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NORM EQUALITIES AND INEQUALITIES FOR TRIDIAGONAL PERTURBED TOEPLITZ OPERATOR MATRICES*

Jiajie Wang[1], Yanpeng Zheng[2,†] and Zhaolin Jiang[1,†]

Abstract Tridiagonal perturbed Toeplitz operator matrices is a class of important structured matrices. In this paper, we present several norm equalities and inequalities for this class of matrices. The special norms we consider include the usual operator norm and the Schatten p-norms. Moreover, pinching type inequalities are also discussed for general weakly unitarily invariant norms. The proofs feature the special structure of tridiagonal perturbed Toeplitz operator matrices.

Keywords Tridiagonal perturbed Toeplitz operator matrix, unitarily invariant norm, equality, inequality.

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1. Introduction

Tridiagonal and tridiagonal perturbed Toeplitz operator matrices play an important role in describing or solving problems of cutting-edge technology [16, 24, 25]. Block tridiagonal and block tridiagonal perturbed Toeplitz operator matrices are of great importance in the surface anomalous Hall effect [11, 32], Markov chain [19], physics [6, 18, 20, 26, 33] and to obtain general properties is of great utility. In general, there is a lot of research on tridiagonal operator matrices or related operator matrices, like their determinants, inverses, and spectra et al. [8–10, 21–23, 27–31].

The problem of estimating the norms of some operator matrices appearing in various fields has attracted some authors to study [1, 13, 17]. Bhatia and Kittaneh [2] used the Schatten p-norm version of the circulant operator matrix to extend Clarkson's inequality to several operators. They [3] had also given norm inequalities for partitioned operators and an application. Bhatia et al. [4] considered pinchings and norms of scaled triangular matrices. Besides, Jiang and Xu [15] considered norm estimates of ω-circulant operator matrices. Jiang et al. presented the norm equalities and inequalities for three circulant operator matrices in [14].

†The corresponding author.
Email: 937262674@qq.com(J. Wang), zhengyanpeng0702@sina.com(Y. Zheng), jzh1208@sina.com(Z. Jiang)
[1]School of Mathematics and Statistics, Linyi University, Linyi, 276000, China
[2]School of Automation and Electrical Engineering, Linyi University, Linyi, 276000, China
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Denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $A \in \mathcal{B}(\mathcal{H})$, let $\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$ and $\|A\| = \sup\{|\langle Ax, y\rangle| : x, y \in \mathcal{H}, \|x\| = \|y\| = 1\}$ be the numerical radius and the usual operator norm of A, respectively. Let $\oplus^n \mathcal{H}$ denote the direct sum of n copies of $\mathcal{H}$.

The weakly unitarily invariant norm τ [7] on $\mathcal{B}(\mathcal{H})$ is defined by $\tau(A) = \tau(UAU^*)$ for $A \in \mathcal{B}(\mathcal{H})$ and unitary operator $U \in \mathcal{B}(\mathcal{H})$. The Schatten p-norms $\|\cdot\|_p$ $(1 \leq p < \infty)$ [7], defined as $\|A\|_p = (\mathrm{tr}|A|^p)^{1/p}$, where $|A| = (A^*A)^{1/2}$, are unitarily invariant.

Now we define the operator matrix (or the partitioned operator) $A = [A_{jk}]$ in $\mathcal{B}(\oplus^n \mathcal{H})$ by

$$Ax = \begin{pmatrix} \Sigma_{k=1}^n A_{1k} x_k \\ \vdots \\ \Sigma_{k=1}^n A_{nk} x_k \end{pmatrix}$$

where every vector $x = (x_1 \cdots x_n)^T \in \oplus^n \mathcal{H}$, and $A_{jk} \in \mathcal{B}(\mathcal{H}), j, k = 1, 2, \ldots, n$.

The pinching inequality [5, 7] for weakly unitarily invariant norms says that

$$\tau\left(\oplus_{j=1}^n A_{jj}\right) \leq \tau(A), \tag{1.1}$$

where $A = [A_{jk}]$.

