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EIGENVALUE PROBLEM FOR A NABLA FRACTIONAL DIFFERENCE EQUATION WITH DUAL NONLOCAL BOUNDARY CONDITIONS

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Abstract In this work, we study the existence of positive solutions for the non-local boundary value problem for a finite nabla fractional difference equation with a parameter $\beta > 0$

$$\begin{cases} -\big(\nabla^{\alpha}_{\rho(a)} u\big)(t) = \beta f(t, u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = g_1(u), \quad u(b) = g_2(u). \end{cases}$$

With the help of properties of the Green's functions and appropriate conditions on the non-linear part of the difference equation, we are able to construct the eigenvalue intervals of the considered boundary value problem using Guo–Krasnoselskii fixed point theorem on a suitable cone. Finally, we provide a couple of examples to demonstrate the applicability of established results.

Keywords Nabla fractional difference, eigenvalue problem, parameter, fixed point, positive solution.

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1. Introduction

Over the last few decades, the theory of fractional calculus has been extensively developed due to its properties, generalizing many results of differential calculus and its non-local nature of fractional derivatives. The contributions of several mathematicians over the span of three centuries have resulted in a robust theory of fractional differential equations for the functions of a real variable. Its roots can be traced back to the Leibniz letter dated "30th September 1695". Today fractional calculus has been successfully used for mathematical modeling in the fields of medical sciences, computational biology, economics, physics and several areas of engineering in the past three decades. For further applications and historical literature, we refer here to a few classical texts on fractional calculus by Miller–Ross [22], Samko et al. [25], Podlubny [24] and Kilbas et al. [21].

On the other side of the coin, nabla fractional calculus is a branch of mathematics that deals with arbitrary order differences and sums in the backward sense. The theory of nabla fractional calculus is relatively young, with the most prominent

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works done in the past decade. The notion of nabla fractional difference and sum can be traced back to the work of Gray and Zhang [9], and Miller and Ross [23]. In this line, Atici & Eloe [3] developed nabla fractional Riemann–Liouville difference operator, initiated the study of nabla fractional initial value problem and established exponential law, product rule, and nabla Laplace transform. Following their works, the contributions of several mathematicians have made the theory of discrete fractional calculus a fruitful field of research in science and engineering. We refer here to a recent monograph by Goodrich & Peterson [7] and the references therein, which is an excellent source for all those who wish to work in this field.

The study of boundary value problems (BVPs) has a long past and can be followed back to the work of Euler and Taylor on vibrating strings. On the discrete fractional side, there is a sudden growth of interest in the development of nabla fractional BVPs. Many authors have studied nabla fractional BVPs recently. To name a few, Ahrendt [2], Goar [6], and Ikram [14] worked with self-adjoint Caputo nabla BVPs. Brackins [5] studied a particular class of self-adjoint Riemann–Liouville nabla BVPs and derived the Green's function associated with it along with a few of its properties. Gholami et al. [10] obtained the Green's function for a non-homogeneous Riemann–Liouville nabla BVP with Dirichlet boundary conditions. Jonnalagadda [11, 15–19] analyzed some qualitative properties of two-point non-linear Riemann–Liouville nabla BVPs associated with a variety of boundary conditions. Goodrich [8] has analyzed FBVP with a non-local condition in the delta case. Han [12], and Sun [26] have analyzed the existence and non-existence of positive solutions to a discrete eigenvalue problem with conjugate conditions and non-local conditions, respectively using Guo–Krasnoselskii fixed point theorem on cone in the delta case, to the best of our knowledge very recently authors in [11] have analyzed solution of nabla FBVP with non-local conditions.

We consider the following boundary value problem with dual non-local conditions with parameter $\beta > 0$

$$\begin{cases} -(\nabla^{\alpha}_{\rho(a)}u)(t) = \beta f(t, u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = g_1(u), \quad u(b) = g_2(u), \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_3, 1 < \alpha < 2,\ f : \mathbb{N}^{b}_{a+2} \times \mathbb{R} \to \mathbb{R}^{+} \cup \{0\}$ and the functionals $g_1, g_2 : \mathbb{R} \to \mathbb{R}$. The present article is organized as follows: Section 2 contains a few preliminaries on nabla fractional calculus. In Section 3, we construct the Green's function corresponding to (1.1) and state a few of its properties. In Section 4, we study the existence of at least one positive solution of (1.1) using the Guo–Krasnoselskii fixed point theorem on cones. In Section 5, we obtain sufficient conditions on the existence of a unique solution for the proposed class of boundary value problems using the contraction mapping theorem. Finally, we conclude this article with a few examples.

2. Preliminaries

Denote the set of all real numbers and positive integers by $\mathbb{R}$ and $\mathbb{Z}^{+}$, respectively. We use the following notations, definitions, and known results of nabla fractional calculus [7]. Assume empty sums and products are 0 and 1, respectively.

Definition 2.1. For $a \in \mathbb{R}$, the sets $\mathbb{N}_a$ and $\mathbb{N}_a^b$, where $b - a \in \mathbb{Z}^+$, are defined by

$$\mathbb{N}_a = \{a, a+1, a+2, ...\}, \quad \mathbb{N}_a^b = \{a, a+1, a+2, ..., b\}.$$

Definition 2.2. We define the backward jump operator, $\rho : \mathbb{N}_{a+1} \longrightarrow \mathbb{N}_a$, by

$$\rho(t) = t - 1, \quad t \in \mathbb{N}_{a+1}.$$

Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by $\big(\nabla u\big)(t) = u(t) - u(t-1)$, for $t \in \mathbb{N}_{a+1}$, and the N^{th}-order nabla difference of u is defined recursively by $\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1}u\big)\Big)(t)$, for $t \in \mathbb{N}_{a+N}$.

Definition 2.3 (See [7])**.** Let $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, the generalized rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)}.$$

Here $\Gamma(\cdot)$ denotes the Euler gamma function. Also, if $t \in \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, then we use the convention that $t^{\overline{r}} = 0$.

