Journal of Applied Analysis and Computation
Volume 13, Number 2, April 2023, 1070–1087
Website:http://www.jaac-online.com
DOI:10.11948/20220491

PROJECTION SYNCHRONIZATION OF FUNCTIONAL FRACTIONAL-ORDER NEURAL NETWORKS WITH VARIABLE COEFFICIENTS

Lili Jia[1], Zongxin Lei[2], Changyou Wang[2,†], Yuqian Zhou[2], Tao Jiang[3], Yuanhua Du[2] and Qiuyan Zhang[2]

Abstract In this paper, the projection synchronization problem of functional fractional-order neural networks with variable coefficients and Caputo derivatives is studied. Firstly, a simple global projection synchronization scheme is designed according to the open-loop and adaptive feedback control. Secondly, by constructing a suitable Lyapunov function and utilizing the properties of delayed fractional-order differential inequalities, some criteria for the global projective synchronization of the variable coefficient functional neural networks with Caputo derivatives are obtained. Finally, a numerical example with many numerical simulations is employed to demonstrate the correctness and validity of the proposed method in this paper.

Keywords Projection synchronization, fractional calculus, functional neural networks, variable coefficient.

MSC(2010) 93D20.

1. Introduction

In 1967, Mandelbrot [21] published an article mentioning the use of fractional dimensional ideas to consider real-world problems. In 1982, Mandelbrot [22] applied fractional-order calculus in his study of Brownian motion and pointed out the existence of fractional-order derivative in many fields of science and engineering, leading to a boom in the study of fractional-order calculus. Because the fractional-order calculus accumulates the overall information of a function in a weighted way, while the integer-order calculus is determined by the local characteristics of the function, the fractional-order derivative can describe the phenomena existing in nature more objectively and has better memorability, and the system with fractional-order derivative is more likely to reach stability. So fractional calculus is introduced into the neural network model to form a fractional-order neural networks, which has been widely studied because of its extensive applications in natural science and engineering technology [14, 17, 25–27, 41]. In addition, because there are inevitably

†The corresponding author. Email: wangchangyou417@163.com(C. Wang)
[1]Dianchi College of Yunnan University, Kunming, Yunnan 650228, China
[2]College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, China
[3]School of Intelligent Medicine, Chengdu University of Traditional Chinese Medicine, Chengdu 610075, China

time delays in many network fields, a real neural network model should contain time delays. Therefore, the dynamic behavior of neural networks with fractional-order derivative and time delay, which should be regarded as the functional fractional-order neural networks (FFNNs), has been widely researched [28, 42, 44, 45].

Synchronization which is one of the important dynamic behaviors of neural networks has attracted more and more scholars' attention. It includes complete synchronization [30, 31], Lag synchronization [8], anti-synchronization [23], cluster synchronization [13], quasi-uniform synchronization [38], etc. Because projection synchronization can achieve faster communication speed based on its proportional characteristics [32], so research on the projection synchronization problem of neural networks has extremely important theoretical and practical significance. Scholars have achieved many different methods to research projective synchronization of neural networks, such as adaptive control [37, 43], linear-nonlinear feedback control [6, 33], impulsive control [11], pinning control [12, 46], and so on. Nowadays, more and more scholars and engineers pay attention to the projection synchronization of FFNNs. For example, in 2018, Zhang et al. [47] studied the projective synchronization of FFNNs based on the comparison principle, and obtained some criteria ensuring the projective synchronization of the FFNNs via using fractional-order differential inequalities. In 2019, Gu et al. [9] discussed the memristor-based FFNNs and obtained several feedback control strategies ensuring the projective synchronization of the fractional-order memristor-based neural networks. In 2020, Guo et al. [10] studied the quasi-projective synchronization of complex-valued FFNNs, and established a quasi-projective synchronization criterion by using some inequality techniques. In 2021, Wang et al. [34] researched the projective synchronization of complex-valued FFNNs, achieved two projective synchronization criteria vie employing Lyapunov stability theory and designing feedback controller and adaptive controller. In 2022, Liu et al. [20] discussed the projective synchronization of FFNNs with mixed time delay, and obtained several criteria ensuring the projective synchronization for the delayed system vie introducing an extended Halanay inequality.

As we all know, when we establish mathematical models to describe objective practical problems, it is difficult to obtain accurate values of model parameters. Therefore, in view of the fact that parameter uncertainty is likely to knock out the synchronization, stability or many other performances of nonlinear systems, we cannot ignore the effect of parameter uncertainty when we discuss the dynamic characteristics of nonlinear systems. Very recently, many scholars have paid attention to the synchronization problem of FFNNs with parameter uncertainty. For example, in 2018, Yang et al. [40] studied the synchronization problem of memristor-based complex-valued FFNNs with parameter nondeterminacy and achieved several criteria to sure the global asymptotically synchronization for the delayed neural networks. In 2020, He et al. [15] studied the matrix projection synchronization problem of FFNNs with different time scales based on fractional-order Barbalat theory and Lyapunov-Krasovskii generalization as well as properties of fractional-order differential inequality. In 2021, Huang et al. [16] discussed the stability of complex-valued FFNNs with uncertain parameters, and established several criteria to ensure the robustly stable of the equilibrium point for the addressed system. In 2022, Li et al. [19] studied the robust stability and projective synchronization of FFNNs with uncertain parameters by using general Halanay inequality and free-weighting method. In particular, in 2020, Wang et al. [35] discussed the global synchronization for FFNNs with variable coefficients vie using the properties of delayed differential inequalities, developing some new analysis method and building appropriate Lyapunov function.

However, FFNNs with variable coefficients is rarely studied in the existing work. Because the self inhibition rate of neurons in the neural networks and the connection weight between neurons should be a function of time, not a constant, FFNNs with variable coefficients can better simulate the interaction between neurons in the neural network system. Therefore, inspired by the previous work, the purpose of this article is to deal with the projective synchronization problem of FFNNs with variable coefficients. By constructing the Lyapunov function, the asymptotic stability of the error system is verified, and some sufficient conditions to ensure the global projective synchronization of the new neural networks are achieved. The remainder of this article is organized as follows. In Section 2, some definitions and lemmas are presented, and the model description is given. In Section 3, the global projective synchronization controllers are designed, and some criteria ensuring the global projective synchronization for the new FFNNs are afforded. In Section 4, a numerical example with many numerical simulations is employed to verify the feasibility of the theoretical results. Finally, the conclusions are summarized.

