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NEW OSCILLATION CRITERIA FOR A CLASS OF HIGHER-ORDER NEUTRAL FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES*

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Abstract In this paper, we study the higher-order neutral functional dynamic equations of the form

$$L_n y(t) + q(t) f(|y(\theta(t))|^{\beta} sgn(y(\theta(t)))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}},$$

on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$, where

$$L_1 y(t) = [y(t) + r(t) y(\tau(t))]^{\Delta}, \ L_{i+1} y(t) = [p_i(t) |L_i y(t)|^{\alpha_i} sgn(L_i y(t))]^{\Delta},$$

α_i, $1 \leq i \leq n-1$ and β are positive constants, p_i, $1 \leq i \leq n-1$ and q are rd-continuous functions from $[t_0, \infty)_{\mathbb{T}}$ to $[0, \infty)$ and $r \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, [0, 1))$. The functions $\tau, \theta \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$ satisfy $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \theta(t) = \infty$. Criteria are established for the oscillation of solutions for both even and odd order cases. The obtained results here generalize and improve some known results for oscillation of the corresponding higher-order ordinary differential equations [13], but the proof of these counterparts are quite different from the literature. Finally, some interesting examples are given to illustrate the versatility of our main results.

Keywords Oscillation, nonlinear dynamic equation, higher-order equation, neutral functional dynamic equation, time scale.

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1. Introduction

In this paper, we establish some sufficient conditions for oscillation of the following higher-order neutral functional dynamic equation

$$L_n y(t) + q(t) f(|y(\theta(t))|^{\beta} sgn(y(\theta(t)))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale with $\sup \mathbb{T} = \infty$, $t_0 \in \mathbb{T}$ is a constant, $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$ and

$$L_1 y(t) = [y(t) + r(t) y(\tau(t))]^{\Delta},$$

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$$L_{i+1}y(t) = [p_i(t)|L_i y(t)|^{\alpha_i} sgn(L_i y(t))]^{\Delta}, \quad i = 1, 2, \cdots, n-1.$$

By a solution of (1.1), we mean a nontrivial real-valued function $y \in \mathrm{C}^1_{\mathrm{rd}}([T_y, \infty)_{\mathbb{T}})$ with $T_y \in [t_0, \infty)_{\mathbb{T}}$, which has the property that $L_i y(t) \in \mathrm{C}^1_{\mathrm{rd}}([T_y, \infty)_{\mathbb{T}})$ for $0 \leq i \leq n$ and satisfies (1.1) on $[T_y, \infty)_{\mathbb{T}}$, where $\mathrm{C}^1_{\mathrm{rd}}$ is the space of differentiable functions whose derivative is rd-continuous. We exclude from our consideration those solutions of (1.1) which vanish identically in some neighborhoods of infinity. A solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. For more details on time scales, we refer the reader to Bohner and Peterson [4].

Throughout this paper, we assume that the following conditions are satisfied.

(H_1) $q \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$ and $q \not\equiv 0$ on $[t_1, \infty)_{\mathbb{T}}$ for any $t_1 \in [t_0, \infty)_{\mathbb{T}}$;

(H_2) $p_i \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ satisfies $\int_{t_0}^{\infty} \left(\frac{1}{p_i(s)}\right)^{\frac{1}{\alpha_i}} \Delta s = \infty$, $1 \leq i \leq n-1$;

(H_3) $r \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, [0, 1))$, $\tau, \theta \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \theta(t) = \infty$;

(H_4) $f \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ satisfies $f(u)/u \geq M$ for $u \neq 0$, where M is a positive constant.