Definition 2.4 (See [7])**.** Let t, $a \in \mathbb{R}$ and $\mu \in \mathbb{R} \setminus \{\dots, -2, -1\}$. The μ^{th}-order nabla fractional Taylor monomial is given by

$$H_\mu(t, a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)},$$

provided the right-hand side exists.

We observe the following properties of the nabla fractional Taylor monomials.

Lemma 2.1 (See [14])**.** *Let $\mu > -1$ and $s \in \mathbb{N}_a$. Then the following hold:*

1. *If $t \in \mathbb{N}_{\rho(s)}$, then $H_\mu(t, \rho(s)) \geq 0$ and if $t \in \mathbb{N}_s$, then $H_\mu(t, \rho(s)) > 0$.*
2. *If $t \in \mathbb{N}_s$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is an increasing function of s.*
3. *If $t \in \mathbb{N}_{s+1}$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of t.*
4. *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of s.*
5. *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu \geq 0$, then $H_\mu(t, \rho(s))$ is a non-decreasing function of t.*
6. *If $t \in \mathbb{N}_s$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is a increasing function of t.*
7. *If $0 < v \leq \mu$, then $H_v(t, a) \leq H_\mu(t, a)$, for each fixed $t \in \mathbb{N}_a$.*

Lemma 2.2. *Let a, b be two real numbers such that $0 < a \leq b$ and $1 < \alpha < 2$. Then $\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}$ is a decreasing function of s for $s \in \mathbb{N}_0^{a-1}$.*

Proof. It is enough to show that $\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}\right) < 0$.

Consider

$$\begin{aligned}
&\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}\right) \\
&= \frac{-(b-s)^{\overline{\alpha-1}}(\alpha-1)(a-\rho(s))^{\overline{\alpha-2}} + (a-s)^{\overline{\alpha-1}}(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}}
\end{aligned}$$

$$= \frac{(\alpha-1)\left(-(b-s)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}+(a-s)(a-\rho(s))^{\overline{\alpha-2}}(b-\rho(s))^{\overline{\alpha-2}}\right)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}}$$
$$= \frac{(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}(-b+s+a-s)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}}$$
$$= \frac{(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}(a-b)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}}.$$

Since $b > a$, it follows from Lemma 2.1 that $\nabla_s\left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}\right) < 0$. The proof is complete. □

Definition 2.5 (See [7]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u is given by

$$\left(\nabla_a^{-\nu}u\right)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s))u(s), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.6 (See [7]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \nu \le N$. The ν^{th}-order Riemann–Liouville nabla difference of u is given by

$$\left(\nabla_a^{\nu}u\right)(t) = \left(\nabla^N\left(\nabla_a^{-(N-\nu)}u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

Theorem 2.1 (See [7]). *Assume $v > 0$ and $N-1 < v \le N$. Then a general solution of $\nabla_a^v x(t) = 0$ is given by*

$$x(t) = c_1(t-a)^{\overline{v-1}} + c_2(t-a)^{\overline{v-2}} + \cdots + c_N(t-a)^{\overline{v-N}}, \quad \textit{for} \quad t \in \mathbb{N}_a.$$

3. Green's Function

In this section, we construct the Green's function for the boundary value problem (1.1) and derive a few properties of the same, which will be used in the rest of the article.

Theorem 3.1 (See [5]). *The nabla fractional boundary value problem*

$$\begin{aligned} -\left(\nabla_{\rho(a)}^{\alpha}u\right)(t) &= h(t), \quad t \in \mathbb{N}_{a+2}^{b}, \\ u(a) &= u(b) = 0, \end{aligned} \tag{3.1}$$

where $a, b \in \mathbb{R}$ with $b-a \in \mathbb{N}_2, 1 < \alpha < 2$ and $h : \mathbb{N}_{a+2}^b \to \mathbb{R}$, has the unique solution

$$u(t) = \sum_{s=a+2}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_a^b, \tag{3.2}$$

where the Green's function $G(t,s)$ is given by

$$G(t,s) = \begin{cases} G_1(t,s) = \dfrac{H_{\alpha-1}(t,a)}{H_{\alpha-1}(b,a)}H_{\alpha-1}(b,\rho(s)), & t \in \mathbb{N}_a^{s-1}, \\ G_2(t,s) = \dfrac{H_{\alpha-1}(t,a)}{H_{\alpha-1}(b,a)}H_{\alpha-1}(b,\rho(s)) - H_{\alpha-1}(t,\rho(s)), & t \in \mathbb{N}_s^b. \end{cases} \tag{3.3}$$

Lemma 3.1. *The equivalent form of the homogeneous nabla fractional boundary value problem with non-local conditions*

$$\begin{cases} -(\nabla^{\alpha}_{\rho(a)} w)(t) = 0, \quad \textit{for} \quad t \in \mathbb{N}^{b}_{a+2}, \\ w(a) = g_1(w), \quad w(b) = g_2(w), \end{cases} \tag{3.4}$$

is given by

$$w(t) = g_1(w)\Big(\frac{b-t}{b-a}\Big)\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} + g_2(w)\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}, \quad t \in \mathbb{N}^{b}_{a}. \tag{3.5}$$

Proof. From Theorem 2.1, the general solution of the equation $-(\nabla^{\alpha}_{\rho(a)} w)(t) = 0$, is given by

$$w(t) = c_1(t-a+1)^{\overline{\alpha-1}} + c_2(t-a+1)^{\overline{\alpha-2}}, \qquad t \in \mathbb{N}^{b}_{a}, \tag{3.6}$$

where c_1 and c_2 are arbitrary constants. Using $w(a) = g_1(w)$ and $w(b) = g_2(w)$, respectively in (3.6), we have

$$\begin{aligned} \frac{g_1(w)}{\Gamma(\alpha-1)} &= c_1(\alpha-1) + c_2, \\ g_2(w) &= c_1(b-a+1)^{\overline{\alpha-1}} + c_2(b-a+1)^{\overline{\alpha-2}}. \end{aligned}$$