For the operator norm and the Schatten p-norms, the inequality (1.1) becomes

$$\max\{\|A_{jj}\| : j = 1, 2, \ldots, n\} \leq \|A\| \tag{1.2}$$

and

$$\left(\sum_{j=1}^n \|A_{jj}\|_p^p\right)^{1/p} \leq \|A\|_p \tag{1.3}$$

for $1 \leq p < \infty$, respectively. As shown in [12], the equality (1.3) holds for $1 < p < \infty$ if and only if A is block-diagonal, that is, if and only if $A_{jk} = 0$, for $j \neq k$.

2. Norm equalities for special tridiagonal perturbed Toeplitz operator matrices

The main results of this section are some norm equalities for tridiagonal perturbed Toeplitz operator matrices. Some consequences are presented when the norm or the tridiagonal perturbed Toeplitz operator matrix is specified.

Theorem 2.1. *Let*

$$\mathfrak{D} = \begin{pmatrix} M+N & N & 0 & \cdots & 0 \\ N & M & N & \ddots & \vdots \\ 0 & N & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & M & N \\ 0 & \cdots & 0 & N & M+N \end{pmatrix}_{n \times n} \tag{2.1}$$

be a tridiagonal perturbed Toeplitz operator matrix in $\mathcal{B}(\mathcal{H}^{(n)})$, where $M, N \in \mathcal{B}(\mathcal{H})$. Then for every weakly unitarily invariant norm $\tau(\cdot)$ such that

$$\tau(\mathfrak{D}) = \tau\Big(\bigoplus_{s=1}^{n}[M - (2\cos\frac{s\pi}{n})N]\Big). \tag{2.2}$$

Proof. Let

$$U = \sqrt{\frac{2}{n}} \begin{pmatrix} \cos\frac{(n-1)\pi}{2n} & \cos\frac{3(n-1)\pi}{2n} & \cos\frac{5(n-1)\pi}{2n} & \cdots & \cos\frac{(2n-1)(n-1)\pi}{2n} \\ \cos\frac{(n-2)\pi}{2n} & \cos\frac{3(n-2)\pi}{2n} & \cos\frac{5(n-2)\pi}{2n} & \cdots & \cos\frac{(2n-1)(n-2)\pi}{2n} \\ \cos\frac{(n-3)\pi}{2n} & \cos\frac{3(n-3)\pi}{2n} & \cos\frac{5(n-3)\pi}{2n} & \cdots & \cos\frac{(2n-1)(n-3)\pi}{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \cdots & \frac{\sqrt{2}}{2} \end{pmatrix}_{n\times n} \otimes I, \tag{2.3}$$

where I is the identity operator in $\mathcal{B}(\mathcal{H})$ and $\otimes$ is the direct (Kronecker) product of two matrices. It can be seen that the column vectors of the $n \times n$ matrix given in the definition of U form an orthonormal set of vectors. Then it is easy to prove that the operator matrix U is a unitary operator in $\mathcal{B}(\mathcal{H}^{(n)})$.

Noting that the operator matrix U is a unitary operator in $\mathcal{B}(\mathcal{H}^{(n)})$ and special construction of the operator matrix $\mathfrak{D}$, it follows that the operator matrix U such that

$$U\mathfrak{D}U^* = \mathrm{diag}\big(M - (2\cos\frac{\pi}{n})N, M - (2\cos\frac{2\pi}{n})N, \cdots, M - (2\cos\frac{n\pi}{n})N\big). \tag{2.4}$$

According to the invariance property of weakly unitarily invariant norms and the equation (2.4), we get the weakly unitarily invariant norms $\tau(\cdot)$ of the operator matrix $\mathfrak{D}$ satisfy the equation (2.2). □

Specifying the norm equality in Theorem 2.1 to the usual operator norm and to the Schatten p-norms, we obtain the following equalities.

1.
$$\omega(\mathfrak{D}) = \max\big\{\omega\big(M - (2\cos\frac{s\pi}{n})N\big) : s = 1, 2, \ldots, n\big\}.$$

In particular, (letting $n = 3$), we have

$$\omega(\mathfrak{D}) = \max\{\omega(M - N), \omega(M + N), \omega(M + 2N)\}.$$

2.
$$\|\mathfrak{D}\| = \max\{\|M - (2\cos\frac{s\pi}{n})N\| : s = 1, 2, \ldots, n\}.$$

In particular, (letting $n = 3$), we have

$$\|\mathfrak{D}\| = \max\{\|M - N\|, \|M + N\|, \|M + 2N\|\}.$$

3.
$$\|\mathfrak{D}\|_p = \left(\sum_{s=1}^{n} \Big\|M - (2\cos\frac{s\pi}{n})N\Big\|_p^p\right)^{\frac{1}{p}}$$

for $1 \le p < \infty$.