Remark 1.1. As we all know, there is no general rule in the design of synchronization controller of neural network, which brings difficulties and challenges to the design of controller. In addition, it is also very difficult to construct a Lyapunov function satisfying the stability theorem of the delay fractional-order differential system, which requires many tests. The innovations and contributions of this article are summarized as follows: (1) In order to more preferably represent the interaction between neurons, the variable-parameters are introduced into the known FFNNs to form a new neural networks model. (2) Utilizing the properties of delayed fractional-order differential inequalities and constructing a suitable Lyapunov function as well as developing some new analysis methods, some criteria for the global projective synchronization of the new neural networks model is established. (3) Based on the fact that Caputo fractional derivative is a generalization of integral derivative, the results obtained in this article are not only applicable for fractional-order systems but also can be regarded as extensions of integer-order ones. (4) Compared with the results in [9,10,19,35,47], the results obtained in this article are more general, which will greatly expand the application scope of FFNNs.

2. Model Statement and Preliminaries

In this section, we will give some preparatory knowledge. A description of FFNNs with variable coefficients is also given in this section.

Definition 2.1 ([24])**.** The Gamma function is defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt.$$

It is obvious that the Gamma function satisfies $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Definition 2.2 ([24])**.** The Caputo fractional-order derivative of function $x(t)$ is defined as

$$D_{0,t}^{\alpha} x(t) = D_{0,t}^{-(n-\alpha)} \frac{d^n}{dt^n} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds,$$

where α represents the order of the derivative, and $n-1<\alpha\leq n$. The fractional-order derivative D^α discussed in this paper refers to the Caputo derivative $D^\alpha_{0,\,t}$.

Lemma 2.1 ([36]). *Suppose that $x(t)=(x_1(t),\cdots,x_n(t))^T\in R^n$ is a differentiable vector-valued function and $P\in R^{n\times n}$ is a symmetric positive matrix. Then, for any time instant $t\geq 0$, we have*

$$\frac{1}{2}D^\alpha[x^T(t)Px(t)]\leq(x^T(t)P)D^\alpha x(t),$$

where $0<\alpha<1$. In particular, if $P=E$ is an identity matrix, then we have

$$\frac{1}{2}D^\alpha[x^T(t)x(t)]\leq x^T(t)D^\alpha x(t).$$

Lemma 2.2 ([3]). *For Caputo fractional-order differential system*

$$D^\alpha x(t)=f(t,x(t),x(t-\tau)),\tag{2.1}$$

where $x\in R^n$, $0<\alpha<1$. Suppose that $w_1(s),w_2(s)$ are continuous nondecreasing functions, $w_1(s)$ and $w_2(s)$ are positive for $s>0$, and $w_1(0)=w_2(0)$, $w_2(s)$ strictly increasing. If there is a continuously differentiable function $V:R\times R^n\to R$, such that

$$w_1(\|x\|)\leq V(t,x)\leq w_2(\|x\|),$$

for $t\in R$, $x\in R^n$, and there exist two constants $p,q>0$, with $p<q$ such that

$$D^\alpha V(t,x(t))\leq -qV(t,x(t))+p\sup_{-\tau\leq\theta\leq 0}V(t+\theta,x(t+\theta)),\quad t>t_0,$$

then the Caputo fractional-order differential system (2.1) is globally uniformly asymptotically stable.

In this paper, we discuss the projection synchronization problem of variable coefficient FFNNs as a master system, which is described by

$$D^\alpha x_i(t)=-a_i(t)x_i(t)+\sum_{j=1}^n b_{ij}(t)f_j(x_j(t))+\sum_{j=1}^n c_{ij}(t)g_j(x_j(t-\tau))+I_i,\tag{2.2}$$

or written as a vector form

$$D^\alpha x(t)=-A(t)x(t)+B(t)f(x(t))+C(t)g(x(t-\tau))+I,\tag{2.3}$$

where each element in $A(t)$, $B(t)$ and $C(t)$ is required to be a positive bounded function, $0<\alpha<1$, $t\geq 0$,$i=1,\cdots,n$, n represents the number of neurons, $x(t)=(x_1(t),\cdots,x_n(t))^T\in R^n$ represents the state variable of the neuron at time t, $\tau>0$ denotes the communication delay of the neuron, $A(t)$ is the rate at which the i-th neuron returns to a resting state without being connected to the neural network, $B(t)$ denotes the influential strength of the j-th neuron to the i-th neural at time t, and $C(t)$ indicates the influential strength of the j-th neuron to the i-th neural at time $t-\tau$, $f(x(t))=(f_1(x_1(t)),\cdots,f_n(x_n(t)))^T$ and $g(x(t-\tau))=(g_1(x_1(t-\tau)),\cdots,g_n(x_n(t-\tau)))^T$ represent the excitation functions of the neuron at time t and $t-\tau$, respectively, and $I=(I_1,\cdots,I_n)^T$ is the external input vector.

The slave system is given by

$$D^{\alpha}y_i(t) = -a_i(t)y_i(t)+\sum_{j=1}^{n} b_{ij}(t)f_j(y_j(t))+\sum_{j=1}^{n} c_{ij}(t)g_j(y_j(t-\tau))+I_i+u_i(t), \quad (2.4)$$

or written as a vector form

$$D^{\alpha}y(t) = -A(t)y(t) + B(t)f(y(t)) + C(t)g(y(t-\tau)) + I + u(t), \qquad (2.5)$$

here $y(t) = (y_1(t), \cdots, y_n(t))^T \in R^n$ *denotes the state variable of the slave system, and* $u(t) = (u_1(t), \cdots, u_n(t))^T$ *represents the synchronous controller to be designed, the other parameters represent the same meaning as those given in system (2.3).*

Remark 2.1. When $A(t) = e^{pt}\mathrm{diag}(a_1, a_2, \cdots, a_n)$, $B(t) = e^{pt}(b_{ij})_{n\times n}$, and $C(t) = e^{pt}(c_{ij})_{n\times n}$, the model (2.3) is reduced to the model in literature [35]. When $A(t) = \mathrm{diag}(a_1, a_2, \cdots, a_n)$, $B(t) = (b_{ij})_{n\times n}$, and $C(t) = (c_{ij})_{n\times n}$, the model (2.3) is reduced to the model in literature [9,23,47]. Moreover, if $\tau = 0$, the model (2.3) is further reduced to the model in literature [1,4,39].

