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KAM THEOREM AND ISO-ENERGETIC KAM THEOREM ON POISSON MANIFOLD

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Abstract In present paper, we give KAM theorem and iso-energetic KAM theorem for Hamiltonian system on $n-$dimensional Poisson manifold (M,Π) with $rank\ \Pi = 2r$ everywhere, where Π is given a bivector field, $2r < n$.

Keywords KAM theorem, Iso-energetic KAM theorem, Poisson manifold.

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1. Introduction

The present paper concerns KAM theorem and iso-energetic KAM theorem on n-dimensional Poisson Manifold (M,Π), where $rank\ M = 2r$ everywhere, $2r \leq n$, Π is a given bivector field.

On symplectic manifold, i.e., $2r = n$, the celebrated KAM theory duo to Kolmogorov ([19]), Arnold ([1]) and Moser ([29]) asserts the persistence of Lagrangian invariant tori for nearly integrable system, which answers certain stability of the planetary systems.

It is well known that a symplectic manifold is a smooth manifold with a non-degenerate closed differential 2-form. When $2r < n$, the 2-form on symplectic manifold is degenerate. A general manifold, i.e., Poisson manifold, need to be considered. For an n-dimensional Poisson manifold (M,Π), in local coordinate, Hamilton's equations take the form

$$\frac{dx}{dt} = J(x)\nabla H(x),$$

where $x \in M$, $J(x) = (J_{i,j} = \{x_i, x_j\})_{n\times n}$ is called Poisson structure matrix. The rank of Poisson manifold (M,Π) at x equals the rank of the structure matrix $J(x)$, which is independent of the choice of coordinate. Since the Poisson bracket is skew-symmetrical, i.e. for F, $H \in M$, $\{F,H\} = -\{H,F\}$, the rank of a Poisson manifold at any point is always an even integer, which is the reason why consider a Poisson manifold with rank $2r$. Defined the Hamiltonian vector field by $\mathcal{X}_H := \{\cdot, H\}$, where $H \in C^\infty(M)$. Let us retrospect some basic definitions [20,32].

Definition 1.1. A Hamiltonian sysytem $H(x)$ on n-dimension Poisson manifold (M,Π) with rank $2r$ everywhere is integrable, if there are functions f_1, $\cdots$, f_{l-1} satisfied

(1) f_1, $\cdots$, f_{l-1}, $H(x)$ are independent;

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(2) $f_1, \cdots, f_{l-1}, H(x)$ are in involution (pairwise);

(3) $r + l = n$.

Definition 1.2. Let $\mathcal{X}_{f_1}, \cdots, \mathcal{X}_{f_l}$ be vector fields on Poisson manifold M. An integral submanifold of $\{\mathcal{X}_{f_1}, \cdots, \mathcal{X}_{f_l}\}$ is a manifold $N \subset M$ whose tangent space $TN|_x$ is spanned by the vectors $\{\mathcal{X}_{f_1}|_x, \cdots, \mathcal{X}_{f_l}|_x\}$ for each $x \in N$.

Assume the integral manifold $\mathcal{F}_x$ of $\mathcal{X}_{f_1}, \cdots, \mathcal{X}_{f_l}$, passing through x, is compact. Then, in the neighbourhood of $\mathcal{F}_x$, the Poisson structure and integrable Hamiltonian have simple form, which could be stated as follow [20]:

Lemma 1.1. *Let (M, Π) an n-dimensional Poisson manifold with rank $\Pi = 2r$ everywhere. Denote $F = (f_1, \cdots, f_l)$ an integrable system on (M, Π). Assume that, for $x \in M$,*

(1) $d_x f_1 \wedge \cdots \wedge d_x f_l \neq 0$,

(2) The integral manifold $\mathcal{F}_x$ of $\mathcal{X}_{f_1}, \cdots, \mathcal{X}_{f_l}$, passing through x, is compact.

Then, in a neighborhood U of $\mathcal{F}_x$, there is a coordinate transformation $\phi : (\theta_1, \cdots, \theta_r, I_1, \cdots, I_l) \mapsto x$ such that in new coordinate

(1) U is diffeomorphism $T^r \times B^l$;

(2) Poisson structure is $\Pi = \sum\limits_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \sigma_i}$;

(3) Hamiltonians $f_1, \cdots, f_l$ only depend on coordinates $I_1, \cdots, I_l$.

Remark 1.1. Here, we call $I = (I_1, \cdots, I_l)$ and $\theta = (\theta_1, \cdots, \theta_r)$ action coordinate and angle coordinate, respectively. According to definition 1.1, the number of action is bigger than the angle.

Remark 1.2. In action-angle coordinate, the motion equation of integrable Hamiltonian system on (M, Π) could be written as follows:

$$
\begin{aligned}
\begin{pmatrix} \frac{d\theta}{dt} \\ \frac{dI}{dt} \end{pmatrix} &= \begin{pmatrix} 0_{r\times r} & I_{r\times r} & 0_{r\times(n-2r)} \\ -I_{r\times r} & 0_{r\times r} & 0_{r\times(n-2r)} \\ 0_{(n-2r)\times r} & 0_{(n-2r)\times r} & 0_{(n-2r)\times(n-2r)} \end{pmatrix} \begin{pmatrix} \frac{\partial H(I)}{\partial \theta} \\ \frac{\partial H(I)}{\partial I} \end{pmatrix} \\
&= \begin{pmatrix} \omega \\ 0_{l\times 1} \end{pmatrix},
\end{aligned}
$$

where ω is the first r components of $\frac{\partial H(I)}{\partial I}$. According to Weyl Theorem, if $\omega = (\omega_1, \cdots, \omega_r)$ is rationally independent, the phase trajectories are everywhere dense on tori.

The integral submanifolds of Hamiltonian vector fields $\mathcal{X}_{f_1}, \cdots, \mathcal{X}_{f_l}$, are leaves of a foliation. Denote $\mathcal{F}_x$ the leaf through $x \in M$, which is an invariant manifold. When $\mathcal{F}_x$ is compact, it is diffeomorphic to an r-dimensional torus, which is called standard Liouville torus.

The study about the influence of small Hamiltonian perturbations on an integrable Hamiltonian system was called by Poincaré the basic problem of the dynamics ([2, 45]). At present Poincaré's basic problem of the dynamics continues to occupy one of the most important places in the theory of dynamical systems ([45]). Then, for Hamiltonian system on Poisson manifold, what is the influence of small perturbation on the foliation of integrable Hamiltonian?

Consider a nearly integrable Hamiltonian system on (M, Π), a n-dimensional Poisson manifold with rank $\Pi = 2r$ everywhere,

$$H(x) = H_1(x) + \varepsilon P(x), \tag{1.1}$$

where $H_1(x)$ is integrable, $P(x)$ is non-integrable, $x \in M$, ε is a small parameter. According the Lemma 1.1, in the neighborhood of integrable manifold $\mathcal{F}_x$, passing through x, there is a transformation $\phi : (I, \theta) \to x$ such that (1.1) is changed to

$$H \circ \phi = H(I, \theta) = H_1(I) + \varepsilon P(I, \theta), \tag{1.2}$$

where $\theta = (\theta_1, \cdots, \theta_r) \in T^r$, $p = (p_1, \cdots, p_{n-r}) \in G \subset B^{n-r}$, G is a bounded closed region.

To state our main results, we make the following assumptions.

(R) There exists an $N > 1$ such that

$$rank\{\partial_I^\alpha \omega : 0 \leq |\alpha| \leq N, \quad \forall I \in G\} = r,$$

where ω is the first r components of $\frac{\partial H_1(I)}{\partial I}$.

