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IDENTIFYING STRONG ELLIPTICITY VIA BOUNDS ON THE MINIMUM *M*-EIGENVALUE OF ELASTICITY *Z*-TENSORS*

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Abstract M-eigenvalues of fourth-order partially symmetric tensors play an important role in the nonlinear elastic material analysis. In this paper, we establish sharp upper and lower bounds on the minimum M-eigenvalue via extreme eigenvalue of the symmetric matrices extracted from elasticity Z-tensors without irreducible conditions, which improves some existing results. Based on the lower bound estimations for the minimum M-eigenvalue, we provide some checkable sufficient or necessary conditions for the strong ellipticity of elasticity Z-tensors. Numerical examples are given to demonstrate the proposed results.

Keywords Elasticity Z-tensors, minimum M-eigenvalue, upper and lower bounds, strong ellipticity.

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1. Introduction

A fourth-order real tensor $\mathcal{A}$ is called a partially symmetric tensor, denoted by $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$, if

$$a_{ijkl} = a_{jikl} = a_{ijlk},\ i, j, k, l \in N = \{1, 2, \cdots, n\}.$$

We know from [1–6, 14–16, 18, 23, 24, 27, 28] that it is the most well-known tensor among fourth-order tensors and comes from the following bi-quadratic homogeneous polynomial optimization problem:

$$\begin{aligned} &\min_{x,y} f(x, y) = \mathcal{A}x^2y^2 = \sum_{i,k\in N}\sum_{j,l\in N} a_{ijkl}x_ix_jy_ky_l \\ &s.t.\ \ x^\top x = 1, y^\top y = 1,\ x, y \in \mathbb{R}^n. \end{aligned} \tag{1.1}$$

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The optimization problem introduced by Dahl *et al.* [3] arises from the nonlinear elastic material analysis. For example, a fourth-order partially symmetric tensor with $n = 2$ or 3, called the elasticity tensor, can be used in the two/three-dimensional field equations for a homogeneous compressible nonlinearly elastic material without body forces [1, 16, 18]. In elastic material analysis, elastic materials with the strong ellipticity can keep good properties [1,16,28]. Based on tensor characterizations of elastic material, Han *et al.* [6] and Qi *et al.* [15] pointed out that strong ellipticity condition holds if and only if the optimal value of (1.1) is positive. To further characterize the strong ellipticity condition, Han *et al.* [6] introduced M-eigenvalue of an elasticity tensor as follows. For $\lambda \in \mathbb{R}, x, y \in \mathbb{R}^n$, if

$$\begin{cases} \mathcal{A}xy^2 = \lambda x, \quad \mathcal{A}x^2y = \lambda y; \\ x^\top x = 1, \quad y^\top y = 1, \end{cases}$$

where $(\mathcal{A}xy^2)_i = \sum\limits_{j,k,l \in N} a_{ijkl}x_j y_k y_l$, $(\mathcal{A}x^2y)_l = \sum\limits_{i,j,k \in N} a_{ijkl}x_i x_j y_k$, then the scalar λ is called an M-eigenvalue of the tensor $\mathcal{A}$, and x and y are called left and right M-eigenvectors associated with λ.

Thus, the strong ellipticity condition holds for an elasticity tensor if and only if its minimum M-eigenvalue is positive. However, it is not easy to compute the minimum M-eigenvalues due to the complexity of the M-eigenvalue problem [10, 13]. Thus, some researchers turned to investigating structured tensors, such as nonnegative tensors and M-tensors [4,24–26]. Particularly, Ding *et al.* [4] introduced a structured partially symmetric tensor named elasticity Z-tensors and elasticity M-tensors as follows.

Definition 1.1. Tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ is called an elasticity Z-tensor if there exists a nonnegative tensor $\mathcal{B} \in \mathbb{E}_{4,n}$ and a real number s such that

$$\mathcal{A} = s\mathcal{I}_M - \mathcal{B},$$

where $\mathcal{I}_M = (e_{ijkl}) \in \mathbb{E}_{4,n}$ is called elasticity identity tensor with its entries

$$e_{ijkl} = \begin{cases} 1, \text{ if } i = j \text{ and } k = l \\ 0, \text{ otherwise.} \end{cases}$$

If $s \geq \rho_M(\mathcal{B})$, we call $\mathcal{A}$ an elasticity M-tensor. Further, if $s > \rho_M(\mathcal{B})$, then we call $\mathcal{A}$ a nonsingular elasticity M-tensor.