In particular, (letting $n=3$), we have

$$\|\mathfrak{D}\|_p=\left(\|M-N\|_p^p+\|M+N\|_p^p+\|M+2N\|_p^p\right)^{\frac{1}{p}}.$$

Here we give some special cases of Theorem 2.1.

1. If M=0, then

$$\omega(\mathfrak{D})=2\max\{|\cos\frac{s\pi}{n}|\,\omega(N):s=1,2,\dots,n\},$$
$$\|\mathfrak{D}\|=2\max\{|\cos\frac{s\pi}{n}|\,\|N\|:s=1,2,\dots,n\},$$

and

$$\|\mathfrak{D}\|_p=2\left(\sum_{s=1}^{n}|\cos\frac{s\pi}{n}|^p\right)^{\frac{1}{p}}\|N\|_p$$

for $1\leq p<\infty$.

2. If N=0, then

$$\omega(\mathfrak{D})=\omega(M),\quad \|\mathfrak{D}\|=\|M\|,$$

and

$$\|\mathfrak{D}\|_p=n^{\frac{1}{p}}\|M\|_p$$

for $1\leq p<\infty$.

3. If M=N, then

$$\omega(\mathfrak{D})=\max\{|1-2\cos\frac{s\pi}{n}|\,\omega(M):s=1,2,\dots,n\},$$
$$\|\mathfrak{D}\|=\max\{|1-2\cos\frac{s\pi}{n}|\,\|M\|:s=1,2,\dots,n\},$$

and

$$\|\mathfrak{D}\|_p=\left(\sum_{s=1}^{n}|1-2\cos\frac{s\pi}{n}|^p\right)^{\frac{1}{p}}\|M\|_p$$

for $1\leq p<\infty$.

4. If N=iM, then

$$\omega(\mathfrak{D})=\max\{\omega((1-(2\cos\frac{s\pi}{n})i)M):s=1,2,\dots,n\},$$
$$\|\mathfrak{D}\|=\max\{|1-(2\cos\frac{s\pi}{n})i|\,\|M\|:s=1,2,\dots,n\},$$

and

$$\|\mathfrak{D}\|_p=\left(\sum_{s=1}^{n}|1-(2\cos\frac{s\pi}{n})i|^p\right)^{\frac{1}{p}}\|M\|_p$$

for $1\leq p<\infty$.

Theorem 2.2. *Let*

$$\mathfrak{Q} = \begin{pmatrix} M+\rho^{\frac{k_1+k_2}{2}}N & \rho^{k_2}N & 0 & \cdots & 0 \\ \rho^{k_1}N & M & \rho^{k_2}N & \ddots & \vdots \\ 0 & \rho^{k_1}N & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & M & \rho^{k_2}N \\ 0 & \cdots & 0 & \rho^{k_1}N & M+\rho^{\frac{k_1+k_2}{2}}N \end{pmatrix}_{n\times n} \tag{2.5}$$

be a tridiagonal perturbed Toeplitz operator matrix in $\mathcal{B}(\mathcal{H}^{(n)})$*, where* $M, N \in \mathcal{B}(\mathcal{H})$*,* $\rho = e^{\frac{i\varphi}{n}}$ *is the* n^{th} *root of* $e^{i\varphi}$*,* $0 < \varphi \le 2\pi, i = \sqrt{-1}$*,* k_1 *and* k_2 *are both positive integers. Then for every weakly unitarily invariant norm* $\tau(\cdot)$ *such that*

$$\tau(\mathfrak{Q}) = \tau(\bigoplus_{s=1}^{n}[M - (2\rho^{\frac{k_1+k_2}{2}}\cos\frac{s\pi}{n})N]). \tag{2.6}$$

Proof. Let $R\in\mathcal{B}(\mathcal{H}^{(n)})$ and $R=\mathrm{diag}\big(I, \rho^{\frac{k_2-k_1}{2}}I, \ldots, \rho^{\frac{(k_2-k_1)(n-2)}{2}}I, \rho^{\frac{(k_2-k_1)(n-1)}{2}}I\big)$. Then it is easy to prove that the operator matrix R is a unitary operator in $\mathcal{B}(\mathcal{H}^{(n)})$, i.e., $RR^* = I$.

