Journal of Applied Analysis and Computation
Volume 13, Number 2, April 2023, 715–733
Website:http://www.jaac-online.com
DOI:10.11948/20220040

WAVE PROPAGATION FOR A DISCRETE DIFFUSIVE VACCINATION EPIDEMIC MODEL WITH BILINEAR INCIDENCE*

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Abstract The aim of the current paper is to study the existence of traveling wave solutions for a vaccination epidemic model with bilinear incidence. The existence result is determined by the basic reproduction number $\Re_0$. More specifically, the system admits nontrivial traveling wave solutions when $\Re_0 > 1$ and $c \geq c^*$, where c^* is the critical wave speed. We also found that the traveling wave solution is connecting two different equilibria by constructing Lyapunov functional. Lastly, we give some biological explanations from the perspective of epidemiology.

Keywords Traveling wave solution, vaccination model, lattice dynamical system, Schauder's fixed point theorem, Lyapunov functional.

MSC(2010) 92D30, 35C07, 35K57.

1. Introduction

Vaccination is critical for the prevention and control of infectious diseases. Vaccinators can achieve immunity by having the immune system recognize foreign substances, antibodies are then screened and generated to produce antibodies against the pathogen or similar pathogen, and then giving the injected individual a high level of disease resistance. In [11], Liu et al. proposed the following system with continuous vaccination strategy:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta_1 S(t)I(t) - \alpha S(t) - \mu S(t), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \alpha S(t) - \beta_2 V(t)I(t) - (\delta + \mu)V(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta_1 S(t)I(t) + \beta_2 V(t)I(t) - \gamma I(x,t) - \mu I(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \delta V(t) + \gamma I(t) - \mu R(t), \end{cases} \tag{1.1}$$

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*This research was supported by the National Natural Science Foundation of China (Nos. 12271401, 12101309), the Natural Science Foundation of Tianjin, China(No.22JCYBJC00080), the Fundamental Research Funds for the Colleges and Universities in Heilongjiang Province (No. 2022-KYYWF-1113) and the Heilongjiang Provincial Key Laboratory of the Theory and Computation of Complex Systems.

where $S(t),\ V(t),\ I(t)$ and $R(t)$ are the densities of susceptible, vaccinated, infective and removed individuals at time t, respectively. The parameters of model (1.1) are biologically explained as in Table 1.

Table 1. Biological meaning of parameters in model (1.1).

Parameter	Interpretation
Λ	Recruitment rate
β_1	Transmission rate between infectious and susceptible individuals
β_2	Transmission rate between infectious and vaccinated individuals
α	The vaccinated rate
μ	Natural death rate
γ	Recovery rate
δ	Rate at which a vaccinating individual obtains immunity

In [11], the authors shown that the disease-free equilibrium for model (1.1) is globally asymptotically stable if the basic reproduction number is less than one, while if the number is greater than one, then a positive endemic equilibrium exists which is globally asymptotically stable. Since then, the epidemic models with vaccination have attracted the attention of many scholars. Kuniya [8] extended the study in [11] to a multi-group case, and then studied the global stability by using the graph-theoretic approach and Lyapunov method. Considering the effect of age, three vaccination epidemic models with age structure are proposed in [3, 14, 15], and the global stabilities are studied. For more recent studies on the vaccination epidemic models, we refer to [7,13,16] and the references therein.

With the increasing trend of globalization and mobility of people, the spatial structure of human density and location has a significant impact on the spread of diseases. It is necessary to investigate the role of diffusion in the epidemic modeling. Mathematically, Laplacian operator in the reaction-diffusion systems usually used to study the diffusive infectious disease model, since it could describe the random diffusion of each individual in the adjacent space. On the other hand, nonlocal operator could describe the long range diffusion on the whole habitat [10]. In the study of local and nonlocal diffusive epidemic models, there is a solution called traveling wave solution. Viewing from infectious diseases perspective, the existence of traveling wave solutions for epidemic model implies that the disease can be invaded [9]. Up to now, there have been many studies on the traveling wave solutions for local and nonlocal diffusive epidemic models (see, for example, [4, 6, 17–21, 25, 28]). By considering both vaccination and spatial diffusion, Xu et al. [22] studied a local diffusive SVIR model, where the global dynamics on bounded domain and traveling wave solutions on unbounded domain for the model were studied. Meanwhile, the problem of traveling wave solution for two different SVIR models with nonlocal diffusion were investigated in [24, 30].

Unlike local and nonlocal diffusive, there is another diffusion in infectious disease modeling, which is discrete diffusion. In fact, epidemic model with discrete diffusion can be regarded as lattice system, such system is better to describe the epidemic model with patch structure [12]. Recently, Chen et al. [2] proposed a lattice SIR epidemic model:

$$\begin{cases} \dfrac{\mathrm{d}S_n(t)}{\mathrm{d}t} = [S_{n+1}(t) + S_{n-1}(t) - 2S_n(t)] + \mu - \beta S_n(t)I_n(t) - \mu S_n(t), \\ \dfrac{\mathrm{d}I_n(t)}{\mathrm{d}t} = d[I_{n+1}(t) + I_{n-1}(t) - 2I_n(t)] + \beta S_n(t)I_n(t) - (\gamma+\mu)I_n(t), \end{cases} \tag{1.2}$$

where $n \in \mathbb{Z}$. S_n and I_n denote densities of susceptible and infectious individuals at time t and niche n. β is the disease transmission rate. 1 (normalized) and d denote the random migration parameters for each compartments. Chen et al. shown that system (1.2) admits traveling wave solutions when $\Re_0 > 1$ and $c \geq c^*$. More recently, the traveling wave solutions for (1.2) was proved to be converged to the endemic equilibrium by Zhang et al [26]. Model (1.2) is an SIR model with constant recruitment (i.e. the constant Λ), and the existence of traveling wave solutions for the discrete diffusive epidemic model without constant recruitment was studied in [5, 23, 27, 29]. However, to our best knowledge, there are only a few studies focus on the problem of traveling wave solutions for discrete diffusive epidemic models, especially for the model with constant recruitment.

Based on the above facts, in order to study the role of vaccination and patch structure in the disease modeling, we consider a discrete diffusive vaccination epidemic model as follows

$$\begin{cases} \dfrac{\mathrm{d}S_n(t)}{\mathrm{d}t} = d_1[S_{n+1} - 2S_n + S_{n-1}](t) + \Lambda - \beta_1 S_n(t)I_n(t) - (\alpha+\mu)S_n(t), \\ \dfrac{\mathrm{d}V_n(t)}{\mathrm{d}t} = d_2[V_{n+1} - 2V_n + V_{n-1}](t) + \alpha S_n(t) - \beta_2 V_n(t)I_n(t) - (\delta+\mu)V_n(t), \\ \dfrac{\mathrm{d}I_n(t)}{\mathrm{d}t} = d_3[I_{n+1} - 2I_n + I_{n-1}](t) + (\beta_1 S_n(t) + \beta_2 V_n(t))I_n(t) - (\gamma+\mu)I_n(t), \\ \dfrac{\mathrm{d}R_n(t)}{\mathrm{d}t} = d_4[R_{n+1} - 2R_n + R_{n-1}](t) + \delta V_n(t) + \gamma I_n(t) - \mu R_n(t), \end{cases} \tag{1.3}$$

where S_n, V_n, I_n and R_n denote susceptible, vaccinated, infectious and removed individuals. d_i, $(i = 1, 2, 3, 4)$ are the diffusive rates. The biological significance of the parameters of (1.3) are the same as those in (1.1).

The current paper devotes to study the existence of traveling wave solutions for system (1.3) with bilinear incidence. In fact, there are very few studies on traveling wave solutions for the epidemic model with bilinear incidence and the main difficulty is the boundedness of traveling wave solutions [12]. On the other hand, introducing the constant recruitment (i.e. Λ in model (1.3)) will bring much more complexity in mathematical analysis than the system without constant recruitment. Moreover, it is difficult to obtain the behaviour of traveling wave solutions at $+\infty$ for such model (see, for example, [2]). One motivation of this paper is to show the convergence of traveling wave solutions for lattice epidemic model (1.3). To gain this purpose, we will construct an appropriate Lyapunov functional for the wave form equations corresponding to lattice dynamical system (1.3). To do this, we prove the persistence of traveling wave solutions, which is crucial to guarantee the Lyapunov functional has a lower bound. We should point out that, for different models, the construction of Lyapunov functional is also different and requires technique. Biologically, since the vaccination has an effect of decreasing the basic reproduction number in [11], we want to study how vaccination affects the speed of traveling wave solution.

The organization of this paper is as follows. In Section 2, we give some preliminaries results. Section 3 devote to study the existence of traveling wave solutions for system (1.3) by applying Schauder's fixed point theorem. In Section 4, we show the boundedness of traveling wave solutions. Furthermore, we show the convergence of traveling wave solutions in Section 5. Finally, there is a brief discussion and some explanations from the perspective of epidemiology will be given in Section 6.

