Journal of Applied Analysis and Computation
Volume 13, Number 2, April 2023, 644–654
Website:http://www.jaac-online.com
DOI:10.11948/20210405

THE GROUND STATE SOLUTIONS FOR CRITICAL FRACTIONAL PROBLEMS WITH STEEP POTENTIAL WELL*

Lan-Dan Deng[1] and Yan-Ying Shang[1,†]

Abstract In this paper we investigate the existence of ground state solutions for a class of critical fractional problems. Under suitable assumptions of nonlinear terms and parameters, we get the existence of the ground states solutions.

Keywords fractional Schrödinger problem, ground state solutions, steep potential well.

MSC(2010) 35J60, 35J40.

1. Introduction

In this paper we investigate the existence of ground state solutions for the following critical fractional Schrödinger equation

$$(-\Delta)^s u + (1+\mu a(x))u = f(u) + u^{2_s^*-2}u, \qquad x \in \mathbb{R}^3, \tag{1.1}$$

where $\mu > 0$, $2_s^* = \frac{6}{3-2s}$ is the fractional critical Sobolev exponent, $(-\Delta)^s$ is the fractional laplacian operator defined by a normalization constant as

$$(-\Delta)^s u(x) = -\frac{1}{2}C(3,s)\int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{\mid y \mid^{3+2s}} dy, \qquad \forall x \in \mathbb{R}^3,$$

where $C(3,s) = \left(\int_{\mathbb{R}^3} \frac{1-\cos(\zeta_1)}{\zeta^{3+2s}}\right)^{-1}$, $\zeta = (\zeta_1, \zeta_2, \zeta_3)$. More details on the fractional laplace operator $(-\Delta)^s$ and fractional Sobolev spaces can be referred to [10]. We consider the more general form of the equation (1.1)

$$(-\Delta)^s u + V(x)u = f(x,u), \qquad x \in \mathbb{R}^3.$$

When $s = 1$, $V(x)$ is steep potential, Clapp and Ding [7] established the existence and multiplicity of solutions when $f(x,u) = \lambda u + u^{2^*-1}$. After Bartsch and Wang in [5] firstly introduced the steep potential for which $V(x) = 1 + \lambda a(x)$, many researchers have done similar researches. For example, T. Bartsch et al. [4] considered the positive solutions for nonlinear Schrödinger equations. L.F. Yin et al. in [20] got existence and concentration of ground state solutions by variational

†The corresponding author. Email: shangyan@swu.edu.cn(Y. Shang)
1School of Mathematics and Statistics, Southwest University, 400715, China
*The authors were supported by National Natural Science Foundation of China(11971393).

method for $f(x,u) = |u|^4\ u + \lambda u^{q-2}u$. For some other important results, one can refer to [1, 2, 6, 9, 13, 15] and the references therein.

When $s \in (0,1)$, X.D. Fang et al. [12] considered the ground state solution and multiple solutions, where V and $f(x,u)$ are periodic, asymptotically linear and satisfying a monotonicity condition. In [21], H. Zhang et al. considered the superlinear fractional Schrödinger equation where V and f are asymptotically periodic in x. F. Patricio et al. in [11] got the positive solution where $f(x,u)$ is superlinear and subcritical growth with respect to u. J. Zhang et al. in [22] studied the critical case and obtained the existence of ground state solutions by establishing Pohožave type identity when $s \in (\frac{3}{4}, 1)$.

Inspired by the works described above and [3, 8, 14, 18], in this paper, we discuss the fractional critical problem under the following conditions:

(f_1) $f \in C(\mathbb{R},\mathbb{R})$, $\lim\limits_{t\to 0} \frac{f(t)}{t} = 0$;

(f_2) $\lim\limits_{|t|\to\infty} \frac{f(t)}{|t|^{2^*-2}t} = 0$;

(f_3) $\lim\limits_{t\to\infty} \frac{F(t)}{t^{\frac{2s}{3-3s}}} = \infty$, when $s \in [\frac{3}{4}, 1)$ and $F(t) = \int_0^t f(x)dx$;

(a_1) $a \in C(\mathbb{R}^3,\mathbb{R})$, $a(x) \geq 0$ for all $x \in \mathbb{R}^3$, there exists $a_0 > 0$ such that

$$\text{meas}\{x \in \mathbb{R} : a(x) \leq a_0\} < \infty;$$

(a_2) $a(x) \in C^1(\mathbb{R}^3,\mathbb{R})$, $(\nabla a(x), x) \in L^\infty(\mathbb{R}) \cup L^{\frac{2s}{3}}(\mathbb{R})$, let $V(x) = 1 + \mu a(x)$ satisfy $3V(x) - (\nabla V(x), x) \geq 0, x \in \mathbb{R}^3$;

(a_3) let $\Omega := \text{int}a^{-1}(0)$ be non-empty.