Multiplying the both sides of the equation (2.5) by R from the left and by R^* from the right yields respectively, we obtain

$$R\mathfrak{Q}R^* = \begin{pmatrix} M+\rho^{\frac{k_1+k_2}{2}}N & \rho^{\frac{k_1+k_2}{2}}N & 0 & \cdots & 0 \\ \rho^{\frac{k_1+k_2}{2}}N & M & \rho^{\frac{k_1+k_2}{2}}N & \ddots & \vdots \\ 0 & \rho^{\frac{k_1+k_2}{2}}N & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & M & \rho^{\frac{k_1+k_2}{2}}N \\ 0 & \cdots & 0 & \rho^{\frac{k_1+k_2}{2}}N & M+\rho^{\frac{k_1+k_2}{2}}N \end{pmatrix}_{n\times n}. \tag{2.7}$$

According to the invariance property of weakly unitarily invariant norms, the equation (2.7) and the Theorem 2.1, we get the weakly unitarily invariant norms $\tau(\cdot)$ of the operator matrix $\mathfrak{Q}$ satisfy the equation (2.6). □

Corollary 2.1. *Let* $\mathfrak{Q} \in \mathcal{B}(\mathcal{H}^{(n)})$ *be given as in (2.5). If* $\varphi = 2\pi$*,* $k_1 = 1$ *and* $k_2 = n-1$*, then, we have*

$$\tau(\mathfrak{Q}) = \tau(\bigoplus_{s=1}^{n}[M + (2\cos\frac{s\pi}{n})N]).$$

Corollary 2.2. *Let* $\mathfrak{Q} \in \mathcal{B}(\mathcal{H}^{(n)})$ *be given as in (2.5). If* $k_1 = k$ *and* $k_2 = 2n-k$*, then, we have*

$$\tau(\mathfrak{Q}) = \tau(\bigoplus_{s=1}^{n}[M - (2e^{i\varphi}\cos\frac{s\pi}{n})N]).$$

Corollary 2.3. *Let* $\mathfrak{Q} \in \mathcal{B}(\mathcal{H}^{(n)})$ *be given as in (2.5). If* $k_1 = k$ *and* $k_2 = n-k$*, then, we have*

$$\tau(\mathfrak{Q}) = \tau(\bigoplus_{s=1}^{n}[M - (2e^{\frac{i\varphi}{2}}\cos\frac{s\pi}{n})N]).$$

Theorem 2.3. *Let*

$$\mathfrak{S} = \begin{pmatrix} M+\sqrt{\rho_1\rho_2}N & \rho_2 N & 0 & \cdots & 0 \\ \rho_1 N & M & \rho_2 N & \ddots & \vdots \\ 0 & \rho_1 N & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & M & \rho_2 N \\ 0 & \cdots & 0 & \rho_1 N & M+\sqrt{\rho_1\rho_2}N \end{pmatrix}_{n\times n} \tag{2.8}$$

be a tridiagonal perturbed Toeplitz operator matrix in $\mathcal{B}(\mathcal{H}^{(n)})$, where $M, N \in \mathcal{B}(\mathcal{H})$, the ρ_1 and ρ_2 are both complex numbers and $|\rho_1| = |\rho_2|$. Then for every weakly unitarily invariant norm $\tau(\cdot)$ such that

$$\tau(\mathfrak{S}) = \tau(\bigoplus_{s=1}^{n}[M - (2\sqrt{\rho_1\rho_2}\cos\frac{s\pi}{n})N]). \tag{2.9}$$

Proof. Let $T \in \mathcal{B}(\mathcal{H}^{(n)})$ and $T = \operatorname{diag}\big(I, (\frac{\rho_2}{\rho_1})^{\frac{1}{2}}I, \ldots, (\frac{\rho_2}{\rho_1})^{\frac{n-2}{2}}I, (\frac{\rho_2}{\rho_1})^{\frac{n-1}{2}}I\big)$. Then it is easy to prove that T is a unitary operator in $\mathcal{B}(\mathcal{H}^{(n)})$, i.e., $TT^* = I$.