Definition 2.3 ([2]). If there is a non-zero constant β such that, for any two solutions $x(t)$ and $y(t)$ of the master-slave systems (2.3) and (2.5) with different initial values, it holds that $\lim_{t\to\infty} \|y(t) - \beta x(t)\| = 0$, then the slave system (2.5) is called to be globally asymptotically projective synchronized to the master system (2.3).

Remark 2.2. When $\beta = 0$, the projective synchronization problem is equivalent to that the system (2.5) is stabilized to the origin. When $\beta = 1$ and -1, the projective synchronization problem is equivalent to complete synchronization and anti-synchronization, respectively.

Assumption 2.1. *The neuron excitation function* $g_j(\cdot), f_j(\cdot)$ *satisfies the Lipschitz continuity in the field of real numbers, that is, there are constants* $h_j > 0$, $l_j > 0$ $(j = 1, 2, \cdots, n)$ *such that*

$$|g_j(y) - g_j(x)| \le h_j\,|y - x|\,, |f_j(y) - f_j(x)| \le l_j\,|y - x|\,,$$

for any $y \neq x \in R$.

3. Projective synchronization

In this section, we study the projective synchronization of master-slave systems (2.3) and (2.5) for FFNNs with variable coefficients, which is equivalent to discuss the stability of error systems. So now we only need to establish an appropriate controller to prove the stability of the error system under the control scheme.

Set the error vector

$$e_i(t) = y_i(t) - \beta x_i(t), \quad i = 1, \cdots, n,$$

then the error system is as follows

$$\begin{aligned} D^\alpha e_i(t) = & -a_i(t)y_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(y_j(t)) + \sum_{j=1}^n c_{ij}(t)g_j(y_j(t-\tau)) + I_i + u_i(t) \\ & -\beta\left\{-a_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n c_{ij}(t)g_j(x_j(t-\tau)) + I_i\right\} \\ = & -a_i(t)y_i(t) + \beta a_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)[f_j(y_j(t)) - \beta f_j(x_j(t))] \\ & + \sum_{j=1}^n c_{ij}(t)[g_j(y_j(t-\tau)) - \beta g_j(x_j(t-\tau))] + (1-\beta)I_i + u_i(t), \end{aligned} \tag{3.1}$$

or written as a vector form

$$\begin{aligned} D^\alpha e(t) = & -A(t)e(t) + B(t)[f(y(t)) - \beta f(x(t))] \\ & + C(t)[g(y(t-\tau)) - \beta g(x(t-\tau))] + u(t) + (1-\beta)I, \end{aligned} \tag{3.2}$$

where $e(t) = (e_1(t), \cdots, e_n(t))^T \in R^n$, and β is a projective coefficient.

Through analysis, it is easy to know that if we prove the asymptotic stability of the zero solution of the error system (3.2), it is equivalent to us proving the asymptotic projective synchronization of the master system (2.3) and the slave system (2.5).

The controller $u_i(t),\ i = 1, \cdots, n$, selected as follows

$$\begin{aligned} u_i(t) = & \ s_i(t) + w_i(t), \\ s_i(t) = & \sum_{j=1}^n b_{ij}(t)[\beta f_j(x_j(t)) - f_j(\beta x_j(t))] \\ & + \sum_{j=1}^n c_{ij}(t)[\beta g_j(x_j(t-\tau)) - g_j(\beta x_j(t-\tau)) + (\beta-1)I_i, \\ w_i(t) = & -k_i[y_i(t) - \beta x_i(t)], \end{aligned} \tag{3.3}$$

where k_i are the positive constant depends on $A(t), B(t), C(t),$ and $h_j, l_j, j=1,2,\cdots,n$.

Remark 3.1. The controller $u_i(t)$ is a hybrid controller, where $s_i(t)$ is an open loop controller and $w_i(t)$ is an adaptive feedback controller.

Let

$$p = \max_{i=1}^n\{\sum_{j=1}^n h_j \sup_{t_0\le t<+\infty}\{c_{ij}(t)\}\},$$

$$q = \min_{i=1}^n\{2\inf_{t_0\le t<+\infty}\{a_i(t)\}+2k_i-\sum_{j=1}^n \sup_{t_0\le t<+\infty}\{c_{ij}(t)\}h_j-2\sum_{j=1}^n l_j \sup_{t_0\le t<+\infty}\{b_{ij}(t)\}\}.$$

Theorem 3.1. *Under Assumptions 2.1, if there exist appropriate positive constants $k_i,\ i = 1, \cdots, n$, such that $q > p > 0$, then the variable coefficient FFNNs (2.5) with controller (3.3) can be globally projective synchronized onto (2.3) in the finite time.*

Proof. Setting up a suitable Lyapunov function

$$V(t) = \frac{1}{2}e(t)^T e(t).$$

According to Lemma 2.1, we take the fractional-order derivative of $V(t)$, such that

$$\begin{aligned}
D^\alpha V(t) =& D^\alpha \left[\frac{1}{2}e(t)^T e(t)\right] \\
&\leq e(t) D^\alpha e(t) \\
&= \sum_{i=1}^n e_i(t)\{-a_i(t)y_i(t) + \beta a_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)\left[f_j(y_j(t)) - f_j(\beta x_j(t)\right] \\
&\quad + \sum_{j=1}^n c_{ij}(t)[g_j(y_j(t-\tau)) - g_j(\beta x_j(t-\tau)) - k_i[y_i(t) - \beta x_i(t)]\} \\
&= \sum_{i=1}^n e_i(t)\{(-a_i(t) - k_i)\, e_i(t) + \sum_{j=1}^n b_{ij}(t)\left[f_j(y_j(t)) - f_j(\beta x_j(t)\right] \\
&\quad + \sum_{j=1}^n c_{ij}(t)[g_j(y_j(t-\tau)) - g_j(\beta x_j(t-\tau))\}.
\end{aligned} \tag{3.4}$$