(K) Assume

$$\det \frac{\partial^2 H_1}{\partial I^2}(I) \neq 0, \forall I \in G.$$

(Iso) Assume

$$\det \begin{pmatrix} \frac{\partial^2 H_1}{\partial I^2} & \frac{\partial H_1}{\partial I} \\ (\frac{\partial H_1}{\partial I})^T & 0 \end{pmatrix} \neq 0, \forall I \in G.$$

Our main results can be stated as follows.

Theorem 1.1. *Consider nearly integrable Hamiltonian (1.1) on (M, Π), n-dimensional Poisson manifold with rank $\Pi = 2r$ everywhere.*

1). *Assume* (**R**) *hold. Then there exist a $\Delta_0 > 0$ and a family of Cantor sets $G_\varepsilon \subset G$, $0 < \varepsilon \leq \Delta_0$, such that for any $I \in G_\varepsilon$ the unperturbed torus T_I persists and gives rise to an analytic, Diophantine, invariant $r-$torus of the perturbed system with small perturbed frequency $\omega_\varepsilon(I)$. Moreover, the Lebesgue measure $|G \setminus G_\varepsilon| \to 0$ as $\varepsilon \to 0$.*

2). *Assume* (**R**) *and* (**K**) *hold on G. Then there exist a $\Delta_0 > 0$ and a family of Cantor sets $G_\varepsilon \subset G$, $0 < \varepsilon \leq \Delta_0$, such that for any $I \in G_\varepsilon$ the unperturbed Diophantine tori will persist and give rise to perturbed tori preserving corresponding unperturbed toral frequencies.*

3). *Let $\Sigma = \{I : H_1(I) = c\}$ be a given energy surface. Assume* (**Iso**) *and* (**R**) *on Σ. Then there exist a $\Delta_0 > 0$ and a family of Cantor sets $\Sigma_\varepsilon \subset \Sigma$, $0 < \varepsilon \leq \Delta_0$, such that for any $I \in \Sigma_\varepsilon$ the unperturbed Diophantine tori will persist and give rise to perturbed tori keeping the same energy and maintaining the frequency ratio. Moreover, $|\Sigma \setminus \Sigma_\varepsilon| \to 0$ as $\varepsilon \to 0$.*

Remark 1.3. A torus is called Diophantine torus, if, for some constants $c > 0$ and $\tau > 0$, the frequency $\hat{\omega}$ on torus satisfy that $|\langle k, \hat{\omega} \rangle| > \frac{1}{c|k|^\tau}$ for all integer vectors $k \neq 0$.

Remark 1.4. According to the proof, present paper only showes the preservation of frequency ratio on a given high energy surface.

Remark 1.5. The first part of main results implies that there exists foliation for nearly integrable Hamiltonian on n-dimensional Poisson manifold with rank $2r$.

Remark 1.6. For nearly integrable Hamiltonian on n-dimensional Poisson manifold with rank $2r$, the second part of main results tell us not only the existence of foliation, but also the preservation of frequency between two leaves.

Remark 1.7. The third part of main results tell us the existence of foliation on a given high energy surface and the preservation of frequency ratio between two leaves.

The paper is organized as follows. In section 2, for an abstract Hamiltonian system on an n-dimensional Poisson manifold (M, Π) with rank $\Pi = 2r$ everywhere, we give a KAM theorem and a iso-energetical KAM theorem. In section 3, we prove our main results using the theorem in section 2.

2. Abstract Hamiltonian

Consider a family of real analytic Hamiltonian systems with the following action-angle form:

$$H(y, x) = H_1(y) + \varepsilon P(y, x), \tag{2.1}$$

where $(x, y) \in U$, U is a complex neighbourhood $\{(x, y) : |Imx| < r, dist(y, G) < \beta\}$ of $T^r \times G$, $G \subset R^l$ is a bounded closed region and ε is a small parameter.

When $r = l$, the celebrated KAM theory asserts the persistence of Diophantine tori under Kolmogorov nondegenerate condition, i.e. $\det \partial_I^2 H_1 \neq 0$. For further study about the persistence of lower dimensional invariant tori, see [8,16–18,26,27,30,34–36,39,41,50], especially, for resonant invariant tori, see [7,10,23,24,37,44]. For KAM theorem on multiscale Hamiltonian system, refer to [11,21,38,40,46–48,51].

For a long time, one has been trying to establish the KAM type results for nonsymmetric Hamiltonian systems, i.e. $l \neq r$. When $l < r$ and $l + r$ is even, the system is co-isotropic one, for which we refer the reader to [6,12,13,31,33,49]. When $l+n$ is odd, which is a challenge problem ([25,28,42]), [22] studied the persistence of invariant tori under the background of Poisson manifold, which could be applied to the perturbation of three-dimensional incompressible fluid flows ([3,28,32]). Also see [4,9]. For KAM theory on atropic tori, refer to [14,15,42,43].

Consider a parameterized Hamiltonian system of the following form:

$$\begin{aligned} \mathcal{H} &= \mathcal{N}(y, \xi) + \varepsilon \mathcal{P}(x, y, \xi), \\ \mathcal{N} &= e(\xi) + \langle \Omega(\xi), y \rangle + \langle y, A(\xi) y \rangle + \sum_{j=3}^{m} h_j(\xi) y^j, \end{aligned} \tag{2.2}$$

defined on $D(r, s) = \{(x, y) : |Im\ x| < r,\ |y| < s\}$, a $(r, s)-$complex neighborhood of $T^r \times \{0\} \subset T^r \times R^l$, $l > r$, where $\mathcal{P} = \varepsilon P(x, y, \xi)$, $\xi \in \Lambda = \{\xi : |\xi| \leq \delta_1\} \subset R^d$ and ε defined as above. Denote $\bar{\Lambda} = \{\xi \in C^d : |\xi - \Lambda| \leq \bar{\eta}\}$.

To state the results for (2.2), we need the following assumptions.

(A1) There is an $N > 1$ such that

$$rank\{\partial_\xi^\alpha \omega : 0 \leq |\alpha| \leq N\} = r,$$

where ω is the first r components of Ω.

(A2) Assume

$$\det(A + \partial_y^2 \sum_{j=3}^{m} h_j(\xi) y^j) \neq 0, \forall \xi \in \Lambda.$$

(A3) Assume

$$\det \begin{pmatrix} A + \partial_y^2 \sum\limits_{j=3}^{m} h_j(\xi) y^j & \Omega(\xi) \\ \Omega^T(\xi) & 0 \end{pmatrix} \neq 0, \forall \xi \in \Lambda.$$

Our results for (2.2) state as follows.

Theorem 2.1. *Consider Hamiltonian* (2.2) *on a* $(l+r)$*-dimensional Poisson manifold* (M, Π) *with Poisson structure matrix*

$$J = \begin{pmatrix} 0_{r\times r} & I_{r\times r} & 0_{r\times(l-r)} \\ -I_{r\times r} & 0_{r\times r} & 0_{r\times(l-r)} \\ 0_{(l-r)\times r} & 0_{(l-r)\times r} & 0_{(l-r)\times(l-r)} \end{pmatrix}.$$

1). *Assume* (**A1**). *Then there exist a* $\Delta_0 > 0$ *and a family of Cantor sets* $\Lambda_\varepsilon \subset \Lambda$, $0 < \varepsilon \leq \Delta_0$, *such that for any* $\xi \in \Lambda_\varepsilon$ *the unperturbed torus* T_ξ *persists and gives rise to an analytic, Diophantine, invariant* $r-$*torus of the perturbed system with small perturbed frequency* $\omega_\varepsilon(\xi)$. *Moreover, the Lebesgue measure* $|\Lambda \setminus \Lambda_\varepsilon| \to 0$ *as* $\varepsilon \to 0$.