It is well known that a unification theory was proposed in [12] by Stephan Hilger, which is called time-scale calculus. Since then, a great number of theoretical issues concerning dynamic equations on time scales have received considerable attention. Many researchers attempt to harmonize the oscillation theory for the continuous and the discrete. The oscillation and nonoscillation of solutions of various equations have been investigated extensively. We refer the reader to the excellent monograph [4], the papers [1, 2, 6, 8–11, 18, 20, 21, 24, 29–31], and the references cited therein. Saker and O'Regan [22] established some new oscillation criteria for the second-order neutral functional dynamic equation

$$[p(t)([x(t) + r(t)x(\tau(t))]^{\Delta})^{\gamma}]^{\Delta} + f(t, x(\theta(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}} \tag{1.2}$$

by means of the generalized Riccati substitution. Deng et al. [5] further studied the generalized Philos-type oscillation criteria of the second-order nonlinear neutral delay dynamic equation

$$[r(t)|(x(t) + p(t)x(g(t)))^{\Delta}|^{\gamma-1}(x(t) + p(t)x(g(t)))^{\Delta}]^{\Delta} + f(t, x(\tau(t))) = 0, \quad \gamma > 0 \tag{1.3}$$

by employing the generalized Riccati technique and the integral averaging technique. When $r(t) = 0$, $\theta(t) = t$ and $\mathbb{T} = \mathbb{R}$, the dynamic equation (1.1) is the half-linear ordinary differential equation

$$L_n x(t) + q(t) f(|x(t)|^{\beta} sgn(x(t))) = 0, \quad t \geq t_0 > 0 \tag{1.4}$$

with

$$L_1 x(t) = x'(t), \;\; L_{i+1}x(t) = [p_i(t)|L_i x(t)|^{\alpha_i} sgn(L_i x(t))]', \;\; i = 1, 2, \cdots, n-1.$$

Jaros [13] showed that (1.4)$(n = 2k)$ is oscillatory under some suitable assumptions. Picking up $n = 2$, Baculikova [3] considered the following noncanonical differential equation with delay argument

$$(r(t)(y'(t))^{\alpha})' + p(t)y^{\beta}(\tau(t)) = 0, \quad t \in [t_0, \infty) \tag{1.5}$$

and established some sufficient conditions for oscillation of (1.5). Hassan [10] established Kamenev-type oscillation criteria for the second-order nonlinear dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + f(t, x(g(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{1.6}$$

Erbe et al. [7] extended this result to the higher-order neutral delay dynamic equation

$$[x(t) + p(t)x(g(t))]^{\Delta^n} + q(t)x(\tau(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{1.7}$$

They proved that (1.7) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{h_{n_k}(t,l)} \int_l^t h_{n_k}(t,\sigma(\xi)) \left(B_1^k(\xi)\psi_k(\xi) - \frac{(\psi_k^{\Delta}(\xi))^2}{4\psi_k(\xi)h_{k-1}(\beta(\xi),s)} \right) \Delta\xi = \infty \tag{1.8}$$

for all large enough $l, s \in \mathbb{T}$ with $\beta(l) > s$, where $h_k(t,s)$ is the generalized Taylor monomials on time scales. For more works about the oscillation criteria for higher-order nonlinear delay dynamic equations in other cases, we refer to [14–17, 19, 23, 25–28].

To the best of our knowledge, no equation such as Eq. (1.1) has been considered for oscillation except the continuous case (1.4). The purpose of this work is to obtain some new sufficient conditions of oscillation for Eq. (1.1), which improve and unify some aforementioned oscillatory results on the topic. Compared with (1.2), the investigation of the higher-order dynamic equation (1.1) is more complicated. We should point out that the traditional methods such as the classical Riccati technique [21, 26, 28] can not be effective for Eq. (1.1). Motivated by [10, 11, 22], we will employ the generalized Riccati technique to study the oscillatory behavior for (1.1). Additionally, the obtained Theorems 3.1,3.2,3.4,3.5 are established under the assumptions (3.1) and (3.2), see also [25–27]. But at the end of Subsections 3.1 and 3.2, we obtain a strong result of oscillation of Eq. (1.1) without these technical assumptions for the cases $\theta(t) > t$ and $\theta(t) \le t$, respectively. So we believe that our results are interesting and meaningful.

The outline of this paper is as follows. In Section 2, we give some basic properties for quasi-Δ-differential operators L_i, $i = 1, 2, \cdots, n$. In Section 3, firstly, we prove some useful auxiliary lemmas and important estimates, which will be used in the proof of our main results. Later, we prove the main results for the case $\theta(t) > t$ in Section 3.1 and the main results for the case $\theta(t) \le t$ in Section 3.2. At last, we present some examples to illustrate the applicability of the main results of this paper.