According to **Assumption 2.1**, (3.4) and the boundedness of matrices $A(t)$, $B(t)$ and $C(t)$, we have

$$\begin{aligned}
D^\alpha V(t) \leq& \sum_{i=1}^n e_i(t)\{(-a_i(t) - k_i)\}e_i(t) + \sum_{i=1}^n |e_i(t)|\{\sum_{j=1}^n \sup_{t_0 \leq t < +\infty}\{b_{ij}(t)\}l_j\,|e_i(t)| \\
&+ \sum_{j=1}^n \sup_{t_0 \leq t < +\infty}\{c_{ij}(t)\}h_j\,|e_i(t-\tau)|\} \\
\leq& \sum_{i=1}^n \left(-\inf_{t_0 \leq t < +\infty}\{a_i(t)\} - k_i\right) e_i^2(t) + \sum_{i=1}^n \sup_{t_0 \leq t < +\infty}\{b_{ij}(t)\}l_j e_i^2(t) \\
&+ \sum_{i=1}^n \sum_{j=1}^n \sup_{t_0 \leq t < +\infty}\{c_{ij}(t)\}h_j\,|e_i(t)|\,|e_i(t-\tau)|.
\end{aligned} \tag{3.5}$$

According to the trigonometric inequality, we can get

$$|e_i(t)|\,|e_i(t-\tau)| \leq \frac{1}{2}|e_i(t)|^2 + \frac{1}{2}|e_i(t-\tau)|^2, \tag{3.6}$$

thus, from (3.5) and (3.6), it holds that

$$\begin{aligned}
D^\alpha V(t) \leq& -\frac{1}{2}\sum_{i=1}^n (2\inf_{t_0 \leq t < +\infty}\{a_i(t)\} + 2k_i - \sum_{j=1}^n \sup_{t_0 \leq t < +\infty}\{c_{ij}(t)\}h_j \\
&- 2\sum_{j=1}^n l_j \sup_{t_0 \leq t < +\infty}\{b_{ij}(t)\})e_i^2(t) + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n h_j \sup_{t_0 \leq t < +\infty}\{c_{ij}(t)\}e_i^2(t-\tau) \\
\leq& -qV(t) + pV(t-\tau).
\end{aligned}$$

So, we can get

$$D^{\alpha}V(t,x(t)) \leq -qV(t,x(t)) + p \sup_{-\tau\leq\theta\leq 0} V(t+\theta, x(t+\theta)).$$

By **Lemma 2.2** and $q > p > 0$, we have that the variable coefficient FFNNs (2.5) with controller (3.3) can be globally projective synchronized onto (2.3) in the finite time. □

4. Numerical simulation

In this section, some numerical simulations are presented to illustrate the validity of the theoretical results established above.

Example 4.1. Consider the following two-dimensional FFNNs with variable coefficients as the master system

$$D^{\alpha}x(t) = -A(t)x(t) + B(t)f(x(t)) + C(t)g(x(t-\tau)) + I, \tag{4.1}$$

and the slave system is described by

$$D^{\alpha}y(t) = -A(t)y(t) + B(t)f(y(t)) + C(t)g(y(t-\tau)) + I + u(t), \tag{4.2}$$

where

$$A(t) = \begin{bmatrix} \cos t + 3 & 0 \\ 0 & \cos t + 3 \end{bmatrix}, B(t) = \begin{bmatrix} 2\cos t & \cos t \\ -\sin t & \cos t \end{bmatrix}, C(t) = \begin{bmatrix} -2\sin t & -2\cos t \\ -\sin t & -\sin t \end{bmatrix},$$
$$u(t) = (u_1(t), u_2(t))^T, \alpha = 0.9, I = (I_1, I_2)^T = (0,0)^T, \tau = 0.01.$$

The activation functions are given by $f(x(t)) = g(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^T$, it is easy to know that the function $f(u), g(u)$ satisfies **Assumption 2.1** when

$$\mathrm{diag}(l_1, l_2) = \mathrm{diag}(h_1, h_2) = \mathrm{diag}(1, 1).$$

According to the **Controller** (3.3), we select the adaptive constants $k_1 = 6, k_2 = 7$ to make $q > p > 0$. Therefore, according to **Theorem 3.1**, the master-slave systems (4.1) and (4.2) can achieve global asymptotic projective synchronization, which is demonstrated in **Figures 1-2**.

In **Figure 1**, it is easy to see that the projective synchronization errors with the projective coefficient $\beta = -2$ and initial values

$$x_1(t) = 1.2, x_2(t) = 3.5, y_1(t) = 2.1, y_2(t) = 1.2, t \in [-0.01, 0], \tag{4.3}$$

converge to zero, which indicate that master-slave systems (4.1) and (4.2) can obtain global asymptotic projective synchronization. And evolutions of the master-slave systems (4.1) and (4.2) are shown in **Figure 2**.

The simulation results for the projective coefficient $\beta = 3$ are shown in **Figures 3-4**. To further verify that the master-slave systems (4.1) and (4.2) can achieve global asymptotic projective synchronization, we choose different initial conditions, different fractional orders and different time delays for numerical simulation, which are shown in **Figures 5-16**.

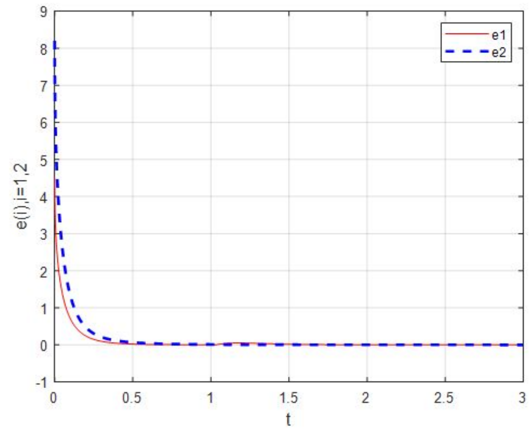

Figure 1. Synchronization errors of the master-slave systems (4.1) and (4.2) with $\beta = -2$ and (4.3).

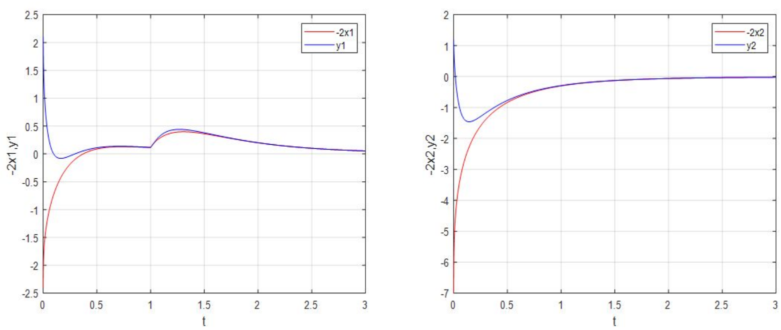

Figure 2. Evolutions of the master-slave systems (4.1) and (4.2) with $\beta = -2$ and (4.3).