2). *Assume* (**A1**) *and* (**A2**) *on* Λ. *Then there exist a* $\Delta_0 > 0$ *and a family of Cantor sets* $\Lambda_\varepsilon \subset \Lambda$, $0 < \varepsilon \leq \Delta_0$, *such that for any* $\xi \in \Lambda_\varepsilon$ *the unperturbed Diophantine tori will persist and give rise to perturbed tori which preserve corresponding unperturbed toral frequency. Moreover, the Lebesgue measure* $|\Lambda \setminus \Lambda_\varepsilon| \to 0$ *as* $\varepsilon \to 0$.

3). *Let* $\Sigma = \{\xi : N = c\}$ *be a given energy surface. Assume* (**A1**) *and* (**A3**) *on* Σ. *Then there exist a* $\Delta_0 > 0$ *and a family of Cantor sets* $\Sigma_\varepsilon \subset \Sigma$, $0 < \varepsilon \leq \Delta_0$, *such that for any* $\xi \in \Sigma_\varepsilon$ *the unperturbed Diophantine tori on* Σ_ε *will persist and give rise to perturbed tori keeping the same energy and maintaining the frequency ratio. Moreover,* $|\Sigma \setminus \Sigma_\varepsilon| \to 0$ *as* $\varepsilon \to 0$.

Remark 2.1. Here we consider the case $l > r$, since we will use this theorem to prove the persistence of invariant tori for nearly Hamiltonian system on Poisson manifold, where $l > r$. Combining the proof of [22], we believe those results for the case $l < r$ also hold.

Remark 2.2. Different from [22], we not only study the existence of invariant tori for nearly integrable Hamiltonian system, but also study the persistence of invariant tori with the same frequency and the persistence of invariant tori with proportionable frequency on a given energy surface.

Remark 2.3. Denote $\acute{A}$ the $r\times r$- principal minor of $A+\partial_y^2\sum_{j=3}^{m} h_j(\xi)y^j$. According to the proof of Lemma 2.1, $(A2)$ could be reduced to:

(A2') $\det \acute{A}\neq 0, \forall \xi\in\Lambda$.

Remark 2.4. According to the proof of Lemma 2.1, $(A3)$ could be reduced to:

(A3') $\det\begin{pmatrix}\acute{A} & \omega\\ \omega^T & 0\end{pmatrix}\neq 0, \forall\xi\in\Lambda.$

2.1. KAM steps

Throughout the paper, unless specified explanation, we shall use the same symbol $|\cdot|$ to denote an equivalent (finite dimensional) vector norm and its induced matrix norm, absolute value of functions, and measure of sets, etc., and denote by $|\cdot|_D$ the supremum norm of functions on a domain D. Also, for any two complex column vectors ξ,ζ of the same dimension, $\langle\xi,\zeta\rangle$ always means $\xi^T\zeta$, i.e. the transpose of ξ times ζ. For the sake of brevity, we shall not specify smoothness orders for functions having obvious orders of smoothness indicated by their derivatives taking.

Let us begin with system (2.2) by regarding it as a Hamiltonian of 0−step, and rewriting it as follows:

$$\begin{aligned}\mathcal{H}_0 &= \mathcal{N}_0(y,\xi)+\varepsilon\mathcal{P}_0(x,y,\xi),\\ \mathcal{N}_0 &= e_0(\xi)+\langle\Omega_0(\xi),y\rangle+h_0(y,\xi),\\ h_0 &= \langle y,A_0(\xi)y\rangle+\hat{h}_0,\\ \hat{h}_0 &= \sum_{j=3}^{m} h_j(\xi)y^j,\end{aligned}\tag{2.3}$$

defined on $D(r_0,s_0)=\{(x,y): |Im\ x|<r_0,\ |y|<s_0\}$, a $(r_0,s_0)-$complex neighborhood of $T^n\times\{0\}\subset T^n\times R^l$, where $\mathcal{P}_0=\varepsilon P(x,y,\xi)$, $\xi\in\Lambda_0\subset R^d$. Moreover, let $\gamma_0=\varepsilon^{\frac{1}{12(7+N)}}$, $s_0=\varepsilon^{\frac{2}{3m}}$, $\mu_0=\varepsilon^{\frac{1}{4}}$, $\bar{\eta}_0=\varepsilon^{\frac{3(1-b)-4(b+\sigma)}{14N}}$, where b and σ are constants to be determined next. Then by Cauchy estimate we have

$$|\partial_\xi^q\mathcal{P}_0|_{D(r_0,s_0)\times\bar{\Lambda}_0}<\frac{\gamma_0^{N+7}s_0^m\mu_0}{\bar{\eta}_0^N},\qquad |q|\le N.$$

Next, we will show the KAM iteration from $\nu-$step to $(\nu+1)-$step. For simplicity, we shall omit the index for all quantities of the $\nu-$th KAM step and use $'+'$ to index all quantities in the $(\nu+1)-$th KAM step. Suppose, at $\nu-$th step, we have obtained the following smooth family of real analytic Hamiltonians

$$\mathcal{H}(x,y,\xi)=\mathcal{N}(y,\xi)+\varepsilon\mathcal{P}(x,y,\xi),\tag{2.4}$$

$$\mathcal{N}(y,\xi)=e(\xi)+\langle\Omega(\xi),y\rangle+h(y,\xi),\tag{2.5}$$

$$h(y,\xi)=\langle y,A(\xi)y\rangle+\hat{h}(y,\xi),$$
$$\hat{h}(y,\xi)=\sum_{j=3}^{m}h_j(\xi)y^j,$$

where $(x,y)\in D(r,s)=\{(x,y):|Im\ x|<r,\ |y|<s\}$, a $(r,s)-$complex neighborhood of $T^r\times\{0\}\subset T^r\times R^l$, $\xi\in\Lambda\subset R^d$. Moreover,

$$|\partial_\xi^q\mathcal{P}|_{D(r,s)\times\bar{\Lambda}}\le\frac{\gamma^{N+7}s^m\mu}{\bar{\eta}^N},\qquad |q|\le N. \tag{2.6}$$

We need to construct a canonical transformation Φ_+, which, on a small phase domain $D(r_+,s_+)$ and a smaller parameter domain Λ_+, transforms (2.4) into a family of Hamiltonians with the following form

$$\mathcal{H}_+=\mathcal{H}\circ\Phi_+=\mathcal{N}_++\varepsilon\mathcal{P}_+$$

enjoying the similar properties to (2.4) but with a much smaller unintegrable perturbation $\mathcal{P}_+$.

All constants below, for simplicity, denoted by c, are positive and independent of the iteration process. Define

$$r_+=\delta r-d(1-\frac{\delta^2}{2})r_0,\qquad s_+=s^{1+b+\sigma},\qquad \gamma_+=\frac{\gamma_0}{4}+\frac{\gamma}{2},$$
$$K_+=([\log\frac{1}{s}]+1)^3,\quad D_+=D(s_+,r_+),\quad \tilde{D}=D(s_0,r_++\frac{5}{8}(r-r_+)),$$
$$D_i=D(is_+,r_++\frac{i-1}{8}(r-r_+)),\ i=1,\cdots,8,\quad \bar{\eta}_+=\bar{\eta}-\frac{\bar{\eta}_0}{2^{\nu+1}},$$

where b, σ, d are chosen so that $1<b\ll\sigma\ll1$, $0<d\ll1$, $2-m(b+\sigma)-\sigma>\frac{3}{2}$, $\delta(1+b+\sigma)>1$ and $\delta=1-d$. Hereafter, we let $\tau>\max\{0,r(r+1)-1,l(l+1)-1,(N+1)N-1\}$ be fixed.