2. Preliminaries

Firstly, the corresponding ordinary differential system for (1.3) is

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta_1 S(t)I(t) - \mu_1 S(t), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \alpha S(t) - \beta_2 V(t)I(t) - \mu_2 V(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta_1 S(t)I(t) + \beta_2 V(t)I(t) - \mu_3 I(t), \end{cases} \tag{2.1}$$

where $\mu_1 = \alpha + \mu$, $\mu_2 = \delta + \mu$, $\mu_3 = \gamma + \mu$ and R-equation is decoupled from other equations. Clearly, system (2.1) has a disease-free equilibrium $E_0 = (S_0, V_0, 0) = \left(\frac{\Lambda}{\mu_1}, \frac{\Lambda\alpha}{\mu_1\mu_2}, 0\right)$. Define

$$\Re_0 = \frac{\beta_1 S_0 + \beta_2 V_0}{\mu_3}$$

as the basic reproduction number. The well known results for (2.1) is the following lemma.

Lemma 2.1 ([11, Theorem 2.1]). *For system (2.1), if $\Re_0 < 1$, E_0 is globally asymptotically stable; if $\Re_0 > 1$, system (2.1) has a globally asymptotically stable positive equilibrium $E^* = (S^*, V^*, I^*)$ satisfies*

$$\begin{cases} \Lambda - \beta_1 S^* I^* - \mu_1 S^* = 0, \\ \alpha S^* - \beta_2 V^* I^* - \mu_2 V^* = 0, \\ \beta_1 S^* I^* + \beta_2 V^* I^* - \mu_3 I^* = 0. \end{cases}$$

Now, we state our purpose of the current paper. Letting $\varsigma = n + ct$ in system (1.3), where c is wave speed, we arrive at

$$\begin{cases} cS'(\varsigma) = d_1 \mathcal{J}[S](\varsigma) + \Lambda - \mu_1 S(\varsigma) - \beta_1 S(\varsigma)I(\varsigma), \\ cV'(\varsigma) = d_2 \mathcal{J}[V](\varsigma) + \alpha S(\varsigma) - \beta_2 V(\varsigma)I(\varsigma) - \mu_2 V(\varsigma), \\ cI'(\varsigma) = d_3 \mathcal{J}[I](\varsigma) + \beta_1 S(\varsigma)I(\varsigma) + \beta_2 V(\varsigma)I(\varsigma) - \mu_3 I(\varsigma), \end{cases} \tag{2.2}$$

for all $\varsigma \in \mathbb{R}$, where $\mathcal{J}[(\cdot)](\varsigma) := (\cdot)(\varsigma + 1) - 2(\cdot)(\varsigma) + (\cdot)(\varsigma - 1)$. We want to find traveling wave solutions satisfying:

$$\lim_{\varsigma \to -\infty} (S(\varsigma), V(\varsigma), I(\varsigma)) = (S_0, V_0, 0), \tag{2.3}$$

and

$$\lim_{\varsigma \to +\infty} (S(\varsigma), V(\varsigma), I(\varsigma)) = (S^*, V^*, I^*). \tag{2.4}$$

2.1. Eigenvalue problem

Linearizing the third equation of (2.2) at the E_0 yields

$$cI'(\varsigma) = d_3\mathcal{J}[I](\varsigma) - \mu_3 I(\varsigma) + (\beta_1 S_0 + \beta_2 V_0)I(\varsigma).$$

Let $I(\varsigma) = e^{\lambda\varsigma}$, we have

$$d_3[e^{\lambda} + e^{-\lambda} - 2] - c\lambda + (\beta_1 S_0 + \beta_2 V_0) - \mu_3 = 0.$$

Denote

$$\Delta(\lambda, c) = d_3[e^{\lambda} + e^{-\lambda} - 2] - c\lambda + (\beta_1 S_0 + \beta_2 V_0) - \mu_3.$$

By some calculations, for $\lambda > 0$ and $c > 0$, we have

$$\begin{aligned}
&\Delta(0, c) = (\beta_1 S_0 + \beta_2 V_0) - (\mu + \gamma), \quad \lim_{c\to+\infty} \Delta(\lambda, c) = -\infty,\\
&\frac{\partial \Delta(\lambda, c)}{\partial \lambda} = d_3[e^{\lambda} - e^{-\lambda}] - c, \quad \frac{\partial \Delta(\lambda, c)}{\partial c} = -\lambda < 0,\\
&\frac{\partial^2 \Delta(\lambda, c)}{\partial \lambda^2} = d_3[e^{\lambda} + e^{-\lambda}] > 0, \quad \left.\frac{\partial \Delta(\lambda, c)}{\partial \lambda}\right|_{(0,c)} = -c < 0.
\end{aligned}$$

Therefore, we arrive at the following lemma on the distribution for the roots of $\Delta(\lambda, c)$.

Lemma 2.2. *Let $\Re_0 > 1$. There exist $c^* > 0$ and $\lambda^* > 0$ such that*

$$\left.\frac{\partial \Delta(\lambda, c)}{\partial \lambda}\right|_{(\lambda^*, c^*)} = 0 \quad and \quad \Delta(\lambda^*, c^*) = 0.$$

Furthermore,

(i) $\Delta(\lambda, c) > 0$ *for all* λ *if* $0 < c < c^*$.

(ii) $\Delta(\lambda, c) = 0$ *has only one positive real root* λ^* *if* $c = c^*$.

(iii) $\Delta(\lambda, c) = 0$ *has two positive real roots* λ_1, λ_2 *with* $\lambda_1 < \lambda^* < \lambda_2$ *if* $c > c^*$.

Before giving the results on the existence of traveling wave solutions, we directly proposed the following theorem of nonexistence of traveling wave solutions. Since the proof are almost the same with those in [2, 26] by using Laplace transform, we omit the details here.

Theorem 2.1. *If $\Re_0 > 1$ and $0 < c < c^*$, then there is no nontrivial traveling wave solution of system (1.3) satisfying the asymptotic boundary conditions (2.3)-(2.4).*

2.2. Sub- and super-solutions

Fix $c > c^*$ and $\Re_0 > 1$, we will show the following lemma.

Lemma 2.3. *For sufficiently small $\varepsilon_i > 0$ and sufficiently large $M_i > 0$ $(i = 1, 2, 3)$, we define the following six functions:*

$$\begin{cases} S^+(\varsigma) = S_0,\\ V^+(\varsigma) = V_0,\\ I^+(\varsigma) = e^{\lambda_1\varsigma}, \end{cases} \qquad \begin{cases} S^-(\varsigma) = \max\{S_0(1 - M_1 e^{\varepsilon_1\varsigma}), 0\},\\ V^-(\varsigma) = \max\{V_0(1 - M_2 e^{\varepsilon_2\varsigma}), 0\},\\ I^-(\varsigma) = \max\{e^{\lambda_1\varsigma}(1 - M_3 e^{\varepsilon_3\varsigma}), 0\}. \end{cases}$$

Then they satisfy

$$\begin{cases} cS^{+\prime}(\varsigma) \geq d_1\mathcal{J}[S^+] + \Lambda - \mu_1 S^+ - \beta_1 S^+ I^-, \\ cV^{+\prime}(\varsigma) \geq d_2\mathcal{J}[V^+] + \alpha S^+ - \beta_2 V^+ I^- - \mu_2 V^+, \\ cI^{+\prime}(\varsigma) \geq d_3\mathcal{J}[I^+] + \beta_1 S^+ I^+ + \beta_2 V^+ I^+ - \mu_3 I^+, \end{cases} \tag{2.5}$$

and

$$\begin{cases} cS^{-\prime}(\varsigma) \leq d_1\mathcal{J}[S^-] + \Lambda - \mu_1 S^- - \beta_1 S^- I^+, & \varsigma \neq \varepsilon_1^{-1} \ln M_1^{-1} := \mathfrak{X}_1, \quad (2.6a) \\ cV^{-\prime}(\varsigma) \leq d_2\mathcal{J}[V^-] + \alpha S^- - \beta_2 V^- I^+ - \mu_2 V^-, & \varsigma \neq \varepsilon_2^{-1} \ln M_2^{-1} := \mathfrak{X}_2, \quad (2.6b) \\ cI^{-\prime}(\varsigma) \leq d_3\mathcal{J}[I^-] + \beta_1 S^- I^- + \beta_2 V^- I^- - \mu_3 I^-, & \varsigma \neq \varepsilon_3^{-1} \ln M_3^{-1} := \mathfrak{X}_3. \quad (2.6c) \end{cases}$$

Proof. The proof of (2.5) are trivial, so we omit the details. Now, we focus on the proof of inequalities (2.6). If $\varsigma > \mathfrak{X}_1$, then equation (2.6a) holds since $S^-(\varsigma) = 0$. If $\varsigma < \mathfrak{X}_1$, then $S^-(\varsigma) = S_0(1 - M_1 e^{\varepsilon_1 \varsigma})$ and

$$\begin{aligned} & d_1\mathcal{J}[S^-](\varsigma) + \Lambda - \mu_1 S^-(\varsigma) - \beta_1 S^-(\varsigma) I^+(\varsigma) - cS^{-\prime}(\varsigma) \\ & \geq e^{\varepsilon_1 \varsigma} S_0 \left[-M_1(d_1 e^{\varepsilon_1} + d_1 e^{-\varepsilon_1} - 2d_1 - \mu_1 - c\varepsilon_1) - \beta_1 e^{\lambda_1 \varsigma} e^{-\varepsilon_1 \varsigma} \right]. \end{aligned}$$