The following is our main result.

Theorem 1.1. *If (f_1)-(f_3) and (a_1)-(a_3) hold, there is a constant $\mu_0 > 0$ such that $\mu > \mu_0$, equation (1.1) has a nontrivial ground state solution.*

Remark 1.1. Without Ambrosetti-Rabinowitz conditions, the boundedness of Palais-Smale sequence is difficult to get. Moreover, the minimum value of the energy functional is greater than zero, which can not be easily obtained by variational method. In order to overcome these difficulties, we establish the Pohozaev type identity and the Nehari-Pohozaev-Palais-Smale sequence.

2. Preliminaries and the functional setting

We now collect some preliminary results for the fractional Laplacian. For any $s \in [\frac{3}{4}, 1)$, the fractional order Sobolev space:

$$H^s(\mathbb{R}^3) = \{u \in L^2 : [u]_{H^s} < \infty\},$$

where $[u]_{H^s}$is the so-called Gagliardo seminorm defined as

$$[u]_{H^s}(\mathbb{R}^3) = \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy\Big)^{\frac{1}{2}}.$$

Then the fractional Sobolev space

$$H^s(\mathbb{R}^3) = \Big\{u \in L^2 : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy + \int_{\mathbb{R}^3} u^2(x)dx < \infty\Big\}.$$

$H^s(\mathbb{R}^3)$ equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^3)} := [u]_{H^s(\mathbb{R}^3)} + |u|_{L^2(\mathbb{R}^3)}.$$

Then $H^s(\mathbb{R}^3)$ is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)}{|x-y|^{3+2s}}dxdy + \int_{\mathbb{R}^3} u(x)v(x)dx.$$

It is well known that the fractional soblev space $H^s(\mathbb{R}^3)$ is continuously embedding into L^q where $q \in [2, 2_s^*]$, and $H^s(\mathbb{R}^3)$ is compactly embedding into L^q_{loc}, where $q \in [1, 2_s^*)$. Then we define the best fractional Sobolev constants

$$S = \inf_{u\in H^s(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\ (-\Delta)^{\frac{s}{2}} u\ |^2 dx}{\Big(\int_{\mathbb{R}^3} |\ u\ |^{2_s^*} dx\Big)^{\frac{2}{2_s^*}}}.$$

Moreover, for any $\mu > 0$, we define the following space

$$E = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} \mu a(x)u^2 < \infty \right\},$$

with the inner product as follows

$$\langle u, \varphi\rangle = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}\varphi dx + \int_{\mathbb{R}^3} (1 + \mu a(x))u\varphi dx,$$

the corresponding norm

$$\| u \| = \Big[\int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u)^2 + (1 + \mu a(x))u^2 dx\Big]^{\frac{1}{2}}.$$

Then the energy functional $I : E \to \mathbb{R}$ is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u)^2 + (1 + \mu a(x))u^2 dx - \int_{\mathbb{R}^3} F(u)dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} u^{2_s^*} dx.$$

It is easy to know that $I(u)$ is well defined and the derivative is given by

$$\langle I'(u), \varphi\rangle = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}\varphi + (1+\mu a(x))u\varphi dx - \int_{\mathbb{R}^3} f(u)\varphi dx - \int_{\mathbb{R}^3} u^{2_s^*-2}u\varphi dx.$$

Thus, u is a solution of (1.1) if and only if u is a critical point of I.

Lemma 2.1 ([13]). *Let $(E, \|\cdot\|_E)$ be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfies $I(0) = 0$ and*
(i) there exists constants $\rho, \kappa > 0$ such that $I\big|_{\partial B_\rho} \geq \kappa$;
(ii) There is an $e \in X\setminus B_\rho$ such that $I(e) < 0$;
Then for any constant $c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I(\gamma(t)) \geq \kappa$, there is a sequence $u_k \in E$ such that $I(u_k) \to c$ and $I'(u_k) \to 0$, where $\Gamma = \{\gamma \in C^1([0, 1], E), \gamma(0) = 0, \gamma(1) = e\}$.

Lemma 2.2 (Mountain Pass Geometry)**.** *E is a fractional Sobolev space, $I \in C'(E,\mathbb{R})$, If (f_1), (f_2) and (a_1) hold, then*
(i) there exists a positive constant ρ such that $I(u) > 0$ for any $\|u\| = \rho$;
(ii) There exists $e \in H^s(\mathbb{R}^3)$ such that $\|e\| > \rho$, and $I(e) < 0$.