2.1.1. Truncation

According to the Taylor-Fourier series, we get

$$\mathcal{P}(x,y)=\sum_{|k|\in Z^n,\ i\in Z_+^n}P_{ki}y^ie^{\sqrt{-1}\langle k,x\rangle}.$$

Let

$$\mathcal{R}=\sum_{|k|\le K_+,\ |i|\le m}P_{ki}y^ie^{\sqrt{-1}\langle k,x\rangle}. \tag{2.7}$$

Using Lemma 2.1 in [22], we have

$$|\partial_\xi^q(\mathcal{P}-\mathcal{R})|_{D_8\times\bar{\Lambda}}\le\frac{c\gamma^{N+7}\mu(s^{(m+1)(1+b+\sigma)}+\frac{s_+^{m+1}}{s})}{\bar{\eta}^N},$$

under assumptions

$$s_+\le\frac{s}{16}, \tag{2.8}$$
$$\int_{K_+}^{\infty}\lambda^{n+N}e^{-\frac{\lambda(r-r_+)}{16}}d\lambda\le s^{(m+1)(1+b+\sigma)}. \tag{2.9}$$

2.1.2. Elimination of Harmonic terms

The next aim is averaging out all harmonic terms of $\mathcal{R}$, i.e., all terms $P_{ki}y^i e^{\sqrt{-1}\langle k,x\rangle}$, $0<|k|\le K_+$, $|i|\le m$.

Let F be a Hamiltonian on Poisson manifold. Then on local action-angle coordinate, (x,y), we have

$$\{\cdot,F\}=\sum_{i=1}^{r}(\frac{\partial\cdot}{\partial x_i}\frac{\partial F}{\partial y_i}-\frac{\partial\cdot}{\partial y_i}\frac{\partial F}{\partial x_i}). \tag{2.10}$$

Denote ϕ_F^1 be the time-1 map of Hamiltonian F. According to [32], ϕ_F^1 preserves the Poisson structure. Write

$$[\mathcal{R}]=\frac{1}{(2\pi)^n}\int_{T^n}\mathcal{R}(x,y)dx. \tag{2.11}$$

Assume that there is a F such that

$$\{\mathcal{N},F\}+\varepsilon(\mathcal{R}-[\mathcal{R}])=0, \tag{2.12}$$

then

$$\bar{\mathcal{H}}_+=\mathcal{H}\circ\phi_F^1=(\mathcal{N}+\varepsilon\mathcal{R})\circ\phi_F^1+\varepsilon(\mathcal{P}-\mathcal{R})\circ\phi_F^1=\bar{\mathcal{N}}_++\bar{\mathcal{P}}_+,$$

where

$$\begin{aligned}
&\bar{\mathcal{P}}_+=\int_0^1\{\mathcal{R}_t,F\}\circ\phi_F^t dt+\varepsilon(\mathcal{P}-\mathcal{R})\circ\phi_F^1,\\
&\bar{\mathcal{N}}_+=\mathcal{N}+\varepsilon[\mathcal{R}],\\
&\mathcal{R}_t=(1-t)\{\mathcal{N},F\}+\varepsilon\mathcal{R}.
\end{aligned}\tag{2.13}$$

Therefore,

$$\bar{\mathcal{H}}_+=\tilde{e}_++\langle\tilde{\Omega}_+,y\rangle+\frac{1}{2}\langle y,\tilde{A}_+y\rangle+\tilde{\hat{h}}_+(y)+\bar{\mathcal{P}}_+,$$

where

$$\begin{aligned}
&\tilde{e}_+=e+P_{00},\quad \tilde{\Omega}_+=\Omega+P_{01},\\
&\tilde{A}_+=A+P_{02},\quad \tilde{\hat{h}}_+(y)=\hat{h}_+(y)+[\hat{R}](y),\\
&[\hat{R}](y)=[R]-P_{00}-\langle P_{01},y\rangle-\frac{1}{2}\langle y,P_{02}y\rangle.
\end{aligned}$$

2.1.3. Homological Equations

We are going to solve homological equations (2.12). Consider the Taylor-Fourier series of $F(x,y)$

$$F(x,y)=\sum_{0<|k|\le K_+,\ |i|\le m}f_{ki}y^i e^{\sqrt{-1}\langle k,x\rangle}. \tag{2.14}$$

Denote $\omega(\xi)$ and $\partial_{\hat{y}}h(y,\xi)$ are the first r components of $\Omega(\xi)$ and $\partial_y h(y,\xi)$, respectively. Combing formulas (2.5), (2.10) and (2.14), we have

$$\begin{aligned}\{N,F\} &= -\sum_{i=1}^{r}\frac{\partial N}{\partial y_i}\frac{\partial F}{\partial x_i}\\ &= -\sum_{\substack{0<|k|\le K_+,\\ |i|\le m}}\sum_{j=1}^{r}\sqrt{-1}k_j\frac{\partial(\Omega(\xi)+h(y,\xi))}{\partial y_j}f_{ki}y^i e^{\sqrt{-1}\langle k,x\rangle}\\ &= -\sum_{\substack{0<|k|\le K_+,\\ |i|\le m}}\sqrt{-1}\langle k,\tilde{\omega}(\xi)+\partial_{\hat{y}}h(y,\xi)\rangle f_{ki}y^i e^{\sqrt{-1}\langle k,x\rangle}. \end{aligned}\tag{2.15}$$

Putting (2.7), (2.11) and (2.15) into (2.12) and comparing coefficients of (2.12) on both sides, formally, we have

$$\sqrt{-1}\langle k,\omega(\xi)+\partial_{\hat{y}}h(y,\xi)\rangle f_{ki}=\varepsilon P_{ki}.\tag{2.16}$$

Denote

$$\begin{aligned}\Lambda_+ &= \{\xi\in\Lambda: |\langle k,\omega\rangle|>\frac{\gamma}{|k|^\tau},\ 0<|k|\le K_+\},\\ L_k &= \langle k,\omega(\xi)+\partial_{\hat{y}}h(y,\xi)\rangle.\end{aligned}$$

Then, on Λ_+,

$$\begin{aligned}|L_k| &= |\langle k,\omega(\xi)+\partial_{\hat{y}}h(y,\xi)\rangle|\\ &\ge \big||\langle k,\omega(\xi)\rangle|-|\langle k,\partial_{\hat{y}}h(y,\xi)\rangle|\big|\\ &\ge c\frac{\gamma}{|k|^\tau},\end{aligned}$$

supposed

$$s\cdot K_+^{\tau+1}=o(\gamma),\tag{2.17}$$

which means that (2.16) is solvable on Λ_+. Moreover, all solutions f_{ki}, $0<|k|\le K_+$, $|i|\le m$, are real analytic on $\Lambda_+\times D_8$.

Inductively,

$$\begin{aligned}|\partial_\xi^q\partial_y^j L_k^{-1}| &\le c|k|^{|j|+|q|-1}\frac{|k|^{\tau(|j|+|q|)}}{\gamma^{|j|+|q|}}\\ &\le \frac{|k|^{\tau(|j|+|q|)+|j|+|q|-1}}{\gamma^{|j|+|q|}}.\end{aligned}$$

Then

$$\begin{aligned}|\partial_\xi^q\partial_y^j f_{ki}| &\le \frac{c|k|^{\tau(|j|+|q|)+|j|+|q|-1}}{\gamma^{|j|+|q|}}\frac{\gamma^{N+7}s^{m-|i|}\mu}{\bar{\eta}^N}e^{-|k|r}\\ &\le c|k|^{\tau(|j|+|q|)+|j|+|q|-1}\frac{s^{m-|i|}\mu}{\bar{\eta}^N}e^{-|k|r}.\end{aligned}\tag{2.18}$$

Therefore,

$$|\partial_\xi^q \partial_y^j \partial_x^z F| \le \sum_{|j|\le 2,\ 0<|k|\le K_+} |k|^z |\partial_y^j(\partial_\xi^q f_{ki} y^i)| e^{|k|(r_+ + \frac{7}{8}(r-r_+))}$$
$$\le \frac{s^{m-|i|}\mu}{\bar\eta^N}\Gamma(r-r_+), \tag{2.19}$$

where $\Gamma(r-r_+) = \sum_{|j|\le 2,\ 0<|k|\le K_+} |k|^{\tau(|j|+|q|)+|j|+|q|-1+|z|} e^{-\frac{|k|(r-r_+)}{8}}$.