Choosing $0 < \varepsilon_1 < \lambda_1$ such that $\varepsilon_1 M_1 = 1$. With the help of L'Hopital's rule, we can obtain that

$$\lim_{\varepsilon_1 \to 0^+} M_1(2 - e^{\varepsilon_1} - e^{-\varepsilon_1}) = \lim_{\varepsilon_1 \to 0^+} \frac{2 - e^{\varepsilon_1} - e^{-\varepsilon_1}}{\varepsilon_1} = \lim_{\varepsilon_1 \to 0^+} \varepsilon_1 (e^{-\varepsilon_1} - e^{\varepsilon_1}) = 0.$$

Since $\varsigma < \mathfrak{X}_1$ and $0 < \varepsilon_1 < \lambda_1$, we have

$$e^{(\lambda_1 - \varepsilon_1)\mathfrak{X}_1} \leq \varepsilon_1 e^{\frac{\lambda_1 - \varepsilon_1}{\varepsilon_1}} \to 0 \quad \text{as} \quad \varepsilon_1 \to 0.$$

Hence, we can claim that (2.6a) holds for ε_1 is small enough. Similarly, (2.6b) is true for $\varsigma \neq \varepsilon_2^{-1} \ln M_2^{-1} := \mathfrak{X}_2$.

Now, we focus on (2.6c), let M_3 satisfy $\frac{1}{\varepsilon_3} \ln M_3 > \max\left\{\frac{1}{\varepsilon_1} \ln M_1, \frac{1}{\varepsilon_2} \ln M_2\right\}$. If $\varsigma > \mathfrak{X}_3$, then (2.6c) holds since $I^-(\varsigma) = 0$. If $\varsigma < \mathfrak{X}_3$, then $I^-(\varsigma) = e^{\lambda_1 \varsigma}(1 - M_3 e^{\varepsilon_3 \varsigma})$, and (2.6c) is equivalent to

$$\begin{aligned} & d_3\mathcal{J}[I^-](\varsigma) + \beta_1 S^-(\varsigma) I^-(\varsigma) + \beta_2 V^-(\varsigma) I^-(\varsigma) - \mu_3 I^-(\varsigma) - cI^{-\prime}(\varsigma) \\ \geq & \ d_3 \left[e^{\lambda_1(\varsigma+1)} \left(1 - M_3 e^{\varepsilon_3(\varsigma+1)}\right) + e^{\lambda_1(\varsigma-1)} \left(1 - M_3 e^{\varepsilon_3(\varsigma-1)}\right) - 2e^{\lambda_1 \varsigma} \left(1 - M_3 e^{\varepsilon_3 \varsigma}\right) \right] \\ & + \beta_1 S_0 e^{\lambda_1 \varsigma} \left(1 - M_1 e^{\varepsilon_1 \varsigma}\right) \left(1 - M_3 e^{\varepsilon_3 \varsigma}\right) + \beta_2 V_0 e^{\lambda_1 \varsigma} \left(1 - M_2 e^{\varepsilon_2 \varsigma}\right) \left(1 - M_3 e^{\varepsilon_3 \varsigma}\right) \\ & - \mu_3 e^{\lambda_1 \varsigma} \left(1 - M_3 e^{\varepsilon_3 \varsigma}\right) - c\lambda_1 e^{\lambda_1 \varsigma} + c(\lambda_1 + \varepsilon_3) e^{(\lambda_1 + \varepsilon_3)\varsigma} \\ \geq & \ e^{\lambda_1 \varsigma} \Delta(\lambda_1, c) - e^{(\lambda_1 + \varepsilon_3)\varsigma} M_3 \Delta(\lambda_1 + \varepsilon_3, c) - \beta_1 S_0 M_1 e^{(\lambda_1 + \varepsilon_1)\varsigma} - \beta_2 V_0 M_2 e^{(\lambda_1 + \varepsilon_2)\varsigma}. \end{aligned}$$

Using the definition of $\Delta(\lambda, c)$ and noticing that $\Delta(\lambda_1 + \varepsilon_3, c) < 0$, then it suffices to show that

$$-M_3 \Delta(\lambda_1 + \varepsilon_3, c) \geq \beta_1 S_0 M_1 e^{(\varepsilon_1 - \varepsilon_3)\varsigma} + \beta_2 V_0 M_2 e^{(\varepsilon_2 - \varepsilon_3)\varsigma},$$

which holds for M_3 is large enough. □

3. Existence of traveling wave solutions

Let $X > \mathfrak{X} > 0$, where $\mathfrak{X} := \max\{-\mathfrak{X}_1, -\mathfrak{X}_2, -\mathfrak{X}_3\}$. Define

$$\Gamma_X := \left\{ (\phi,\varphi,\psi) \in C([-X,X],\mathbb{R}^3) \,\middle|\, \begin{array}{l} S^-(\varsigma) \le \phi(\varsigma) \le S^+(\varsigma),\ V^-(\varsigma) \le \varphi(\varsigma) \le V^+(\varsigma), \\ I^-(\varsigma) \le \psi(\varsigma) \le I^+(\varsigma) \ \text{ for all } \ \varsigma \in [-X,X], \\ \phi(-X) = S^-(-X), \quad \varphi(-X) = V^-(-X), \\ \psi(-X) = I^-(-X). \end{array} \right\}.$$

For any $(\phi,\varphi,\psi) \in C([-X,X],\mathbb{R}^3)$, define

$$\hat{\phi}(\varsigma) = \begin{cases} \phi(X), & \text{for } \varsigma > X, \\ \phi(\varsigma), & \text{for } \varsigma \in [-X,X], \\ S^-(\varsigma), & \text{for } \varsigma < -X, \end{cases} \qquad \hat{\varphi}(\varsigma) = \begin{cases} \varphi(X), & \text{for } \varsigma > X, \\ \varphi(\varsigma), & \text{for } \varsigma \in [-X,X], \\ V^-(\varsigma), & \text{for } \varsigma < -X, \end{cases}$$

and

$$\hat{\psi}(\varsigma) = \begin{cases} \psi(X), & \text{for } \varsigma > X, \\ \psi(\varsigma), & \text{for } \varsigma \in [-X,X], \\ I^-(\varsigma), & \text{for } \varsigma < -X. \end{cases}$$

For $(\phi,\varphi,\psi) \in \Gamma_X$, let

$$\begin{cases} d_1\hat{\phi}(\varsigma+1) + d_1\hat{\phi}(\varsigma-1) + \Lambda + \rho_1\phi(\varsigma) - \beta_1\phi(\varsigma)\psi(\varsigma) := H_1(\psi,\varphi,\psi), \\ d_2\hat{\varphi}(\varsigma+1) + d_2\hat{\varphi}(\varsigma-1) + \alpha\phi(\varsigma) + \rho_2\varphi - \beta_2\varphi(\varsigma)\psi(\varsigma) := H_2(\psi,\varphi,\psi), \\ d_3\hat{\psi}(\varsigma+1) + d_3\hat{\psi}(\varsigma-1) + \beta_1\phi(\varsigma)\psi(\varsigma) + \beta_2\varphi(\varsigma)\psi(\varsigma) := H_3(\psi,\varphi,\psi). \end{cases}$$

Consider the following truncated initial problem:

$$\begin{cases} cS'(\varsigma) + (2d_1+\mu_1+\rho_1)S(\varsigma) = H_1(\psi,\varphi,\psi), \\ cV'(\varsigma) + (2d_2+\mu_2+\rho_2)V(\varsigma) = H_2(\psi,\varphi,\psi), \\ cI'(\varsigma) + (2d_3+\mu_3)I(\varsigma) = H_3(\psi,\varphi,\psi), \\ (S,V,I)(-X) = (S^-,V^-,I^-)(-X), \end{cases} \tag{3.1}$$

where ρ_1 is large enough such that $\rho_1\phi - \beta_1\phi\psi$ is nondecreasing on ϕ and ρ_2 is large enough such that $\rho_2\varphi - \beta_2\varphi\psi$ is nondecreasing on φ. Clearly, system (3.1) has a unique solution $(S_X(\varsigma), V_X(\varsigma), I_X(\varsigma)) \in C([-X,X],\mathbb{R}^3)$. Define

$$\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) : \Gamma_X \to C\left([-X,X],\mathbb{R}^3\right)$$

by

$$S_X(\varsigma) = \mathcal{A}_1(\phi,\varphi,\psi)(\varsigma), \quad V_X(\varsigma) = \mathcal{A}_2(\phi,\varphi,\psi)(\varsigma) \ \text{ and } \ I_X(\varsigma) = \mathcal{A}_3(\phi,\varphi,\psi)(\varsigma).$$

Lemma 3.1. *The operator $\mathcal{A}$ maps Γ_X into itself and it is completely continuous.*