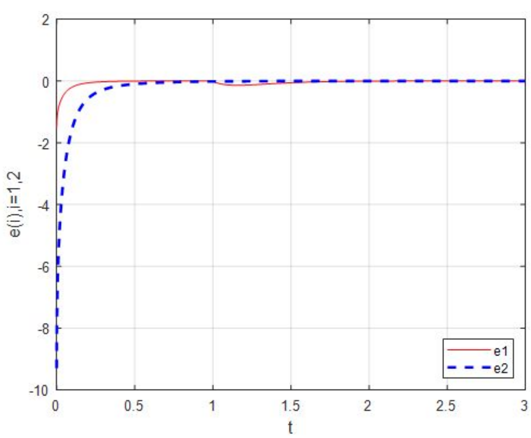

Figure 3. Synchronization errors of the master-slave systems (4.1) and (4.2) with $\beta = 3$ and (4.3).

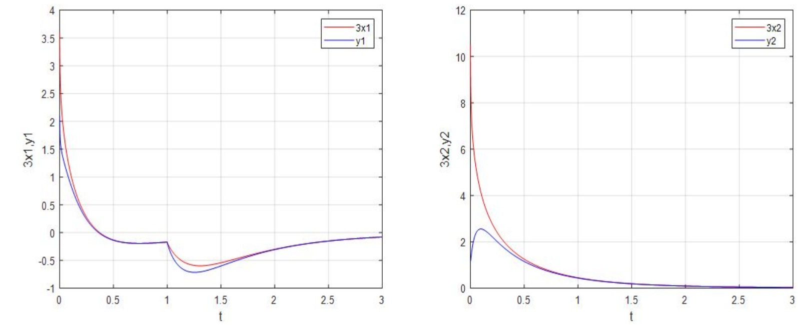

Figure 4. Evolutions of the master-slave systems (4.1) and (4.2) with $\beta = 3$ and (4.3).

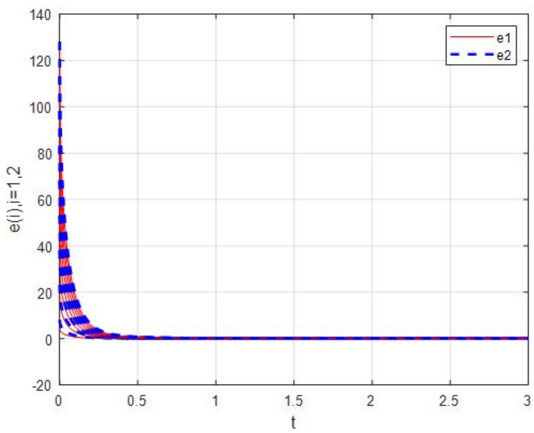

Figure 5. Synchronization errors of the systems (4.1) and (4.2) with different initial values and $\beta = -2$.

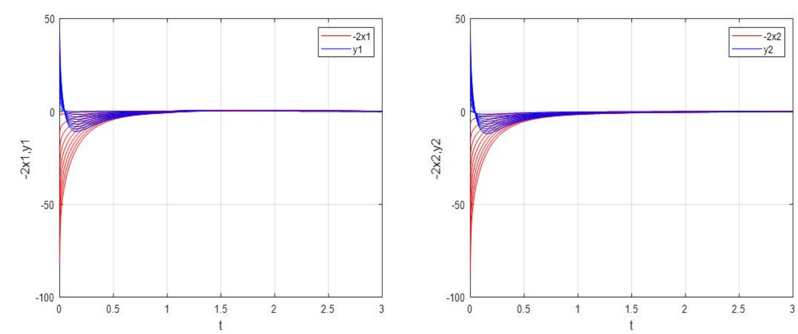

Figure 6. Evolutions of the master-slave systems (4.1) and (4.2) with different initial values and $\beta = -2$.

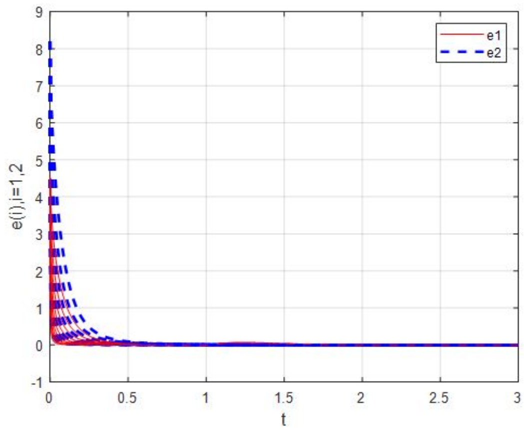

Figure 7. Synchronization errors of the systems (4.1) and (4.2) with different fractional orders and $\beta = -2$.

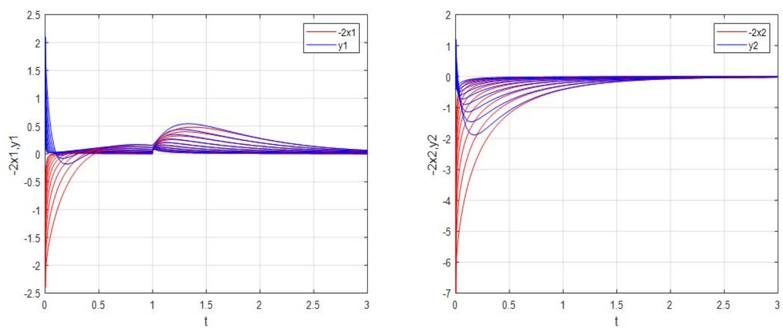

Figure 8. Evolutions of the master-slave systems (4.1) and (4.2) with different fractional orders and $\beta = -2$.