2. Basic properties for quasi-Δ-differential operators

In this section, we mainly show some basic properties for quasi-Δ-differential operators L_i, $i = 1, 2, \cdots, n$. For the convenience of discussion, we define $L_0y(t) = y(t) + r(t)y(\tau(t))$.

Lemma 2.1. *Let $1 \le i \le n-1$. Assume that there exists a $T \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$L_{i+1}y(t) \le 0 \ (or \ \ \ge 0) \quad on \quad [T, \infty)_{\mathbb{T}}, \tag{2.1}$$

and

$$L_{i+1}y(t) \not\equiv 0 \quad on \quad [\widehat{T},\infty)_{\mathbb{T}} \quad for\ any \quad \widehat{T} \in [T,\infty)_{\mathbb{T}}. \tag{2.2}$$

Further assume that $L_{i-1}y(t) > 0$ *(or* < 0*) on* $[T,\infty)_{\mathbb{T}}$*. Then,*

$$L_iy(t) > 0 \ (or \ < 0) \quad on \quad [T,\infty)_{\mathbb{T}}. \tag{2.3}$$

Proof. Suppose that (2.1) and (2.2) are satisfied. We first consider the case where $L_{i+1}y(t) \le 0$ for $t \in [T,\infty)_{\mathbb{T}}$. Let $L_{i-1}y(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. We claim that $L_iy(t) \ge 0$ for $t \in [T,\infty)_{\mathbb{T}}$ by a contradiction argument. If there exists a real number $t_1 \in [T,\infty)_{\mathbb{T}}$ such that $L_iy(t_1) < 0$, using the fact that $p_i(t)|L_iy(t)|^{\alpha_i} sgn(L_iy(t))$ is non-increasing on $[T,\infty)_{\mathbb{T}}$, we obtain

$$p_i(t)|L_iy(t)|^{\alpha_i} sgn(L_iy(t)) \le p_i(t_1)|L_iy(t_1)|^{\alpha_i} sgn(L_iy(t_1)) \quad \text{over} \quad [t_1,\infty)_{\mathbb{T}}.$$

Next, integrating $L_iy(t) = [p_{i-1}(t)|L_{i-1}y(t)|^{\alpha_{i-1}} sgn(L_{i-1}y(t))]^{\Delta}$ from t_1 to t, we have

$$\begin{aligned}
&p_{i-1}(t)|L_{i-1}y(t)|^{\alpha_{i-1}} sgn(L_{i-1}y(t))\\
=&p_{i-1}(t_1)|L_{i-1}y(t_1)|^{\alpha_{i-1}} sgn(L_{i-1}y(t_1)) + \int_{t_1}^{t} L_iy(s)\Delta s\\
\le&p_{i-1}(t_1)L_{i-1}y(t_1)^{\alpha_{i-1}} + p_i^{\frac{1}{\alpha_i}}(t_1)L_iy(t_1)\int_{t_1}^{t}\left(\frac{1}{p_i(s)}\right)^{\frac{1}{\alpha_i}}\Delta s.
\end{aligned}$$

Owing to the assumption (H_2) and $L_iy(t_1) < 0$, we derive that $L_{i-1}y(t)$ is eventually negative. This is a contradiction to $L_{i-1}y(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. Thus, we have $L_iy(t) \ge 0$ on $[T,\infty)_{\mathbb{T}}$.

If $L_iy(t_1) = 0$ for some real number $t_1 \in [T,\infty)_{\mathbb{T}}$, then $p_i(t)|L_iy(t)|^{\alpha_i} sgn(L_iy(t)) \equiv 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$ since $p_i(t)|L_iy(t)|^{\alpha_i} sgn(L_iy(t))$ is nonnegative and non-increasing on $[T,\infty)_{\mathbb{T}}$. Consequently, $L_{i+1}y(t) \equiv 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. This contradicts the assumption (2.2). Hence, $L_iy(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. The cases where $L_{i+1}y(t) \ge 0$ and $L_{i-1}y(t) < 0$ on $[T,\infty)_{\mathbb{T}}$ can be processed in an analogously manner. This finishes the proof of Lemma 2.1. □