Now, solving the above system of equations for c_1 and c_2, we have

$$\begin{aligned} c_1 &= -\frac{g_1(w)(b-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)(b-a)^{\overline{\alpha-1}}} + \frac{g_2(w)}{(b-a)^{\overline{\alpha-1}}}, \\ c_2 &= \frac{g_1(w)}{\Gamma(\alpha-1)} - (\alpha-1)\left[-\frac{g_1(w)(b-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)(b-a)^{\overline{\alpha-1}}} + \frac{g_2(w)}{(b-a)^{\overline{\alpha-1}}}\right]. \end{aligned}$$

Substituting c_1 and c_2 in (3.6), we have

$$\begin{aligned} w(t) &= \left[\frac{g_2(w)}{(b-a)^{\overline{\alpha-1}}} - \frac{g_1(w)(b-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)(b-a)^{\overline{\alpha-1}}}\right](t-a+1)^{\overline{\alpha-1}} \\ &\quad + \left[\frac{g_1(w)}{\Gamma(\alpha-1)} - (\alpha-1)\left[-\frac{g_1(w)(b-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)(b-a)^{\overline{\alpha-1}}} + \frac{g_2(w)}{(b-a)^{\overline{\alpha-1}}}\right]\right](t-a+1)^{\overline{\alpha-2}} \\ &= \frac{g_1(w)(b-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)(b-a)^{\overline{\alpha-1}}}\left[(\alpha-1)(t-a+1)^{\overline{\alpha-2}} - (t-a+1)^{\overline{\alpha-1}}\right] \\ &\quad + \frac{g_2(w)}{(b-a)^{\overline{\alpha-1}}}\left[(t-a+1)^{\overline{\alpha-1}} - (\alpha-1)(t-a+1)^{\overline{\alpha-2}}\right] + \frac{g_1(w)}{\Gamma(\alpha-1)}(t-a+1)^{\overline{\alpha-2}} \\ &= \frac{g_1(w)}{\Gamma(\alpha-1)}\left[(t-a+1)^{\overline{\alpha-2}} - \frac{(b-a+1)^{\overline{\alpha-2}}(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha-1)(b-a)^{\overline{\alpha-1}}}\right] + g_2(w)\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}} \\ &= g_1(w)\Big(\frac{b-t}{b-a}\Big)\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} + g_2(w)\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}. \end{aligned}$$

The proof is complete. □

Lemma 3.2. *w satisfies the following property:*

$$\max_{t\in\mathbb{N}_a^b} w(t) \leq g_1(w) + g_2(w), \tag{3.7}$$

where w is given by (3.5)*.*

Proof. Consider

$$w(t) = \Big(\frac{b-t}{b-a}\Big)\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}g_1(w) + \frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}g_2(w).$$

Clearly, $\left(\frac{b-t}{b-a}\right)$ is a decreasing function with respect to t for $t \in \mathbb{N}_a^b$. It follows from Lemma 2.1 that $\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}$ is a decreasing function of t and $\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}$ is an increasing function of t for $t \in \mathbb{N}_a^b$. Thus, we have

$$\begin{aligned}
&\max_{t\in\mathbb{N}_a^b}\Big(\frac{b-t}{b-a}\Big) = 1,\\
&\max_{t\in\mathbb{N}_b^a}\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} = \frac{(a-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} = 1,\\
&\max_{t\in\mathbb{N}_a^b}\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}} = 1,
\end{aligned}$$

implying that

$$w(t) \leq g_1(w) + g_2(w), \quad \text{for } t \in \mathbb{N}_a^b.$$

The proof is complete. □

Theorem 3.2 (See [15])**.** *Let $1 < \alpha < 2$ and $f : \mathbb{N}_a^b \times \mathbb{R} \to \mathbb{R}$. The equivalent form of* (1.1) *is given by*

$$u(t) = w(t) + \beta\sum_{s=a+2}^{b} G(t,s)f(s,u(s)), \quad t \in \mathbb{N}_a^b,$$

where the Green's function $G(t,s)$ and $w(t)$ are given by (3.3) *and* (3.5)*, respectively.*

Theorem 3.3 (See [5, 15])**.** *The Green's function $G(t,s)$ defined in* (3.3) *satisfies the following properties:*

1. *$G(a,s) = G(b,s) = 0$, for all $s \in \mathbb{N}_{a+1}^b$.*
2. *$G(t,a+1) = 0$, for all $t \in \mathbb{N}_a^b$.*
3. *$G(t,s) > 0$, for all $(t,s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^b$.*
4. *$\max_{t\in\mathbb{N}_{a+1}^{b-1}} G(t,s) = G(s-1,s)$ for all $s \in \mathbb{N}_{a+2}^b$.*
5. *$\sum_{s=a+1}^{b} G(t,s) \leq \Lambda$, for all $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$, where*

$$\Lambda = \Big(\frac{b-a-1}{\alpha\Gamma(\alpha+1)}\Big)\Big(\frac{(\alpha-1)(b-a)+1}{\alpha}\Big)^{\overline{\alpha-1}}. \tag{3.8}$$

4. Eigenvalue Problem

In this section, we show the existence of positive solutions of (1.1) using the Guo–Krasnoselskii fixed point theorem on a suitable cone.

Definition 4.1. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset K of $\mathcal{B}$ is said to be a cone provided,
(i) $au + bv \in K$, for all $u, v \in K$ and all $a, b \geq 0$, and
(ii) $u \in K$ and $-u \in K$ implies $u = 0$.