As we know, an elasticity Z-tensor is a nonsingular elasticity M-tensor if and only if the minimum M-eigenvalue is positive [4]. Thus, the key to verifying the nonsingular elasticity M-tensors is whether the minimum M-eigenvalue is positive. With the help of the theory of multi-linear algebra, researchers turned to investigating eigenvalue inclusion to judge whether an elasticity Z-tensor is a nonsingular elasticity M-tensor. Based on the minimum diagonal entries, He *et al.* [8] proposed some bounds for the minimum M-eigenvalue of elasticity M-tensors under irreducible conditions. Combining the maximum diagonal entries with accurate eigenvector information, Wang *et al.* [21] established sharp bound estimations on the minimum M-eigenvalue of elasticity Z-tensors in the absence of irreducible conditions, and gave the checkable sufficient conditions for the strong ellipticity condition. In virtue of the relationship between M-eigenvalues and the extreme eigenvalues of the corresponding symmetric matrices, Li *et al.* [12] established two-sided bounds of M-eigenvalues. For an elasticity Z-tensor, not only the structural features of the fourth-order tensor should be considered, but also the structural features of the Z-tensor should be explored. Therefore, if we further explore structural characteristics of the Z-tensor and combine the M-eigenvalue estimation technology [12], we may propose tight bounds on the minimum M-eigenvalue of elasticity Z-tensors, and accurately identify whether the elasticity Z-tensor is the nonsingular elasticity M-tensor and strong ellipticity of the elasticity Z-tensor holds. These constitute the main motivation of the paper.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are recalled. In Section 3, we propose an improved upper bound and two lower bounds for the minimum M-eigenvalue via the minimum eigenvalue of the symmetric matrices extracted from elasticity Z-tensors without irreducible conditions. In Section 4, we establish some sufficient conditions to verify whether an elasticity Z-tensor is strong ellipticity and a nonsingular elasticity M-tensor. Numerical examples are proposed to verify the efficiency of the obtained results.

2. Preliminaries

We begin our work by collecting some definitions and important properties of elasticity Z-tensors [4, 8, 15].

Definition 2.1. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be an m-th order n dimensional real square tensor. $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $J \subset \{1, 2, \dots, n\}$ such that $a_{i_1 i_2 \dots i_m} = 0$, $\forall\, i_1 \in J, \forall\, i_2, \dots, i_m \notin J$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ to be irreducible.

Lemma 2.1 (Theorem 1 of [15])**.** *M-eigenvalues always exist. If x and y are left and right M-eigenvectors of $\mathcal{A}$, associated with an M-eigenvalue λ, then $\lambda = \mathcal{A}x^2y^2$.*

Recently, Wang *et al.* [21] proposed new characterizations of the minimum M-eigenvalue and corresponding to M-eigenvectors for elasticity Z-tensors based on Theorem 6 of [4].

Lemma 2.2 (Lemma 2.4 of [21])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor and $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue. Then, there exist nonnegative left and right M-eigenvectors (x, y) corresponding to $\tau_M(\mathcal{A})$ such that*

$$\mathcal{A}xy^2 = \tau_M(\mathcal{A})x, \ \ \mathcal{A}x^2y = \tau_M(\mathcal{A})y.$$

He *et al.* [8] established the lower bound of the minimum M-eigenvalue for an irreducible elasticity M-tensor.

Lemma 2.3 (Theorem 3.1 of [9])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. Then*

$$\tau_M(\mathcal{A}) \geq \max\{\min_{i\in N}\{\alpha_i - R_i(\mathcal{A}), \min_{l\in N}\{\beta_l - C_l(\mathcal{A})\}\},$$

where $\alpha_i = \min\limits_{l\in N} a_{iill}, \beta_l = \min\limits_{i\in N} a_{iill}, R_i(\mathcal{A}) = \gamma_i + r_i(\mathcal{A}), C_l(\mathcal{A}) = \delta_l + c_l(\mathcal{A})$,
$\gamma_i = \max\limits_{l\in N}\{\sum\limits_{j\in N, j\neq i} |a_{ijll}|\}, \delta_l = \max\limits_{i\in N}\{\sum\limits_{k\in N, k\neq l} |a_{iikl}|\}, r_i(\mathcal{A}) = \sum\limits_{j,k,l\in N, k\neq l} |a_{ijkl}|, c_l(\mathcal{A}) =$

$\sum_{i,j,k\in N,i\neq j} |a_{ijkl}|$.

Wang *et al.* [21] pointed out that the bound in Theorem 3.2 is tighter than that of Theorem 3.1 of [8].

Lemma 2.4 (Theorem 3.2 of [21])**.** *Let $\mathcal{A}=(a_{ijkl})\in\mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. Then,*

$$\begin{aligned}\tau_M(\mathcal{A}) &\geq \max\{\min_{i\in N}\{\mu_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\kappa_l - M_l(\mathcal{A})\}\}\\ &\geq \max\{\min_{i\in N}\{\alpha_i - R_i(\mathcal{A}), \min_{l\in N}\{\beta_l - C_l(\mathcal{A})\}\},\end{aligned}$$

where $G_i(\mathcal{A})=\omega_i(\mathcal{A})-\frac{1}{2}r_i(\mathcal{A}), \mu_i=\max_{l\in N}\{a_{iill}\}, \omega_i(\mathcal{A})=\max_{l\in N}\{\mu_i-a_{iill}-\sum_{j\in N,j\neq i} a_{ijll}\}$,

$$\begin{aligned}&r_i(\mathcal{A}) = \sum_{j,k,l\in N,k\neq l} a_{ijkl},\ \ M_l(\mathcal{A}) = m_l(\mathcal{A}) - \frac{1}{2}c_l(\mathcal{A}), \kappa_l = \max_{i\in N}\{a_{iill}\},\\ &m_l(\mathcal{A}) = \max_{i\in N}\{\kappa_l - a_{iill} - \sum_{k\in N,k\neq l} a_{iikl}\}, c_l(\mathcal{A}) = \sum_{i,j,k\in N,i\neq j} a_{ijkl}.\end{aligned}$$

In order to characterize the M-eigenvalue of elasticity Z-tensors by the minimum eigenvalue of the symmetric matrices extracted from elasticity Z-tensors, we end this section with some important results of the symmetric matrices [7].