Lemma 2.2. *Let* $1 \le i \le n-1$*. Assume that* (2.1) *and* (2.2) *are satisfied. Further assume that there exists a* $T \in [t_0,\infty)_{\mathbb{T}}$ *such that*

$$L_iy(t) < 0 \ (or \ > 0) \quad on \quad [T,\infty)_{\mathbb{T}}. \tag{2.4}$$

Then, there exists a $T^* \in [T,\infty)_{\mathbb{T}}$ *such that for* $t \in [T^*,\infty)_{\mathbb{T}}$*,*

$$L_jy(t) < 0 \ (or \ > 0), \ 0 \le j \le i-1. \tag{2.5}$$

Proof. We only consider the case $L_iy(t) > 0$ for $t \in [T,\infty)_{\mathbb{T}}$. If so, we deduce that $L_{i-1}y(t)$ is increasing on $[T,\infty)_{\mathbb{T}}$. Thus, we obtain that either $L_{i-1}y(t) < 0$ on $[T,\infty)_{\mathbb{T}}$ or there is a $T^* \in [T,\infty)_{\mathbb{T}}$ such that $L_{i-1}y(t) > 0$ on $[T^*,\infty)_{\mathbb{T}}$. If the former holds, then Lemma 2.1 implies that $L_iy(t) < 0$ on $[T,\infty)_{\mathbb{T}}$, which contradicts to (2.4). Therefore, $L_{i-1}y(t) > 0$ on $[T^*,\infty)_{\mathbb{T}}$ for some $T^* \in [T,\infty)_{\mathbb{T}}$. Using the same arguments as above, we also have

$$L_jy(t) > 0 \ \text{ for all large } t, \ j = i-2, i-3, \cdots, 0. \tag{2.6}$$

The proof of this lemma is completed. □

Theorem 2.1. *Assume that $y(t)$ is a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Then, there exists a $T^*\in[t_0,\infty)_{\mathbb{T}}$ and an integer $l\in[0,n-1]_{\mathbb{Z}}$ with $n+l$ is odd such that*

$$y(t)L_jy(t)>0 \text{ on } [T^*,\infty)_{\mathbb{T}} \text{ for } j=1,2,\cdots,l, \tag{2.7}$$

and

$$(-1)^{n+j}y(t)L_jy(t)<0 \text{ on } [T^*,\infty)_{\mathbb{T}} \text{ for } j=l+1,l+2,\cdots,n-1. \tag{2.8}$$

Proof. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by $(H_1)-(H_4)$, there is a $T\in[t_0,\infty)_{\mathbb{T}}$ such that $y(t)>0$, $y(\tau(t))>0$ and $y(\theta(t))>0$ for all $t\in[T,\infty)_{\mathbb{T}}$. By (1.1), we have

$$L_ny(t)=-q(t)f(|y(\theta(t))|^{\beta}sgn(y(\theta(t))))\le -Mq(t)|y(\theta(t))|^{\beta}sgn(y(\theta(t)))\le 0$$

on $[T,\infty)_{\mathbb{T}}$, and $L_ny(t)\not\equiv 0$ on $[\hat{T},\infty)_{\mathbb{T}}$ for any $\hat{T}\in[T,\infty)_{\mathbb{T}}$. Noting that

$$L_ny(t)=[p_{n-1}(t)|L_{n-1}y(t)|^{\alpha_{n-1}}sgn(L_{n-1}y(t))]^{\Delta},$$

we infer that $p_{n-1}(t)|L_{n-1}y(t)|^{\alpha_{n-1}}sgn(L_{n-1}y(t))$ is non-increasing on $[T,\infty)_{\mathbb{T}}$. Thus, we have the following three possibilities:

- (a_1) $p_{n-1}(t)|L_{n-1}y(t)|^{\alpha_{n-1}}sgn(L_{n-1}y(t))>0$ for $t\in[T,\infty)_{\mathbb{T}}$;
- (a_2) there exists a $T^*\in(T,\infty)_{\mathbb{T}}$ such that $p_{n-1}(t)|L_{n-1}y(t)|^{\alpha_{n-1}}sgn(L_{n-1}y(t))\equiv 0$ on $[T^*,\infty)_{\mathbb{T}}$;
- (a_3) there exists a $T^*\in(T,\infty)_{\mathbb{T}}$ such that $p_{n-1}(t)|L_{n-1}y(t)|^{\alpha_{n-1}}sgn(L_{n-1}y(t))<0$ on $[T^*,\infty)_{\mathbb{T}}$.

