Journal of Applied Analysis and Computation
Volume 13, Number 2, April 2023, 701–714
Website:http://www.jaac-online.com
DOI:10.11948/20210515

THE MODULUS-BASED MATRIX SPLITTING METHOD WITH INNER ITERATION FOR A CLASS OF NONLINEAR COMPLEMENTARITY PROBLEMS

Changfeng Ma[1,†] and Ting Wang[2]

Abstract In this paper, we propose a modulus-based matrix splitting iteration method with inner iteration for a class of nonlinear complementarity problems. Convergence conditions of the iteration method are analyzed carefully, which shows that the iteration sequence generated by this method converges to a solution of the NCP under certain conditions. Moreover, the convergence conditions of the proposed method are studied when the system matrix is symmetric positive definite or is an H_+-matrix. Theoretical results are supported by the numerical experiments, which implies that the iteration method with inner iteration is more effective and feasible for solving certain nonlinear complementarity problems.

Keywords Nonlinear complementarity problem, matrix splitting iteration method; modulus-based, convergence analysis.

MSC(2010) 90C33.

1. Introduction

In this paper, we consider the following nonlinear complementarity problem (NCP): find the solution $u \in \mathbb{R}^n$ such that

$$u \geq 0, \quad v = F(u) \geq 0, \quad u^T v = 0, \tag{1.1}$$

where the function F has the form of $Au + \Phi(u) + q$, $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a given matrix, $q \in \mathbb{R}^n$ is a given vector, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a given diagonal differentiable mapping, that is, the ith component Φ_i of Φ is a function of ith variable u_i only

$$\Phi_i = \Phi_i(u_i), \quad i = 1, 2, \cdots, n.$$

Obviously, if Φ is a linear function, then problem (1.1) will reduce to a linear complementarity problem (LCP). The linear complementarity problems arise in many scientific computing and engineering application areas. Such as the linear and quadratic programming, the Nash equilibrium point of a bimatrix game,

†The corresponding author. Email: mcf@fzfu.edu.cn(C. Ma)
[1]School of Big Data & Key Laboratory of Digital Technology and Intelligent Computing, Fuzhou University of International Studies and Trade, Fuzhou 350202, China
[2]School of Mathematics and Informatics, Fujian Normal University, Fuzhou 350117, China

the contact problems, the free boundary problems, and the network equilibrium problems; for more details, please see [9, 20]. There are many efficient methods to solve the linear complementarity problem. For example, the projected iteration method [1], the general fixed-point iterations [10,23], and the matrix multi-splitting iterations [2–4]. Recently, a new method called modulus-based matrix splitting iteration method for linear complementarity problem attracts a great deal of attention. Bai presented a modulus-based matrix splitting iteration methods for solving the LCP in [5]. Numerical experiments have shown that the modulus-based iteration methods are superior to the projected relaxation methods as well as the modified modulus iteration method. Furthermore, Zhang proposed a two-step modulus-based matrix splitting iteration methods in [28] and the two-stage multisplitting iteration methods using modulus-based matrix splitting as inner iteration in [30] for linear complementarity problems.

When Φ is a nonlinear function, we will get down to solve the NCP. The nonlinear complementarity problems have various important applications in many fields, for example, economic equilibrium problems and static traffic flow equilibrium problems; for more details, please see [11, 21]. Also, many researchers have presented a lot of algorithms, such as linearized projected relaxation methods [12], nonlinear multi-splitting iteration methods [6, 7], multigrid methods [31] and domain decomposition methods [8, 15]. Besides, inexact semismooth Newton methods [16] have been developed to solve (1.1). Similarly, some modulus-based iteration methods have been developed to solve NCP. For example, Xia and Li [24] presented some modulus-based matrix splitting methods and established the convergence theories. Huang and Ma [13] proposed a class of modulus-based matrix splitting methods for NCP and introduced several analysis tools and techniques. Literature [14] present Modulus-based matrix splitting iteration methods for a class of implicit complementarity problems. For more works about the modulus-based iteration method see [17–19, 26, 32]. Recently, Xie at al. [25] proposed a two-step modulus-based matrix splitting iteration method for a class of nonlinear complementarity problems. Motivated by these ideas of the modulus-based matrix splitting iteration algorithms, in this paper, we propose a modulus-based matrix splitting iterative method with inner iteration for the NCP (1.1). And we discuss the convergence conditions of the method. Numerical experiments are presented to verify the theoretical results and illustrate the effectiveness of the proposed method.

Due to the necessity of the following analysis, we should introduce some necessary notations and definitions. Let $A = (a_{ij}), B = (b_{ij})$ be two real $m \times n$ matrices. We call $A \geq B$ $(A > B)$ if $a_{ij} \geq b_{ij}$ $(a_{ij} > b_{ij})$ holds for all $1 \leq i \leq m, 1 \leq j \leq n$. If O is a null matrix and $A \geq O$ $(A > O)$, then A is called a nonnegative (positive) matrix. Let $|A| = (|a_{ij}|) \in \mathbb{R}^{m\times n}$ be the absolute value of the matrix A, A^T be its transpose, A^{-1} be its inverse, $\rho(A)$ be its spectral radius, and $\|A\|_2$ denotes 2-norm of matrix A.