Proof. (i) By (f_1), (f_2), there are constants $\eta \in (0,1)$, $C_\eta > 0$ such that

$$|\, f(u)\,|\leq \eta\ |\,u\,| + C_\eta\ |\,u\,|^{2_s^*-1}\ . \tag{2.1}$$

From $a(x) > 0$, for any $\mu > 0$, we have

$$\int_{\mathbb{R}^3} u^2 dx \leq \|\,u\,\|^2\ . \tag{2.2}$$

By (2.2) and Hölder inequality, for any $\tau \in (2, 2_s^*)$, we have

$$\int_{\mathbb{R}^3} u^\tau dx \leq S^{\frac{2_s^*(2-\tau)}{2(2_s^*-2)}}\ \|\,u\,\|^\tau\ . \tag{2.3}$$

Together with (2.1), (2.2) and (2.3), we get

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\Big((-\Delta)^{\frac{s}{2}}u\Big)^2 + (1+\mu a(x))u^2 dx - \int_{\mathbb{R}^3} F(u)dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} u^{2_s^*}dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^3}\Big((-\Delta)^{\frac{s}{2}}u\Big)^2 + (1+\mu a(x))u^2 dx - \frac{\eta}{2}\int_{\mathbb{R}^3} u^2 dx - \frac{C_\eta}{2_s^*}\int_{\mathbb{R}^3} u^{2_s^*}dx \\ &\geq \frac{1-C_1}{2}\ \|\,u\,\|^2 - \frac{C_\mu+1}{2_s^*}S^{-\frac{2_s^*}{2}}\ \|\,u\,\|^{2_s^*}\ . \end{aligned}$$

Then there is a constant $\rho \in (0,1)$ such that $I(u) > 0$ for all $\|\, u\,\| = \rho$.
(ii) We choose a fixed $u \in E \setminus \{0\}$, for $t \to +\infty$, then

$$\begin{aligned} I(tu) &= \frac{t^2}{2}\int_{\mathbb{R}^3}\Big((-\Delta)^{\frac{s}{2}}u\Big)^2 + (1+\mu a(x))u^2 dx - \int_{\mathbb{R}^3} F(tu)dx - \frac{t^{2_s^*}}{2^{*_s}}\int_{\mathbb{R}^3} u^{2_s^*}dx \\ &\geq \frac{t^2}{2}\int_{\mathbb{R}^3}\Big((-\Delta)^{\frac{s}{2}}u\Big)^2 + (1+\mu a(x))u^2 dx - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3} u^{2_s^*}dx \\ &\to -\infty. \end{aligned}$$

We choose $e = tu$ for t large enough such that $\|\, e\,\| > \rho$, $I(e) < 0$. ☐

Lemma 2.3. *If $\mu > 0$, $(f_1)-(f_3)$ and $(a_1), (a_3)$ hold, then there is a $u_\varepsilon \in E \setminus \{0\}$, such that $0 < c \leq \max\limits_{t\geq 0} I(tu_\varepsilon) < \frac{1}{3}S^{\frac{3}{2s}}$.*

Proof. From [17], we know that the best Soblev constant can be attained by the function $\tilde{u} = \frac{\kappa}{(\mu^2+|x-x_0|^2)^{\frac{3-2s}{2}}}$ with $\kappa \in \mathbb{R}\setminus\{0\}$, $\mu > 0$ and the fixed $x_0 \in \mathbb{R}^3$. Let $\bar{u} = \frac{\tilde{u}(x)}{\|\tilde{u}\|_{L^{2_s^*}(\mathbb{R}^3)}}$, $u^*(x) = \bar{u}(\frac{x}{S^{\frac{1}{2s}}})$ and $U_\varepsilon(x) = \varepsilon^{-\frac{3-2s}{2}}u^*(\frac{x}{\varepsilon})$. Thus, we have $\|\, u^*\,\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)} = \|U_\varepsilon\|^{2_s^*}_{2_s^*(\mathbb{R}^3)} = S^{\frac{3}{2s}}$. $\psi(x)$ is a truncated function defined as

$$\psi(x) = \begin{cases} 1 & \text{if } x \in B_r, \\ 0 & \text{if } x \in B_r^c. \end{cases}$$

Let $u_\varepsilon(x)=\psi(x)U_\varepsilon(x)$, it is easy to get $\mid u_\varepsilon(x)\mid\leq\mid U_\varepsilon(x)\mid\leq C\varepsilon^{\frac{3-2s}{2}}$, and

$$\int_{\mathbb{R}^3}\mid(-\Delta)^{\frac{s}{2}}u_\varepsilon\mid^2dx\leq S^{\frac{3}{2s}}+o(\varepsilon^{3-2s}), \tag{2.4}$$
$$\int_{\mathbb{R}^3}\mid u_\varepsilon\mid^{2_s^*}dx=S^{\frac{3}{2s}}+o(\varepsilon^3).$$