If the case (a_3) occurs, then that is to say, $L_{n-1}y(t)<0$ on $[T^*,\infty)_{\mathbb{T}}$. Lemma 2.2 implies that $L_0y(t)<0$ for all large $t\in[T^*,\infty)_{\mathbb{T}}$. One gets a contradiction, since $y(t)$ is eventually positive. Hence the case (a_3) does not happen. Also, the case (a_2) is not possible since $L_ny(t)\not\equiv 0$ on $[T^*,\infty)_{\mathbb{T}}$. Thus, the case (a_1) holds.

In view of $L_{n-1}y(t)=[p_{n-2}(t)|L_{n-2}y(t)|^{\alpha_{n-2}}sgn(L_{n-2}y(t))]^{\Delta}$ and $L_{n-1}y(t)>0$ on $[T,\infty)_{\mathbb{T}}$, we infer that $p_{n-2}(t)|L_{n-2}y(t)|^{\alpha_{n-2}}sgn(L_{n-2}y(t))$ is increasing on $[T,\infty)_{\mathbb{T}}$ and exactly one of the following is true:

- (b_1) there exists a $T^*\in(T,\infty)_{\mathbb{T}}$ such that $L_iy(t)>0$ on $[T^*,\infty)_{\mathbb{T}}$ for all $i=1,2,\cdots,n-2$;
- (b_2) there exist a $T^*\in[T,\infty)_{\mathbb{T}}$ and an integer $j\in\{1,2,\cdots,n-3\}$ such that $L_jy(t)<0$ on $[T^*,\infty)_{\mathbb{T}}$.

If the case (b_1) holds, then the conclusions of this lemma are obtained. If the case (b_2) holds, then there exists a smallest integer $m\in\{1,2,\cdots,n-3\}$ with $m+n$ is odd, such that $(-1)^{n+j}L_jy(t)<0$ on $[T^*,\infty)_{\mathbb{T}}$ for $m\le j\le n$. Noting that $L_my(t)=[p_{m-1}(t)|L_{m-1}y(t)|^{\alpha_{m-1}}sgn(L_{m-1}y(t))]^{\Delta}>0$, we obtain that either $L_{m-1}y(t)<0$ on $[T^*,\infty)_{\mathbb{T}}$ or $L_{m-1}y(t)>0$ on $[T^{**},\infty)_{\mathbb{T}}$ for some $T^{**}\in[T^*,\infty)_{\mathbb{T}}$. If $L_{m-1}y(t)>0$ on $[T^{**},\infty)_{\mathbb{T}}$ holds, then it follows from Lemma 2.2 that $L_jy(t)>0$ for $j\in\{1,2,\cdots,m-2\}$. If $L_{m-1}y(t)<0$ on $[T^*,\infty)_{\mathbb{T}}$, then the same arguments as in the proof $L_{n-1}y(t)>0$ imply that $L_{m-2}y(t)>0$ on $[T^*,\infty)_{\mathbb{T}}$. It is a contradiction to the definition of m. We get the desired results and complete the proof. □

3. Main results

Lemma 3.1. *Assume that either*

$$\int_{t_0}^{\infty} q(t)\Delta t = \infty \tag{3.1}$$

or

$$\int_{t_0}^{\infty} q(t)\Delta t < \int_{t_0}^{\infty} \rho_2(t)\Delta t = \infty, \tag{3.2}$$

and $(H_1)-(H_4)$ *hold, where*

$$\rho_0(t) = q(t), \quad \rho_k(t) = \left[\frac{1}{p_{n-k}(t)}\int_t^{\infty} \rho_{k-1}(s)\Delta s\right]^{\frac{1}{\alpha_{n-k}}}, \quad k = 1, 2, \cdots, n-1.$$