Lemma 2.5. *Let $P=(p_{ij})\in\mathbb{R}^{[n]\times[n]}$ be a real symmetric matrix and $\lambda_{\min}(P)$ (or $\lambda_{\max}(P)$) denote the minimal (or maximal) eigenvalue of P. Then,*

$$\lambda_{min}(P) = \min_{x^\top x=1,x\in\mathbb{R}^n}(x^\top Px) \leq x^\top Px \leq \max_{x^\top x=1,x\in\mathbb{R}^n}(x^\top Px) = \lambda_{max}(P).$$

3. Bounds for the minimum M-eigenvalue of elasticity Z-tensors

In this section, inspired by Z-eigenvalue and M-eigenvalue intervals [8–12, 17, 19–22], we propose sharp lower and upper bounds on the minimum M-eigenvalue of elasticity Z-tensors based on the minimum eigenvalues of symmetric matrices.

Theorem 3.1. *Let $\mathcal{A}=(a_{ijkl})\in\mathbb{E}_{4,n}$ be an elasticity tensor and $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue. Then,*

$$\tau_M(\mathcal{A}) \leq \min\{\min_{i\in N}\mu_{ii}, \frac{\omega}{n}\},$$

where μ_{ii} is the minimum eigenvalue of the symmetric matrix $\mathcal{A}(i,i,:,:)$, ω is the minimum eigenvalue of the symmetric matrix $\sum_{i,j\in N}\mathcal{A}(i,j,:,:)$ and $\mathcal{A}(i,j,:,:)$ is the symmetric matrix by fixing i and j indices of $\mathcal{A}$.

Proof. Let $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue of $\mathcal{A}$. It follows from Lemma 2.1 that

$$\tau_M(\mathcal{A}) = \min_{x,y}\{f_{\mathcal{A}}(x,y) = \mathcal{A}x^2y^2 : x^\top x = 1 \text{ and } y^\top y = 1\}. \tag{3.1}$$

On the one hand, for a feasible solution $\tilde{x} = (0, \ldots, \tilde{x}_i = 1, \ldots, 0)$ and $y^\top y = 1$, from Lemma 2.4, we can get

$$\begin{aligned}\tau_M(\mathcal{A}) &= \min_{x,y} \mathcal{A}x^2y^2 \leq \min_{y} \mathcal{A}\tilde{x}^2y^2 = \min_{y} \sum_{k,l\in N} a_{iikl}y_ky_l \\ &= \min_{y} y^\top \mathcal{A}(i,i,:,:)y = \mu_{ii}. \end{aligned} \tag{3.2}$$

Further, $\tau_M(\mathcal{A}) \leq \min_{i\in N} \mu_{ii}$ holds.

On the other hand, setting a feasible solution $\bar{x} = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$ and $y^\top y = 1$, one has

$$\begin{aligned}\tau_M(\mathcal{A}) &= \min_{x,y} \mathcal{A}x^2y^2 \leq \min_{y} \mathcal{A}\bar{x}^2y^2 = \frac{\min\limits_{y} \sum\limits_{i,j\in N} \sum\limits_{k,l\in N} a_{ijkl}y_ky_l}{n} \\ &= \frac{\min\limits_{y} y^\top \sum\limits_{i,j\in N} \mathcal{A}(i,j,:,:)y}{n} = \frac{\omega}{n}. \end{aligned} \tag{3.3}$$

Equation (3.3), in conjunction with (3.2), provides upper bounds on the minimum M-eigenvalue. $\square$

In the following, we show the results of Theorem 3.1 improve the results of Theorem 3.1 of [21].

Lemma 3.1 (Theorem 3.1 of [21])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor and $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue. Then,*

$$\tau_M(\mathcal{A}) \leq \min\{\min_{i,l\in N} a_{iill}, \frac{\sum\limits_{i\in N} S_i(\mathcal{A})}{n^2}\},$$

where $S_i(\mathcal{A}) = \sum\limits_{j,k,l\in N} a_{ijkl}$.