2.1.4. Preservation of Frequency or Frequency ratio.

Consider the transformation

$$\phi : x \to x,\ y \to y + y_*.$$

Then

$$\begin{aligned}\mathcal{H}_+ &= \bar{\mathcal{H}}_+ \circ \phi \\ &= e_+ + \langle \Omega_+, y\rangle + \frac{1}{2}\langle y, A_+ y\rangle + \hat h_+(y) + \mathcal{P}_+,\end{aligned}$$

where

$$\begin{aligned}
e_+ &= \tilde e_+ + \langle \tilde\Omega_+, y_*\rangle + \frac{1}{2}\langle y_*, \tilde A_+ y_*\rangle + \hat h(y_*) + [\hat R](y_*),\\
\Omega_+ &= \tilde\Omega_+ + \tilde A_+ y_* + \partial_y \hat h(y_*),\\
A_+ &= \tilde A_+ + \partial_y^2 \hat h(y_*) + \partial_y^2[\hat{\mathcal{R}}](y_*),\\
\hat h_+ &= \hat h(y+y_*) - \hat h(y_*) - \langle \partial_y \hat h(y_*), y\rangle - \frac{1}{2}\langle y, \partial_y^2 \hat h(y_*) y\rangle + [\hat R](y+y_*)\\
&\quad - [\hat R](y_*) - \langle \partial_y[\hat R](y_*), y\rangle - \frac{1}{2}\langle y, \partial_y^2[\hat R](y_*) y\rangle,\\
\mathcal{P}_+ &= \bar{\mathcal{P}}_+ \circ \phi + \langle \partial_y[\hat R](y_*), y\rangle.
\end{aligned}$$

Lemma 2.1. *Denote* $\hat A_0 = A + \partial_y^2 \hat h$.

(1). Assume that $\hat A_0$ is nonsingular. Then there is a y_* such that $\Omega_+ = \Omega$. Moreover, $|\partial_\xi^q y_*| \le \frac{c\gamma^{N+7} s^{m-1}\mu}{\bar\eta^N}$, $|q| \le N$.

(2). Assume that $\begin{pmatrix} \hat A_0 & \Omega^T \\ \Omega & 0 \end{pmatrix}$ is nonsingular. Then, on a given energy surface, $\{y, h(y) = \tilde e_+\}$, there is a y_* such that $\Omega_+ = t\Omega$. Moreover, $|\partial_\xi^q y_*| \le \frac{c\gamma^{N+7} s^{m-1}\mu}{\bar\eta^N}$, $|q| \le N$.

Proof. Consider the first part of this Lemma, first. Assume that there is a y_* such that

$$\tilde A_+(\xi) y_* + \partial_y \hat h(y_*) = -P_{01}, \tag{2.20}$$

which means the preservation of frequency during KAM step.

Denote $M_* = \max_{\xi\in\Lambda_0}|(A_0+\partial_y^2\hat{h})^{-1}|+1,\ M^* = \max_{\substack{|l|\le d,|j|\le m+5,\\ |y|\le s_0,\xi\in\Lambda_0}}|\partial_\xi^l\partial_y^j h_0(y,\xi)|$. We could make s_0 small such that

$$M_*(M^*+1)s_0 < \frac{1}{8}.$$

Denote

$$B(y,\xi) = \tilde{A}_+ + \int_0^1 \frac{\partial^2\hat{h}(\theta y,\xi)}{\partial y^2}d\theta.$$

Then (2.20) is changed to

$$B(y_*,\xi)y_* = -P_{01}. \tag{2.21}$$

Since $|\tilde{A}_+ - A_0|_\Lambda \le \mu_0^{\frac{1}{2}}$ (according to the definition of $\tilde{A}_+$) and $|\frac{\partial^2\hat{h}}{\partial y^2}|_{D(s)} \le (M^*+1)s$, we have

$$\begin{aligned}|\hat{A}_0 - B(y_*)| &\le |\hat{A}_0 - A_0| + |A_0-\tilde{A}_+| + |B(y_*)-\tilde{A}_+|\\ &\le s_0^2 + \mu_0^{\frac{1}{2}} + (M^*+1)s_0\\ &\le \frac{1}{2M_*}.\end{aligned}$$

Therefore, $B(y_*)$ is nonsingular and

$$|B^{-1}(y_*)| \le \frac{|\hat{A}_0^{-1}|}{1-|\hat{A}_0-B(y_*)||\hat{A}_0^{-1}|} \le 2M_*.$$

Hence,

$$|y_*| \le 2M_*|P_{01}| \le 2M_*\gamma^{N+7}s^{m-1}\mu.$$

Moreover, with Cauchy estimate, we have

$$|\partial_\xi^q y_*| \le \frac{c\gamma^{N+7}s^{m-1}\mu}{\bar{\eta}^N},$$

for $|q|\le N$.

Next, we consider the second part. Assume that there is a (y_*,t_*) such that

$$\begin{cases}\langle\tilde{\Omega}_+,y\rangle + \frac{1}{2}\langle y,\tilde{A}_+y\rangle + \hat{h}(y) + [\hat{R}](y) = 0,\\ \tilde{A}_+(\xi)y + \partial_y\hat{h}(y) + t\Omega(\xi) + P_{01} = 0,\end{cases} \tag{2.22}$$

which means the preservation of frequency ratio during KAM step on a given energy surface if $\langle\partial_y[\hat{R}](y_*),y\rangle$ is small enough that could be put into the new perturbation.

Denote $\breve{a} = \int_0^1 \tilde{A}_+\theta y_* + \partial_y\hat{h}(\theta y_*) + \partial_y[\hat{R}](\theta y_*)d\theta$. Rewrite (2.22) as follows:

$$\begin{pmatrix} B(y_*,\xi) & \tilde{\Omega}_+\\ \tilde{\Omega}_+ + \breve{a} & 0\end{pmatrix}\begin{pmatrix} y_*\\ t_*\end{pmatrix} = \begin{pmatrix} -P_{01}\\ 0\end{pmatrix}, \tag{2.23}$$

which implies that formally

$$\begin{pmatrix} y_* \\ t_* \end{pmatrix} = \begin{pmatrix} B(y_*,\xi) & \tilde{\Omega}_+ \\ \tilde{\Omega}_+ + \breve{a} & 0 \end{pmatrix}^{-1} \begin{pmatrix} -P_{01} \\ 0 \end{pmatrix}. \tag{2.24}$$