Let $A \in \mathbb{R}^{n\times n}$ be a real matrix, its comparison matrix is $\langle A\rangle = (\langle a\rangle_{ij}) \in \mathbb{R}^{n\times n}$, where

$$\langle a\rangle_{ij} = \begin{cases} |a_{ij}|, & \text{for } i = j \\ -|a_{ij}| & \text{for } i \neq j \end{cases} \quad i, j = 1, 2, \cdots, n.$$

A square matrix $A \in \mathbb{R}^{n\times n}$ is called an M-matrix if all of its off-diagonal entries are non-positive and $A^{-1} \geq O$, an H-matrix if its comparison $\langle A\rangle$ is an M-matrix, and an H_+-matrix if it is an H-matrix and all of its diagonal entries are positive.

Let $A, B \in \mathbb{R}^{n\times n}$ be two M-matrix, $\Omega \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix, and $C \in \mathbb{R}^{n\times n}$. Then, $A \leq B$ implies $B^{-1} \leq A^{-1}$, and $A \leq C \leq \Omega$ implies that C is an M-matrix. If A is an H_+-matrix, then $O \leq |A^{-1}| \leq \langle A\rangle^{-1}$.

Let $A = M - N$. If M is nonsingular, then the splitting $A = M - N$ is called a splitting of the matrix A; if $\rho(M^{-1}N) < 1$, then the splitting $A = M - N$ is called a convergent splitting; if $\langle M\rangle - |N|$ is an M-matrix, then the splitting $A = M - N$ is called an H-splitting; if $\langle A\rangle = \langle M\rangle - |N|$, then the splitting $A = M - N$ is called an H-compatible splitting. If $A = M - N$ is an H-splitting, then $\rho(M^{-1}N) \leq \rho(\langle M\rangle^{-1}|N|) < 1$.

To our knowledge, there are few results about modulus-based matrix splitting method with inner iteration for nonlinear complementarity problems. In this paper, we establish the modulus-based matrix splitting iteration method for NCP (1.1) and discuss convergence of the proposed method. Some numerical experiments are also given to verify the validity of the the proposed algorithms .

The rest of this paper is organized as follows. In Section 2, we establish the modulus-based matrix splitting iteration method. The convergence analysis of the new iterative method is given in Section 3. Moreover, numerical experiments are presented to illustrate the effectiveness of the iteration method in Section 4. Finally, we draw a brief conclusion in Section 5.

2. Modulus-based matrix splitting iteration methods

In this section, we will give the new algorithm with inner iteration, which is based on the equivalent fixed-point equation of the NCP (1.1). Firstly, we will begin with a useful lemma.

Lemma 2.1 ([25]). *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, $\Omega \in \mathbb{R}^{n\times n}$ be a nonnegative diagonal matrix, h be a positive number. Then the following statements hold true:*
(i) if (u,v) is a solution of (1.1), then $x = \frac{1}{h}(u - \Omega^{-1}v)$ satisfies the implicit fixed-point equation

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \frac{2}{h}(q + \Phi(\tfrac{h}{2}(|x| + x))); \tag{2.1}$$

(ii) if x satisfies the implicit fixed-point equation (2.1), then

$$u = \frac{h}{2}(|x| + x) \quad \text{and} \quad v = \frac{h}{2}\Omega(|x| - x) \tag{2.2}$$

is the solution of (1.1).

By adding inner iteration in the above method, we obtain the new method. Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, $\Omega \in \mathbb{R}^{n\times n}$ be a nonnegative diagonal matrix , h be a positive number. Giving an initial vector $x^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, 2, \cdots$, compute $x^{k+1} \in \mathbb{R}^n$ by solving the linear system:

$$(\Omega + M)x^{(k,j+1)} = Nx^{(k,j)} + (\Omega - A)|x^{(k,j)}| - \frac{2}{h}(q + \Phi(u^{(k)})), \quad j = 0, 1, \cdots, l_k,$$

and we set

$$x^{(k,0)} = \frac{1}{h}(u^{(k)} - \Omega^{-1}v^{(k)}),$$

where

$$u^{(k)} = \frac{h}{2}(|x^{(k)}| + x^{(k)}), \quad v^{(k)} = \frac{h}{2}\Omega(|x^{(k)}| - x^{(k)}).$$

After inner iterations, we set

$$x^{(k+1)} = x^{(k,l_k+1)}.$$

More specifically, we give the following Method 2.1.

Method 2.1 (Modulus-based matrix splitting iteration method)**.**

Step 1: Give $\varepsilon > 0$, select an initial $x^{(0)} \in \mathbb{R}^n$. set k:=0;

Step 2: Find the solution $u^{(k+1)}$:

(1) Calculate the initial vector

$$x^{(k,0)} = \frac{1}{h}(u^{(k)} - \Omega^{-1}v^{(k)}), \tag{2.3}$$

where

$$u^{(k)} = \frac{h}{2}(|x^{(k)}| + x^{(k)}), \quad v^{(k)} = \frac{h}{2}\Omega(|x^{(k)}| - x^{(k)}).$$

(2) Iterative computing $x^{(k+1)} \in \mathbb{R}^n$ by solving the equations

$$(\Omega+M)x^{(k,j+1)} = Nx^{(k,j)}+(\Omega-A)|x^{(k,j)}|-\frac{2}{h}(q+\Phi(u^{(k)})), \quad j = 0, 1, \cdots, l_k. \tag{2.4}$$

Set $x^{(k+1)} := x^{(k,l_k+1)}$.

(3) Compute $u^{k+1} \in \mathbb{R}^n$

$$u^{(k+1)} = \frac{h}{2}(|x^{(k+1)}| + x^{(k+1)}). \tag{2.5}$$

Step3: If Res$(u^{(k)}) = \| \min(u^{(k)}, F(u^{(k)}))\|_2 < \varepsilon$, then stop; otherwise, set k:=k+1 and return to Step 2.