Definition 4.2. An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Lemma 4.1 (See [1]). *[Guo–Krasnoselskii fixed point theorem] Let $\mathcal{B}$ be a Banach space and $K \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume further that $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ is a completely continuous operator. If either*

1. $\|Ty\| \leq \|y\|$ *for* $y \in K \cap \partial\Omega_1$ *and* $\|Ty\| \geq \|y\|$ *for* $y \in K \cap \partial\Omega_2$*; or*
2. $\|Ty\| \geq \|y\|$ *for* $y \in K \cap \partial\Omega_1$ *and* $\|Ty\| \leq \|y\|$ *for* $y \in K \cap \partial\Omega_2$*;*

holds, then T has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We make use of the following lemmas, which will be used later in the proof of our main result. This results have recently appeared in [11] and the same has be proved here for the completeness of article.

Lemma 4.2. *There exists a number $\lambda \in (0, 1)$, such that*

$$\min_{t \in \mathbb{N}_c^d} G(t, s) \geq \lambda \max_{t \in \mathbb{N}_a^b} G(t, s) = \lambda G(s-1, s), \tag{4.1}$$

where, $c, d \in \mathbb{N}_{a+1}^{b-1}$, such that $c = a + \left\lceil \frac{b-a+1}{4} \right\rceil$ and $d = a + 3\left\lfloor \frac{b-a+1}{4} \right\rfloor$.

Proof. By using the properties of Green's function and Taylor monomials from Definition 2.4, Lemma 2.1 and Theorem 3.3, respectively.

Consider, for $s \in \mathbb{N}_{a+2}^b$,

$$\frac{G(t,s)}{G(s-1,s)} = \begin{cases} \dfrac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s > t, \\[2ex] \dfrac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \dfrac{(t-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \leq t. \end{cases}$$

Now, for $s > t$ and $c \leq t \leq d$, $G_1(t, s)$ is an increasing function with respect to t. Then, we have

$$\min_{t \in \mathbb{N}_c^d} G_1(t, s) = G_1(c, s) = \frac{(c-a)^{\overline{\alpha-1}}(b-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)(b-a)^{\overline{\alpha-1}}}.$$

For $t \geq s$ and $c \leq t \leq d$, $G_2(t, s)$ is an decreasing function with respect to t. Then, we have

$$\min_{t \in \mathbb{N}_c^d} G_2(t, s) = G_2(d, s) = \frac{(d-a)^{\overline{\alpha-1}}(b-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)(b-a)^{\overline{\alpha-1}}} - \frac{(d-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}.$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) = \begin{cases} G_2(d,s), & \text{for } s \in \mathbb{N}_{a+2}^{c}, \\ \min\{G_2(d,s), G_1(c,s)\}, & \text{for } s \in \mathbb{N}_{c+1}^{d-1}, \\ G_1(c,s), & \text{for } s \in \mathbb{N}_d^b, \end{cases}$$
$$= \begin{cases} G_2(d,s), & \text{for } s \in \mathbb{N}_{a+2}^{r}, \\ G_1(c,s), & \text{for } s \in \mathbb{N}_r^b, \end{cases}$$

where $c < r < d$. Consider

$$\frac{\min_{t\in\mathbb{N}_c^d} G(t,s)}{G(s-1,s)} = \begin{cases} \dfrac{(d-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \dfrac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s\in\mathbb{N}_{a+2}^{r}, \\ \dfrac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \in \mathbb{N}_r^b. \end{cases}$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \lambda(s) \max_{t\in\mathbb{N}_a^b} G(t,s), \tag{4.2}$$

where

$$\lambda(s) = \min\left[\frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, \frac{(d-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}\right].$$

Let for $s \in \mathbb{N}_r^b$, denote by

$$\begin{aligned} \lambda_1(s) =& \frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} \\ \geq& \frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}. \end{aligned}$$

Similarly, for $s \in \mathbb{N}_{a+2}^r$, we take

$$\lambda_2(s) = \frac{1}{(s-a-1)^{\overline{\alpha-1}}}\left[(d-a)^{\overline{\alpha-1}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}}\right].$$

By Lemma 2.2, we see that $\frac{(d-s+1)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}}$ is a decreasing function for $s \in \mathbb{N}_{a+2}^r$. Then

$$\begin{aligned} \lambda_2(s) \geq& \frac{1}{(s-a-1)^{\overline{\alpha-1}}}\left[(d-a)^{\overline{\alpha-1}} - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}\right] \\ >& \frac{1}{(d-a)^{\overline{\alpha-1}}}\left[(d-a)^{\overline{\alpha-1}} - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}\right]. \end{aligned}$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \lambda \max_{t\in\mathbb{N}_a^b} G(t,s), \tag{4.3}$$

where

$$\lambda = \min\left[\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}, 1-\frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}(d-a)^{\overline{\alpha-1}}}\right]. \tag{4.4}$$

Since $G_1(c,s) > 0$ and $G_2(d,s) > 0$, we have $\lambda(s) > 0$ for all $s \in \mathbb{N}_{a+2}^{b}$, implying $\lambda > 0$. It would be suffice to prove that one of the terms $\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}, 1 - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}(d-a)^{\overline{\alpha-1}}}$ is less then 1. It follows from Lemma 2.1 that

$$\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}} < 1.$$

Therefore, we conclude that $\lambda \in (0,1)$. The proof is complete. □

Lemma 4.3. *There exists a number* $\lambda_0 \in (0,1)$ *such that*

$$\min_{t\in\mathbb{N}_c^d} w(t) \geq \lambda_0 \max_{t\in\mathbb{N}_a^b} w(t), \tag{4.5}$$

where w *is given by* (3.5).