Multiplying the both sides of the equation (2.8) by T from the left and by T^* from the right yields respectively, we get

$$T\mathfrak{S}T^* = \begin{pmatrix} M+\sqrt{\rho_1\rho_2}N & \sqrt{\rho_1\rho_2}N & 0 & \cdots & 0 \\ \sqrt{\rho_1\rho_2}N & M & \sqrt{\rho_1\rho_2}N & \ddots & \vdots \\ 0 & \sqrt{\rho_1\rho_2}N & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & M & \sqrt{\rho_1\rho_2}N \\ 0 & \cdots & 0 & \sqrt{\rho_1\rho_2}N & M+\sqrt{\rho_1\rho_2}N \end{pmatrix}_{n\times n}. \tag{2.10}$$

According to the invariance property of weakly unitarily invariant norms, the equation (2.10) and the Theorem 2.1, we have the desired result, i.e., the equation (2.9) for the weakly unitarily invariant norms $\tau(\cdot)$ of the operator matrix $\mathfrak{S}$. □

3. Pinching type inequalities for tridiagonal perturbed Toeplitz operator matrices

In this section we discuss pinching type inequalities for tridiagonal perturbed Toeplitz operator matrices. We also show that the equalities conditions in these norm inequalities are intimately connected to the class of tridiagonal perturbed Toeplitz operator matrices.

Theorem 3.1. *Let*

$$\mathfrak{W} = \begin{pmatrix} M+\frac{1}{2}(N+L) & N & 0 & \cdots & 0 \\ L & M & N & \ddots & \vdots \\ 0 & L & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & M & N \\ 0 & \cdots & 0 & L & M+\frac{1}{2}(N+L) \end{pmatrix}_{n\times n} \tag{3.1}$$

be a tridiagonal perturbed Toeplitz operator matrix in $\mathcal{B}(\mathcal{H}^{(n)})$, where $L, M, N \in \mathcal{B}(\mathcal{H})$. Then for every weakly unitarily invariant norm $\tau(\cdot)$ such that

$$\tau(\mathfrak{W}) \geq \tau(\bigoplus_{s=1}^{n}[M-(\cos\frac{s\pi}{n})(N+L)]). \tag{3.2}$$

Specially, the inequality (3.2) with equality if and only if $N = L$.

Proof. Let $Y \in \mathcal{B}(\mathcal{H}^{(n)})$ and

$$Y = \begin{pmatrix} 0 & 0 & \cdots & 0 & I \\ 0 & \cdots & 0 & I & 0 \\ \vdots & .\cdot^{\cdot} & I & 0 & 0 \\ 0 & .\cdot^{\cdot} & .\cdot^{\cdot} & .\cdot^{\cdot} & \vdots \\ I & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}. \tag{3.3}$$

Then it is easy to prove that Y is a unitary operator in $\mathcal{B}(\mathcal{H}^{(n)})$ and

$$\mathfrak{W}+Y\mathfrak{W}Y^* = \begin{pmatrix} 2M+(N+L) & N+L & 0 & \cdots & 0 \\ N+L & 2M & N+L & \ddots & \vdots \\ 0 & N+L & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2M & N+L \\ 0 & \cdots & 0 & N+L & 2M+(N+L) \end{pmatrix}_{n\times n}. \tag{3.4}$$

By the equation (3.4) and Theorem 2.1, we obtain

$$\tau(\mathfrak{W}+Y\mathfrak{W}Y^*) = \tau(\bigoplus_{s=1}^{n}[2M-(2\cos\frac{s\pi}{n})(N+L)]). \tag{3.5}$$

Now, from the equation (3.5), the invariance property of weakly unitarily invariant norms and the triangle inequality, we have

$$\tau(\mathfrak{W}) \geq \tau(\bigoplus_{s=1}^{n}[M-(\cos\frac{s\pi}{n})(N+L)]). \qquad \square$$

Specifying the norm equality in Theorem 3.1 to the usual operator norm and to the Schatten p-norms, we obtain the following corollary.