Proof. Firstly, we show that $\mathcal{A}$ maps Γ_X into Γ_X. If $\varsigma \in (\mathfrak{X}_1, X)$, then $S^-(\varsigma) = 0$ and it is a sun-solution of the first equation of (3.1). If $\varsigma \in (-X, \mathfrak{X}_1)$, then $S^-(\varsigma) = S_0(1 - M_1 e^{\varepsilon_1 \varsigma})$. Using the definition of constant ρ_1 and Lemma 2.3, we have

$$\begin{aligned}
&cS^{-'} + (2d_1 + \mu_1 + \rho_1)S^- - d_1(\hat{\phi}(\varsigma + 1) + \hat{\phi}(\varsigma - 1)) - \Lambda - \rho_1\phi + \beta_1\phi\psi \\
\le& cS^{-'} - d_1\mathcal{J}[S^-](\varsigma) - \Lambda + \mu_1 S^- + \beta S^- I^+ \\
\le& 0,
\end{aligned}$$

which implies that $S^-(\varsigma) = S_0(1 - M_1 e^{\varepsilon_1 \varsigma})$ is a sub-solution to the first equation of (3.1). Thus $S^-(\varsigma) \le S_X(\varsigma)$ for any $\varsigma \in [-X, X]$. On the other hand,

$$\begin{aligned}
&cS^{+'} + (2d_1 + \mu_1 + \rho_1)S^+ - d_1(\hat{\phi}(\varsigma + 1) + \hat{\phi}(\varsigma - 1)) - \Lambda - \rho_1\phi - \beta_1\phi\psi \\
\ge& \beta_1 S_0 I^- \\
\ge& 0,
\end{aligned}$$

thus $S^+(\varsigma) = S_0$ is a super-solution to the first equation of (3.1), which gives us $S_X(\varsigma) \le S_0$ for any $\varsigma \in [-X, X]$. With some similar arguments as above, we can obtain that

$$V^-(\varsigma) \le V_X(\varsigma) \le V^+(\varsigma) \text{ and } I^-(\varsigma) \le I_X(\varsigma) \le I^+(\varsigma) \quad \text{for all} \quad \varsigma \in [-X, X],$$

this means that $\mathcal{A}$ maps Γ_X into itself.

Next, we focus on the second part of Lemma 3.1. For $i = 1, 2$, suppose that $(\phi_i(\varsigma), \varphi_i(\varsigma), \psi_i(\varsigma)) \in \Gamma_X$ with

$$S_{X,i}(\varsigma) = \mathcal{A}_1(\phi_i(\varsigma), \varphi_i(\varsigma), \psi_i(\varsigma)), \quad V_{X,i}(\varsigma) = \mathcal{A}_2(\phi_i(\varsigma), \varphi_i(\varsigma), \psi_i(\varsigma)),$$

and

$$I_{X,i}(\varsigma) = \mathcal{A}_3(\phi_i(\varsigma), \varphi_i(\varsigma), \psi_i(\varsigma)).$$

Direct calculation yields

$$S_X(\varsigma) = S^-(-X)e^{-\frac{2d_1+\mu_1+\rho_1}{c}(\varsigma+X)} + \frac{1}{c}\int_{-X}^{\varsigma} e^{\frac{2d_1+\mu_1+\rho_1}{c}(\tau-\varsigma)} H_1(\phi, \varphi, \psi)(\tau)\mathrm{d}\tau,$$

$$V_X(\varsigma) = V^-(-X)e^{-\frac{2d_2+\mu_2+\rho_2}{c}(\varsigma+X)} + \frac{1}{c}\int_{-X}^{\varsigma} e^{\frac{2d_2+\mu_2+\rho_2}{c}(\tau-\varsigma)} H_2(\phi, \varphi, \psi)(\tau)\mathrm{d}\tau$$

and

$$I_X(\varsigma) = I^-(-X)e^{-\frac{2d_3+\mu_3}{c}(\varsigma+X)} + \frac{1}{c}\int_{-X}^{\varsigma} e^{\frac{2d_3+\mu_3}{c}(\tau-\varsigma)} H_3(\phi, \varphi, \psi)(\tau)\mathrm{d}\tau.$$

For $i = 1, 2$ and any $(\phi_i, \varphi_i, \psi_i) \in \Gamma_X$, we have

$$\begin{aligned}
&|\phi_1(\varsigma)\psi_1(\varsigma) - \phi_2(\varsigma)\psi_2(\varsigma)| \\
\le& |\phi_1(\varsigma)\psi_1(\varsigma) - \phi_1(\varsigma)\psi_2(\varsigma)| + |\phi_1(\varsigma)\psi_2(\varsigma) - \phi_2(\varsigma)\psi_2(\varsigma)| \\
\le& S_0 \max_{\varsigma\in[-X,X]} |\psi_1(\varsigma) - \psi_2(\varsigma)| + e^{\lambda_1 X} \max_{\varsigma\in[-X,X]} |\phi_1(\varsigma) - \phi_2(\varsigma)|.
\end{aligned}$$

Hence,

$$\begin{aligned} &c(S'_{X,1}(\varsigma) - S'_{X,2}(\varsigma)) + (2d_1 + \mu_1)(S_{X,1}(\varsigma) - S_{X,2}(\varsigma)) \\ &\le \beta_1 S_0 \max_{\varsigma \in [-X,X]} |\psi_1(\varsigma) - \psi_2(\varsigma)| + \left(2d_1 + \beta_1 e^{\lambda_1 X}\right) \max_{\varsigma \in [-X,X]} |\phi_1(\varsigma) - \phi_2(\varsigma)|. \end{aligned}$$

Using the explicit solutions $(S_X(\varsigma), V_X(\varsigma), I_X(\varsigma))$, similar arguments to V_X and I_X, we know that the operator $\mathcal{A}$ is continuous. Moreover, S'_X, V'_X and I'_X are bounded by (3.1). Thus, the operator $\mathcal{A}$ is completely continuous. □

By using Schauder's fixed point theorem, there exists $(S_X, V_X, I_X) \in \Gamma_X$ such that

$$(S_X(\varsigma), V_X(\varsigma), I_X(\varsigma)) = \mathcal{A}(S_X, V_X, I_X)(\varsigma)$$

for $\varsigma \in [-X, X]$. Next, we give some prior estimates for (S_X, V_X, I_X). Define

$$C^{1,1}([-X,X]) = \{\upsilon \in C^1([-X,X]) \mid \upsilon, \upsilon' \text{are Lipschitz continuous}\}$$

with

$$\|\upsilon\|_{C^{1,1}([-X,X])} = \max_{x\in[-X,X]} |\upsilon| + \max_{x\in[-X,X]} |\upsilon'| + \sup_{\substack{x,y\in[-X,X] \\ x\neq y}} \frac{|\upsilon'(x) - \upsilon'(y)|}{|x-y|}.$$

Lemma 3.2. *There exists constant $\mathcal{C}(\mathcal{X}) > 0$ such that*

$$\|S_X\|_{C^{1,1}([-\mathcal{X},\mathcal{X}])} \le \mathcal{C}(\mathcal{X}), \quad \|V_X\|_{C^{1,1}([-\mathcal{X},\mathcal{X}])} \le \mathcal{C}(\mathcal{X}) \quad \text{and} \quad \|I_X\|_{C^{1,1}([-\mathcal{X},\mathcal{X}])} \le \mathcal{C}(\mathcal{X})$$

for any $\mathcal{X} < X$.

Proof. Since (S_X, V_X, I_X) is the fixed point of $\mathcal{A}$, one has

$$\begin{cases} cS'_X(\varsigma) = d_1\hat{S}_X(\varsigma+1) + d_1\hat{S}_X(\varsigma-1) - (2d_1+\mu_1)S_X(\varsigma) + \Lambda - \beta_1 S_X(\varsigma)I_X(\varsigma), \\ cV'_X(\varsigma) = d_2\hat{V}_X(\varsigma+1) + d_2\hat{V}_X(\varsigma-1) - (2d_2+\mu_2)V_X(\varsigma) + \alpha S_X(\varsigma) - \beta_2 V_X(\varsigma)I_X(\varsigma), \\ cI'_X(\varsigma) = d_3\hat{I}_X(\varsigma+1) + d_3\hat{I}_X(\varsigma-1) - (2d_3+\mu_3)I_X(\varsigma) + (\beta_1 S_X(\varsigma) + \beta_2 V_X(\varsigma))I_X(\varsigma), \end{cases} \tag{3.2}$$

where

$$\hat{S}_X(\varsigma) = \begin{cases} S_X(X), \text{ for } \varsigma > X, \\ S_X(\varsigma), \quad \text{for } \varsigma \in [-X,X], \\ S^-(\varsigma), \quad \text{for } \varsigma < -X, \end{cases} \quad \hat{V}_X(\varsigma) = \begin{cases} V_X(X), \text{ for } \varsigma > X, \\ V_X(\varsigma), \quad \text{for } \varsigma \in [-X,X], \\ V^-(\varsigma), \quad \text{for } \varsigma < -X, \end{cases}$$

and

$$\hat{I}_X(\varsigma) = \begin{cases} I_X(X), \text{ for } \varsigma > X, \\ I_X(\varsigma), \quad \text{for } \varsigma \in [-X,X], \\ I^-(\varsigma), \quad \text{for } \varsigma < -X. \end{cases}$$