Remark 2.1. Method 2.1 is called the stationary iteration method if the number of inner iterations l_k keeps fixed at each outer iteration step, and the nonstationary iteration method if the number of inner iterations l_k changes with the outer iteration index k. Note that the inner iterations do not need to communicate in the actual implementation of Method 2.1. Generally speaking, if we increase the number of inner iterations, the number of outer iterations will decrease. This may lead to the reduction of the total computing time. We hope the decrement of communication time is greater than the increment of computing time for the inner iterations by choosing l_k suitably, which is available. So adding inner iteration may greatly improve the computing time for solving the NCP (1.1)

Remark 2.2. In fact, from Step 2 in Method 2.1, the initial vector $x^{(k,0)} \in \mathbb{R}^n$ can be obtained by the following formulation

$$x^{(k,0)} = \frac{1}{h}(u^{(k)} - \Omega^{-1}v^{(k)}).$$

Assume $(u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$ is a solution of the NCP (1.1), x^* is a solution of the equation (2.1), according to Lema 2.1, we acquire

$$x^* = \frac{1}{h}(u^* - \Omega^{-1}v^*),$$

where,

$$v^{(k)} = F(u^{(k)}) = Au^{(k)} + \Phi(u^{(k)}) + q, \qquad v^* = F(u^*) = Au^* + \Phi(u^*) + q.$$

Suppose that there exists a nonnegative matrix $B \in \mathbb{R}^{n\times n}$ such that

$$|\Phi(u) - \Phi(v)| \le B|u - v|, \qquad for\ all\ u, v \in \mathbb{R}^n. \tag{2.6}$$

Then, we obtain

$$\begin{aligned}
|x^{(k,0)} - x^*| &= |\frac{1}{h}(u^{(k)} - \Omega^{-1}v^{(k)}) - \frac{1}{h}(u^* - \Omega^{-1}v^*)| \\
&= \frac{1}{h}|u^{(k)} - u^* - \Omega^{-1}(v^{(k)} - v^*)| \\
&= \frac{1}{h}|u^{(k)} - u^* - \Omega^{-1}(Au^{(k)} - Au^* + \Phi(u^{(k)}) - \Phi(u^*))| \\
&\le \frac{1}{h}(|u^{(k)} - u^*| + \Omega^{-1}|A||u^{(k)} - u^*| + \Omega^{-1}B|u^{(k)} - u^*|) \\
&= \frac{1}{h}(I + \Omega^{-1}|A| + \Omega^{-1}B)|u^{(k)} - u^*|.
\end{aligned} \tag{2.7}$$

From the aforementioned formula, if $k \to \infty, u^{(k)} \to u^*, then\ x^{(k,0)} \to x^*$. So according to (2.7) , we can obtain a better initial value $x^{(k,0)}$, which is the main purpose we formulate the inner iteration. In fact, numerical results in section 4 suggest the convergence performance of Method 2.1 is better than the method of [25] with less iteration steps and CPU time.

In Method 2.1, let $A = D - L - U$, where D, $-L$, and $-U$ are diagonal, strictly lower-triangular and strictly upper-triangular matrix of A, respectively. Then we have

(a) When $M = A,\ N = O$, and Ω be a positive diagonal matrix,then Method 2.1 reduces to the following modulus-based splitting method with inner iteration (MSI):

$$(\Omega + A)x^{(k,j+1)} = (\Omega - A)|x^{(k,j)}| - \frac{2}{h}(q + \Phi(u^{(k)})), \quad j = 0, 1, \cdots, l_k,$$

with $u^{(k)} = \frac{h}{2}(|x^{(k)}| + x^{(k)})$;

(b) When $M = D,\ N = L + U$, and Ω be a positive diagonal matrix, then Method 2.1 gives the following modulus-based Jacobi method with inner iteration (MJI):

$$(\Omega + D)x^{(k,j+1)} = (L + U)x^{(k,j)} + (\Omega - A)|x^{(k,j)}| - \frac{2}{h}(q + \Phi(u^{(k)})), \quad j = 0, 1, \cdots, l_k,$$

with $u^{(k)} = \frac{h}{2}(|x^{(k)}| + x^{(k)})$;

(c) When $M = D - L,\ N = U$, and Ω be a positive diagonal matrix, then Method 2.1 gives the following modulus-based Gauss-Seidel method with inner iteration (MGSI):

$$(\Omega + D - L)x^{(k,j+1)} = Ux^{(k,j)} + (\Omega - A)|x^{(k,j)}| - \frac{2}{h}(q + \Phi(u^{(k)})), \quad j = 0, 1, \cdots, l_k,$$

with $u^{(k)} = \frac{h}{2}(|x^{(k)}| + x^{(k)})$;

(d) When $M = (1/\alpha)D - L$, $N = (1/\alpha - 1)D + U$, and Ω be a positive diagonal matrix, then Method 2.1 gives the following modulus-based successive overrelaxation method with inner iteration (MSORI):

$$\begin{aligned}(\alpha\Omega + D - \alpha L)x^{(k,j+1)} =&[(1-\alpha)D + \alpha U)]x^{(k,j)} + \alpha(\Omega - A)|x^{(k,j)}| \\ &- \frac{2\alpha}{h}(q + \Phi(u^{(k)})),\ j = 0, 1, \cdots, l_k,\end{aligned}$$

with $u^{(k)} = \frac{h}{2}(|x^{(k)}| + x^{(k)})$;