Corollary 3.1. *Let $L, M, N \in \mathcal{B}(\mathcal{H})$ and*

$$\mathfrak{W} = \begin{pmatrix} M + \frac{N+L}{2} & N & 0 & \cdots & 0 \\ L & M & N & \ddots & \vdots \\ 0 & L & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & M & N \\ 0 & \cdots & 0 & L & M + \frac{N+L}{2} \end{pmatrix}_{n \times n}$$

be a tridiagonal perturbed Toeplitz operator matrix in $\mathcal{B}(\mathcal{H}^{(n)})$. Then
(i)

$$\omega(\mathfrak{W}) \geq \max\{\omega(M - (\cos \frac{s\pi}{n})(N + L)) : s = 1, 2, \ldots, n\}$$

with equality when $N = L$.
(ii)

$$\|\mathfrak{W}\| \geq \max\{\|M - (\cos \frac{s\pi}{n})(N + L)\| : s = 1, 2, \ldots, n\}$$

with equality when $N = L$.
(iii)

$$\|\mathfrak{W}\|_p \geq \left(\sum_{s=1}^{n} \left\| M - (\cos \frac{s\pi}{n})(N + L) \right\|_p^p \right)^{\frac{1}{p}}$$

for $1 < p < \infty$ with equality if and only if $N = L$.

Proof. In view of Theorem 2.1, it is enough to prove the "only if" part of (iii). Assume that

$$\|\mathfrak{W}\|_p^p = \sum_{s=1}^{n} \left\| M - (\cos \frac{s\pi}{n})(N + L) \right\|_p^p$$

for $1 < p < \infty$ and consider the same U in the equation (2.3), then we have

$$\|\mathfrak{W}\|_p^p = \|U\mathfrak{W}U^*\|_p^p,$$

i.e.,

$$\left\| \begin{bmatrix} M + \frac{N+L}{2} & N & 0 & \cdots & 0 \\ L & M & N & \ddots & \vdots \\ 0 & L & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & M & N \\ 0 & \cdots & 0 & L & M + \frac{N+L}{2} \end{bmatrix}_{n \times n} \right\|_p^p$$

$$= \left\| \begin{bmatrix} M-(N+L)\cos\frac{\pi}{n} & W_{12} & \cdots & \cdots & W_{1n} \\ -W_{12} & \ddots & \ddots & \ddots & W_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & W_{n-1,n} \\ -W_{1n} & \cdots & \cdots & -W_{n-1,n} & M-(N+L)\cos\frac{n\pi}{n} \end{bmatrix}_{n\times n} \right\|_p^p$$

$$= \sum_{s=1}^{n} \left\| M-(N+L)\cos\frac{s\pi}{n} \right\|_p^p,$$

where $W_{ij} = (\frac{1-(-1)^{i+j}}{n}\sum_{k=1}^{n-1}\cos\frac{(2k-1)(n-i)\pi}{2n}\cos\frac{(2k+1)(n-j)\pi}{2n})(N-(-1)^{i+j-1}L)$, $(i=1,2,\ldots,n-1; j=i+1,i+2,\ldots,n-1)$, and $W_{in} = (\frac{1-(-1)^{i+n}}{\sqrt{2n}}\cos\frac{(n-i)\pi}{2n})(N-(-1)^{i+n-1}L)$, $(i=1,2,\ldots,n-1)$. Now employing (3), we conclude that $U\mathfrak{W}U^*$ must be block diagonal, i.e., $N-L=0$, and hence $N=L$, as required. $\square$

4. Norm equalities and pinching type inequalities for anti-tridiagonal perturbed Hankel operator matrices

In this section, we discuss norm equalities and pinching type inequalities for anti-tridiagonal perturbed Hankel operator matrices.