Corollary 3.1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor. Then*

$$\min\{\min_{i\in N} \mu_{ii}, \frac{\omega}{n}\} \leq \min\{\min_{i,l\in N} a_{iill}, \frac{\sum\limits_{i\in N} S_i(\mathcal{A})}{n^2}\}.$$

Proof. It follows from (3.2) that

$$\mu_{ii} = \min_{y} y^\top \mathcal{A}(i,i,:,:)y \leq \min_{l\in N} a_{iill},$$

and

$$\min_{i\in N} \mu_{ii} \leq \min_{i,l\in N} a_{iill}.$$

Meanwhile,

$$\begin{aligned}\frac{\omega}{n} &= \frac{\min\limits_{y} y^\top \sum\limits_{i,j\in N} \mathcal{A}(i,j,:,:)y}{n} = \min_{y} \mathcal{A}\bar{x}^2y^2 \leq \mathcal{A}\bar{x}^2\bar{y}^2 \\ &= \sum_{i,j,k,l\in N} a_{ijkl}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}} = \frac{\sum\limits_{i\in N} S_i(\mathcal{A})}{n^2},\end{aligned}$$

where $(\bar{x},\bar{y}) = (\frac{1}{\sqrt{n}},\cdots,\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}},\cdots,\frac{1}{\sqrt{n}})$.

Consequently,

$$\min\{\min_{i\in N}\mu_{ii},\frac{\omega}{n}\} \le \min\{\min_{i,l\in N} a_{iill},\frac{\sum\limits_{i\in N} S_i(\mathcal{A})}{n^2}\}. \qquad \square$$

In what follows, we establish sharp lower bounds for the minimum M-eigenvalue of elasticity Z-tensors.

Theorem 3.2. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor and $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue. Then,*

$$\tau_M(\mathcal{A}) \ge \max\{\min_{i\in N}\mu_i(\mathcal{A}),\min_{l\in N}\kappa_l(\mathcal{A})\},$$

where $\mu_i(\mathcal{A})$ is the minimum eigenvalue of the symmetric matrix $\sum\limits_{j\in N}\mathcal{A}(i,j,:,:)$ and $\kappa_l(\mathcal{A})$ is the minimum eigenvalue of the symmetric matrix $\sum\limits_{k\in N}\mathcal{A}(:,:,k,l)$.

Proof. Let $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue of $\mathcal{A}$. It follows from Lemma 2.2 that there exist nonnegative left and right M-eigenvectors (x,y) corresponding to $\tau_M(\mathcal{A})$. Setting $x_p = \max\limits_{i\in N}\{x_i\}$, by $x^\top x = 1$, one has $0 < x_p \le 1$. Since $\mathcal{A}$ is an elasticity Z-tensor, then $a_{ijkl} \le 0$ for all $i,j,k,l \in N$ except for $i = j$ and $k = l$. Recalling the p-th equation of $\tau_M(\mathcal{A})x = \mathcal{A}xy^2$, from Lemma 2.5, we deduce

$$\begin{aligned}
\tau_M(\mathcal{A})x_p &= \sum_{j,k,l\in N} a_{pjkl}x_jy_ky_l \ge \sum_{j,k,l\in N} a_{pjkl}x_py_ky_l \\
&= (y^\top \sum_{j\in N}\mathcal{A}(p,j,:,:)y)x_p \ge \min_{y^\top y=1}(y^\top \sum_{j\in N}\mathcal{A}(p,j,:,:)y)x_p \\
&= \mu_p(\mathcal{A})x_p,
\end{aligned}$$

that is,

$$\tau_M(\mathcal{A}) \ge \mu_p(\mathcal{A}). \tag{3.4}$$

On the other hand, setting $y_t = \max\limits_{l\in N}\{y_l\}$, from the t-th equation of $\tau_M(\mathcal{A})y = \mathcal{A}x^2y$ and Lemma 2.5, one has

$$\begin{aligned}
\tau_M(\mathcal{A})y_t &= \sum_{i,j,k\in N} a_{ijkt}x_ix_jy_k \ge \sum_{i,j,k\in N} a_{ijkt}x_ix_jy_t \\
&= (x^\top \sum_{k\in N}\mathcal{A}(:,:,k,t)x)y_t \ge \min_{x^\top x=1}(x^\top \sum_{k\in N}\mathcal{A}(:,:,k,t)x)y_t \\
&= \kappa_t(\mathcal{A})y_t,
\end{aligned}$$

which implies

$$\tau_M(\mathcal{A}) \ge \kappa_t(\mathcal{A}). \tag{3.5}$$

By (3.4) and (3.5), we obtain the desired results. $\square$

Now, we are at a position to prove that the lower bound in Theorem 3.2 is sharper than that of Theorem 3.2 of [21].

Corollary 3.2. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor. Then*

$$\max\{\min_{i\in N} \mu_i(\mathcal{A}), \min_{l\in N} \kappa_l(\mathcal{A})\} \geq \max\{\min_{i\in N}\{\mu_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\kappa_l - M_l(\mathcal{A})\}\},$$

where $\mu_i, \kappa_l, G_i(\mathcal{A})$ and $M_l(\mathcal{A})$ are defined in Lemma 2.4.