(e) When $M = 1/\alpha(D - \beta L)$, $N = 1/\alpha[(1-\alpha)D + (\alpha - \beta)L + \alpha U]$, and Ω be a positive diagonal matrix, then Method 2.1 gives the following modulus-based accelerated overrelaxation method with inner iteration (MAORI):

$$\begin{aligned}(\alpha\Omega + D - \beta L)x^{(k,j+1)} =& [(1-\alpha)D + (\alpha - \beta)L + \alpha U]x^{(k,j)} \\ &+\alpha(\Omega - A)|x^{(k,j)}| - \frac{2\alpha}{h}(q + \Phi(u^{(k)})),\ j = 0, 1, \cdots, l_k,\end{aligned}$$

(f) When $M = H(A), N = S(A)$, where $H(A) = 1/2(A + A^T)$, $S(A) = -1/2(A - A^T)$, and Ω be a positive diagonal matrix,then Method 2.1 gives the following modulus-based HSS method with inner iteration (MHSSI):

$$(\Omega + H(A))x^{(k,j+1)} = S(A)x^{(k,j)} + (\Omega - A)|x^{(k,j)}| - \frac{2}{h}(q + \Phi(u^{(k)})),\ \ j = 0, 1, \cdots, l_k,$$

with $u^{(k)} = \frac{h}{2}(|x^{(k)}| + x^{(k)})$.

3. Convergence analysis

In this section, we will discuss the convergence conditions of the proposed method. In this paper, we fix the number of the inner iterations with l.

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a positive definite matrix , assume that $\Omega \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, h is a positive constant. Define*

$$G_1 = |(\Omega + A)^{-1}(\Omega - A)|, \qquad G_2 = |(\Omega + A)^{-1}|,$$

and

$$G = 2\sum_{i=0}^{l} G_1^i G_2, \qquad Z = G_1^{l+1}(I + \Omega^{-1}|A| + \Omega^{-1}B) + GB.$$

If $\rho(Z) < 1$, then the iteration sequence $\{u^k\}_{k=0}^{\infty} \subseteq \mathbb{R}_+^n$ generated by Method 2.1 converges to a solution $u^* \in \mathbb{R}_+^n$ of the NCP (1.1) for any initial vector $x^0 \in \mathbb{R}^n$.

Proof. Assume $(u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$ is a solution of the NCP (1.1), according to Lema 2.1, we obtain $x^* = \frac{1}{h}(u^* - \Omega^{-1}v^*)$ is a solution of the equation (2.1). From the former section, we have

$$u^{(k+1)} = \frac{h}{2}(|x^{(k+1)}| + x^{(k+1)}),$$

and

$$u^* = \frac{h}{2}(|x^*| + x^*),$$

so, we obtain

$$\begin{aligned} |u^{(k+1)} - u^*| &= |\frac{h}{2}(|x^{(k+1)}| + x^{(k+1)}) - \frac{h}{2}(|x^*| + x^*)| \\ &\leq \frac{h}{2}||x^{(k+1)}| - |x^*|| + \frac{h}{2}|x^{(k+1)} - x^*| \\ &\leq h|x^{(k+1)} - x^*|. \end{aligned} \tag{3.1}$$

From the Method (2.1), we obtain

$$(\Omega + A)x^{(k,j+1)} = (\Omega - A)|x^{(k,j)}| - \frac{2}{h}(q + \Phi(u^{(k)})), \quad j = 0, 1, \cdots, l,$$

and

$$(\Omega + A)x^* = (\Omega - A)|x^*| - \frac{2}{h}(q + \Phi(u^*)).$$

So

$$x^{(k,j+1)} = (\Omega + A)^{-1}[(\Omega - A)|x^{(k,j)}| - \frac{2}{h}(q + \Phi(u^{(k)}))], \quad j = 0, 1, \cdots, l, \tag{3.2}$$

$$x^* = (\Omega + A)^{-1}[(\Omega - A)|x^*| - \frac{2}{h}(q + \Phi(u^*))]. \tag{3.3}$$

Combining (3.2) and (3.3), we can obtain

$$\begin{aligned} |x^{(k,j+1)} - x^*| &= |(\Omega + A)^{-1}[(\Omega - A)(|x^{(k,j)}| - |x^*|) - \frac{2}{h}(\Phi(u^{(k)}) - \Phi(u^*))]| \\ &\leq |(\Omega + A)^{-1}(\Omega - A)||x^{(k,j)} - x^*| + \frac{2}{h}|(\Omega + A)^{-1}|B|u^{(k)} - u^*| \\ &= G_1|x^{(k,j)} - x^*| + \frac{2}{h}G_2B|u^{(k)} - u^*|. \end{aligned} \tag{3.4}$$

Then,

$$\begin{aligned} |x^{(k+1)} - x^*| &= |x^{(k,l+1)} - x^*| \\ &\leq G_1|x^{(k,l)} - x^*| + \frac{2}{h}G_2B|u^{(k)} - u^*| \\ &\leq G_1[G_1|x^{(k,l-1)} - x^*| + \frac{2}{h}G_2B|u^{(k)} - u^*|] + \frac{2}{h}G_2B|u^{(k)} - u^*| \\ &= G_1^2|x^{(k,l-1)} - x^*| + \frac{2}{h}(G_1G_2 + G_2)B|u^{(k)} - u^*| \\ &\leq G_1^{l+1}|x^{(k,0)} - x^*| + \frac{2}{h}(\sum_{i=0}^{l} G_1^iG_2)B|u^{(k)} - u^*|. \end{aligned} \tag{3.5}$$