Since $\det \begin{pmatrix} A & x \\ y^T & a \end{pmatrix} = a \det A - y^T adj A x$, where A is an $n \times n$-matrix, x, $y \in R^n$, we have

$$\begin{aligned} \begin{pmatrix} B(y_*,\xi) & \tilde{\Omega}_+ \\ \tilde{\Omega}_+ + \breve{a} & 0 \end{pmatrix}^{-1} &= \frac{adj \begin{pmatrix} B(y_*,\xi) & \tilde{\Omega}_+ \\ \tilde{\Omega}_+ + \breve{a} & 0 \end{pmatrix}}{\det \begin{pmatrix} B(y_*,\xi) & \tilde{\Omega}_+ \\ \tilde{\Omega}_+ + \breve{a} & 0 \end{pmatrix}} \\ &= \frac{adj \begin{pmatrix} B(y_*,\xi) & \tilde{\Omega}_+ \\ (\tilde{\Omega}_+ + \breve{a})^T & 0 \end{pmatrix}}{(\tilde{\Omega}_+ + \breve{a})^T adj B(y_*,\xi) \tilde{\Omega}_+} \\ &= \frac{adj \begin{pmatrix} B(y_*,\xi) & \tilde{\Omega}_+ \\ (\tilde{\Omega}_+ + \breve{a})^T & 0 \end{pmatrix}}{(\Omega + P_{01} + \breve{a})^T adj B(y_*,\xi)(\Omega + P_{01})} \\ &= \frac{adj \begin{pmatrix} B(y_*,\xi) & \tilde{\Omega}_+ \\ (\tilde{\Omega}_+ + \breve{a})^T & 0 \end{pmatrix}}{\breve{b}}, \end{aligned}$$

where $\breve{b} = \Omega^T adj B(y_*,\xi)\Omega + \Omega^T adj B(y_*,\xi) P_{01} + (P_{01} + \breve{a}) adj B(y_*,\xi)\Omega + (P_{01} + \breve{a}) adj B(y_*,\xi) P_{01}$. Hence,

$$\begin{aligned} \left| \begin{pmatrix} y_* \\ t_* \end{pmatrix} \right| &\le \left(\frac{1}{\Omega^T adj B(y_*,\xi)\Omega} + O(\mu_0)\right) \left| adj \begin{pmatrix} B(y_*,\xi) & \tilde{\Omega}_+ \\ \tilde{\Omega}_+ + \breve{a} & 0 \end{pmatrix} \begin{pmatrix} -P_{01} \\ 0 \end{pmatrix} \right| \\ &\le \gamma^{N+7} s^{m-1} \mu. \end{aligned}$$

Moreover, for $|q| \le N$,

$$|\partial_\xi^q y_*| \le \frac{c\gamma^{N+7} s^{m-1} \mu}{\bar{\eta}^N}.$$

□

2.1.5. Estimate for New Hamiltonian

By the estimate of $|\partial_\xi^q y_*|$ together with definitions of e_+, Ω_+ and A_+, we have

$$|\partial_\xi^q (e_+ - e)| \le \frac{c\gamma^{N+7} s\mu}{\bar{\eta}^N},$$

$$|\partial_\xi^q(\Omega_+ - \Omega)| \le \frac{c\gamma^{N+7}s\mu}{\bar{\eta}^N},$$
$$|\partial_\xi^q(A_+ - A)| \le \frac{c\gamma^{N+7}\mu}{\bar{\eta}^N}.$$

Denote $\phi_{F_1}^t$, $\phi_{F_2}^t$ as the components of ϕ_F^t in x, y planes, respectively, and let X_F be the vector field defined by

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0_{r\times r} & I_{r\times r} & 0_{r\times(n-2r)} \\ -I_{r\times r} & 0_{r\times r} & 0_{r\times(n-2r)} \\ 0_{(n-2r)\times r} & 0_{(n-2r)\times r} & 0_{(n-2r)\times(n-2r)} \end{pmatrix} \begin{pmatrix} \frac{\partial F(x,y)}{\partial x} \\ \frac{\partial F(x,y)}{\partial y} \end{pmatrix}. \tag{2.25}$$

Then $\phi_F^t = id + \int_0^t X_F \circ \phi_F^\lambda d\lambda$. For any $(y,x) \in D_3$, we let $t_* = \sup\{t \in [0,1] : \phi_F^t(y,x) \in D_4\}$. Since $s_+ \le \frac{s}{16}$, we get $D_4 \subset D_*$, where $D_4 = D(4s_+, r_+ + \frac{3}{8}(r - r_+))$, $D_* = D(\frac{s}{2}, r_+ + \frac{6}{7}(r - r_+))$. Further, we have

$$\begin{aligned} |\phi_{F_1}^t(y,x)| &= |x| + |\int_0^t (I_{r\times r}, 0_{r\times(n-2r)})F_y \circ \phi_F^\lambda d\lambda| \\ &\le \frac{2}{8}(r - r_+) + \frac{c\mu\Gamma(r-r_+)}{\bar{\eta}^N} \le r_+ + \frac{3}{8}(r - r_+), \\ |\phi_{F_2}^t(y,x)| &= |y| + |\int_0^t \begin{pmatrix} -I_{r\times r} \\ 0_{(n-2r)\times r} \end{pmatrix} F_x \circ \phi_F^\lambda d\lambda| \\ &\le s_+ + \frac{cs^m\mu\Gamma(r-r_+)}{\bar{\eta}^N} \le 4s_+, \end{aligned}$$

supposed

$$\frac{c\mu\Gamma(r-r_+)}{\bar{\eta}^N} < \frac{1}{8}(r - r_+), \tag{2.26}$$

$$\frac{cs^2\mu\Gamma(r-r_+)}{\bar{\eta}^N} < 3s_+, \tag{2.27}$$

which implies that $\phi_F^t(y,x) \in D_4$ for all $0 \le t \le t_*$. Then $\phi_F^t : D_3 \to D_4$ for all $0 \le t \le 1$. Therefore, $\phi_F^1 : D_+ \to D(s,r)$. It follows that $|\phi_F^t - id|_{\tilde{D}} \le \frac{c\mu\Gamma(r-r_+)}{\bar{\eta}^N}$.

With Gronwall Inequality and

$$D\phi_F^t = Did + \int_0^t (I \cdot D^2F) \circ \phi_F^\lambda \cdot D\phi_F^\lambda d\lambda,$$

where $I = \begin{pmatrix} 0_{r\times r} & I_{r\times r} & 0_{r\times(n-2r)} \\ -I_{r\times r} & 0_{r\times r} & 0_{r\times(n-2r)} \\ 0_{(n-2r)\times r} & 0_{(n-2r)\times r} & 0_{(n-2r)\times(n-2r)} \end{pmatrix}$, we have

$$\begin{aligned} |D\phi_F^t - Did| &\le \int_0^t e^{\int_s^t |-I\cdot D^2F\circ\phi_F^r|dr}| - I \cdot D^2F \circ \phi_F^s|ds \\ &\le \frac{c\mu\Gamma(r-r_+)}{\bar{\eta}^N}. \end{aligned} \tag{2.28}$$

Lemma 2.2. *Assume*

$$\Delta_+ < \frac{\gamma_+^{N+7} s_+^m \mu_+}{\bar{\eta}_+^N}, \tag{2.29}$$

where

$$\Delta_+ = \frac{\gamma^{N+7} s^{2m-1} \mu^2 + \mu^2 s^{2m} \Gamma^2(r - r_+) + \gamma^{N+7} \mu (s^{(m+1)(1+b+\sigma)} + \frac{s_+^{m+1}}{s})}{\bar{\eta}^N}.$$

Then there is a constant c such that

$$|\partial_\xi^q P_+|_{D_+ \times \bar{\Lambda}_+} \leq c \frac{\gamma_+^{N+7} s_+^m \mu_+}{\bar{\eta}_+^N}.$$

Proof. According to the definition of $[\hat{R}]$ and Lemma 2.1, we deduce

$$|\partial_\xi^q \big(\langle \partial_y [\hat{R}](y_*), y \rangle\big)| \leq \frac{c \gamma^{N+7} s^{2m-1} \mu^2}{\bar{\eta}^N}.$$

Directly, combining (2.10), (2.13), (2.19) and Lemma 2.1,

$$|\partial_\xi^q \int_0^1 \{R_t, F\} \circ \phi_F^t dt \circ \phi| \leq \frac{c \mu^2 s^{2m} \Gamma^2 (r - r_+)}{\bar{\eta}^N}.$$

Besides

$$|\partial_\xi^q (\mathcal{P} - \mathcal{R}) \circ \phi_F^1 \circ \phi| \leq \frac{c \gamma^{N+7} \mu}{\bar{\eta}^N} (s^{(m+1)(1+b+\sigma)} + \frac{s_+^{m+1}}{s}),$$

finally,

$$|\partial_\xi^q P_+| \leq \Delta_+ \leq \frac{\gamma_+^{N+7} s_+^m \mu_+}{\bar{\eta}_+^N}.$$