Proof. Since $\mathcal{A}$ is an elasticity Z-tensor, we can obtain $a_{ijkl} \leq 0$ for all $i, j, k, l \in N$ except for $i = j$ and $k = l$ and

$$\begin{aligned}
\mu_i - G_i(\mathcal{A}) &= \mu_i - \omega_i(\mathcal{A}) + \frac{1}{2} r_i(\mathcal{A}) \\
&= \mu_i - \max_{l\in N}\{\mu_i - a_{iill} - \sum_{j\in N, j\neq i} a_{ijll}\} + \frac{1}{2} \sum_{j,k,l\in N, k\neq l} a_{ijkl} \\
&= \min_{l\in N}\{\sum_{j\in N} a_{ijll}\} + \frac{1}{2} \sum_{j,k,l\in N, k\neq l} a_{ijkl}.
\end{aligned}$$

By Gersghorin theorem [7], for a matrix $Q = (q_{kl}) \in \mathbb{R}^{n\times n}$, there exists $k \in N$ such that

$$\lambda_{min}(Q) \geq q_{kk} - r_k(Q) \geq \min_{k\in N} q_{kk} - r_k(Q) = \min_{l\in N} q_{ll} - r_k(Q),$$

where $r_k(Q) = \sum_{l\in N, l\neq k} |q_{kl}|$. Thus, for the symmetric matrix $\sum_{j\in N} \mathcal{A}(i, j, :, :)$, there exists $k \in N$ such that

$$\begin{aligned}
\mu_i(\mathcal{A}) = \lambda_{min}(\sum_{j\in N} \mathcal{A}(i, j, :, :)) &\geq \min_{l\in N}\{\sum_{j\in N} a_{ijll}\} - \sum_{l\in N, l\neq k} |\sum_{j\in N} a_{ijkl}| \\
&= \min_{l\in N}\{\sum_{j\in N} a_{ijll}\} + \sum_{j,l\in N, l\neq k} a_{ijkl}.
\end{aligned} \tag{3.6}$$

Since $\mathcal{A}$ is partially symmetric, it holds that

$$\frac{1}{2}(a_{ijkl} + a_{ijlk}) = a_{ijkl}, \forall l \in N, l \neq k$$

and

$$\sum_{j,l\in N, l\neq k} a_{ijkl} \geq \frac{1}{2} \sum_{j,k,l\in N, l\neq k} a_{ijkl}. \tag{3.7}$$

Summing (3.6) and (3.7), one has

$$\mu_i(\mathcal{A}) \geq \mu_i - G_i(\mathcal{A}), \forall i \in N. \tag{3.8}$$

Following the similar arguments to the proof of (3.8), we deduce

$$\kappa_l(\mathcal{A}) \geq \kappa_l - M_l(\mathcal{A}), \forall l \in N. \tag{3.9}$$

It follows from (3.8) and (3.9) that the desired results hold. □

Next, we shall obtain sharp lower bound on the minimum M-eigenvalue by choosing x_p as a component with the largest modulus and x_q as an arbitrary component of left M-eigenvector x.

Theorem 3.3. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor and $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue. Then*

$$\tau_M(\mathcal{A}) \geq \max\{\min_{i\in N} \max_{v\in N, v\neq i} \xi_{i,v}(\mathcal{A}), \min_{l\in N} \max_{u\in N, u\neq l} \rho_{l,u}(\mathcal{A})\},$$

where

$$\xi_{i,v}(\mathcal{A}) = \frac{1}{2}[\mu_{vv} + \mu_i^v(\mathcal{A}) - (\Delta_{i,v}(\mathcal{A}))^{\frac{1}{2}}],$$
$$\Delta_{i,v}(\mathcal{A}) = (\mu_{vv} - \mu_i^v(\mathcal{A}))^2 + 4\mu_{iv}\mu_v^v(\mathcal{A}), \mu_i^v(\mathcal{A}) = \lambda_{min}(\sum_{j\in N, j\neq v} \mathcal{A}(i,j,:,:)),$$
$$\rho_{l,u}(\mathcal{A}) = \frac{1}{2}[\kappa_{uu} + \kappa_l^u(\mathcal{A}) - (\theta_{l,u}(\mathcal{A}))^{\frac{1}{2}}],$$
$$\theta_{l,u}(\mathcal{A}) = (\kappa_{uu} - \kappa_l^u(\mathcal{A}))^2 + 4\kappa_{ul}\kappa_u^u(\mathcal{A}), \kappa_l^u(\mathcal{A}) = \lambda_{min}(\sum_{k\in N, k\neq u} \mathcal{A}(:,:,k,l)),$$

μ_{iv} denotes the minimum eigenvalue of the symmetric matrix $\mathcal{A}(i,v,:,:)$ with $v \neq i$ and κ_{ul} denotes the minimum eigenvalue of the symmetric matrix $\mathcal{A}(:,:,u,l)$ with $u \neq l$.