Since $0 \le S_X(\varsigma) \le S_0$, $0 \le V_X(\varsigma) \le V_0$ and $0 \le I_X(\varsigma) \le e^{\lambda_1\mathcal{X}}$ for all $\varsigma \in [-\mathcal{X}, \mathcal{X}]$, it follows from (3.2) that

$$|S'_X(\varsigma)| \le \frac{4d_1+\mu_1}{c}S_0 + \frac{\Lambda}{c} + \frac{\beta_1 S_0}{c}e^{\lambda_1\mathcal{X}},$$

$$|V_X'(\varsigma)| \leq \frac{4d_2+\mu_2}{c}V_0 + \frac{\alpha S_0}{c} + \frac{\beta_2 V_0}{c}e^{\lambda_1 \mathcal{X}},$$

and

$$|I_X'(\varsigma)| \leq \frac{4d_3+\mu_3+(\beta_1 S_0+\beta_2 V_0)}{c}e^{\lambda_1 \mathcal{X}}.$$

Hence,

$$\|S_X\|_{C^1([-\mathcal{X},\mathcal{X}])} \leq C_1(\mathcal{X}),\ \ \|V_X\|_{C^1([-\mathcal{X},\mathcal{X}])} \leq C_1(\mathcal{X}) \text{ and } \|I_X\|_{C^1([-\mathcal{X},\mathcal{X}])} \leq C_1(\mathcal{X}),$$

for some constant $C_1(\mathcal{X}) > 0$. It follows from the proof of Lemma 2.4 in [23] that $|\hat{S}_X(\varsigma+1) - \hat{S}_X(\eta+1)| \leq C_1(\mathcal{X})|\varsigma-\eta|$ and $|\hat{S}_X(\varsigma-1) - \hat{S}_X(\eta-1)| \leq C_1(\mathcal{X})|\varsigma-\eta|$ for all $\varsigma, \eta \in [-\mathcal{X},\mathcal{X}]$. Furthermore

$$\begin{aligned}
&|\beta_1 S_X(\varsigma)I_X(\varsigma) - \beta_1 S_X(\eta)I_X(\eta)| \\
&\leq |\beta_1 S_X(\varsigma)I_X(\varsigma) - \beta_1 S_X(\varsigma)I_X(\eta)| + |\beta_1 S_X(\varsigma)I_X(\eta) - \beta_1 S_X(\eta)I_X(\eta)| \\
&\leq \beta_1 C_1(\mathcal{X})\left(|S_X(\varsigma) - S_X(\eta)| + |I_X(\varsigma) - I_X(\eta)|\right)
\end{aligned}$$

for all $\varsigma, \eta \in [-\mathcal{X},\mathcal{X}]$. Thus, $\|S_X\|_{C^{1,1}([-\mathcal{X},\mathcal{X}])} \leq \mathcal{C}(\mathcal{X})$ for some constant $\mathcal{C}(\mathcal{X}) > 0$. Similarly,

$$\|V_X\|_{C^{1,1}([-\mathcal{X},\mathcal{X}])} \leq \mathcal{C}(\mathcal{X}) \text{ and } \|I_X\|_{C^{1,1}([-\mathcal{X},\mathcal{X}])} \leq \mathcal{C}(\mathcal{X}).$$

for any $\mathcal{X} < X$. □

Choosing $\{X_n\}_{n=1}^{+\infty}$ be an increasing sequence such that $X_n > \max\{\mathfrak{X}, \mathcal{X}\}$ and $X_n \to +\infty$ as $n \to +\infty$ for all $n \in \mathbb{N}$, where $\mathcal{X}$ is from Lemma 3.2. Denote $(S_n, V_n, I_n) \in \Gamma_{X_n}$ be the solution of system (3.1). For any $N \in \mathbb{N}$, with the help of Lemma 3.2 and following from the standard arguments in [26], we know that the sequences (S_n, V_n, I_n), (S_n', V_n', I_n') and (S_n'', V_n'', I_n'') are uniformly bounded in $[-X_n, X_n]$ for $n \geq N$. By the Arzela-Ascoli theorem, we can use a diagonal process to extract a subsequence, denoted by $\{S_{n_k}\}_{k\in\mathbb{N}}$, $\{V_{n_k}\}_{k\in\mathbb{N}}$ and $\{I_{n_k}\}_{k\in\mathbb{N}}$ such that

$$S_{n_k} \to S,\ V_{n_k} \to V, I_{n_k} \to I,\ S_{n_k}' \to S', V_{n_k}' \to V' \text{ and } I_{n_k}' \to I' \text{ as } k \to +\infty$$

uniformly in any compact subinterval of $\mathbb{R}$, for some functions S, V and I in $C^1(\mathbb{R})$. Thus, (S, V, I) is solution for system (2.2) with

$$S^- \leq S(\varsigma) \leq S^+,\ \ V^- \leq V(\varsigma) \leq V^+,\ \ I^- \leq I(\varsigma) \leq I^+,\ \ \forall \varsigma \in \mathbb{R}.$$

Up to know, we only obtain the existence of traveling wave solutions, the boundedness of this solution will be proved in the following section.

4. Boundedness of traveling wave solutions

Lemma 4.1. *The functions $S(\varsigma)$, $V(\varsigma)$ and $I(\varsigma)$ satisfy*

$$0 < S(\varsigma) < S_0,\ \ 0 < V(\varsigma) < V_0 \text{ and } I(\varsigma) > 0 \text{ in } \mathbb{R}.$$

Proof. Firstly, to show $S(\varsigma) > 0$. If there exists some ς_0 such that $S(\varsigma_0) = 0$, then $d_1\mathcal{J}[S](\varsigma_0) \geq 0$ and $S'(\varsigma_0) = 0$. Due to (2.2), we have

$$0 = d_1\mathcal{J}[S](\varsigma_0) + \Lambda > 0,$$

which is a contradiction. Similarly, we have $V(\varsigma) > 0$ in $\mathbb{R}$.

Next, if there is ς_1 such that $I(\varsigma_1) = 0$ and $I(\varsigma) > 0$ for $\varsigma < \varsigma_1$. From the third equation of (2.2), we have

$$I(\varsigma_1 + 1) + I(\varsigma_1 - 1) = 0.$$

Consequently, $I(\varsigma_1+1) = I(\varsigma_1-1) = 0$ since $I(\varsigma) \geq 0$ in $\mathbb{R}$, which is a contradiction.

Lastly, we show that $S(\varsigma) < S_0$, if there exists ς_2 such that $S(\varsigma_2) = S_0$, one has that

$$0 = d_1\mathcal{J}[S](\varsigma_2) - \beta_1 S(\varsigma_2)I(\varsigma_2) < 0.$$

This contradiction leads to $S(\varsigma) < S_0$. Similarly, we have $V(\varsigma) < V_0$ for all $\varsigma \in \mathbb{R}$. □

Now, we show the following four claims.

Claim I. The functions $\frac{I(\varsigma\pm1)}{I(\varsigma)}$ is bounded in $\mathbb{R}$.

To show this claim, we denote $\kappa := (2d_3 + \mu_3)/c$ and $U(\varsigma) := e^{\kappa\varsigma}I(\varsigma)$, one has that

$$cU'(\varsigma) = e^{\kappa\varsigma}cI'(\varsigma) + (\mu_3 + 2d_3)I(\varsigma) > 0.$$

From the monotonicity of $U(\varsigma)$, we have

$$\frac{I(\varsigma - 1)}{I(\varsigma)} < e^{\kappa}, \quad \forall\varsigma \in \mathbb{R}.$$

Direct calculation yields

$$\begin{aligned}
[e^{\kappa\varsigma}I(\varsigma)]' &= \frac{1}{c}e^{\kappa\varsigma}\left[d_3I(\varsigma+1) + d_3I(\varsigma-1) + (\beta_1S(\varsigma) + \beta_2V(\varsigma))I(\varsigma)\right] \\
&> \frac{d_3}{c}e^{\kappa\varsigma}I(\varsigma+1).
\end{aligned} \tag{4.1}$$

Integrating (4.1) over $[\varsigma, \varsigma+1]$ and using the monotonicity of $e^{\kappa\varsigma}$, one has

$$\begin{aligned}
e^{\kappa(\varsigma+1)}I(\varsigma+1) &> e^{\kappa\varsigma}I(\varsigma) + \frac{d_3}{c}\int_{\varsigma}^{\varsigma+1} e^{\kappa s}I(s+1)\mathrm{d}s \\
&> e^{\kappa\varsigma}I(\varsigma) + \frac{d_3}{c}\int_{\varsigma}^{\varsigma+1} e^{\kappa(\varsigma+1)}I(\varsigma+1)e^{-\kappa}\mathrm{d}s \\
&= e^{-\kappa}e^{\kappa(\varsigma+1)}\left[I(\varsigma) + \frac{d_3}{c}I(\varsigma+1)\right].
\end{aligned}$$