From (3.5) and (3.1), we have

$$\begin{aligned} |u^{(k+1)} - u^*| &\leq h|x^{(k+1)} - x^*| \\ &= h[G_1^{l+1}|x^{(k,0)} - x^*| + 2(\sum_{i=0}^{l} G_1^iG_2)B|u^{(k)} - u^*|] \end{aligned}$$

$$\leq [(G_1^{l+1}(I+\Omega^{-1}|A|+\Omega^{-1}B)+2\sum_{i=0}^{l}G_1^iG_2B)]|u^{(k)}-u^*|$$
$$= Z|u^{(k)}-u^*|.$$

If $\rho(Z)<1$, then the iteration sequence $\{u^k\}_{k=0}^{\infty}\subseteq\mathbb{R}_+^n$ generated by Method 2.1 converges to a solution $u^*\in\mathbb{R}_+^n$ of the NCP (1.1). This completes the proof. □

Theorem 3.2. *Let A be a symmetric positive definite matrix and A is a positive matrix. Assume $\Omega=\omega I\in\mathbb{R}^{n\times n}$ is a positive diagonal matrix and h is a positive constant. Let $\mu_{\min}$ and $\mu_{\max}$ are the minimum and maximum eigenvalues of matrix A, respectively. Let $\lambda=\rho(B)$, and we denote $a=\frac{\mu_{\min}}{\omega}$, $b=\frac{\mu_{\max}}{\omega}$, $c=\frac{\lambda}{\omega}$. If*

$$\frac{c}{a}<1 \quad and \quad \frac{c}{a}+\Big(1+b+c-\frac{c}{a}\Big)\Big(\frac{1-a}{1+a}\Big)^{l+1}<1,$$

then the iteration sequence $\{u^k\}_{k=0}^{\infty}\subseteq\mathbb{R}_+^n$ generated by Method 2.1 converges to a solution $u^*\in\mathbb{R}_+^n$ of the NCP (1.1) for any initial vector $x^0\in\mathbb{R}^n$.

Proof. From Theorem 3.1, we know we only need to derive the condition $\rho(Z)<1$, where

$$Z=G_1^{l+1}(I+\Omega^{-1}|A|+\Omega^{-1}B)+GB,\quad G=2\sum_{i=0}^{l}G_1^iG_2.$$

We define that

$$\eta_1=\|(\Omega+A)^{-1}(\Omega-A)\|_2,\qquad \eta_2=\|(\Omega+A)^{-1}\|_2,$$
$$\theta=\|I+\Omega^{-1}|A|+\Omega^{-1}B\|_2,\qquad \sigma=\|GB\|_2.$$

Because $A>O$ is symmetric positive definite and $\Omega=\omega I$, we have

$$\eta_1=\|(\omega I+A)^{-1}(\omega I-A)\|_2=\frac{\omega-\mu_{\min}}{\omega+\mu_{\min}}=\frac{1-a}{1+a},$$
$$\eta_2=\|(\omega I+A)^{-1}\|_2=\frac{1}{\omega+\mu_{\min}},$$
$$\theta=\|I+(\omega I)^{-1}A+(\omega I)^{-1}B\|_2\leq 1+\frac{\mu_{\max}}{\omega}+\frac{\lambda}{\omega}=1+b+c,$$
$$\sigma=\|GB\|_2=\left\|2\sum_{i=0}^{l}G_1^iG_2B\right\|_2\leq 2\|G_2\|_2\|B\|_2\left\|\sum_{i=0}^{l}G_1^i\right\|_2$$
$$\leq 2\frac{\lambda}{\omega+\mu_{\min}}\left[\frac{1-(\frac{1-a}{1+a})^{l+1}}{1-\frac{1-a}{1+a}}\right]$$
$$=\frac{2c}{1+a}\frac{1+a}{2a}\left(1-\left(\frac{1-a}{1+a}\right)^{l+1}\right)$$
$$=\frac{c}{a}\left(1-\left(\frac{1-a}{1+a}\right)^{l+1}\right).$$

Then we obtain

$$\rho(Z)\leq\eta_1^{l+1}\theta+\sigma$$

$$\leq \left(\frac{1-a}{1+a}\right)^{l+1}(1+b+c)+\frac{c}{a}\left(1-\left(\frac{1-a}{1+a}\right)^{l+1}\right)$$
$$= \frac{c}{a}+\left(1+b+c-\frac{c}{a}\right)\left(\frac{1-a}{1+a}\right)^{l+1}.$$

The proof is completed. □

Now we give the convergence analysis of Method 2.1 when the system $A \in \mathbb{R}^{n\times n}$ is an H_+-matrix.

Theorem 3.3. *Let A be an H_+-matrix and $A = M-N$ be an H-compatible splitting of the matrix A, that is, $\langle A\rangle = \langle M\rangle - |N|$. Assume that M is a positive definite matrix, Ω is a positive diagonal matrix, and h is a positive constant. Let $\lambda = \rho(B)$, $\Psi_1 = (\Omega+\langle M\rangle)^{-1}(|\Omega - M| + 2|N|)$, $\Psi_2 = (\Omega+\langle M\rangle)^{-1}$.*

If $\frac{2\rho(\Psi_2)}{1-\rho(\Psi_1)}\lambda < 1$, then the iteration sequence $\{u^k\}_{k=0}^{\infty} \subseteq \mathbb{R}^n_+$ generated by Method 2.1 converges to a solution $u^* \in \mathbb{R}^n_+$ of the NCP(1.1) for any initial vector $x^0 \in \mathbb{R}^n$.