Proof. Let $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue with nonnegative left and right M-eigenvectors (x, y). Set $x_p = \max_{i\in N}\{x_i\} > 0$. Since $\mathcal{A}$ is an elasticity Z-tensor, then $a_{ijkl} \leq 0$ for all $i, j, k, l \in N$ except for $i = j$ and $k = l$. By the p-th equation of $\tau_M(\mathcal{A})x = \mathcal{A}xy^2$ and from Lemma 2.4, for any $q \in N, q \neq p$, one has

$$\begin{aligned}
\tau_M(\mathcal{A})x_p &= \sum_{j,k,l\in N} a_{pjkl}x_jy_ky_l = \sum_{j,k,l\in N, j\neq q} a_{pjkl}x_jy_ky_l + \sum_{k,l\in N} a_{pqkl}x_qy_ky_l \\
&\geq \sum_{j,k,l\in N, j\neq q} a_{pjkl}x_py_ky_l + \sum_{k,l\in N} a_{pqkl}x_qy_ky_l \\
&= (y^\top \sum_{j\in N, j\neq q} \mathcal{A}(p,j,:,:)y)x_p + \sum_{k,l\in N} a_{pqkl}x_qy_ky_l \\
&\geq \lambda_{min}(\sum_{j\in N, j\neq q} \mathcal{A}(p,j,:,:))x_p + \lambda_{min}(\mathcal{A}(p,q :,:))x_q \\
&= \mu_p^q(\mathcal{A})x_p + \mu_{pq}x_q,
\end{aligned}$$

equivalently,

$$(\mu_p^q(\mathcal{A}) - \tau_M(\mathcal{A}))x_p \leq -\mu_{pq}x_q. \tag{3.10}$$

Since $x_q \geq 0$, we now break up the argument into two cases.
Case 1. $x_q > 0$. Recalling the q-th equation of $\tau_M(\mathcal{A})x = \mathcal{A}xy^2$, we deduce

$$\begin{aligned}
\tau_M(\mathcal{A})x_q &= \sum_{j,k,l\in N} a_{qjkl}x_jy_ky_l = \sum_{j,k,l\in N, j\neq q} a_{qjkl}x_jy_ky_l + \sum_{k,l\in N} a_{qqkl}x_qy_ky_l \\
&\geq \sum_{j,k,l\in N, j\neq q} a_{qjkl}x_py_ky_l + \sum_{k,l\in N} a_{qqkl}x_qy_ky_l \\
&= (y^\top \sum_{j\in N, j\neq q} \mathcal{A}(q,j,:,:)y)x_p + \sum_{k,l\in N} a_{qqkl}x_qy_ky_l
\end{aligned}$$

$$
\begin{aligned}
&\geq \lambda_{min}(\sum_{j\in N, j\neq q} \mathcal{A}(q,j,:,:))x_p + \lambda_{min}(\mathcal{A}(q,q\,:,:))x_q \\
&= \mu_q^q(\mathcal{A})x_p + \mu_{qq}x_q,
\end{aligned}
$$

that is,

$$(\mu_{qq} - \tau_M(\mathcal{A}))x_q \leq -\mu_q^q(\mathcal{A})x_p. \tag{3.11}$$

Multiplying (3.10) with (3.11) yields

$$(\mu_p^q(\mathcal{A}) - \tau_M(\mathcal{A}))(\mu_{qq} - \tau_M(\mathcal{A})) \leq \mu_{pq}\mu_q^q(\mathcal{A}), \tag{3.12}$$

equivalently,

$$\tau_M(\mathcal{A})^2 - [\mu_p^q(\mathcal{A}) + \mu_{qq}]\tau_M(\mathcal{A}) + \mu_{qq}\mu_p^q(\mathcal{A}) - \mu_{pq}\mu_q^q(\mathcal{A}) \leq 0. \tag{3.13}$$

Solving (3.13) for $\tau_M(\mathcal{A})$, one has

$$\tau_M(\mathcal{A}) \geq \frac{1}{2}[\mu_{qq} + \mu_p^q(\mathcal{A}) - (\Delta_{p,q}(\mathcal{A}))^{\frac{1}{2}}],$$

where

$$\Delta_{p,q}(\mathcal{A}) = (\mu_{qq} - \mu_p^q(\mathcal{A}))^2 + 4\mu_{pq}\mu_q^q(\mathcal{A}).$$

From the arbitrariness of q, we obtain

$$\tau_M(\mathcal{A}) \geq \max_{q\in N, q\neq p} \frac{1}{2}[\mu_{qq} + \mu_p^q(\mathcal{A}) - (\Delta_{p,q}(\mathcal{A}))^{\frac{1}{2}}].$$

Further,

$$\tau_M(\mathcal{A}) \geq \min_{i\in N} \max_{v\in N, v\neq i} \frac{1}{2}[\mu_{vv} + \mu_i^v(\mathcal{A}) - (\Delta_{i,v}(\mathcal{A}))^{\frac{1}{2}}]. \tag{3.14}$$

Case 2. $x_q = 0$. Recalling (3.10), we have $\tau_M(\mathcal{A}) \geq \mu_p^q(\mathcal{A})$, which implies that $\tau_M(\mathcal{A}) \geq \mu_p^q(\mathcal{A})$ is a solution of (3.12).