Hence,

$$[e^{\kappa\varsigma}I(\varsigma)]' > \left(\frac{d_3}{c}\right)^2 e^{-2\kappa}e^{\kappa(\varsigma+1)}I(\varsigma+1). \tag{4.2}$$

Integrating (4.2) from $\varsigma - \frac{1}{2}$ to ς yields

$$\frac{I\left(\varsigma + \frac{1}{2}\right)}{I(\varsigma)} < 2\left(\frac{c}{d_3}\right)^2 e^{\frac{3}{2}\kappa}, \quad \forall\varsigma \in \mathbb{R}.$$

Similarly, integrating (4.2) over $[\varsigma, \varsigma + \frac{1}{2}]$, we have

$$\frac{I(\varsigma+1)}{I\left(\varsigma + \frac{1}{2}\right)} < 2\left(\frac{c}{d_3}\right)^2 e^{\frac{3}{2}\kappa}, \quad \forall\varsigma \in \mathbb{R}.$$

Thus
$$\frac{I(\varsigma+1)}{I(\varsigma)}=\frac{I\left(\varsigma+\frac{1}{2}\right)}{I(\varsigma)}\frac{I(\varsigma+1)}{I\left(\varsigma+\frac{1}{2}\right)}<4\left(\frac{c}{d_3}\right)^4 e^{3\kappa},\quad \forall\varsigma\in\mathbb{R}.$$

Claim II. $\frac{I'(\varsigma)}{I(\varsigma)}$ is bounded in $\mathbb{R}$.

This claim is true because Claim I and the third equation of (2.2).

Choose a sequence $\{c_k, S_k, V_k, I_k\}$ of the traveling wave solutions for (1.3) in a compact subinterval of $(0,\infty)$, we have the following claim.

Claim III. For a sequence $\{\varsigma_k\}$, we have $S(\varsigma_k)\to 0$ and $V(\varsigma_k)\to 0$ as $k\to+\infty$ provided that $I(\varsigma_k)\to+\infty$ as $k\to+\infty$.

By way of contradiction, let ς_k be a subsequence of $\{\varsigma_k\}_{k\in\mathbb{N}}$ with $I_k(\varsigma_k)\to+\infty$ and $S_k(\varsigma_k)\geq\varepsilon$ as $k\to+\infty$ in $\mathbb{R}$ for all $k\in\mathbb{N}$. Let $\tilde{c}>0$ be the lower bound of $\{c_k\}$ and we have
$$S_k'(\varsigma)\leq\frac{2S_0+\Lambda}{\tilde{c}}:=\delta_0 \ \text{ in } \ \mathbb{R}.$$
We further denote $\delta=\frac{\varepsilon}{\delta_0}$, one has that
$$S_k(\varsigma)\geq\frac{\varepsilon}{2},\quad \forall\varsigma\in[\varsigma_k-\delta,\varsigma_k] \ \text{ and } \ \forall k\in\mathbb{N}.$$
Thanks to Claim II, there exists some $C_0>0$ such that
$$\frac{I_k(\varsigma_k)}{I_k(\varsigma)}=\exp\left\{\int_{\varsigma}^{\varsigma_k}\frac{I_k'(\sigma)}{I_k(\sigma)}\mathrm{d}\sigma\right\}\leq e^{C_0\delta},\quad \forall\varsigma\in[\varsigma_k-\delta,\varsigma_k]$$
for all $k\in\mathbb{N}$. Thus
$$\min_{\varsigma\in[\varsigma_k-\delta,\ \varsigma_k]} I_k(\varsigma)\geq e^{-C_0\delta}I_k(\varsigma_k),$$
which give us
$$\min_{\varsigma\in[\varsigma_k-\delta,\varsigma_k]} I_k(\varsigma)\to+\infty \ \text{ as } \ k\to+\infty.$$
Recalling the first equation of (2.2), we have
$$\max_{\varsigma\in[\varsigma_k-\delta,\varsigma_k]} S_k'(\varsigma)\leq\delta_0-\frac{\beta_1\varepsilon}{2}\min_{\varsigma\in[\varsigma_k-\delta,\varsigma_k]} I_k(\varsigma)\to-\infty \ \text{ as } \ k\to+\infty.$$
Then,
$$S_k'(\varsigma)\leq-\frac{2S_0}{\delta},\quad \forall k\geq K \ \text{ and } \ \varsigma\in[\varsigma_k-\delta,\varsigma_k],$$
for some $K>0$. Thus, we have $S_k(\varsigma_k)\leq -S_0$, $\forall k\geq K$, which is a contradiction with $S_k'(\varsigma)\leq\delta_0$ and $S_k(\varsigma)\geq\frac{\epsilon}{2}$ for $\varsigma\in[\varsigma_k-\delta,\varsigma_k]$, $k\in\mathbb{N}$. Similarly, we can show that $V_k(\varsigma_k)\to 0$ as $k\to+\infty$.

Claim IV. If $\limsup_{\varsigma\to+\infty} I(\varsigma)=+\infty$, then $\lim_{\varsigma\to+\infty} I(\varsigma)=+\infty$.

With a similar arguments in [2, Lemma 3.4], we know that Claim IV is true. We are now in position to show the boundedness of $I(\varsigma)$ by using Claim I-IV.

Lemma 4.2. *$I(\varsigma)$ is bounded in $\mathbb{R}$.*

Proof. Suppose that $\limsup_{\varsigma\to+\infty} I(\varsigma)=+\infty$, then it follows from Claim III and Claim IV that $\lim_{\varsigma\to+\infty}(S(\varsigma),V(\varsigma))=(0,0)$. Denote $\theta(\varsigma)=\frac{I'(\varsigma)}{I(\varsigma)}$, we have
$$c\theta(\varsigma)=d_3e^{\int_{\varsigma}^{\varsigma+1}\theta(s)\mathrm{d}s}+d_3e^{\int_{\varsigma}^{\varsigma-1}\theta(s)\mathrm{d}s}-(2d_3+\mu_3)+\beta_1S(\varsigma)+\beta_2V(\varsigma).$$

By using [1, Lemma 3.4], the finite limit of $\theta(\varsigma)$ at $+\infty$ exists and denoted by κ, which is satisfying

$$\Upsilon(\kappa, c) := d_3\left(e^{\kappa} + e^{-\kappa} - 2\right) - c\kappa - \mu_3 = 0.$$

Clearly, $\Upsilon(\kappa, c) = 0$ has a unique positive real root κ_0. From Lemma 2.2, we have

$$d_3\left(e^{\lambda_2} + e^{-\lambda_2} - 2\right) - c\lambda_2 - \mu_3 < 0,$$

Recall the definition of λ_1 and λ_2, we have $\lambda_2 < \kappa_0$. Since $\lim_{\varsigma\to+\infty} \theta(\varsigma) = \kappa_0$, then there exists $\tilde{\varsigma}$ such that

$$I(\varsigma) \geq Ce\left(\frac{\lambda_2 + \kappa_0}{2}\right)\varsigma \quad \text{for all} \quad \varsigma \geq \tilde{\varsigma},$$

with some constant C, which contradicts with $I(\varsigma) \leq e^{\lambda_1\varsigma}$ in $\mathbb{R}$ and $\lambda_1 < \kappa_0$. □

Since $I(\varsigma)$ is bounded in $\mathbb{R}$, we can assume that $I(\varsigma) < C$ for some constant $C > 0$. Then it is easy to verify that $S^0 := \frac{\Lambda}{\mu_1+\beta_1 C}$ and $V^0 := \frac{\alpha S^0}{\mu_2+\beta_2 C}$ are sub-solutions for the first and second equations of (2.4), which means that $S(\varsigma) > S^0$ and $V(\varsigma) > V^0$ in $\mathbb{R}$. The following lemma is to show that $I(\varsigma)$ cannot approach 0.

Lemma 4.3. *There holds* $\liminf_{\varsigma\to+\infty} I(\varsigma) > 0$.

Proof. The proof of this lemma is similar with that in [2]. We only need to show that if $I(\varsigma) \leq \varepsilon_0$ for $\varepsilon_0 > 0$ is small enough, then $I'(\varsigma) > 0$ for all $\varsigma \in \mathbb{R}$. If not, we assume that there is no such ε_0. Then there exist a sequence of speed $c_k \in (a, b)$, where a and b are two positive constants with $a < b$, a sequence of solutions $\{(S_k, V_k, I_k)\}$ with speed c_k and $0 < S_k < S_0$, $0 < V_k < V_0$, $I_k > 0$ in $\mathbb{R}$, and a sequence of real number ς_k such that $I_k(\varsigma_k) \to 0$ as $k \to +\infty$ and $I_k'(\varsigma_k) \leq 0$ for all $k \in \mathbb{N}$. Up to a shift of the origin, one can assume without loss of generality that $\varsigma_k = 0$ for all $k \in \mathbb{N}$. With some similar arguments in [2, Lemma 3.8], we know that $S_\infty = S_0$ and $V_\infty = V_0$.