By a direct calculation, there exists a κ_0 such that

$$\begin{aligned}\int_{\mathbb{R}^3}\mid u_\varepsilon\mid^2dx&\leq\int_{B_{2r}(0)}\mid U_\varepsilon(x)\mid^2dx\\&=\varepsilon^{-(3-2s)}\int_{B_{2r}(0)}\frac{\kappa_0^2}{(\mu^2+\mid\frac{x}{\varepsilon S^{\frac{1}{2s}}}\mid)^{3-2s}}\\&\leq C\varepsilon^{2s}\int_{B_{2r}(0)}^{\frac{2r}{\mu S^{\frac{1}{2s}}\varepsilon}}\frac{t^2}{(1+t^2)^{3-2s}}dt\\&=\begin{cases}O\left(\varepsilon^{2s}\mid\log\varepsilon\mid\right), & s=\frac{3}{4},\\ O\left(\varepsilon^{3-2s}\right), & s\in(\frac{3}{4},1).\end{cases}\end{aligned}\tag{2.5}$$

From Lemma 2.2, there exists $t_\varepsilon>0$ such that $I(t_\varepsilon u_\varepsilon)=\max\limits_{t\geq0}I(tu_\varepsilon)$, for $I'(t_\varepsilon u_\varepsilon)=0$ and (a_3), we have

$$\begin{aligned}0&=t_\varepsilon\int_\Omega|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2+t_\varepsilon\int_\Omega u_\varepsilon^2dx-\int_\Omega f(t_\varepsilon u_\varepsilon)u_\varepsilon dx-t_\varepsilon^{2_s^*-1}\\&\leq t_\varepsilon\int_\Omega|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2+t_\varepsilon\int_\Omega u_\varepsilon^2dx-t_\varepsilon^{2_s^*-1}.\end{aligned}$$

Then we get that

$$t^{2_s^*-2}\leq\int_\Omega\mid(-\Delta)^{\frac{s}{2}}u_\varepsilon\mid^2dx+\int_\Omega u_\varepsilon^2dx,$$

combine with (2.4) and (2.5), one get that $t_\varepsilon\leq C$, where C is a constant independent of ε. For some constant $C_1>0$, one gets $t_\varepsilon\geq C_1>0$ for $\varepsilon>0$ small. Otherwise, there is a sequence $\varepsilon_n\to0$ as $n\to\infty$ such that $t_{\varepsilon_n}\to0$ as $n\to\infty$, along a subsequence, we get $t_{\varepsilon_n}u_{\varepsilon_n}\to0$ in E as $n\to\infty$, therefore

$$0<c\leq\max_{t\geq0}I(tu_{\varepsilon_n}(x))=I(t_{\varepsilon_n}u_{\varepsilon_n}(x))\to0,$$

it is a contradiction. From (f_3) and [17], for any $M>0$, there is $T_M>0$ such that $t\in[T_M,+\infty]$, one gets

$$\int_{|x-x_0|<\varepsilon}F(u_\varepsilon)\geq\begin{cases}CM\varepsilon^{2s}\mid ln\varepsilon\mid, & s=\frac{3}{4},\\ CM\varepsilon^{3-3s}, & s\in(\frac{3}{4},1).\end{cases}$$

Hence, for ε small enough, we have

$$F(x_0)=\begin{cases}Mx_0^2, & s=\frac{3}{4},\\ Mx_0^{\frac{2s}{3-3s}}, & s\in(\frac{3}{4},1).\end{cases}$$

Together with (2.5), one has

$$\lim_{\varepsilon\to 0^+}\frac{\int_{|x-x_0|<\varepsilon}F(u_\varepsilon)dx}{\int_\Omega |u_\varepsilon(x)|^2\,dx}=+\infty. \tag{2.6}$$