Proof. Clearly, $\left(\frac{b-t}{b-a}\right)$ is a decreasing function with respect to t for $t \in \mathbb{N}_a^b$. It follows from Lemma 2.1 that $\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}$ is a decreasing function of t and $\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}$ is an increasing function of t for $t \in \mathbb{N}_a^b$. Then, there exists $M_1, M_2 > 0$ such that

$$\begin{aligned}
&\min_{t\in\mathbb{N}_c^d}\left(\frac{b-t}{b-a}\right)\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} = M_1,\\
&\min_{t\in\mathbb{N}_c^d}\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}} = M_2 < 1.
\end{aligned}$$

Take $\lambda_0 = \min(M_1, M_2)$. Clearly $\lambda_0 \in (0,1)$. Thus, for all $t \in \mathbb{N}_c^d$, we have

$$\begin{aligned}
w(t) \geq& \min_{t\in\mathbb{N}_c^d}\left[\left(\frac{b-t}{b-a}\right)\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}\right] g_1(w) + \min_{t\in\mathbb{N}_c^d}\left[\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}\right] g_2(w)\\
=& M_1 g_1(w) + M_2 g_2(w)\\
\geq& \lambda_0 g_1(w) + \lambda_0 g_2(w)\\
=& \lambda_0 \left[\max_{t\in\mathbb{N}_a^b}\left(\frac{b-t}{b-a}\right)\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}\right] g_1(w) + \lambda_0\left[\max_{t\in\mathbb{N}_a^b}\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}\right] g_2(w)\\
\geq& \lambda_0 \max_{t\in\mathbb{N}_a^b} w(t).
\end{aligned}$$

The proof is complete. □

Lemma 4.4. *If* f *is non-negative, then there exists a constant* $\overline{\lambda} \in (0,1)$ *such that*

$$\begin{aligned}
&\min_{t\in\mathbb{N}_c^d}\Big[\sum_{s=a+2}^{b} G(t,s) f(t,u(s))\Big] + \min_{t\in\mathbb{N}_c^d} w(t)\\
\geq& \overline{\lambda} \max_{t\in\mathbb{N}_a^b}\Big[\sum_{s=a+2}^{b} G(t,s) f(t,u(s))\Big] + \overline{\lambda} \max_{t\in\mathbb{N}_a^b} w(t).
\end{aligned} \tag{4.6}$$

Proof. It follows from Lemma 4.2 and Lemma 4.3 that

$$\begin{aligned}
&\min_{t\in\mathbb{N}_c^d}\Big[\sum_{s=a+2}^{b} G(t,s)f(s,u(s))\Big] + \min_{t\in\mathbb{N}_c^d} w(t)\\
&\geq \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d}[G(t,s)]f(s,u(s)) + \lambda_0 \max_{t\in\mathbb{N}_a^b} w(t)\\
&\geq \lambda \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_a^b}[G(t,s)]f(s,u(s)) + \lambda_0 \max_{t\in\mathbb{N}_a^b} w(t)\\
&\geq \overline{\lambda} \max_{t\in\mathbb{N}_a^b} \sum_{s=a+2}^{b} G(t,s)f(s,u(s)) + \overline{\lambda} \max_{t\in\mathbb{N}_a^b} w(t),
\end{aligned}$$

where $\overline{\lambda} = \min(\lambda, \lambda_0) \in (0,1)$. The proof is complete. □

We know from Theorem 3.2 that the equivalent form of (1.1) is given by

$$u(t) = w(t) + \beta \sum_{s=a+2}^{b} G(t,s)f(s,u(s)), \quad t \in \mathbb{N}_a^b,$$

where the Green's function $G(t,s)$ and $w(t)$ are given by (3.3) and (3.5), respectively.

Note that any solution $u : \mathbb{N}_a^b \to \mathbb{R}$ of (1.1) can be viewed as a real $(b-a+1)$-tuple vector. Consequently, $u \in \mathbb{R}^{b-a+1}$. Define the operator $T_\beta : \mathbb{R}^{b-a+1} \to \mathbb{R}^{b-a+1}$ by

$$\big(T_\beta u\big)(t) = w(t) + \beta \sum_{s=a+2}^{b} G(t,s)f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{4.7}$$

Clearly, u is a fixed point of T_β if and only if u is a solution of (1.1). We use the fact that $\mathcal{B} = \mathbb{R}^{b-a+1}$ is a Banach space equipped with the maximum norm

$$\|u\| = \max_{t\in\mathbb{N}_a^b} |u(t)|,$$

for any $u \in \mathcal{B}$. We define the cone K by

$$K = \left\{ u \in \mathcal{B} : \; u(t) \geq 0 \text{ and } \min_{t\in\mathbb{N}_c^d} u(t) \geq \overline{\lambda}\, \|u(t)\| \right\}. \tag{4.8}$$

It follows from Lemma 4.4 that

$$\begin{aligned}
\min_{t\in\mathbb{N}_c^d}(T_\beta u)(t) \geq& \min_{t\in\mathbb{N}_c^d}\left[\beta \sum_{s=a+2}^{b} G(t,s)f(s,u(s))\right] + \min_{t\in\mathbb{N}_c^d} w(t)\\
\geq& \beta\overline{\lambda} \max_{t\in\mathbb{N}_b^a} \sum_{s=a+2}^{b} G(t,s)f(s,u(s)) + \overline{\lambda} \max_{t\in\mathbb{N}_a^b} w(t)\\
\geq& \overline{\lambda}\|T_\beta u\|.
\end{aligned}$$

Clearly $(T_\beta u)(t) \geq 0$ whenever $u \in K$ for all $t \in \mathbb{N}_a^b$. Thus, $T_\beta : K \to K$. Since T_β is a summation operator defined on a discrete finite set, it is trivially completely continuous. We state here the following hypotheses, which will be used later.

(F1) $f(t,u) = h(t)g(u)$ where h is a positive function defined on $\mathbb{N}_a^b$ and g is a non-negative function defined on $\mathbb{R}$.

(F2) $\lim\limits_{u\to 0^+} \frac{g(u)}{u} = 0, \quad \lim\limits_{u\to+\infty} \frac{g(u)}{u} = +\infty.$

(F3) $\lim\limits_{u\to 0^+} \frac{g(u)}{u} = +\infty, \quad \lim\limits_{u\to+\infty} \frac{g(u)}{u} = 0.$

(g1) There exists a number $r_1 > 0$ such that $g_1(u), g_2(u) \leq \frac{r_1}{3}$, whenever $0 \leq u \leq r_1$.