Proof. From (3.1), we can obtain

$$|u^{(k+1)} - u^*| \leq h|x^{(k+1)} - x^*|,$$

and similar to the proof of Theorem 3.1, we have

$$x^{(k+1)} = x^{(k,l+1)}$$
$$= (\Omega+M)^{-1}Nx^{(k,l)}+(\Omega+M)^{-1}(\Omega-A)|x^{(k,l)}|-\frac{2}{h}(\Omega+M)^{-1}(q+\Phi(u^{(k)})),$$
$$x^* = (\Omega+M)^{-1}Nx^* + (\Omega+M)^{-1}(\Omega-A)|x^*| - \frac{2}{h}(\Omega+M)^{-1}(q+\Phi(u^*)).$$

As

$$\langle A\rangle = \langle M\rangle - |N|,$$

it holds that

$$\langle A\rangle \leq \langle M\rangle \leq diag(M).$$

So we can obtain $M \in \mathbb{R}^{n\times n}$ is an H_+-matrix and

$$|(\Omega+M)^{-1}| \leq (\Omega+\langle M\rangle)^{-1}.$$

Then,

$$\begin{aligned}
&|x^{(k+1)} - x^*| \\
&\leq |(\Omega+M)^{-1}||N||x^{(k,l)} - x^*| + |(\Omega+M)^{-1}||\Omega - A||x^{(k,l)} - x^*| \\
&\quad + \frac{2}{h}|(\Omega+M)^{-1}||\Phi(u^{(k)}) - \Phi(u^*)| \\
&\leq |(\Omega+M)^{-1}||N||x^{(k,l)} - x^*| + |(\Omega+M)^{-1}||\Omega - M||x^{(k,l)} - x^*| \\
&\quad + |(\Omega+M)^{-1}||N||x^{(k,l)} - x^*| + \frac{2}{h}|(\Omega+M)^{-1}||\Phi(u^{(k)}) - \Phi(u^*)| \\
&\leq (\Omega+\langle M\rangle)^{-1}(|\Omega - M| + 2|N|)|x^{(k,l)} - x^*| + \frac{2}{h}(\Omega+\langle M\rangle)^{-1}B|u^{(k)} - u^*| \\
&= \Psi_1|x^{(k,l)} - x^*| + \frac{2}{h}\Psi_2 B|u^{(k)} - u^*|
\end{aligned}$$

$$\begin{aligned}
&\leq \Psi_1\left[\Psi_1|x^{(k,l-1)}-x^*|+\frac{2}{h}\Psi_2 B|u^{(k)}-u^*|\right]+\frac{2}{h}\Psi_2 B|u^{(k)}-u^*|\\
&=\Psi_1^2|x^{(k,l-1)}-x^*|+\frac{2}{h}(\Psi_1\Psi_2+\Psi_2)B|u^{(k)}-u^*|\\
&\leq \Psi_1^{l+1}|x^{(k,0)}-x^*|+\frac{2}{h}(\sum_{i=0}^{l}\Psi_1^i\Psi_2)B|u^{(k)}-u^*|. \qquad (3.6)
\end{aligned}$$

So,

$$|u^{(k+1)}-u^*|\leq\left[\Psi_1^{l+1}(I+\Omega^{-1}|A|+\Omega^{-1}B)+2\sum_{i=0}^{l}\Psi_1^i\Psi_2 B\right]|u^{(k)}-u^*|. \quad (3.7)$$

We denote

$$\hat{Z}=\Psi_1^{l+1}(I+\Omega^{-1}|A|+\Omega^{-1}B)+2\sum_{i=0}^{l}\Psi_1^i\Psi_2 B.$$

We know that $\rho(\Psi_1)<1$(Theorem 4.3 in Bai [5]), then $\Psi_1^{l+1}\to 0$ when $l\to\infty$. Therefore, in the matrix $\hat{Z}$, we only need to consider $\rho(2\sum_{i=0}^{l}\Psi_1^i\Psi_2 B)$. By the condition

$$\frac{2\rho(\Psi_2)}{1-\rho(\Psi_1)}\lambda<1,$$

we have

$$\begin{aligned}
\rho(\hat{Z})\approx\rho\left(2\sum_{i=0}^{l}\Psi_1^i\Psi_2 B\right)&\leq\|2(\Psi_1^l\Psi_2+\ldots+\Psi_1^2\Psi_2+\Psi_1\Psi_2+\Psi_2\|_2\rho(B)\\
&\leq\|2(I-\Psi_1)^{-1}\Psi_2\|_2\rho(B)\\
&\leq\frac{2\rho(\Psi_2)}{1-\rho(\Psi_1)}\lambda<1.
\end{aligned}$$

Then the proof is completed. □

4. Numerical results

In this section, we will perform two numerical examples to illustrate the theoretical results and the effectiveness of the proposed iteration method for solving the NCP (1.1). All numerical experiments are carried out in MATLAB 2010a on a personal computer. We compare the proposed methods with the methods in literature [13] and [25] in terms of the elapsed CPU times in seconds (denoted as 'Time') and the number of iteration steps (denoted as 'Iter'). In actual computations, the initial point is chosen $x^{(0)}=(1,1,\cdots,1)^T$. For MSORI and MSOR [25] we choose $\alpha=0.4$. And the stopping condition is the current iterations satisfy $Res\leq 10^{-5}$ or the number of the prescribed iteration steps $k_{\max}=1000$, where the Res is defined by

$$Res=\left\|\min\left(F(u^{(k)}),u^{(k)}\right)\right\|_2.$$

Example 4.1. Consider NCP(1.1) with the following coefficient sub-matrices, let $F(u) = Au + \Phi(u) + q$,

$$A = \begin{pmatrix} H & -I & & \\ -I & H & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & H \end{pmatrix} \in \mathbb{R}^{n\times n}$$

with

$$H = \begin{pmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m},$$

$\Phi_i(u_i) = u_i/(1+u_i)$ and a diagonal mapping $\Phi(u) = (\Phi_i(u_i))$ and $q = (-1, 1, -1, 1, \cdots)^T$.