Moreover we get that

$$\int_{|x-x_0|<\varepsilon}F(u_\varepsilon)dx\geq C\int_{|x-x_0|<\varepsilon}|u_\varepsilon|^2\geq C\int_\Omega |u_\varepsilon|^2\,dx, \tag{2.7}$$

by (a_2), (2.5), (2.6), (2.7) and $t_\varepsilon\leq C$, we consider

$$\begin{aligned}
I(t_\varepsilon u_\varepsilon) &= \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\,dx+\frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}(1+\mu a(x))u_\varepsilon^2dx\\
&\quad -\int_{\mathbb{R}^3}F(t_\varepsilon u_\varepsilon)dx-\frac{t_\varepsilon^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}\,dx\\
&=\frac{t_\varepsilon^2}{2}\int_\Omega|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\,dx+\frac{t_\varepsilon^2}{2}\int_\Omega u_\varepsilon^2dx-\int_\Omega F(t_\varepsilon u_\varepsilon)dx-\frac{t_\varepsilon^{2_s^*}}{2_s^*}\int_\Omega|u_\varepsilon|^{2_s^*}\,dx\\
&\leq \max_{t\geq 0}\Big(\frac{t^2}{2}\int_\Omega|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\,dx+\frac{t^2}{2}\int_\Omega u_\varepsilon^2dx\Big)\\
&\quad -\int_\Omega F(t_\varepsilon u_\varepsilon)dx+\frac{t_\varepsilon^{2_s^*}}{2_s^*}\int_\Omega|u_\varepsilon|^{2_s^*}\,dx\\
&\leq \frac{1}{3}S^{\frac{3}{2s}}+o(\varepsilon^{\frac{1}{2}})-\int_{|x-x_0|\leq\varepsilon}F(t_\varepsilon u_\varepsilon)dx+C\int_\Omega|u_\varepsilon|^2\,dx\\
&\leq \frac{1}{3}S^{\frac{3}{2s}}.
\end{aligned}$$

□

Lemma 2.4. *Assume $(f_1)-(f_3)$, (a_1) and (a_3) hold, let $\{u_k\}\subset H^s(\mathbb{R})$ be a sequence such that $I(u_k)\to c$ ($c<\frac{1}{3}S^{\frac{3}{2s}}$), $I'(u_k)\to 0$, then u_k is bounded.*

Proof. If we assume that $\|u_k\|\to\infty$, let $v_k=\frac{u_k}{\|u_k\|}$, then $v_k\rightharpoonup v$ weakly in $H^s(\mathbb{R})$, $v_k\to v$ a.e. in $\mathbb{R}$.

If $v(x)=0$ a.e. in E for $x\in\mathbb{R}^3$, by $(f_1),(f_2)$, there is a constant $\omega\in(0,\frac{1}{\theta}-\frac{1}{2_s^*})$ satisfied $\theta\in(4,2_s^*)$ and a constant $C_\omega>0$ such that

$$\Big|\frac{1}{\theta}f(u_k)u_k-F(u_k)\Big|\leq\omega|u_k|^{2_s^*}+C_\omega|u_k|^2,$$

then

$$\Big(I(u_k)-\frac{1}{\theta}\langle I'(u_k)u_k\rangle\Big)\geq\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|u_k\|^2-C_\omega\int_{\mathbb{R}^3}|u_k|^2\,dx,$$

thus, we derive that

$$\frac{1}{\|u_k\|^2}\Big(I(u_k)-\frac{1}{\theta}\langle I'(u_k)u_k\rangle\Big)\geq\Big(\frac{1}{2}-\frac{1}{\theta}\Big)-C_\omega\int_{\mathbb{R}^3}|v_k|^2\,dx.$$

We have known that $I(u_k)\to c$, $I'(u_k)\to 0$ and $\|u_k\|\to\infty$, then through a simple calculation, we get $0\geq(\frac{1}{2}-\frac{1}{\theta})$, it is a contradiction.

If $v(x) \neq 0$, let $\Omega = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, by $v_k(x) = \frac{u_k}{\|u_k\|} \to v(x)$ for any $x \in \Omega$, we have $\lim\limits_{n\to\infty} u_k(x) \to \infty$, from (f_3), there is a constant $K, C > 0$, such that $F(u) \geq Ku^2 + C$ and by Fatou's lemma

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\| u_k \|} = \lim_{n\to\infty} \frac{I(u_k)}{\| u_k \|} \\
&= \lim_{n\to\infty} \left[\frac{1}{2} - \frac{1}{\| u_k \|^2} \int_{\mathbb{R}^3} F(u)dx - \frac{1}{2_s^* \| u_k \|^2} \int_{\mathbb{R}^3} u^{2_s^*} dx \right] \\
&\leq \lim_{n\to\infty} \left[\frac{1}{2} - \int_{\mathbb{R}^3} \frac{F(u_k)}{u_k^2} v_k^2 dx \right] \\
&\leq \frac{1}{2} - K \int_{\mathbb{R}^3} v^2 dx,
\end{aligned}$$

when K is big enough, it is a contradiction. Thus, u_k is bounded. $\square$

Lemma 2.5. *If (a_1), (a_3), $(f_1) - (f_3)$ hold, there is a constant $\mu_0 > 0$, for any $\mu > \mu_0$, there holds $u \neq 0$ and $I'(u) = 0$.*