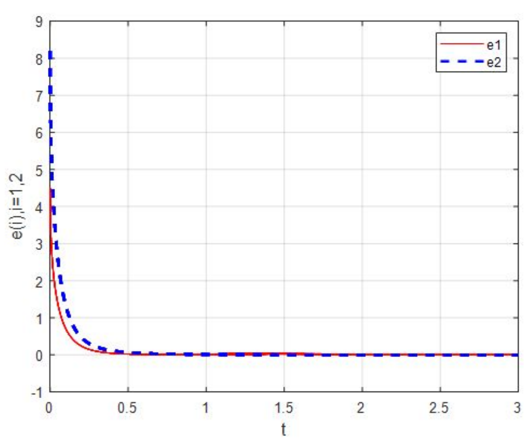

Figure 9. Synchronization errors of the systems (4.1) and (4.2) with different delays and $\beta = -2$.

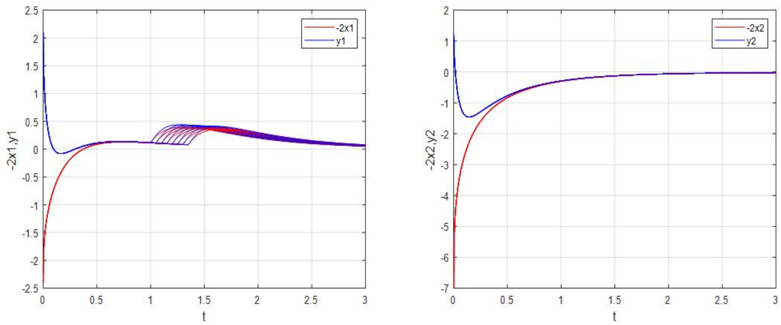

Figure 10. Evolutions of the master-slave systems (4.1) and (4.2) with different delays and $\beta = -2$.

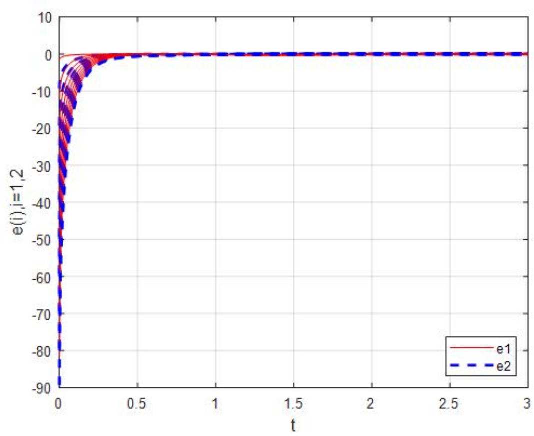

Figure 11. Synchronization errors of the systems (4.1) and (4.2) with different initial values and $\beta = 3$.

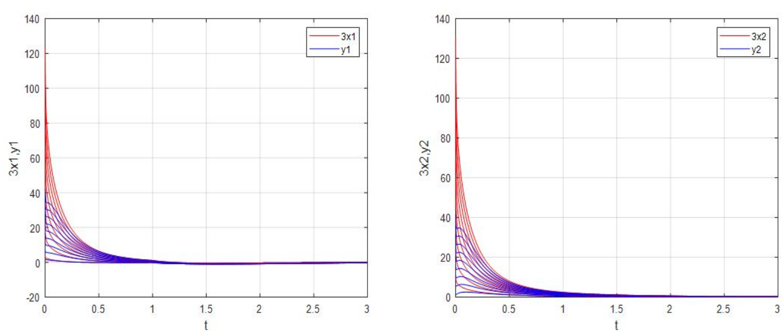

Figure 12. Evolutions of the master-slave systems (4.1) and (4.2) with different initial values and $\beta = 3$.

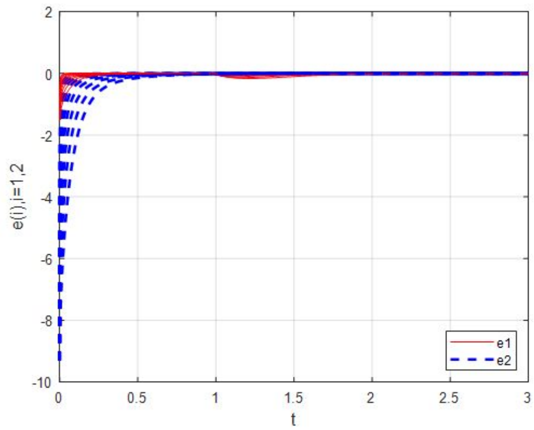

Figure 13. Synchronization errors of the systems (4.1) and (4.2) with different fractional orders and $\beta = 3$.

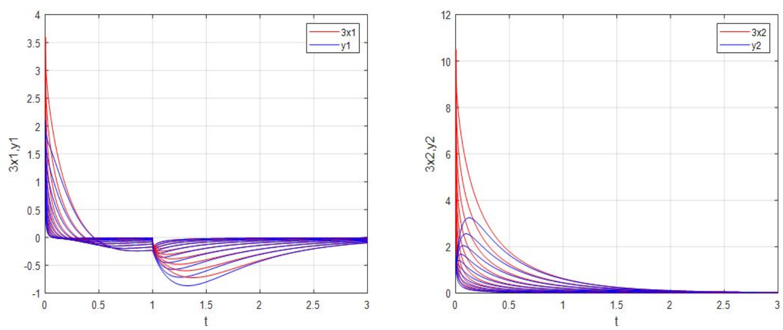

Figure 14. Evolutions of the master-slave systems (4.1) and (4.2) with different fractional orders and $\beta = 3$.

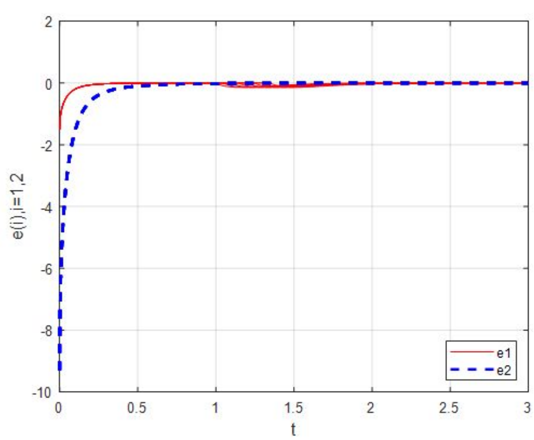

Figure 15. Synchronization errors of the systems (4.1) and (4.2) with different delays and $\beta = 3$.