Let $\pi_k(\varsigma) := \frac{I_k(\varsigma)}{I_k(0)}$ for $k \in \mathbb{N}$ and $\varsigma \in \mathbb{R}$, we have

$$\pi_k'(\varsigma) = \frac{I_k'(\varsigma)}{I_k(0)} = \frac{I_k'(\varsigma)}{I_k(\varsigma)}\pi_k(\varsigma).$$

From Claim II, we have that the sequence $\{I_k'/I_k\}$ is bounded in $\mathbb{R}$, then π_k and π_k' are locally bounded in $\mathbb{R}$. Recall the third equation in (2.2) and note that S_k and V_k are bounded in $C^1_{loc}(\mathbb{R})$, which means that π_k'' is locally bounded. Due to Arzela-Ascoli Theorem, up to extraction of a subsequence, π_k converge to a nonnegative function π_∞ in $C^1_{loc}(\mathbb{R})$, which is satisfy

$$c_\infty\pi_\infty'(\varsigma) = d_3\mathcal{J}[\pi_\infty](\varsigma) + (\beta_1 S_0 + \beta_2 V_0)\pi_\infty(\varsigma) - \mu_3\pi_\infty(\varsigma)$$

in $\mathbb{R}$. One can have $\pi_\infty(\varsigma) > 0$ in $\mathbb{R}$. Indeed, if there is a ς_0 such that $\pi_\infty(\varsigma_0) = 0$, then $\pi_\infty'(\varsigma_0) = 0$ and

$$0 = d_3(\pi_\infty(\varsigma_0 + 1) + \pi_\infty(\varsigma_0 - 1)).$$

Thus $\pi_\infty(\varsigma_0 + 1) = \pi_\infty(\varsigma_0 - 1) = 0$, it follows that $\pi_\infty(\varsigma_0 + \tau) = 0$ for all $\tau \in \mathbb{Z}$. Recall that $c_\infty\pi_\infty'(\varsigma) \geq -(\mu_3 + 2d_3)\pi_\infty(\varsigma)$, then the map $\varsigma \mapsto \pi_\infty(\varsigma)e^{\frac{(\mu_3+2d_3)\varsigma}{c_\infty}}$ is nondecreasing. Since it vanishes at $\varsigma_0 + \tau$ for all $m \in \mathbb{Z}$, one can conclude that $\pi_\infty = 0$ in $\mathbb{R}$, which is a contradiction with $\pi_\infty(0) = 1$.

Denote $P(\varsigma) := \frac{\pi'_\infty(\varsigma)}{\pi_\infty(\varsigma)}$, one has that

$$c_\infty P(\varsigma) = d_3 e^{\int_\varsigma^{\varsigma+1} P(s)\mathrm{d}s}\mathrm{d}y + d_3 e^{\int_\varsigma^{\varsigma-1} P(s)\mathrm{d}s}\mathrm{d}y - 2d_3 + \beta_1 S_0 + \beta_2 V_0 - \mu_3. \tag{4.3}$$

Using [1, Lemma 3.4], $P(\varsigma)$ has finite limits $\omega_\pm$ and satisfy

$$c_\infty \omega_\pm = d_3\left(e^{\omega_\pm} + e^{-\omega_\pm} - 2\right) + \beta_1 S_0 + \beta_2 V_0 - \mu_3.$$

By Lemma 2.2, we know that $\omega_\pm > 0$ and $\pi'_\infty(\pm\infty)$ are positive. Moreover, one can have that $\pi'_\infty(\varsigma) > 0$ for all $\varsigma \in \mathbb{R}$. In fact, if there exists some ς^* such that $P(\varsigma^*) = \inf_{\mathbb{R}} P(\varsigma)$, then $P(\varsigma^*) = 0$. Differentiating (4.3) yields

$$c_\infty P'(\varsigma) = d_3(P(\varsigma+1) - P(\varsigma))\frac{\pi_\infty(\varsigma+1)}{\pi_\infty(\varsigma)} + d_3(P(\varsigma-1) - P(\varsigma))\frac{\pi_\infty(\varsigma-1)}{\pi_\infty(\varsigma)}.$$

It follows that

$$P(\varsigma^*) = P(\varsigma^*+1) = P(\varsigma^*-1).$$

Hence $P(\varsigma^*) = P(\varsigma^* + \kappa)$ for all $\kappa \in \mathbb{Z}$. Then,

$$\inf_{\mathbb{R}} P(\varsigma) \geq \min\{P(+\infty), P(-\infty)\} > 0.$$

Furthermore,

$$0 < \pi'_\infty(0) = \lim_{k\to+\infty} \pi'_k(0) = \lim_{k\to+\infty} \frac{I'_k(0)}{I_k(0)}.$$

Thus, $I'_k(0) > 0$, which contradicts with the fact that $I'_k(0) \leq 0$. □

5. Convergence of the traveling wave solutions

In this section, we show the convergence of traveling wave solutions.

Theorem 5.1. *If $\Re_0 > 1$, then for each $c > c^*$, system (1.3) has a traveling wave solution $(S(\varsigma), V(\varsigma), I(\varsigma))$ satisfying conditions (2.3) and (2.4).*

Proof. In what following, we use (S, V, I) short for $(S(\varsigma), V(\varsigma), I(\varsigma))$. Define the following four functionals

$$W_1(\varsigma) = cS^* g\left(\frac{S}{S^*}\right) + cV^* g\left(\frac{V}{V^*}\right) + cI^* g\left(\frac{I}{I^*}\right),$$
$$W_2(\varsigma) = \int_0^1 g\left(\frac{S(\varsigma-\sigma)}{S^*}\right)\mathrm{d}\sigma - \int_{-1}^0 g\left(\frac{S(\varsigma-\sigma)}{S^*}\right)\mathrm{d}\sigma,$$
$$W_3(\varsigma) = \int_0^1 g\left(\frac{V(\varsigma-\sigma)}{V^*}\right)\mathrm{d}\sigma - \int_{-1}^0 g\left(\frac{V(\varsigma-\sigma)}{V^*}\right)\mathrm{d}\sigma,$$

and

$$W_4(\varsigma) = \int_0^1 g\left(\frac{I(\varsigma-\sigma)}{I^*}\right)\mathrm{d}\sigma - \int_{-1}^0 g\left(\frac{I(\varsigma-\sigma)}{I^*}\right)\mathrm{d}\sigma,$$

where $g(x) = x - 1 - \ln x$. The derivative of $W_1(\varsigma)$ is calculated as follows

$$\frac{\mathrm{d}W_1(\varsigma)}{\mathrm{d}\varsigma} = \left(1 - \frac{S^*}{S}\right) d_1 \mathcal{J}[S](\varsigma) + \left(1 - \frac{V^*}{V}\right) d_2 \mathcal{J}[V](\varsigma) + \left(1 - \frac{I^*}{I}\right) d_3 \mathcal{J}[I](\varsigma) + \Sigma(\varsigma),$$

where

$$\begin{aligned}\Sigma(\varsigma) = & \left(1 - \frac{S^*}{S}\right)(\Lambda - \mu_1 S - \beta_1 SI) + \left(1 - \frac{V^*}{V}\right)(\alpha S - \beta_2 VI - \mu_2 V) \\ & + \left(1 - \frac{I^*}{I}\right)((\beta_1 S + \beta_2 V)I - \mu_3 I).\end{aligned}$$

Since (S^*, V^*, I^*) is the endemic equilibrium of system (1.3) and $\mu_1 = \mu + \alpha$, one has

$$\begin{aligned}\Sigma(\varsigma) = & \mu S^* \left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) + \mu_2 V^* \left(3 - \frac{S^*}{S} - \frac{V}{V^*} - \frac{SV^*}{S^*V}\right) \\ & - \beta_1 S^* I^* \left[g\left(\frac{S^*}{S}\right) + g\left(\frac{S}{S^*}\right)\right] \\ & - \beta_2 V^* I^* \left[g\left(\frac{S^*}{S}\right) + g\left(\frac{SV^*}{S^*V}\right) + g\left(\frac{V}{V^*}\right)\right].\end{aligned}$$

Using the fact that $\mu_2 V^* = \alpha S^* - \beta_2 V^* I^*$ and

$$g\left(\frac{S^*}{S}\right) + g\left(\frac{SV^*}{S^*V}\right) + g\left(\frac{V}{V^*}\right) = \frac{S^*}{S} + \frac{SV^*}{S^*V} + \frac{V}{V^*} - 3,$$

we have

$$\begin{aligned}\Sigma(\varsigma) = & \mu S^* \left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) + \alpha S^* \left(3 - \frac{S^*}{S} - \frac{V}{V^*} - \frac{SV^*}{S^*V}\right) \\ & - \beta_1 S^* I^* \left[g\left(\frac{S^*}{S}\right) + g\left(\frac{S}{S^*}\right)\right].\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{\mathrm{d}W_2(\varsigma)}{\mathrm{d}\varsigma} = & \frac{\mathrm{d}}{\mathrm{d}\varsigma}\left[\int_0^1 g\left(\frac{S(\varsigma-\sigma)}{S^*}\right)\mathrm{d}\sigma - \int_{-1}^0 g\left(\frac{S(\varsigma-\sigma)}{S^*}\right)\mathrm{d}\sigma\right] \\ = & \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\varsigma} g\left(\frac{S(\varsigma-\sigma)}{S^*}\right)\mathrm{d}\sigma - \int_{-1}^0 \frac{\mathrm{d}}{\mathrm{d}\varsigma} g\left(\frac{S(\varsigma-\sigma)}{S^*}\right)\mathrm{d}\sigma \\ = & -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\sigma} g\left(\frac{S(\varsigma-\sigma)}{S^*}\right)\mathrm{d}\sigma + \int_{-1}^0 \frac{\mathrm{d}}{\mathrm{d}\sigma} g\left(\frac{S(\varsigma-\sigma)}{S^*}\right)\mathrm{d}\sigma \\ = & 2g\left(\frac{S}{S^*}\right) - g\left(\frac{S(\varsigma-1)}{S^*}\right) - g\left(\frac{S(\varsigma+1)}{S^*}\right).\end{aligned}$$