Let $y(t)$ *be a nonoscillatory solution of Eq.* (1.1) *on* $[t_0,\infty)_{\mathbb{T}}$. *Then there exists a sufficiently large* $T^* \in [t_0,\infty)_{\mathbb{T}}$, *such that* $L_n y(t) \le 0$ *for any* $t \in [T^*,\infty)_{\mathbb{T}}$. *Moreover,*

$$y(t)L_j y(t) > 0, \quad t \in [T^*,\infty)_{\mathbb{T}}, \quad j = 0, 1, \cdots, n-1 \tag{3.3}$$

holds when $n \in 2\mathbb{N}$, *and either* (3.3) *holds or* $\lim_{t\to\infty} y(t) = 0$ *when* $n \in 2\mathbb{N}+1$.

Proof. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by $(H_1)-(H_4)$, there is a $T \in [t_0,\infty)_{\mathbb{T}}$ such that $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\theta(t)) > 0$ for all $t \in [T,\infty)_{\mathbb{T}}$ and Theorem 2.1 holds.

When $n \in 2\mathbb{N}$, l must be an odd integer and $L_1 y(t) = [L_0 y(t)]^{\Delta} > 0$ on $[T,\infty)_{\mathbb{T}}$. Consequently,

$$\lim_{t\to\infty} L_0 y(t) \text{ exists and is positive, or } \lim_{t\to\infty} L_0 y(t) = \infty. \tag{3.4}$$

We claim that $l = n-1$ by a contradiction argument. Assume not, then $L_{n-1}y(t) > 0$, $L_{n-2}y(t) < 0$ and $L_{n-3}y(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. In view of $L_0 y(t) = y(t) + r(t)y(\tau(t))$ and (3.4), we derive that there exists a $T^* \in [T,\infty)_{\mathbb{T}}$ such that $y(\theta(t)) \ge c$ on $[T^*,\infty)_{\mathbb{T}}$ for some constant $c > 0$. From (1.1), we have

$$L_n y(t) = -q(t)f(|y(\theta(t))|^{\beta} sgn(y(\theta(t)))) \le -Mc^{\beta}q(t) \ \text{ on } \ [T^*,\infty)_{\mathbb{T}}.$$

If (3.1) holds, then integrating the above inequality from T^* to t, we obtain that for $t \in [T^*,\infty)_{\mathbb{T}}$,

$$0 < p_{n-1}(t)L_{n-1}y(t)^{\alpha_{n-1}} \le p_{n-1}(T^*)L_{n-1}y(T^*)^{\alpha_{n-1}} - Mc^{\beta}\int_{T^*}^{t} q(s)\Delta s,$$

which is a contradiction to assumption (3.1). Thus, $l = n-1$ and (3.3) hold.

If (3.2) holds, then integrating $L_n y(t) = [p_{n-1}(t)|L_{n-1}y(t)|^{\alpha_{n-1}} sgn(L_{n-1}y(t))]^{\Delta}$ over $[t,\infty)_{\mathbb{T}}$, we get that for $t \in [T^*,\infty)_{\mathbb{T}}$,

$$-p_{n-1}(t)L_{n-1}y(t)^{\alpha_{n-1}} \le -Mc^{\beta}\int_t^{\infty} q(s)\Delta s.$$

Next, integrating $L_{n-1}y(t) = [p_{n-2}(t)|L_{n-2}y(t)|^{\alpha_{n-2}} sgn(L_{n-2}y(t))]^{\Delta}$ from t to τ, $t \in [T^*,\infty)_{\mathbb{T}}$, and using the above inequality, we have

$$-p_{n-2}(\tau)[-L_{n-2}y(\tau)]^{\alpha_{n-2}} + p_{n-2}(t)[-L_{n-2}y(t)]^{\alpha_{n-2}} \ge (cM^{1/\beta})^{\frac{\beta}{\alpha_{n-1}}}\int_t^{\tau} \rho_1(s)\Delta s,$$

and so

$$-[p_{n-3}(t)L_{n-3}y(t)^{\alpha_{n-3}}]^{\Delta}=-L_{n-2}y(t)\geq (cM^{1/\beta})^{\frac{\beta}{\alpha_{n-1}\alpha_{n-2}}}\rho_2(t).$$

Finally, integrating the last inequality on $[T^*,\infty)_{\mathbb{T}}$, we obtain

$$\infty>p_{n-3}(T^*)L_{n-3}y(T^*)^{\alpha_{n-3}}\geq (cM^{1/\beta})^{\frac{\beta}{\alpha_{n-1}\alpha_{n-2}}}\int_{T^*}^{\infty}\rho_2(t)\Delta t,$$

which contradicts (3.2). Hence $l=n-1$ and (3.3) hold.