(g2) There exists a number $r_2^* > 0$ such that $g_1(u), g_2(u) \geq \frac{r_2^*}{3\overline{\lambda}}$, whenever $r_2 \leq u \leq r_2^*$.

(g3) There exists a number $r_3 > 0$ such that $g_1(u), g_2(u) \geq \frac{r_3}{3\overline{\lambda}}$, whenever $0 \leq u \leq r_3$.

(g4) There exists a number $r_2 > 0$ such that $g_1(u), g_2(u) \leq \frac{r_2}{3}$, whenever $r_2 \leq u \leq \frac{r_2}{\overline{\lambda}}$.

Denote by

$$G = \max_{(t,s)\in\mathbb{N}_a^b\times\mathbb{N}_{a+2}^b} G(t,s),$$

$$H = \max_{t\in\mathbb{N}_a^b} h(t) \text{ and } h = \min_{t\in\mathbb{N}_a^b} h(t).$$

Theorem 4.1. *Assume that the conditions* $(F1)$-$(F2)$ *and* $(g1)$-$(g2)$ *hold good. If there exists a sufficiently small positive constant* δ *and a sufficiently large constant* M *such that* $H(b-a-1)\delta < h(b-a-1)M$ *holds for each*

$$\beta \in \left[(Gh(b-a-1)M)^{-1}, (GH(b-a-1)\delta)^{-1}\right], \tag{4.9}$$

then (1.1) *has at least one positive solution.*

Proof. By condition $(F2)$, there exists $r_1 > 0$ and a sufficiently small constant $\frac{\delta}{3} > 0$ such that

$$g(u) \leq \frac{\delta r_1}{3}, \quad \text{whenever} \quad 0 < u \leq r_1. \tag{4.10}$$

Set $\Omega_1 = \{u \in \mathcal{B} : \|u\| < r_1\}$. Thus, by (4.9), (4.10), $(g1)$, and Lemma 3.2, for $u \in K$ with $\|u\| = r_1$, we have

$$\begin{aligned}
\|T_\beta u\| &\leq \max_{t\in\mathbb{N}_b^a} \left|\beta \sum_{s=a+2}^{b} G(t,s)f(s,u(s))\right| + \max_{t\in\mathbb{N}_a^b} |w(t)| \\
&\leq \beta \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_b^a} [G(t,s)]\, h(s)g(u) + g_1(u) + g_2(u) \\
&\leq \beta GH \frac{\delta r_1}{3}(b-a-1) + \frac{r_1}{3} + \frac{r_1}{3} \\
&\leq r_1 = \|u\|.
\end{aligned}$$

Therefore, $\|T_\beta u\| \leq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$. Similarly, by condition $(F2)$, we can find $0 < r_1 < r_2$ and a sufficient large constant M such that

$$g(u) \geq \frac{Mr_2}{3\overline{\lambda}^2}, \text{ for } u \geq r_2. \tag{4.11}$$

Set $r_2^* = \frac{r_2}{\overline{\lambda}} > r_2$ and $\Omega_2 = \{u \in \mathcal{B} : \|u\| < r_2^*\}$. Then, for $u \in K$ with $\|u\| = r_2^*$, we have

$$\min_{t\in\mathbb{N}_c^d} u(t) \geq \overline{\lambda}\|u\| = \overline{\lambda} r_2^*,$$

implying that $u(t) \geq r_2$ for $t \in \mathbb{N}_a^b$. Therefore, by (4.9), (4.11), $(g2)$ and Lemma 4.3, we have

$$\begin{aligned}
\|T_\beta u\| &\geq \min_{t\in\mathbb{N}_c^d} |T_\beta u(t)| \\
&\geq \min_{t\in\mathbb{N}_c^d} \left[\beta \sum_{s=a+2}^{b} G(t,s)f(s,u(s))\right] + \min_{t\in\mathbb{N}_c^d} w(t) \\
&\geq \beta \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_d^c} [G(t,s)] f(s,u(s)) + \overline{\lambda}(g_1(u)+g_2(u)) \\
&\geq \overline{\lambda}\beta \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_b^a} [G(t,s)]\, h(s)g(u) + \overline{\lambda}(g_1(u)+g_2(u)) \\
&\geq \overline{\lambda}\beta G h \frac{Mr_2}{3\overline{\lambda}^2}(b-a-1) + \overline{\lambda}\left(\frac{r_2^*}{3\overline{\lambda}} + \frac{r_2^*}{3\overline{\lambda}}\right) \\
&\geq \frac{r_2^*}{3} + \frac{2r_2^*}{3} \\
&= r_2^* = \|u\|.
\end{aligned}$$

Thus, we conclude that $\|T_\beta u\| \geq \|u\|$, for $u \in \partial\Omega_2 \cap \mathcal{K}$. By part (1) of Lemma 4.1, we conclude that T_β has a fixed point u_0 in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, satisfying $r_1 < \|u_0\| < r_2^*$. The proof is complete. □

