{-5}	6.0507×10^{-6}	1.8462×10^{-6}
0.8	2.0818×10^{-4}	5.8754×10^{-5}	1.6603×10^{-5}	4.6985×10^{-6}	1.3317×10^{-6}
0.9	2.0718×10^{-4}	5.4730×10^{-5}	1.4458×10^{-5}	3.8213×10^{-6}	1.0106×10^{-6}
1.0	2.0363×10^{-4}	5.3432×10^{-5}	1.3359×10^{-5}	3.3400×10^{-6}	8.3502×10^{-7}

6. Conclusion

Fractional differential equations are the best way to model many real-world physical phenomena. Apart from modelling, solution strategies and their repercussions are

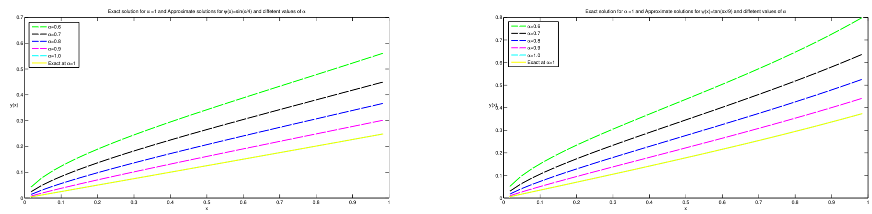

Figure 4. Approximate results for $\gamma = 1$, $J = 6$ and various choices of $\psi(\varkappa)$.

essential for determining critical points where a significant divergence or bifurcation begins. As a result, high-precision solutions are always required.The ψ-Caputo fractional derivatives give additional flexibility to mathematical models, and the ψ-Caputo derivative has the ability to extract hidden features of real-world phenomena. This paper introduces a computational method for solving a class of fractional differential equations involving the ψ-Caputo fractional derivative based on a new operational-matrix of fractional integration, the ψ-HW operational-matrix.The method's convergence is demonstrated, and the numerical experiments presented in Section 6 confirm the effectiveness of this approach. The method can also be applied to other wavelet bases, such as Legendre, Chebyshev, and Gegenbauer wavelets.This approach can be applied to boundary value problems in FDEs as well as fractional partial differential equations.

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