Proof. By Lemma 2.1, Lemma 2.2 and Lemma 2.4, we get that there is a bounded sequence $\{u_k\} \subset E$, along a subsequence still written as $\{u_k\}$. There exists a $u \in E$ such that $u_k \rightharpoonup u$, $I(u) \to c < \frac{1}{3} S^{\frac{3}{2s}}$ and $I'(u_k) \to 0$. For any $\varphi \in C^\infty(\mathbb{R})$, we have $\langle I'(u_k), \varphi \rangle = 0$, and

$$\begin{aligned}
&\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_k (-\Delta)^{\frac{s}{2}} \varphi + (1+\mu a(x)) u_k \varphi dx \to \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + \Big(1+\mu a(x)\Big) u \varphi dx, \\
&\int_{\mathbb{R}^3} f(u_k)\varphi dx - \int_{\mathbb{R}^3} u_k^{2_s^*-2} u \varphi dx \to \int_{\mathbb{R}^3} f(u)\varphi dx - \int_{\mathbb{R}^3} u^{2_s^*-2} u \varphi dx.
\end{aligned}$$

So we derive that $I'(u) = 0$.

Suppose that $u = 0$, then $u_k \rightharpoonup 0$, $B_r(y)$ be a bounded ball, we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} | u |^2 dx \geq 0.$$

Next, we will discuss two cases to show that this hypothesis does not hold.

If $\lim\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^3} \int_{B_r(y)} | u |^2 dx = 0$, by Lions Lemma([19], Lemma 1.21), we can derive that $\int_{|x|\leq r} | u_k |^p dx \to 0$, where $p \in (2, 2_s^*)$. From (f_1) and (f_2), there is a $\varepsilon > 0$ satisfying $C_\varepsilon > 0$ such that $| F(u_k) |\leq \varepsilon(u_k^2 + u_k^{2_s^*}) + C_\varepsilon | u_k |^q$, $q \in (2, 2_s^*)$. By Lebesgue dominated convergence theorem, there is

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_k)dx = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^3} f(u_k)u_k dx = 0.$$

Thus

$$\begin{aligned}
I(u_k) &= \frac{1}{2} \| u_k \|^2 - \int_{\mathbb{R}^3} F(u_k)dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} u_k^{2_s^*} dx \\
&= \frac{1}{2} \| u_k \|^2 - \frac{1}{2_s^*} \int_{\mathbb{R}^3} u_k^{2_s^*} dx + o_k(1),
\end{aligned}$$

and

$$o_k(1) = I'(u_k) = \| u_k \|^2 - \int_{\mathbb{R}^3} f(u_k)u_k dx - \int_{\mathbb{R}^3} u_k^{2_s^*} dx$$
$$= \| u_k \|^2 - \int_{\mathbb{R}^3} u_k^{2_s^*} dx. \tag{2.8}$$

We take a subsequence, which is still recorded as $\{u_k\}$. Then there exists $l \geq 0$ such that

$$\lim_{k\to\infty} \| u_k \|^2 = \lim_{k\to\infty} | u_k |_{2_s^*}^{2_s^*} = l. \tag{2.9}$$

Suppose $l = 0$, then $\| u_k \| \to 0$ in E, $I(u_k) \to 0$. But we have already known that $I(u_k) \to c > 0$, there is a contradiction. Therefore, $l > 0$ holds. By $I(u_k) \to c > 0$, (2.8) and (2.9), we get

$$3c \geq l + 1, \tag{2.10}$$

by the definition of S, we derive that

$$\| u_k \|^2 \geq \int_{\mathbb{R}^3} \Big((-\Delta)^{\frac{s}{2}} u_k \Big)^2 dx \geq S \Big(\int_{\mathbb{R}^3} | u_k |^{2_s^*} \Big)^{\frac{2}{2_s^*}},$$

which could get $l \geq S^{\frac{3}{2s}}$ as $k \to \infty$, combining with (2.10), we derive that $c \geq \frac{1}{3} S^{\frac{3}{2s}}$, a contradiction.