Theorem 4.1. *Let*

$$\mathfrak{V} = \begin{pmatrix} 0 & \cdots & 0 & N & M+N \\ \vdots & \iddots & N & M & N \\ 0 & \iddots & \iddots & N & 0 \\ N & M & \iddots & \iddots & \vdots \\ M+N & N & 0 & \cdots & 0 \end{pmatrix}_{n\times n} \tag{4.1}$$

be an anti-tridiagonal perturbed Hankel operator matrix in $\mathcal{B}(\mathcal{H}^{(n)})$, where $M, N \in \mathcal{B}(\mathcal{H})$. Then for every weakly unitarily invariant norm $\tau(\cdot)$ such that

$$\tau(\mathfrak{V}) = \tau(\bigoplus_{s=1}^{n}(-1)^{s+1}[M-(2\cos\frac{s\pi}{n})N]), \tag{4.2}$$

where n is odd, and

$$\tau(\mathfrak{V}) = \tau(\bigoplus_{s=1}^{n}(-1)^{s}[M-(2\cos\frac{s\pi}{n})N]), \tag{4.3}$$

where n is even.

Proof. Multiplying the both sides of the equation (4.1) by using the same U in the equation (2.3) from the left and by U^* from the right yields respectively, when n is odd, we have

$$U\mathfrak{V}U^* = \bigoplus_{s=1}^{n}(-1)^{s+1}[M - (2\cos\frac{s\pi}{n})N], \tag{4.4}$$

when n is even, we have

$$U\mathfrak{V}U^* = \bigoplus_{s=1}^{n}(-1)^{s}[M - (2\cos\frac{s\pi}{n})N]. \tag{4.5}$$

Hence, from the invariance property of weakly unitarily invariant norms, the equation (4.4) and the equation (4.5), we get the desired result, i.e., the equation (4.2) and the equation (4.3) for the weakly unitarily invariant norms $\tau(\cdot)$ of the operator matrix $\mathfrak{V}$. □

Theorem 4.2. *Let*

$$\mathfrak{X} = \begin{pmatrix} 0 & \cdots & 0 & N & M+\frac{N+L}{2} \\ \vdots & \iddots & N & M & L \\ 0 & \iddots & \iddots & L & 0 \\ N & M & \iddots & \iddots & \vdots \\ M+\frac{N+L}{2} & L & 0 & \cdots & 0 \end{pmatrix}_{n\times n} \tag{4.6}$$

be an anti-tridiagonal perturbed Hankel operator matrix in $\mathcal{B}(\mathcal{H}^{(n)})$, where $L, M, N \in \mathcal{B}(\mathcal{H})$. Then for every weakly unitarily invariant norm $\tau(\cdot)$ such that

$$\tau(\mathfrak{X}) \geq \tau\big(\bigoplus_{s=1}^{n}(-1)^{s+1}[M - (\cos\frac{s\pi}{n})(N+L)]\big), \tag{4.7}$$

where n is odd and

$$\tau(\mathfrak{X}) \geq \tau\big(\bigoplus_{s=1}^{n}(-1)^{s}[M - (\cos\frac{s\pi}{n})(N+L)]\big), \tag{4.8}$$

where n is even.

Specially, the inequalities (4.7) and (4.8) with equality if and only if $N = L$.

Proof. By using the same Y in the equation (3.3) and the equation (4.6), it is easy to prove that

$$\mathfrak{X} + Y\mathfrak{X}Y^* = \begin{pmatrix} 0 & \cdots & 0 & N+L & 2M+N+L \\ \vdots & \iddots & N+L & 2M & N+L \\ 0 & \iddots & \iddots & N+L & 0 \\ N+L & 2M & \iddots & \iddots & \vdots \\ 2M+N+L & N+L & 0 & \cdots & 0 \end{pmatrix}_{n\times n}. \tag{4.9}$$

Hence, from the equation (4.9) and Theorem 4.1, we have the result that when n is odd

$$\tau(\mathfrak{X} + Y\mathfrak{X}Y^*) = \tau\big(\bigoplus_{s=1}^{n}(-1)^{s+1}[2M - (2\cos\frac{s\pi}{n})(N+L)]\big), \tag{4.10}$$

and when n is even

$$\tau(\mathfrak{X} + Y\mathfrak{X}Y^*) = \tau\big(\bigoplus_{s=1}^{n}(-1)^{s}[2M - (2\cos\frac{s\pi}{n})(N+L)]\big). \tag{4.11}$$

Now, from the equation (4.10), the equation (4.11), the invariance property of weakly unitarily invariant norms and the triangle inequality, we obtain the desired results like the inequalities (4.7) and (4.8). □

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