When $n\in 2\mathbb{N}+1$, we infer from Theorem 2.1 that l is an even integer. Thus, $L_1y(t)>0$ or $L_1y(t)<0$, which means that $\lim_{t\to\infty}L_0y(t)\geq 0$. We claim that $\lim_{t\to\infty}L_0y(t)\neq 0$ implies that $l=n-1$. By a similar argument as above, we get a contradiction to (3.1) or (3.2). Thus, $\lim_{t\to\infty}L_0y(t)=0$. Noting that $0<y(t)\leq L_0y(t)$, we find $\lim_{t\to\infty}y(t)=0$. This ends the proof. □

Lemma 3.2. *Assume that either* (3.1) *or* (3.2) *holds. Let $y(t)$ be a nonoscillatory solution of Eq.* (1.1) *which satisfies* (3.3) *on $[T,\infty)_{\mathbb{T}}$. Then, there exists a sequence $\{T_i\}_{i=1}^{n}\subseteq (T,\infty)_{\mathbb{T}}$ with $T_{j+1}\in (T_j,\infty)_{\mathbb{T}}$, $j=1,2,\cdots,n-1$ such that*

$$\frac{p_j(t)L_jy(t)^{\alpha_j}}{R_{n-j-1}(t,T_{n-j-1})}\ \text{is non-increasing on}\ [T_{n-j-1},\infty)_{\mathbb{T}}\ \text{for}\ 0\leq j\leq n-2, \quad (3.5)$$

and for any $t\in[T_{n-j-1},\infty)_{\mathbb{T}}$,

$$p_j(t)L_jy(t)^{\alpha_j}\geq\left[\frac{p_{j+1}(t)L_{j+1}y(t)^{\alpha_{j+1}}}{R_{n-j-2}(t,T_{n-j-2})}\right]^{\frac{1}{\alpha_{j+1}}}R_{n-j-1}(t,T_{n-j-1}),\ j=0,1,\cdots,n-2, \quad (3.6)$$

where $p_0(t):=1$ and

$$R_0(t,T_0)=1,\ R_{i+1}(t,T_{i+1})=\int_{T_{i+1}}^{t}\left[\frac{R_i(s,T_i)}{p_{n-i-1}(s)}\right]^{\frac{1}{\alpha_{n-i-1}}}\Delta s,\ i=0,1,\cdots,n-2. \quad (3.7)$$

Proof. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by $(H_1)-(H_4)$, there is a $T\in[t_0,\infty)_{\mathbb{T}}$ such that $y(t)>0$, $y(\tau(t))>0$ and $y(\theta(t))>0$ for all $t\in[T,\infty)_{\mathbb{T}}$. Then, it follows from (3.3) that for $t\in[T_1,\infty)_{\mathbb{T}}$ with $T_1\in(T,\infty)_{\mathbb{T}}$,

$$\begin{aligned}p_{n-2}(t)L_{n-2}y(t)^{\alpha_{n-2}}&\geq\int_{T_1}^{t}\frac{[p_{n-1}(s)L_{n-1}y(s)^{\alpha_{n-1}}]^{\frac{1}{\alpha_{n-1}}}}{p_{n-1}^{\frac{1}{\alpha_{n-1}}}(s)}\Delta s\\&\geq p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t)L_{n-1}y(t)R_1(t,T_1).\end{aligned}$$