□

2.2. Iteration Lemma

Next, we will give an Iteration Lemma which insures process of infinite KAM steps. Let r_0, s_0, γ_0, μ_0, Λ_0, $\mathcal{H}_0$, $\mathcal{N}_0$, e_0, Ω_0, $\mathcal{P}_0$, $\tilde{D}_0 = D(r_0, \beta_0)$, $D_0 = D(r_0, s_0)$ and $\Phi_0 = id$ be given as above. For $\nu = 1, 2, \cdots$, let

$$s_\nu = s_{\nu-1}^{1+b+\sigma}, \quad \mu_\nu = c_0 s_{\nu-1}^\sigma \mu_{\nu-1}, \quad \gamma_\nu = \gamma_0 (1 - \sum_{i=1}^{\nu} \frac{1}{2^{i+1}}), \quad K_\nu = ([\log \frac{1}{s_{\nu-1}}] + 1)^3,$$

$$\Delta_\nu = \gamma_{\nu-1}^{N+7} s_{\nu-1}^{2m-1} \mu_{\nu-1}^2 + \mu_{\nu-1}^2 s_{\nu-1}^{2m} \Gamma^2 (r_{\nu-1} - r_\nu) + \gamma_{\nu-1}^{N+7} \mu (s_{\nu-1}^{(m+1)(1+b+\sigma)} + \frac{s_\nu^{m+1}}{s_{\nu-1}}),$$

$$\Lambda_\nu = \{\xi \in \Lambda_{\nu-1} : |\langle k, \omega_{\nu-1}(\xi) \rangle| > \frac{\gamma_{\nu-1}}{|k|^\tau}, 0 < |k| \leq K_\nu\}, \qquad \bar{\eta}_\nu = \bar{\eta}_{\nu-1} - \frac{\bar{\eta}_0}{2^{\nu+1}},$$

$$D_\nu = D(r_\nu, s_\nu), \quad \tilde{D}_\nu = D(r_\nu + \frac{7}{8}(r_{\nu-1} - r_\nu), s_0),$$

where c_0 is the maximum among c mentioned above.

Lemma 2.3. *Assume $\mu_0 = \mu_0(\varepsilon)$ is sufficiently small. The following hold for all $\nu = 0, 1, \cdots$:*

(1)

$$|e_\nu - e_0|_{\Lambda_\nu},\ |\Omega_\nu - \Omega_0|_{\Lambda_\nu},\ |\omega_\nu - \omega_0|_{\Lambda_\nu},\ |h_\nu - h_0|_{\Lambda_\nu} \le 2\gamma_0^{N+7}\mu_*,$$
$$|e_{\nu+1} - e_\nu|_{\Lambda_\infty},\ |\Omega_{\nu+1} - \Omega_\nu|_{\Lambda_\infty},\ |\omega_{\nu+1} - \omega_\nu|_{\Lambda_\infty},\ |h_{\nu+1} - h_\nu|_{\Lambda_\infty} \le \frac{\gamma_0^{N+7}\mu_*}{2^{\nu+1}},$$
$$|\partial_\xi^q \mathcal{P}_\nu|_{D_\nu \times \bar{\Lambda}_\nu} \le \frac{\gamma_\nu^{N+7} s_\nu^2 \mu_\nu}{\bar{\eta}_\nu^N};$$

(2) $\Phi_{\nu+1} : \tilde{D}_{\nu+1} \times \Lambda_{\nu+1} \to \tilde{D}_\nu$ *is canonical and real analytic with respect to* $(y, x) \in \tilde{D}_{\nu+1}$, $\xi \in \Lambda_{\nu+1}$. *Moreover,* $\mathcal{H}_{\nu+1} = \mathcal{H}_\nu \circ \Phi_{\nu+1}$, *and, on* $\tilde{D}_{\nu+1} \times \Lambda_{\nu+1}$,

$$|\Phi_{\nu+1} - id|, |D\Phi_{\nu+1} - Did|, |D^i\Phi_{\nu+1}| \le \frac{\mu_*}{2^{\nu+1}}, \quad 2 \le i \le m;$$

(3) $\Lambda_{\nu+1} = \{\xi \in \Lambda_\nu : |\langle k, \omega_\nu(\xi)\rangle| > \frac{\gamma_\nu}{|k|^\tau}, K_\nu < |k| \le K_{\nu+1}\}$.

Proof. The lemma will be proved by verifying conditions (2.8), (2.9), (2.17), (2.26), (2.27) and (2.29) for all $\nu = 0, 1, \cdots$.

Directly,

$$\mu_\nu = c_0^\nu \mu_0 s_0^{\frac{\sigma((1+b+\sigma)^\nu - 1)}{b+\sigma}},$$
$$s_\nu = s_0^{(1+b+\sigma)^\nu}.$$

Then $s_{\nu+1} = s_\nu s_0^{(1+b+\sigma)^\nu(b+\sigma)} \le s_\nu s_0^{b+\sigma} \le \frac{s_\nu}{16}$, which implies that (2.8) holds. Let $E_\nu = \frac{r_\nu - r_{\nu+1}}{8} = \frac{\delta^{\nu+2}\gamma_0(1-\delta)}{16}$ and using $\delta(1+b+\sigma) > 1$, we get

$$\frac{E_\nu}{2}\log\frac{1}{s_\nu} = \frac{\delta^{\nu+2}\gamma_0(1-\delta)}{32}\log s_0^{-(1+b+\sigma)^\nu} \ge -\frac{\gamma_0\delta^2(1-\delta)}{32}\log s_0 \ge 1.$$

Therefore

$$\log(n+N+1)! + 3(n+N)\log([\log\frac{1}{s_\nu}] + 1) - \frac{E_\nu}{2}([\log\frac{1}{s_\nu}] + 1)^3$$
$$\le -(m+1)(1+b+\sigma)\log\frac{1}{s_\nu}.$$

Thus

$$\int_{K_{\nu+1}}^\infty \lambda^{n+N} e^{-\frac{\lambda E_\nu}{2}} d\lambda \le (n+N+1)! K_{\nu+1}^{n+N} e^{-\frac{K_{\nu+1}E_\nu}{2}} \le s_{\nu+1}^{m+1},$$

i.e. (2.9) holds. Obviously,

$$s_\nu K_{\nu+1}^{\tau+1} \le s_0^{(1+b+\sigma)^\nu}(\log\frac{1}{s_0^{(1+b+\sigma)^\nu}} + 2)^{3(\tau+1)}.$$

When $\beta > 0$, $\xi > 1$ and $c > 1$ are constant, $x^\beta(\log\frac{1}{x} + c)^\xi \to 0$. Therefore, (2.17) holds.

Let $l_0 = b$, $\eta = 8 + n + 4[\tau] + 4$, where $[\tau]$ is the integral part of τ. Combining

$$\Gamma_\nu = \sum_{|j|\le 2,\ 0<|k|\le K_{\nu+1}} |k|^{\tau(|j|+|q|+1)+|j|+|q|+|z|} e^{-\frac{|k|(r-r_+)}{8}} \le \frac{\eta!}{E_\nu^\eta}$$

and

$$\frac{\mu_\nu^{l_0}}{E_\nu^{\eta+1}} = \Big(\frac{16}{\gamma_0(1-\delta)\delta^{\nu+2}}\Big)^{\eta+1} c_0^\nu \mu_0^{l_0} s_0^{\frac{\sigma}{b+\sigma}((1+b+\sigma)^\nu - 1)} \le c_* \mu_0^{l_0} \Big(\frac{s_0^\sigma c_0}{\delta^{\eta+1}}\Big)^\nu,$$

when ε_0 small enough, we get

$$\frac{c_0 \mu_\nu \Gamma_\nu}{E_\nu \bar{\eta}_\nu^N} \le c_0 \eta! \frac{\mu_\nu^{l_0}}{E_\nu^{\eta+1}} \le 1,$$

i.e. (2.26) holds. (2.27) holds obviously, since $\frac{c_0 s_\nu \mu_\nu \Gamma_\nu}{s_{\nu+1}\bar{\eta}_\nu^N} \le 3$.