Example 4.2. Consider NCP(1.1) with the following coefficient sub-matrices, let $F(u) = Au + \Phi(u) + q$,

$$A = \begin{pmatrix} H & -0.5I & & \\ -1.5I & H & \ddots & \\ & \ddots & \ddots & -0.5I \\ & & -1.5I & H \end{pmatrix} \in \mathbb{R}^{n\times n}$$

with

$$H = \begin{pmatrix} 4 & -0.5 & & \\ -1.5 & 4 & \ddots & \\ & \ddots & \ddots & -0.5 \\ & & -1.5 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m},$$

$\Phi_i(u_i) = \arctan(u_i)$ and a diagonal mapping $\Phi(u) = (\Phi_i(u_i))$ and $q = (1, -1, 1, -1, \cdots)^T$.

We can obtain many matrix splitting iteration methods from Method 2.1 by using different splittings of A. In our experiments, we fix $\Omega = I, h = 1$, and compare the MSI, MGSI, MSORI, MHSSI method with the MS [25], MGS [25], MSOR [25], MHSS [13] iteration method.

From the above two Tables, we can see that our methods are effective for solving the above two examples. From Table 1 and Table 2, we can see our iteration methods with inner iteration perform better than those methods without inner iteration, since the new methods require much less CPU time and iteration number to achieve the stopping criterion. In fact, the effect is remarkable after adding inner iteration. Particularly when the former methods can not converge, after adding inner iteration, the new methods become converge.

Table 1. Numerical results about different methods for Example 4.1

n	Method	Iter	Time	Res	Method	Iter	Time	Res
100	MS	77	0.0417	8.6620e-06	MSI	10	0.0333	2.8371e-06
	MGS	391	0.0421	9.8112e-06	MGSI	26	0.0256	7.8702e-06
	MSOR	53	0.0173	8.5827e-06	MSORI	10	0.0141	7.4299e-06
	MHSS	77	0.0458	8.6620e-06	MHSSI	10	0.0369	2.8371e-06
400	MS	79	0.5451	8.7537e-06	MSI	10	0.3060	4.4681e-06
	MGS	671	1.8414	9.5661e-06	MGSI	40	0.5384	6.3014e-06
	MSOR	56	0.1831	8.2767e-06	MSORI	11	0.1714	4.3404e-06
	MHSS	79	0.6412	8.7537e-06	MHSSI	10	0.3050	4.4681e-06
900	MS	80	3.6709	9.0515e-06	MSI	10	1.9176	5.7249e-06
	MGS	1000	14.3667	9.9814e-06	MGSI	53	3.7039	6.1517e-06
	MSOR	56	0.1831	8.2767e-06	MSORI	11	0.1714	4.3404e-06
	MHSS	80	3.9958	9.0515e-06	MHSSI	10	2.1171	5.7249e-06
1600	MS	81	15.6848	8.8625e-06	MSI	10	7.6258	6.7526e-06
	MGS	-	-	-	MGSI	65	14.3370	8.7758e-06
	MSOR	58	2.7291	9.7445e-06	MSORI	11	2.5403	8.1612e-06
	MHSS	81	16.7732	8.8625e-06	MHSSI	10	8.3997	6.7526e-06

Table 2. Numerical results about different methods for Example 4.2

n	Method	Iter	Time	Res	Method	Iter	Time	Res
100	MS	546	0.2006	9.8821e-06	MSI	17	0.0374	7.7040e-06
	MGS	-	-	-	MGSI	17	0.0153	9.3190e-06
	MSOR	53	0.0215	8.2387e-06	MSORI	12	0.0140	6.8383e-06
	MHSS	-	-	-	MHSSI	17	0.1302	5.1212e-06
400	MS	549	5.2339	9.9188e-06	MSI	21	0.7151	9.0279e-06
	MGS	-	-	-	MGSI	18	0.2885	9.3914e-06
	MSOR	54	0.1756	9.3452e-06	MSORI	13	0.1621	3,9452e-06
	MHSS	-	-	-	MHSSI	20	2.8495	7.7944e-06
900	MS	552	38.5520	9.8842e-06	MSI	23	5.8906	9.6237e-06
	MGS	-	-	-	MGSI	19	1.4808	6.3334e-06
	MSOR	55	0.9120	9.3502e-06	MSORI	13	0.8103	4.7669e-06
	MHSS	-	-	-	MHSSI	23	22.8536	9.5198e-06
1600	MS	554	203.7181	9.9802e-06	MSI	25	36.1733	64331e-06
	MGS	-	-	-	MGSI	19	4.1329	7.5848e-06
	MSOR	56	2.6809	8.8514e-06	MSORI	13	2.4609	5.5052e-06
	MHSS	-	-	-	MHSSI	27	129.2965	4.9000e-06

5. Conclusions

In this paper, we propose a modulus-based iteration method with inner iteration for nonlinear complementarity problem (1.1). This method is based on presenting the NCP (1.1) as an implicit fix-point equation. The main purpose of our work is improving the convergence performance of the modulus-based matrix splitting iteration method by adding inner iteration. From the results of the numerical tests, we can see that the proposed Method 2.1 is effective, robust, and scalable solver for the NCP (1.1).

Acknowledgements. The authors are grateful to the anonymous referees for their

useful comments and suggestions.