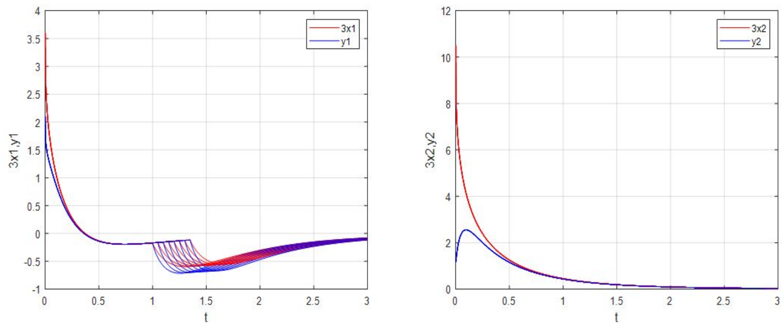

Figure 16. Evolutions of the master-slave systems (4.1) and (4.2) with different delays and $\beta = 3$.

As can be seen from the **Figures 5-16**, when different initial values, fractional orders, delays and projective coefficients are selected, the final error curves of the master-slave systems (4.1) and (4.2) tend to zero according to the **Controller** (3.3). That is to say, based on the **Controller** (3.3), the initial value, fractional order, delay and projection coefficient have no effect on the result of synchronization of master-slave systems (4.1) and (4.2), but have an effect on the synchronization process and synchronization time.

5. Conclusion

This article deals with the projective synchronization problem of FFNNs with variable coefficients and Caputo derivatives. The introduction of variable coefficients into existing neural network models brings two challenges to our research work. First, the construction of a new appropriate controller; The second is the construction of a new Lyapunov function satisfying the Razumikhin-type stability theorems for functional fractional-order differential systems. Based on the properties of delay fractional-order differential inequalities and constructing the Lyapunov function some effective criteria are derived to ensure the global projective synchronization of the master-slave systems obtained the addressed FFNNs. In addition, the correctness of the theoretical results is verified by numerical simulation. It should be pointed out that there is no research on projection synchronization of FFNNs with variable coefficients in the existing literature. And the anti-synchronization, complete synchronization of the addressed FFNNs are some special cases of results obtained in this article. Therefore, the results in this article are more meaningful than those in the literature [9,10,19,35,47]. Synchronization problem of more general FFNNs will be our future research topic. In future work, the new synchronous control method will also be extended to fuzzy systems [7], singular time-delay systems [5], uncertain stochastic memory systems [18,29], and others.

Acknowledgments

This work is supported by the Yunnan Provincial Key Program of Curriculum Ideology and Politics (Research on the ideological and political teaching system of college mathematics curriculum integrated with the educational concept of “San Quan Yu

Ren"), the General Projects of Local Science and Technology Development Funds Guided by the Central Government (No. 2022ZYD0005), the Key Project of Scientific Research and Innovation Team of Chengdu University of Information Technology (No. KYTD202226), the National Natural Science Foundation of China (Nos. 11901063, 12101090), the Scientific Research Fund of Sichuan Provincial Science & Technology Department of China (Nos. 2021YFG0133, 2022YFS0565, 2023NSFSC0071), and the Scientific Research Fund of Chengdu University of Information Technology of China (No. KYTZ201820).

Data Availability. No data were used to support this study.

Conflicts of Interest. The author declares that there is no conflict of interest regarding the publication of this paper.

Author Contributions. L. Jia, Z. Lei, C. Wang, Y. Zhou, T. Jiang, Y. Du and Q. Zhang contributed equally to each part of this work. All authors have read and agreed to the published version of the manuscript.

References

[1] H. Bao and J. Cao, *Projective synchronization of fractional-order memristor-based neural networks,* Neural Networks, 2015, 63, 1–9.

[2] S. Baluni, V. K. Yadav and S. Das, *Quasi projective synchronization of time varying delayed complex valued Cohen-Grossberg neural networks,* Information Sciences, 2022, 612, 231–240.

[3] B. Chen and J. Chen, *Razumikhin-type stability theorems for functional fractional-order differential systems and applications,* Applied Mathematics and Computation, 2015, 254, 63–69.

[4] C. Chen and Z. Ding, *Projective synchronization of nonidentical fractional-Order memristive neural networks,* Discrete Dynamics in Nature and Society, 2019, 2019. Article ID: 8743482.

[5] J. Chen, C. Lin, et al., *Mixed H_∞ and passive control for singular systems with time delay via static output feedback,* Applied Mathematics and Computation, 2017, 293, 244–253.

[6] J. Chen, L. Jiao, et al., *Projective synchronization with different scale factors in a driven-response complex network and its application in image encryption,* Nonlinear Analysis-Real World Applications, 2010, 11, 3045–3058.

[7] X. Du, H. Zhao and X. Chang, *Unknown input observer design for fuzzy systems with uncertainties,* Applied Mathematics and Computation, 2015, 266, 108–118.

[8] W. Guo, *Lag synchronization of complex networks via pinning control,* Nonlinear Analysis-Real World Applications, 2011, 12, 2579–2585.

[9] Y. Gu, Y. Yu and H. Wang, *Projective synchronization for fractional-order memristor-based neural networks with time delays,* Neural Computing and Applications, 2019, 31, 6039–6054.

[10] R. Guo, W. Lv and Z. Zhang, *Quasi-projective synchronization of stochastic complex-valued neural networks with time-varying delay and mismatched parameters,* Neurocomputing, 2020, 415, 184–192.

[11] M. Hu, Y. Yang and Z. Xu, *Impulsive control of projective synchronization in chaotic systems,* Physics Letters A, 2008, 372, 3228–3233.

[12] M. Hu, Z. Xu and Y. Yang, *Projective cluster synchronization in drive-response dynamical networks,* Physica A: Statistical Mechanics and its Applications, 2008, 387, 3759–3768.

[13] A. Hu, J. Cao, et al., *Cluster synchronization in directed networks of non-identical systems with noises via random pinning control,* Physica A: Statistical Mechanics and Its Applications, 2014, 395, 537–548.

[14] Y. Huang, S. Chen, et al., *Coexistence and local Mittag-Leffler stability of fractional-order recurrent neural networks with discontinuous activation functions,* Chinese Physics B, 2019, 28. Article ID: 040701.

[15] J. He, F. Chen, et al., *Global adaptive matrix-projective synchronization of delayed fractional-order competitive neural network with different time scales,* Neural Computing and Applications, 2020, 32, 12813–2826.