If we assume that $\lim\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^3} \int_{B_r(y)} | u |^2 \, dx > 0$, there is a positive constant $\alpha > 0$ such that

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} | u |^2 dx = \alpha > 0.$$

For convenience, let $D_r = \{x \in \mathbb{R}^3 \setminus B_r : a(x) \geq a_0\}$, $A_r = \{x \in \mathbb{R}^3 \setminus B_r : a(x) < a_0\}$, moreover, $\mathrm{meas}(A_r) \to 0$ as $r \to \infty$ by (a_1), one has

$$\limsup_{k\to\infty} \int_{D_r} u_k^2 \leq \limsup_{k\to\infty} \frac{1}{1+\mu a_0} \int_{\mathbb{R}^3} (1 + \mu a_k) u_k^2 dx$$
$$\leq \frac{1}{1+\mu a_0} \limsup_{k\to\infty} \| u_k \|^2$$
$$\leq \frac{C}{1+\mu a_0}, \tag{2.11}$$

where $\| u_k \|^2 \leq C$ is independent of k and μ, then take $\mu \geq \frac{4C}{\alpha a_0}$ such that $\int_{D_r} u_k^2 \leq \frac{C}{1+\mu a_0} \leq \frac{C}{\mu a_0} \leq \frac{\alpha}{4}$ uniformly in k. Moreover, we verify that

$$\int_{A_r} u_k^2 dx \leq \Big(\int_{A_r} u_k^q dx \Big)^{\frac{2}{q}} \Big(\int_{A_r} 1 dx \Big)^{\frac{q-2}{q}}$$
$$\leq \| u_k \|^2 \, (\mathrm{meas}(A_r))^{\frac{q-2}{q}}$$
$$\leq C (\mathrm{meas}(A_r))^{\frac{q-2}{q}} \to 0, \tag{2.12}$$

where $q \in (2, 2_s^*]$ and uniformly in k. From $u_k \to 0$ in $L_{loc}^p(\mathbb{R}^3)$ with $q \in (2, 2_s^*]$, (2.11) and (2.12), as $r \to \infty$, we have

$$\alpha = \lim_{k\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r} u_k^2 dx \leq \limsup_{k\to\infty} \int_{\mathbb{R}^3} u^2 dx$$

$$
\begin{aligned}
&= \limsup_{k\to\infty}\left(\int_{B_r} |u_k|^2\, dx + \int_{B_r^c} |u_k|^2\, dx\right)\\
&= \limsup_{k\to\infty}\left(\int_{D_r} u_k^2 dx + \int_{A_r} u_k^2 dx\right)\\
&\le \frac{\alpha}{4}.
\end{aligned}
$$

This is a contradiction. Hence, there exists $u \neq 0$ such that $u_k \rightharpoonup u$ in E and $I'(u) = 0$, the proof is finished. □

Proof of Theorem 1.1. By Lemma 2.2 - Lemma 2.5, there is a bounded sequence $\{u_k\} \subset E$ such that $I(u_k) \to c < \frac{1}{3}S^{\frac{3}{2s}}$, $I'(u_k) \to 0$. We take a subsequence still denoted by $\{u_k\}$, then $u_k \rightharpoonup u$ in E, $I(u_k) \to c < \frac{1}{3}S^{\frac{3}{2s}}$, $I'(u_k) \to 0$, $I(u) \to c < \frac{1}{3}S^{\frac{3}{2s}}$ and $I'(u) \to 0$. Let

$$
\mathcal{N} = \Big\{u \in E \setminus \{0\} : I'(u) = 0\Big\}, \quad m = \inf_{\mathcal{N}} I(u).
$$

By the argument above, we know that $\mathcal{N}$ is not empty. Now we claim that $m > 0$. We establish the Pohozaev type identity of equation (1.1) as follows

$$
\begin{aligned}
P(u) := \frac{3-2s}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2\, dx + \frac{3}{2}\int_{\mathbb{R}^3}(1+\mu a(x))u^2 dx\\
-3\int_{\mathbb{R}^3} F(u)dx - \frac{3}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\, dx = 0.
\end{aligned}
\tag{2.13}
$$

Moreover we have

$$
\begin{aligned}
I(u) &= I(u) - \frac{1}{2_s^*}I'(u)\\
&= (\frac{1}{2} - \frac{1}{2_s^*})\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 + (1+\mu a(x))u^2 dx\\
&\quad + \frac{1}{2_s^*}\int_{\mathbb{R}^3} f(u)u dx - \int_{\mathbb{R}^3} F(u)dx.
\end{aligned}
\tag{2.14}
$$

Combined with (2.13) (2.14) and (a_2), for $s \in [\frac{3}{4}, 1)$, $u \neq 0$, we have

$$
\begin{aligned}
I(u) &= \frac{4s-3}{6}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(1+\mu a(x))u^2 dx\\
&\quad -\frac{1}{6}\int_{\mathbb{R}^3}(\nabla(1+\mu a(x)), x)u^2 dx + \frac{1}{2_s^*}\int_{\mathbb{R}^3}(f(u)u + u^{2_s^*})dx\\
&\ge \frac{4s-3}{6}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 + \frac{1}{2_s^*}\int_{\mathbb{R}^3}(f(u)u + u^{2_s^*})dx\\
&> 0.
\end{aligned}
$$

Thus, we get $m > 0$. Assume $u_k \rightharpoonup u$ weakly in E, similar to the proof of Lemma 2.5, we derive $I(u) = m$ and $I'(u) = 0$ with $u \neq 0$ from $I(u_k) \to m < \frac{1}{3}S^{\frac{3}{2}}$ and $I'(u_k) = 0$, so m is attained by u. The proof is completed. □

Acknowledgment

The author would like to thank the anonymous referee of this paper for very helpful comments and suggestions.