On the other hand, with $y_t = \max_{l\in N}\{y_l\}$ and the t-th equation of $\tau_M(\mathcal{A})y = \mathcal{A}x^2y$, it is clear that

$$\tau_M(\mathcal{A})y_t = \sum_{i,j,k\in N, k\neq t} a_{ijkt}x_ix_jy_k + \sum_{i,j\in N} a_{ijtt}x_ix_jy_t.$$

Following the similar arguments to the proof of (3.14), we deduce

$$\tau_M(\mathcal{A}) \geq \min_{l\in N} \max_{u\in N, u\neq l} \frac{1}{2}[\kappa_{uu} + \kappa_l^u(\mathcal{A}) - (\theta_{l,u}(\mathcal{A}))^{\frac{1}{2}}]. \tag{3.15}$$

As a consequence, the desired results hold from (3.14) and (3.15). $\square$

Next, we use Example 4.1 of [21] to show the superiority of our results.

Example 3.1. Consider an elasticity Z-tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,3}$ defined by the entries

$$\begin{cases} a_{1111} = a_{2222} = a_{3333} = 5, a_{1122} = a_{1133} = a_{2233} = 6, \\ a_{2211} = a_{3311} = a_{3322} = 7, a_{2123} = a_{1223} = a_{2132} = a_{1232} = -0.2, \\ a_{1112} = a_{1121} = -1, a_{2212} = a_{2221} = -0.5, a_{1222} = a_{2122} = -2, \\ a_{3313} = a_{3331} = -0.5, a_{1333} = a_{3133} = -2, a_{1311} = a_{3111} = -1, \\ a_{2223} = a_{2232} = -0.5, a_{2322} = a_{3222} = -1, a_{2333} = a_{3233} = -2, \\ a_{1213} = a_{1231} = a_{2113} = a_{2131} = -0.2, \\ a_{3132} = a_{3123} = a_{1332} = a_{1323} = -0.2, a_{ijkl} = 0, \quad \text{otherwise}. \end{cases}$$

By computations, we obtain that the minimum M-eigenvalue and corresponding with left and right M-eigenvectors are

$$(\tau_M(\mathcal{A}), \bar{x}, \bar{y}) = (2.5000, (0.7071, 0, 0.7071, 0), (0.7071, 0.7071, 0)).$$

The bounds given in the different literatures are shown in Table 1.

Table 1. Comparisons of the existing results with our methods

References	Bounds
Lemma 2.4 and Theorem 3.1 of [8]	$-0.8000 \leq \tau_M(\mathcal{A}) \leq 5.0000$
Lemma 2.4 and Theorem 3.2 of [8]	$-0.6667 \leq \tau_M(\mathcal{A}) \leq 5.0000$
Theorems 3.1 and 3.2 of [21]	$0.3000 \leq \tau_M(\mathcal{A}) \leq 3.4000$
Theorems 3.1 and 3.3 of [21]	$0.3900 \leq \tau_M(\mathcal{A}) \leq 3.4000$
Theorems 3.1 and 3.2	$1.9241 \leq \tau_M(\mathcal{A}) \leq 2.7643$
Theorems 3.1 and 3.3	$2.0779 \leq \tau_M(\mathcal{A}) \leq 2.7643$

4. Identifying strong ellipticity and elasticity M-tensors

In this section, we establish some sufficient or necessary conditions for identifying an elasticity M-tensor and strong ellipticity based on the conclusions in Theorems 3.1-3.3.

Theorem 4.1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor. If*

$$\max\{\min_{i\in N} \mu_i(\mathcal{A}), \min_{l\in N} \kappa_l(\mathcal{A})\} > 0, \tag{4.1}$$

then strong ellipticity of $\mathcal{A}$ is satisfied, and $\mathcal{A}$ is a nonsingular elasticity M-tensor.

Proof. It follows from Theorem 3.2 and (4.1) that

$$\tau_M(\mathcal{A}) \geq \max\{\min_{i\in N} \mu_i(\mathcal{A}), \min_{l\in N} \kappa_l(\mathcal{A})\} > 0,$$

which implies that strong ellipticity is satisfied, and $\mathcal{A}$ is a nonsingular elasticity M-tensor. □

Theorem 4.2. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor. If*

$$\max\{\min_{i\in N} \max_{v\in N, v\neq i} \xi_{i,v}(\mathcal{A}), \min_{l\in N} \max_{u\in N, u\neq l} \rho_{l,u}(\mathcal{A})\} > 0,$$

then strong ellipticity of $\mathcal{A}$ is satisfied, and $\mathcal{A}$ is a nonsingular elasticity M-tensor.

Proof. Following the similar arguments to the proof of Theorem 4.1, we obtain the desired results. □

With the help of Theorem 3.1, we are now ready to propose a necessary condition of nonsingular elasticity M-tensors or strong ellipticity.

Theorem 4.3. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be a nonsingular elasticity M-tensor or be strong ellipticity. Then*

$$\min_{i\in N} \mu_{ii} > 0 \text{ and } \frac{\omega}{n} > 0, \tag{4.2}$$

where μ_{ii} and ω are defined in Theorem 3.1.