[16] W. Huang, Q. Song and Z. Zhao, *Robust stability for a class of fractional-order complexvalued projective neural networks with neutral-type delays and uncertain parameters,* Neurocomputing, 2021, 450, 399–410.

[17] T. Jiang, Q. Zhang, et al., *Multi-type object tracking based on residual neural network model,* Symmetry, 2022, 14. Article ID: 1689.

[18] J. Luo, W. Tian, et al., *Non-fragile asynchronous H_∞ control for uncertain stochastic memory systems with Bernoulli distribution,* Applied Mathematics and Computation, 2017, 312, 109–128.

[19] M. Li, X. Yang, et al., *Robust asymptotic stability and projective synchronization of time-varying delayed fractional neural networks under parametric uncertainty,* Neural Processing Letters, 2022. https://doi.org/10.1007/s11063-022-10825-6.

[20] P. Liu, M. Kong and Z. Zeng, *Projective synchronization analysis of fractional-order neural networks with mixed time delays,* IEEE Transactions on Cybernetics, 2022, 52, 6798–6808.

[21] B. B. Mandelbrot, *How long is the coast of Britain?* Statistical Self-Similarity and Fractional Dimension. Science, 1967, 156(3775), 636–638.

[22] B. B. Mandelbrot, *The Fractal Geometry of Nature,* New York: W. H. Freeman and Company, 1982.

[23] L. Pan, W. Zhou, et al., *Synchronization and anti-synchronization of new uncertain fractional-order modified unified chaotic systems,* Communications in Nonlinear Science and Numerical Simulation, 2010, 15, 3754–3762

[24] I. Podlubny, *Fractional differential equations,* San Diego: Academic Press, 1999.

[25] S. Qureshi and S. Aziz, *Fractional modeling for a chemical kinetic reaction in a batch reactor via nonlocal operator with power law kernel,* Physica A: Statistical Mechanics and its Applications, 2019, 542. Article ID: 123494.

[26] A. Svecic, D. Roberge and S. Kadoury, *Prediction of inter-fractional radiotherapy dose plans with domain translation in spatiotemporal embeddings,* Cancer Research Journal, 2020, 64. Article ID: 101728.

[27] S. Samarasinghe, *Neural networks for applied sciences and engineering: from fundamentals to complex pattern recognition,* New York: Auerbach Publications, Taylor & Francis Group, 2016.

[28] A. Singh and J. N. Rai, *Stability analysis of fractional order fuzzy cellular neural networks with leakage delay and time varying delays,* Chinese Journal of Physics, 2021, 73, 589–599.

[29] R. Tang, H. Su, et al., *Finite-time synchronization of Markovian coupled neural networks with delays via intermittent quantized control: linear programming approach,* IEEE Transactions on Neural Networks and Learning Systems, 2022, 33, 5268–5278.

[30] S. Wen, S. Chen and W. Guo, *Adaptive global synchronization of a general complex dynamical network with non-delayed and delayed coupling,* Physics Letters A, 2008, 372, 6340–6346.

[31] C. Wang, Q. Yang, et al., *Synchronization analysis of a class of neural networks with multiple time delays,* Journal of Mathematics, 2021, 2021. Article ID: 5573619.

[32] X. Wang and Z. Hu, *Projective synchronization of fractional order chaotic system based on state observer,* International Journal of Modern Physics B, 2012, 26, 304–968.

[33] X. Wu and H. Lu, *Generalized function projective (lag, anticipated and complete) synchronization between two different complex networks with nonidentical nodes,* Communications in Nonlinear Science and Numerical Simulation, 2012, 17, 3005–3021.

[34] S. Wang, H. Zhang, et al., *Finite-time projective synchronization of Caputo type fractional complex-valued delayed neural networks,* Mathematics, 2021, 9. Article ID: 1406.

[35] C. Wang, Q. Yang, et al., *Synchronization analysis of a fractional-order non-autonomous neural network with time delay,* Physica A: Statistical Mechanics and Its Applications, 2020, 549. Article ID: 124176.

[36] H. Wu, L. Wang, et al., *Global Mittag-Leffler projective synchronization for fractional-order neural networks: an LMI-based approach,* Advances in Differences Equations, 2016, 2016. Article ID:132.

[37] J. Xiao, Z. Wang, et al., *Adaptive pinning control for the projective synchronization of drive-response dynamical networks,* Applied Mathematics and Computation, 2012, 219, 2780–2788.

[38] X. Yang, C. Li, et al., *Quasi-uniform synchronization of fractional- order memristor-based neural networks with delay,* Neurocomputing, 2017, 234, 205–215.

[39] J. Yu, C. Hu, et al., *Projective synchronization for fractional neural networks,* Neural Networks, 2014, 49, 87–95.

[40] X. Yang, C. Li, et al., *Synchronization of fractional-order memristor-based complex-valued neural networks with uncertain parameters and time delays,* Chaos, Solitons and Fractals, 2018, 110, 105–123.

[41] H. You, Y. Shen, et al., *Optimal control and parameters design for the fractional-order vehicle suspension system,* Journal of Low Frequency Noise, Vibration and Active Control, 2018, 37, 456–467.

[42] X. Yang, Y. Liu, et al., *Synchronization of coupled time-delay neural networks with mode-dependent average dwell time switching,* IEEE Transactions on Neural Networks and Learning Systems, 2020, 31, 5483–5496.

[43] Q. Zhu, W. Zhou, et al., *Mode-dependent projective synchronization for neutral-type neural networks with distributed time delays,* Neurocomputing, 2014, 140, 97–103.

[44] H. Zhang, R. Ye, et al., *Lyapunov functional approach to stability analysis of Riemann-Liouville fractional neural networks with time-varying delays,* Asian Journal of Control, 2018, 20, 1938–1951.

[45] Z. Zhang, Y. Zhang, et al., *Global asymptotical stability analysis for fractional neural networks with time-varying delays,* Mathematics, 2019, 7. Article ID: 138.

[46] S. Zheng, *Pinning and impulsive synchronization control of complex dynamical networks with non-derivative and derivative coupling,* Journal of the Franklin Institute-Engineering and Applied Mathematics, 2017, 354, 6341–6363.

[47] W. Zhang, J. Cao, et al., *Projective synchronization of fractional-order delayed neural networks based on the comparison principle,* Advances in Difference Equations, 2018, 2018. Article ID: 73.