Similarly,

$$\frac{\mathrm{d}W_3(\varsigma)}{\mathrm{d}\varsigma} = 2g\left(\frac{V}{V^*}\right) - g\left(\frac{V(\varsigma-1)}{V^*}\right) - g\left(\frac{V(\varsigma+1)}{V^*}\right),$$

and

$$\frac{\mathrm{d}W_4(\varsigma)}{\mathrm{d}\varsigma} = 2g\left(\frac{I}{I^*}\right) - g\left(\frac{I(\varsigma-1)}{I^*}\right) - g\left(\frac{I(\varsigma+1)}{I^*}\right).$$

Now, we define a Lyapunov functional as

$$\mathcal{V}(\varsigma) = W_1(\varsigma) + d_1 S^* W_2(\varsigma) + d_2 V^* W_3(\varsigma) + d_3 I^* W_4(\varsigma),$$

and

$$\begin{aligned}
&\frac{\mathrm{d}\mathcal{V}(\varsigma)}{\mathrm{d}\varsigma}\\
=&\mu S^* \left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) + \mu_2 V^* \left(3 - \frac{S^*}{S} - \frac{V}{V^*} - \frac{SV^*}{S^* V}\right)\\
&- \beta_1 S^* I^* \left[g\left(\frac{S^*}{S}\right) + g\left(\frac{S}{S^*}\right)\right] - \beta_2 V^* I^* \left[g\left(\frac{S^*}{S}\right) + g\left(\frac{SV^*}{S^* V}\right) + g\left(\frac{V}{V^*}\right)\right]\\
&- d_2 S^* \left[g\left(\frac{S(\varsigma-1)}{S}\right) + g\left(\frac{S(\varsigma+1)}{S}\right)\right] - d_2 V^* \left[g\left(\frac{V(\varsigma-1)}{V}\right) + g\left(\frac{V(\varsigma+1)}{V}\right)\right]\\
&- d_3 I^* \left[g\left(\frac{I(\varsigma-1)}{I}\right) + g\left(\frac{I(\varsigma+1)}{I}\right)\right].
\end{aligned}$$

Recall that $g(x) \geq 0$ for all $x \geq 0$, then the map $\varsigma \mapsto \mathcal{V}(\varsigma)$ is non-increasing. Choosing $\{\varsigma_k\}_{k \geq 0}$ as an increasing sequence with $\varsigma_k > 0$ and $\varsigma_k \to +\infty$ as $k \to +\infty$. Let

$$\{S_k(\varsigma) = S(\varsigma + \varsigma_k)\}_{k\geq 0}, \quad \{V_k(\varsigma) = V(\varsigma + \varsigma_k)\}_{k \geq 0} \text{ and } \{I_k(\varsigma) = I(\varsigma + \varsigma_k)\}_{k \geq 0}.$$

Since S, V and I have bounded derivatives, then the sequences of functions $\{S_k(\varsigma)\}$, $\{V_k(\varsigma)\}$ and $\{I_k(\varsigma)\}$ converge in $C^\infty_{loc}(\mathbb{R})$ as $k \to +\infty$ by Arzela-Ascoli theorem, up to extraction of a subsequence, we assume that the sequences $\{S_k(\varsigma)\}$, $\{V_k(\varsigma)\}$ and $\{I_k(\varsigma)\}$ convergence to some nonnegative C^∞ functions S_∞, V_∞ and I_∞. Since $\lim_{\varsigma \to -\infty} I(\varsigma) = 0$, so we need to consider the process of approaching negative infinity for $W_4(\varsigma)$ if it is bounded from below. Thanks to Lemma (4.3), we can obtain that $W_4(\varsigma)$ is bounded from below and $\mathcal{V}(S, V, I)(\varsigma)$ is bounded from below, then there exists constant M_0 and some large k such that

$$M_0 \leq \mathcal{V}(S_k, V_k, I_k)(\varsigma) = \mathcal{V}(S, V, I)(\varsigma + \varsigma_k) \leq \mathcal{V}(S, V, I)(\varsigma).$$

Hence, there exists some $\delta \in \mathbb{R}$ such that $\lim\limits_{k \to +\infty} \mathcal{V}(S_k, V_k, I_k)(\varsigma) = \delta$, $\forall \varsigma \in \mathbb{R}$. Using Lebegue dominated convergence theorem, one has that

$$\lim_{k \to +\infty} \mathcal{V}(S_k, V_k, I_k)(\varsigma) = \mathcal{V}(S_\infty, V_\infty, I_\infty)(\varsigma), \ \varsigma \in \mathbb{R}.$$

Thus, $\mathcal{V}(S_\infty, V_\infty, I_\infty)(\varsigma) = \delta$. Recall that $\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\varsigma} = 0$ if and only if $S \equiv S^*$, $V \equiv V^*$ and $I \equiv I^*$, it follows that $(S_\infty, V_\infty, I_\infty) \equiv (S^*, V^*, I^*)$. □

At the last part of this section, we explain the existence of traveling wave solutions under $c = c^*$ by an approximation technique used in [2, Section 4]. To use the methods in [2], we need to verify Lemma 4.1 and Lemma 4.2 in [2]. In fact, these two lemmas are still true for our model, because it only need to focus on the I_n-equation in (2.2). As we can see, the I_n-equation in our paper still meets the inequality proposed on line 3 of [2, Page 2350]. Then other parts of the proofs for critical traveling wave solutions in [2] are still work for our model. Hence, we finish this section with the following remark.

Remark 5.1. For the case $c = c^*$, we can obtain the existence of traveling wave solutions by using a similar approximation technique used in [2, Section 4]. The traveling wave solutions for $c = c^*$ also satisfy (2.3) and (2.4) since the Lyapunov functional is independent of c.

6. Discussion

In this paper, we proposed a discrete diffusive vaccination epidemic model (i.e., system (1.3)). Employing Schauder's fixed point theorem and Lyapunov functional, we obtain the existence of nontrivial positive traveling wave solutions, which is connecting two different equilibrium. Our research examines the conditions (i.e. basic reproduction number) under which an infectious disease can spread, even this disease has a vaccine.

Now we finish this section with some explanations from the perspective of epidemiology. Assume that $(\hat{\lambda}, \hat{c})$ is a root of $\Delta(\lambda, c) = 0$, by some calculations, we obtain

$$\frac{\mathrm{d}\hat{c}}{\mathrm{d}\delta} < 0, \quad \frac{\mathrm{d}\hat{c}}{\mathrm{d}d_3} > 0, \quad \frac{\mathrm{d}\hat{c}}{\mathrm{d}\beta_1} > 0, \quad \frac{\mathrm{d}\hat{c}}{\mathrm{d}\beta_2} > 0 \text{ and } \frac{\mathrm{d}\hat{c}}{\mathrm{d}\Re_0} > 0.$$

here we have used the fact that

$$\Delta(\lambda, c) = d_3[e^{\lambda} + e^{-\lambda} - 2] - c\lambda + (\beta_1 S_0 + \beta_2 V_0) - \mu_3,$$

where $V_0 = \frac{\Lambda\alpha}{\mu_1\mu_2}$ and $\mu_2 = \delta + \mu$. Mathematically, $\hat{c}$ is a decreasing on δ, while $\hat{c}$ is an increasing function on d_3, β_1 and β_2. From the biological point of view, this indicates the following three scenarios:

I. The more successful the vaccination, the slower the disease spreads;

II. The faster the infected individuals move, the faster the disease spreads;

III. The more effective the infections are, the faster the disease spreads.

Accordingly, a good understanding of the movement of the infected individuals and the vaccination rate of susceptible individuals could be important in disease control strategy. In fact, as in the ordinary differential equation case in [11], the basic reproduction number $\Re_0$ is decreasing on δ, while $\Re_0$ is increasing on β_1 and β_2. Compared with [11], our study proposes a new explanation, which is to control the movement of the infected individuals. Another important thing is the effectiveness of vaccination δ is important than the vaccination rate α, which explain the importance of complete vaccination.

Acknowledgements

The authors would like to thank the editor and anonymous reviewers for their valuable comments and suggestions which led to a significant improvement of this study. The authors would also like to thank Professor Shigui Ruan from University of Miami for his suggestions.

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