Proof. Since $\mathcal{A}$ is a nonsingular elasticity M-tensor or is strong ellipticity, we obtain that the minimum M-eigenvalue of $\tau_M(\mathcal{A}) > 0$. This, together with Theorem 3.1 yields

$$0 < \tau_M(\mathcal{A}) \leq \min\{\min_{i\in N} \mu_{ii}, \frac{\omega}{n}\}.$$

Thus, the results hold. □

The following example shows that the results given in Theorems 4.1 and 4.2 can check whether an elasticity Z-tensor is a nonsingular elasticity M-tensor, and verify the strong ellipticity of an elasticity Z-tensor.

Example 4.1. Consider an elasticity Z-tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,3}$ defined by the entries

$$\begin{cases} a_{1111} = a_{2222} = a_{3333} = 5, a_{1122} = a_{1133} = a_{2233} = 6, \\ a_{2211} = a_{3311} = a_{3322} = 7, a_{2123} = a_{1223} = a_{2132} = a_{1232} = -1, \\ a_{1112} = a_{1121} = -1, a_{2212} = a_{2221} = -0.2, a_{1222} = a_{2122} = -2, \\ a_{3313} = a_{3331} = -0.2, a_{1333} = a_{3133} = -2, a_{1311} = a_{3111} = -1, \\ a_{2223} = a_{2232} = -0.2, a_{2322} = a_{3222} = -1, a_{2333} = a_{3233} = -2, \\ a_{1213} = a_{1231} = a_{2113} = a_{2131} = -1, \\ a_{3132} = a_{3123} = a_{1332} = a_{1323} = -1, a_{ijkl} = 0, \quad \text{otherwise}. \end{cases}$$

By computations, the bounds given in the different literatures are shown in the following table.

Table 2. The bounds of the minimum M-eigenvalue in different literatures

References	Bounds
Lemma 2.4 and Theorem 3.1 of [8]	$-5.0000 \le \tau_M(\mathcal{A}) \le 5.0000$
Lemma 2.4 and Theorem 3.2 of [8]	$-4.6214 \le \tau_M(\mathcal{A}) \le 5.0000$
Theorems 3.1 and 3.2 of [21]	$-0.4000 \le \tau_M(\mathcal{A}) \le 2.5333$
Theorems 3.1 and 3.3 of [21]	$-0.3037 \le \tau_M(\mathcal{A}) \le 2.5333$
Theorems 3.1 and 3.2	$0.8000 \le \tau_M(\mathcal{A}) \le 1.8336$
Theorems 3.1 and 3.3	$0.9232 \le \tau_M(\mathcal{A}) \le 1.8336$

From Table 2, the existing results, such as Theorems 3.1-3.2 of [8] and Theorems 3.2-3.3 of [21], cannot verify the strong ellipticity condition of $\mathcal{A}$. Fortunately, we can deduce that $\mathcal{A}$ is a nonsingular elasticity M-tensor and strong ellipticity holds by Theorems 4.1-4.2.

We give the following example to show that Theorem 4.2 is more accurate than Theorem 4.1 in judging the strong ellipticity of elasticity Z-tensors.

Example 4.2. Consider an elasticity Z-tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,2}$ defined by the entries

$$\begin{cases} a_{1111} = 1.9, a_{1122} = a_{2222} = 2.1, a_{2211} = 2, \\ a_{1112} = a_{1121} = a_{1212} = a_{1221} = a_{2112} = a_{2121} = a_{2212} = a_{2221} = -1, \\ a_{ijkl} = 0, \quad \text{otherwise}. \end{cases}$$

By computations, the bounds given in the different literatures are shown in Table 3.

Table 3. Verifying the strong ellipticity of elasticity Z-tensors by different literatures

References	Bounds
Lemma 2.4 and Theorem 3.1 of [8]	$-1.1000 \le \tau_M(\mathcal{A}) \le 1.9000$
Lemma 2.4 and Theorem 3.2 of [8]	$-1.0518 \le \tau_M(\mathcal{A}) \le 1.9000$
Theorems 3.1 and 3.2 of [21]	$-0.1000 \le \tau_M(\mathcal{A}) \le 0.0250$
Theorems 3.1 and 3.3 of [21]	$-0.0362 \le \tau_M(\mathcal{A}) \le 0.0250$
Theorems 3.1 and 3.2	$-0.0025 \le \tau_M(\mathcal{A}) \le 0.0236$
Theorems 3.1 and 3.3	$0.0215 \le \tau_M(\mathcal{A}) \le 0.0236$

By Table 3, only Theorem 4.2 can judge that strong ellipticity of $\mathcal{A}$ holds, and $\mathcal{A}$ is a nonsingular elasticity M-tensor.

5. Conclusions

In this paper, we established sharp lower and upper bounds on the minimum M-eigenvalue of elasticity Z-tensors based on structural characteristics of fourth-order tensors and Z-tensors, which improved the existing results [8,21]. Meanwhile, some checkable sufficient (or necessary) conditions for the strong ellipticity were established via the bound estimations of minimum M-eigenvalue for elasticity Z-tensors.

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