References

[1] Z. Bai, *On the monotone convergence of the projected iteration methods for linear complementarity problem*, Numer. Math., J. Chin. Univer. (Eng. Ser.), 1996, 5, 228–233.

[2] Z. Bai, *On the convergence of the multisplitting methods for the linear complementarity problem*, SIAM. J. Matrix Anal. Appl., 1999, 21, 67–78.

[3] Z. Bai and D. Evans, *Matrix multisplitting methods with applications to linear complementarity problems: parallel asychronous methods*, Int. J. Comput. Math., 2002, 79, 205–232.

[4] Z. Bai and L. Zhang, *Modulus-based synchronous multisplitting iteration methods for linear complementarity problems*, Numer. Linear Algebra Appl., 2013, 20, 425–439.

[5] Z. Bai, *Modulus-based matrix splitting iteration methods for linear complemenarity problems*, Numer. Linear Algebra Appl., 2010, 17, 917–933.

[6] Z. Bai, *The monotone convergence of a class of parallel nonlinear relaxation methods for nonlinear complementarity problems*, Comput. Math. Appl., 1996, 31, 17–33.

[7] Z. Bai, *New comparison theorem for the nonlinear multisplitting relaxation methods for the large sparse nonlinear complementarity problems*, Comput. Math. Appl., 1996, 32, 79–95.

[8] L. Badea, X. Tai and J. Wang, *Convergence rate analysis of a multiplicative Schwarz method for variational inequalities*, SIAM J. Numer. Anal., 2003, 41, 1052–1073.

[9] R. W. Cottle, J. Pang and R. E. Stone, *The Linear Complementarity Problem*, Academic, San Diego, 1992.

[10] J. Dong and M. Jiang, *A modified modulus method for symmetric positive-definite linear complementarity problems*, Numer. Linear Algebra Appl., 2009, 16, 129–143.

[11] M. C. Ferris and J. Pang, *Engineering and economic applications of complementarity problems*, SIAM Review, 1997, 39, 669–713.

[12] F. Facchinei and J. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer Science and Business Media, New York, 2003.

[13] N. Huang and C. Ma, *The modulus-based matrix splitting algorithms for a class of weakly nonlinear complementerity problems*, Numer. Linear Algebra Appl., 2016, 23, 558–569.

[14] J. Hong and C. Li, *Modulus-based matrix splitting iteration methods for a class of implicit complementarity problems*, Numer. Linear Algebra Appl., 2016, 23(4), 629–641.

[15] Y. Jiang and J. Zeng, *A Multiplicative Schwarz algorithm for the nonlinear complementarity problem with an M-function*, Bull. Aust. Soc., 2010, 82, 353–366.

[16] C. Kanzow, *Inexact semismooth Newton methods for large-scale complementarity problems*, Optim. Methods Softw., 2004, 19, 309–325.

[17] B. M. Kwak and J. Y. Kwak, *Binary NCP: a new approach for solving nonlinear complementarity problems*, J. Mech. Sci. Tech., 2021, 35(3), 1161–1166.

[18] W. La Cruz, *A spectral algorithm for large-scale systems of nonlinear monotone equations*, Numer. Algor., 2017, 76(4), 1109–1130 .

[19] F. Mezzadri and E. Galligani, *Modulus-based matrix splitting methods for a class of horizontal nonlinear complementarity problems*, Numer. Algor., 2020, 87(2), 667–687.

[20] K. G. Murty, *Linear Complementarity, Linear and Nonlinear Programming*, Heldermann, Berlin, 1988.

[21] G. H. Meyer, *Free boundary problems with nonlinear source terms*, Numer. Math., 1984, 43, 463–483.

[22] C. Ma, L. Chen and D. Wang, *A globally and superlinearly convergent smoothing Broyden-like method for solving nonlinear complementarity prolem*, Appl. Math. Comput., 2008, 198, 592–604.

[23] J. Pang, *On the convergence of a basic iterative method for the implicit complementarity problems*, J. Optim. Theory Appli., 1982, 37, 149–162.

[24] Z. Xia and C. Li, *Modulus-based matrix splitting iteration methods for a class of nonlinear complementerity problem*, Appl. Math. Comput., 2015, 271, 34–42.

[25] S. Xie and H. Xu, *Two-step modulus-based matrix splitting iteration method for a class of nonlinear complementarity problems*, Linear Algebra Appl., 2016, 494, 1–10.

[26] S. Xie and H. Xu, *An Efficient Class of Modulus-Based Matrix Splitting Methods for Nonlinear Complementarity Problems*, Math. Prob. Engin., 2021, ID 9030547.

[27] N. Zheng and J. Yin, *Convergence of accelerated modulus-based matrix splitting iteration methods for linear complementarity problem with an H_+-matrix*, J. Comput. Appl. Math., 2014, 260, 281–293.

[28] L. Zhang, *Two-step modulus-based matrix splitting iteration methods for linear complementarity problems*, Numer Algor., 2011, 57, 83–99.

[29] L. Zhang and Z. Ren, *Improved convergence theorems of modulus-based matrix splitting iteration methods for linear complementarity problems*, Appl. Math. Lett., 2013, 26, 638–642.

[30] L. Zhang, *Two-stage multispliting iteration methods using modulus-based matrix splitting as inner iteration for linear complementarity problems*, J. Optim. Theory Appl., 2014, 160, 189–203.

[31] Y. Zhang, *Multilevel projection algorithm for solving obstacle problems*, Comput. Math. Appl., 2001, 41, 1505–1513.

[32] X. Zhang and Z. Peng, *A modulus-based nonmonotone line search method for nonlinear complementarity problems*, Appl. Math. Comput., 2020, 387, 125175.