Theorem 4.2. *Assume that the conditions* $(F1)$-$(F3)$ *and* $(g3)$-$(g4)$ *hold good. If there exists a sufficiently large constant* L *such that* $H(b-a-1) < h(b-a-1)L$ *holds for each*

$$\beta \in \left[(Gh(b-a-1)L)^{-1}, (GH(b-a-1))^{-1}\right], \tag{4.12}$$

then (1.1) *has at least one positive solution.*

Proof. By condition $(F3)$, there exists $r_3 > 0$ and a sufficiently large constant $L > 0$ such that $g(u) \geq \frac{Lr_3}{3\overline{\lambda}}$ for $0 < u \leq r_3$. Set $\Omega_1 = \{u \in \mathcal{B} : \|u\| < r_3\}$. Then, for $u \in \Omega_1$, we have

$$\begin{aligned}
\|T_\beta u\| &\geq \min_{t\in\mathbb{N}_c^d} |T_\beta u(t)| \geq \min_{t\in\mathbb{N}_c^d} \left[\beta \sum_{s=a+2}^{b} G(t,s)f(s,u(s))\right] + \min_{t\in\mathbb{N}_c^d} w(t) \\
&\geq \beta \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_d^c} [G(t,s)] f(s,u(s)) + \overline{\lambda}(g_1(u)+g_2(u)) \\
&\geq \overline{\lambda}\beta \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_b^a} [G(t,s)]\, h(s)g(u) + \overline{\lambda}(g_1(u)+g_2(u)) \\
&\geq \overline{\lambda}\beta G h \frac{Lr_3}{3\overline{\lambda}}(b-a-1) + \overline{\lambda}\left(\frac{r_3}{3\overline{\lambda}} + \frac{r_3}{3\overline{\lambda}}\right) \\
&\geq \frac{r_3}{3} + \frac{2r_3}{3} \\
&= r_3 = \|u\|,
\end{aligned}$$

implying that $\|T_\beta u\| \geq \|u\|$, for $u \in \partial\Omega_1 \cap \mathcal{K}$. Now, we consider two cases for the construction of Ω_2.
Case 1: Suppose g is bounded. Then, there exists $R_1 \geq r_2$ such that $g(u) \leq \frac{R_1}{3}$, for $r_2 \leq u \leq \frac{r_2}{\lambda}$. From (4.12), we know that $\beta \leq (GH(b-a-1))^{-1}$. Thus, we have

$$\begin{aligned}
\|T_\beta u\| &= \max_{t\in\mathbb{N}_b^a}\left[\beta\sum_{s=a+2}^{b} G(t,s)f(s,u)\right] + \max_{t\in\mathbb{N}_a^b} w(t)\\
&\leq \beta\sum_{s=a+2}^{b}\max_{t\in\mathbb{N}_b^a}\left[G(t,s)\right]h(s)g(u) + g_1(u) + g_2(u)\\
&\leq \beta GH\frac{R_1}{3}(b-a-1) + \frac{r_2}{3} + \frac{r_2}{3}\\
&\leq \beta GH\frac{R_1}{3}(b-a-1) + \frac{R_1}{3} + \frac{R_1}{3}\\
&\leq R_1 = \|u\|.
\end{aligned}$$

Case 2: Suppose g is unbounded. Then, there exists some constant R_2 and a sufficiently small δ_2 such that $g(u) \leq \frac{\delta_2 u}{3}$ for $u \geq R_2$, and for $0 < u \leq R_2$, $g(u) \leq g(R_2)$. Let $R = \max\{R_1, R_2\}$. Now, we assume that $\Omega_2 = \{u \in \beta : \|u\| < R\}$. So, $g(u) \leq \frac{\delta_2 R}{3}$. Thus, by (4.12), we know that $\beta \leq (GH(b-a-1))^{-1} < (GH\delta_2(b-a-1))^{-1}$. Then, we have

$$\begin{aligned}
\|T_\beta u\| &\leq \max_{t\in\mathbb{N}_b^a}\left[\beta\sum_{s=a+2}^{b} G(t,s)f(s,u)\right] + \max_{t\in\mathbb{N}_a^b} w(t)\\
&\leq \beta\sum_{s=a+2}^{b}\max_{t\in\mathbb{N}_b^a}\left[G(t,s)\right]h(s)g(u) + g_1(u) + g_2(u)\\
&\leq \beta GH\frac{\delta_2 R}{3}(b-a-1) + \frac{r_2}{3} + \frac{r_2}{3}\\
&\leq \beta GH\frac{\delta_2 R}{3}(b-a-1) + \frac{R}{3} + \frac{R}{3}\\
&< R = \|u\|.
\end{aligned}$$

Thus, we have $\|T_\beta u\| \leq \|u\|$ in both the cases for $u \in \partial\Omega_2 \cap \mathcal{K}$. By part (2) of Lemma 4.1, we conclude that T has a fixed point $u \in \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. The proof is complete. □

5. Uniqueness of Solutions

In this section, we present the uniqueness result for the boundary value problem (1.1) using the contraction mapping theorem. We also construct a few examples to illustrate the applicability of established results.

Theorem 5.1 (Contraction Mapping Theorem, see [1]). *Let S be a closed subset of $\mathbb{R}^n$. Assume $T : S \to S$ is a contraction mapping, i.e. there exists a number μ, $0 \leq \mu \leq 1$, such that $\|Tu - Tv\| \leq \mu\|u - v\|$, for all $u, v \in S$. Then, T has a unique fixed point $u_0 \in S$.*

Theorem 5.2. *Assume that $f(t,u)$, $g_1(u)$ and $g_2(u)$ are Lipschitz with respect to u, i.e there exist a', b', $c' > 0$, such that $|f(t,u_1) - f(t,u_2)| \leq a' \|u_1 - u_2\|$, $|g_1(u_1) - g_1(u_2)| \leq b' \|u_1 - u_2\|$ and $|g_2(u_1) - g_2(u_2)| \leq c' \|u_1 - u_2\|$, whenever u_1, $u_2 : \mathbb{R} \to \mathbb{R}$. Then, the boundary value problem (1.1) has a unique solution provided*

$$a'\beta\Lambda + b' + c' < 1 \tag{5.1}$$

holds.