References

[1] C. Alves and M. Souto, *Existence of solution for a class of problem in whole* $\mathbb{R}^{\mathbb{N}}$ *without the Ambrosetti-Rabinowitz condition*, manuscripta math., 2021, 165, 453–468.

[2] C. O. Alves, M. A. S. Souto and M. Montenegro, *Existence of a ground state solution for a nonlinear scalar field equation with critical growth*, Calc. Var. Partial Differential Equations, 2012, 43, 537–554.

[3] D. Albuquerque, C. Josė, F. D. Silva and D. Edcarlos, *Ground states for fractional linear coupled systems via profile decomposition*, Nonlinearity, 2021, 34, 4787–4818.

[4] T. Bartsch and Z. Tang, *Multibump solutions of nonlinear Schrödinger equations with steep potential well and indefinite potential*, Discrete Contin. Dyn. Syst., 2013, 33, 7–26.

[5] T. Bartsch and Z. Wang, *Existence and multiplicity results for superlinear elliptic problems on* $\mathbb{R}^n$, Comm. Partial Differential Equations, 1995, 20, 1725–1741.

[6] T. Bartsch and Z. Wang, *Multiple positive solutions for a nonlinear Schrödinger equation*, Z. Angew. Math. Phys., 2000, 51, 366–384.

[7] M. Clapp and Y. Ding, *Positive solutions of a Schrödinger equation with critical nonlinearity*, Z. Angew. Math. Phys., 2004, 55, 592–605.

[8] X. Chang, *Ground state solutions of asymptotically linear fractional Schrödinger equation*, J. Math. Phys., 2013, 54.

[9] J. M. Do and D. Ferraz, *Concentration-compactness for singular nonlocal Schrödinger equations with oscillatory nonlinearities*, Topol. Methods Nonlinear Anal., 2018, 52, 372–421.

[10] D. N. Eleonora, P. Giampiero and V. Enrico, *Hitchhiker¡¯s guide to the fractional Sobolev spaces*, Bull. Sci. Math., 2012, 136, 521–573.

[11] P. Felmer, A. Quaas and J. Tan, *Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian*, Proc. R. Soc. Edinb. A., 2012, 142, 1237–1262.

[12] X. Fang and Z. Han, *Ground state solution and multiple solutions to asymptotically linear Schrödinger equations*, J. Boundary Value Problems, 2014, 216.

[13] H. Jia, *Concentrating ground state solutions for quasilinear Schrödinger equations with steep potential well*, Appl. Anal., 2021, 100, 3065–3082.

[14] P. Li and Y. Yuan, *Existence and multiplicity of solutions for the fractional Schrödinger equation with steep potential well perturbation*, Int. J. Dyn. Syst. Differ. Equ., 2019, 9, 335–348.

[15] Y. Li and G. Li, *Positive ground state solutions for Choquard equations with lower critical exponent and steep well potential*, Applied Mathematics Letters, 2021, 118, 107151.

[16] J. Mawhin and M. Willem, *Critical point theory and Hamiltonian system*, Springer-Verlag, Berlin, 1989.

[17] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc., 2015, 367, 67–102.

[18] K. Teng, *Multiple solutions for a class of fractional Schrödinger equation in* $\mathbb{R}^N$, Nonlinear Anal. Real World Appl., 2015, 21, 76–86.

[19] M. Willem, *Minimax Theorems*, in Progr. Nonlinear Differential Equations Appl., 1996, 24.

[20] L. Yin, X. Wu and C. Tang, *Existence and concentration of ground state solutions for critical Schrödinger-Poisson equation with steep potential well*, J. Applied Mathematicas and Computation, 2020, 374.

[21] H. Zhang, J. Xu and F. Zhang, *Existence and multiplicity of solutions for superlinear fractional Schrödinger equations in* $\mathbb{R}^n$, J. Math. Phys., 2015, 56, 091502.

[22] J. Zhang, Z. Luo, Y. Ji and W. Shao, *Ground state of Kirchhoff type fractional Schrödinger equations with critical growth*, J. Math. Anal. Appl., 2018, 462, 57–83.