Moreover, by making ε_0 small, we have $c_0 \mu_\nu^{a_0} \Gamma_\nu^3 \le \frac{1}{2^\nu}$ for given $a_0 > 0$. Next for each $\nu \ge 1$

$$\begin{aligned}
|\partial_\xi^l \Delta_{\nu+1}| &\le \big(\gamma_\nu^{N+7} s_\nu^{2m-1} \mu_\nu^2 + \mu_\nu^2 s_\nu^{2m} \Gamma^2(r_\nu - r_{\nu+1}) \\
&\quad + \gamma_\nu^{N+7} \mu \big(s_\nu^{(m+1)(1+b+\sigma)} + \frac{s_{\nu+1}^{m+1}}{s_\nu}\big)\big)/\bar{\eta}_\nu^N \\
&= \big(\gamma_\nu^{N+7} s_\nu^{2m-1} \mu_\nu^2 + \mu_\nu^2 s_\nu^{2m} \Gamma^2(r_\nu - r_{\nu+1}) \\
&\quad + \gamma_\nu^{N+7} \mu s_{\nu+1}^{m+1} (1 + \frac{1}{s_\nu})\big)/\bar{\eta}_\nu^N \\
&\le \big(\gamma_{\nu+1}^{N+7} \frac{\gamma_\nu^{N+7}}{\gamma_{\nu+1}^{N+7}} s_{\nu+1}^m s_\nu^{2m-1-m(1+b+\sigma)-2\sigma} \mu_{\nu+1}^2 \\
&\quad + \gamma_{\nu+1}^{N+7} \mu_{\nu+1}^2 s_{\nu+1}^m \frac{s_\nu^{2m-2\sigma-m(1+b+\sigma)} \Gamma^2(r_\nu - r_{\nu+1})}{\gamma_{\nu+1}^{N+7}} \\
&\quad + \gamma_{\nu+1}^{N+7} \frac{\gamma_\nu^{N+7}}{\gamma_{\nu+1}^{N+7}} \mu_{\nu+1} s_{\nu+1}^m s_\nu^{1+b+\sigma-\sigma} (1 + \frac{1}{s_\nu})\big)/\bar{\eta}_\nu^N \\
&\le \gamma_{\nu+1}^{N+7} s_{\nu+1}^m \mu_{\nu+1} \big(\frac{\gamma_\nu^{N+7}}{\gamma_{\nu+1}^{N+7}} s_\nu^{2m-1-m(1+b+\sigma)-2\sigma} \mu_{\nu+1} \\
&\quad + \mu_{\nu+1} \frac{s_\nu^{2m-2\sigma-m(1+b+\sigma)} \Gamma^2(r_\nu - r_{\nu+1})}{\gamma_{\nu+1}^{N+7}} \\
&\quad + \frac{\gamma_\nu^{N+7}}{\gamma_{\nu+1}^{N+7}} s_\nu^{1+b} (1 + \frac{1}{s_\nu})\big)/\bar{\eta}_\nu^N \\
&\le \frac{\gamma_{\nu+1}^{N+7} s_{\nu+1}^m \mu_{\nu+1}}{\bar{\eta}_{\nu+1}^N},
\end{aligned}$$

i.e. (2.29) holds. □

For brevity we omit the measure estimate of $|\Lambda_0 \setminus \Lambda_*|$ and for details we refer the reader to [5, 22, 38, 39].

3. Proof of Main Theorem

The study about the influence of the small non-integrable Hamiltonian on the foliation of the integrable Hamiltonian on Poisson manifold (M, Π) is closely related to the persistence of invariant tori for Hamiltonian (1.2) on Poisson manifold (M, Π).

Without loss of generality, we assume that there is a closed region $\Lambda \subset R^d$ and a C^{l_0} diffeomorphism $y : \Lambda \to M$ $(= y(\Lambda))$. Let $\xi \in \Lambda$ and consider the transformation: $y \mapsto y + y(\xi)$. Then (1.2) turns into a parameterized Hamiltonian system of the following form:

$$\begin{aligned}\mathcal{H} &= \mathcal{N}(y, \xi) + \varepsilon P(x, y, \xi), \\ \mathcal{N} &= e(\xi) + \langle \Omega(\xi), y \rangle + \frac{1}{2}\langle y, A(\xi) y \rangle + \sum_{|j|=3}^{m} h_j(\xi) y^j + \sum_{|j|>m} h_j(\xi) y^j,\end{aligned} \tag{3.1}$$

where $\xi \in \Lambda = \{\lambda : |\lambda| \leq \delta_1\} \subset R^d$, x, y and ε defined as above. Consider the following symplectic transformation:

$$x \to x, \quad y \to \varepsilon^{\frac{1}{4(m-1)}} y, \quad H \to \varepsilon^{-\frac{1}{4(m-1)}} H,$$

then the Hamiltonian (3.1) is changed to

$$\begin{aligned}\mathcal{H} &= \mathcal{N}(y, \xi, \tilde{\varepsilon}) + \varepsilon^{\frac{m}{4(m-1)}} \tilde{P}(x, y, \xi), \\ \mathcal{N} &= \frac{e(\xi)}{\varepsilon^{\frac{1}{4(m-1)}}} + \langle \Omega(\xi), y \rangle + \frac{\varepsilon^{\frac{1}{4(m-1)}}}{2} \langle y, A(\xi) y \rangle + \sum_{|j|=3}^{m} \varepsilon^{\frac{|j|-1}{4(m-1)}} h_j(\xi) y^j, \\ \tilde{P} &= \varepsilon^{\frac{3m-5}{4(m-1)}} P(x, y, \xi) + \sum_{|j|>m} \varepsilon^{\frac{|j|-m-1}{4(m-1)}} h_j(\xi) y^j.\end{aligned} \tag{3.2}$$

Moreover,

$$|\partial_\xi^l \bar{P}| \leq c \frac{\gamma^{N+7} s^m \mu}{\bar{\eta}^N}, \ |l| \leq N,$$

if $\gamma = \varepsilon^{\frac{m}{48(7+N)(m-1)}}$, $s = \varepsilon^{\frac{1}{6(m-1)}}$, $\mu = \varepsilon^{\frac{m}{16(m-1)}}$, $\bar{\eta} = \varepsilon^{\frac{(3(1-b)-4(b+\sigma))m}{56N(m-1)}}$. Hence, with **Theorem** 2.1 we can get **Theorem** 1.1.

Remark 3.1. Normal forms (3.2) and (2.2) seem a little different. In normal form (3.2), the perturbation is $O(\varepsilon^{\frac{m}{4(m-1)}})$, which is small enough comparing the integrable part. This term $\frac{\varepsilon^{\frac{1}{4(m-1)}}}{2}\langle y, A(\xi) y \rangle + \sum_{|j|=3}^{m} \varepsilon^{\frac{|j|-1}{4(m-1)}} h_j(\xi) y^j$ is bad for the preservation of the frequency and frequency ratio, since the coefficient is related to $\varepsilon^{\frac{1}{4(m-1)}}$. In fact, in our case, this difficulty could be overcome since the coefficient of the perturbation is $\varepsilon^{\frac{m}{4(m-1)}}$. We could achieve the proof of **Theorem** 1.1 step by step using the proof of **Theorem** 2.1.

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