Proof. Consider

$$\begin{aligned}
\|T_\beta u_1 - T_\beta u_2\| =& \max_{t\in\mathbb{N}_a^b} |(T_\beta u_1)(t) - (T_\beta u_2)(t)| \\
&\leq \max_{t\in\mathbb{N}_a^b} \left| \beta \sum_{s=a+2}^{b} G(t,s)[f(s,u_1) - f(s,u_2)] \right| \\
&\quad + \max_{t\in\mathbb{N}_a^b} \left[\Big(\frac{b-t}{b-a}\Big) H_{\alpha-2}(t,\rho(a)) \right] |g_1(u_1) - g_1(u_2)| \\
&\quad + \max_{t\in\mathbb{N}_a^b} \left[\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}} \right] |g_2(u_1) - g_2(u_2)| \\
&\leq \beta \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_a^b} [G(t,s)]\, |f(s,u_1) - f(s,u_2)| + b' \|u_1 - u_2\| \\
&\quad + c' \|u_1 - u_2\| \\
&\leq \beta\Lambda a' \|u_1 - u_2\| + b' \|u_1 - u_2\| + c' \|u_1 - u_2\| \\
&\leq (\beta\Lambda a' + b' + c') \|u_1 - u_2\|.
\end{aligned}$$

Thus, using (5.1) T_β, is a contraction on $\mathbb{R}^{b-a+1}$. Hence, by Theorem 5.1, the result follows. The proof is complete. □

Example 5.1. Suppose $\alpha = 1.1$, $a = 0$, $b = 10$, $f(t,u) = \frac{\beta sin(u)}{15+t}$ with $\beta = 1$, $g_1(u) = \frac{\sum_{s=a}^{b} u(s)}{20}$ and $g_2(u) = \frac{\sum_{s=a}^{b} u(s)}{10}$. Then, (1.1) becomes

$$\begin{cases}
-(\nabla_{\rho(0)}^{1.1} u)(t) = \dfrac{\beta \sin(u)}{15+t}, \quad t \in \mathbb{N}_2^{10}, \\
u(0) = \dfrac{\sum_{s=a}^{b} u(s)}{20}, \quad u(10) = \dfrac{\sum_{s=a}^{b} u(s)}{10}.
\end{cases} \tag{5.2}$$

Clearly, $f(t,u)$, $g_1(u)$ and $g_2(u)$ are Lipschitz with respect to u with Lipschitz constants $\alpha_1 = \frac{1}{15}$, $\beta_1 = \frac{1}{20}$ and $\beta_2 = \frac{1}{10}$, respectively. Then,

$$\Lambda = \Big(\frac{(b-a)(\alpha-1)+1}{\alpha}\Big)^{\overline{\alpha-1}} \Big(\frac{(b-a-1)}{\alpha\Gamma(\alpha+1)}\Big) = 8.05,$$

and

$$(\alpha_1\beta\Lambda + \beta_1 + \beta_2) = 0.6866 < 1.$$

Thus, by Theorem 5.2, the boundary value problem (5.2) has a unique solution.

Example 5.2. Let $\alpha = 1.5,\ a = 1$ and $b = 11$. Then consider the following BVP

$$\begin{cases} -\left(\nabla^{1.5}_{\rho(1)}u\right)(t) = \beta(t+2)e^{-u}, & t \in \mathbb{N}_3^{11}, \\ u(1) = u(11) = 0. \end{cases} \tag{5.3}$$

We have $h(t) = t+2$ and $g(u) = e^{-u}$ for $u \in \mathbb{R}^+$. Take $L = 100$. Also, we have $G \approxeq 1.718,\ H = 13,\ h = 3,\ \lim_{u\to 0^+} \frac{g(u)}{u} = \lim_{u\to 0^+} \frac{e^{-u}}{u} = \infty$ and, $\lim_{u\to +\infty} \frac{g(u)}{u} = \lim_{u\to +\infty} \frac{e^{-u}}{u} = 0$. Then, $H(b-a-1) = 13 \times 9 = 117$ and $h(b-a-1)L = 3 \times 9 \times 100 = 2700$. We see that $H(b-a-1) < h(b-a-1)L$. Therefore, all the conditions of Theorem 4.2 are satisfied. Thus, the boundary value problem (5.3) has at least one positive solution for each $\beta \in [0.00021, 0.00497]$.

Example 5.3. Suppose $\alpha = 1.5,\ a = 0,\ b = 6,\ f(t,u) = \frac{1}{20}u(t)\left(1 + \frac{4-\lambda}{\lambda(1+u^2(t))}\right)$ where λ is given by (4.4), $g_1(u) = \frac{1}{11}u(1) - \frac{1}{23}u(4)$ and $g_2(u) = \frac{1}{9}u(5) - \frac{1}{10}u(2)$. Then, (1.1) becomes

$$\begin{cases} -(\nabla^{1.5}_{\rho(0)}u)(t) = \dfrac{\beta u(t)}{20}\left(1 + \dfrac{4-\lambda}{\lambda(1+u^2(t))}\right), & t \in \mathbb{N}_2^6, \\ u(0) = \frac{1}{11}u(1) - \dfrac{1}{23}u(4), \quad u(6) = \dfrac{1}{9}u(5) - \frac{1}{10}u(2). \end{cases} \tag{5.4}$$

Here $h(t) = 1$ and $g(t) = \frac{1}{20}u(t)\left(1 + \frac{4-\lambda}{\lambda(1+u^2(t))}\right)$. Take $L = 100$. We see that conditions (F1) - (F3) are satisfied, and there exists a number $r_3 > 0$ such that $g_1(u),\ g_2(u) \geq \frac{r_3}{3\lambda}$, whenever $0 \leq u \leq r_3$. Also, there exists a number $r_2 > 0$ such that $g_1(u),\ g_2(u) \leq \frac{r_2}{3}$, whenever $r_2 \leq u \leq \frac{r_2}{\overline{\lambda}}$. By calculations, we obtain that $H = 1,\ h = 1,\ G = 1.2988$ and $\overline{\lambda} = 0.1191$. We observe that $H(b-a-1) < h(b-a-1)L$. Thus, by Theorem 4.2 the boundary value problem (5.4) has at least one positive solution for $\beta \in [0.0015, 0.1540]$.

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