This means that

$$\left[\frac{p_{n-2}(t)L_{n-2}y(t)^{\alpha_{n-2}}}{R_1(t,T_1)}\right]^{\Delta}=\frac{p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t)L_{n-1}y(t)R_1(t,T_1)-p_{n-2}(t)L_{n-2}y(t)^{\alpha_{n-2}}}{p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t)R_1(t,T_1)R_1(\sigma(t),T_1)}\leq 0,$$

and $p_{n-2}(t)L_{n-2}y(t)^{\alpha_{n-2}}/R_1(t,T_1)$ is non-increasing on $[T_1,\infty)_{\mathbb{T}}$. Then, for $t\in[T_2,\infty)_{\mathbb{T}}\subseteq(T_1,\infty)_{\mathbb{T}}$,

$$p_{n-3}(t)L_{n-3}y(t)^{\alpha_{n-3}}\geq\int_{T_2}^{t}\left[\frac{p_{n-2}(s)L_{n-2}y(s)^{\alpha_{n-2}}}{R_1(s,T_1)}\right]^{\frac{1}{\alpha_{n-2}}}\frac{R_1^{\frac{1}{\alpha_{n-2}}}(s,T_1)}{p_{n-2}^{\frac{1}{\alpha_{n-2}}}(s)}\Delta s$$

$$\geq \frac{p_{n-2}^{\frac{1}{\alpha_{n-2}}}(t)L_{n-2}y(t)}{R_1^{\frac{1}{\alpha_{n-2}}}(t,T_1)}R_2(t,T_2),$$

which indicates that

$$\left[\frac{p_{n-3}(t)L_{n-3}y(t)^{\alpha_{n-3}}}{R_2(t,T_2)}\right]^{\Delta}$$
$$=\frac{p_{n-2}^{\frac{1}{\alpha_{n-2}}}(t)L_{n-2}y(t)R_2(t,T_2)-p_{n-3}(t)L_{n-3}y(t)^{\alpha_{n-3}}R_1^{\frac{1}{\alpha_{n-2}}}(t,T_1)}{p_{n-2}^{\frac{1}{\alpha_{n-2}}}(t)R_2(t,T_2)R_2(\sigma(t),T_2)}\leq 0,$$

and $p_{n-3}(t)L_{n-3}y(t)^{\alpha_{n-3}}/R_2(t,T_2)$ is non-increasing on $[T_2,\infty)_{\mathbb{T}}$. Repeating the above process, we conclude that for $t\in[T_{n-1},\infty)_{\mathbb{T}}\subseteq(T_{n-2},\infty)_{\mathbb{T}}$,

$$L_0y(t)\geq\int_{T_{n-1}}^{t}\left[\frac{p_1(s)L_1y(s)^{\alpha_1}}{R_{n-2}(s,T_{n-2})}\right]^{\frac{1}{\alpha_1}}\frac{R_{n-2}^{\frac{1}{\alpha_1}}(s,T_{n-2})}{p_1^{\frac{1}{\alpha_1}}(s)}\Delta s$$
$$\geq\frac{p_1^{\frac{1}{\alpha_1}}(t)L_1y(t)}{R_{n-2}^{\frac{1}{\alpha_1}}(t,T_{n-2})}R_{n-1}(t,T_{n-1}),$$

which implies that

$$\left[\frac{L_0y(t)}{R_{n-1}(t,T_{n-1})}\right]^{\Delta}=\frac{p_1^{\frac{1}{\alpha_1}}(t)L_1y(t)R_{n-1}(t,T_{n-1})-L_0y(t)R_{n-2}^{\frac{1}{\alpha_1}}(t,T_{n-2})}{p_{n-2}^{\frac{1}{\alpha_{n-2}}}(t)R_2(t,T_2)R_2(\sigma(t),T_2)}\leq 0,$$

and $L_0y(t)/R_{n-1}(t,T_{n-1})$ is non-increasing on $[T_{n-1},\infty)_{\mathbb{T}}$. The proof is completed. □

Lemma 3.3. *Assume that either* (3.1) *or* (3.2) *holds. Let $y(t)$ be a nonoscillatory solution of Eq.* (1.1) *which satisfies* (3.3) *on* $[t_0,\infty)_{\mathbb{T}}$*. Then, there exists a constant* $T\in[t_0,\infty)_{\mathbb{T}}$ *such that*

$$L_ny(t)+Mq(t)[1-r(\theta(t))]^{\beta}L_0y(\theta(t))^{\beta}\leq 0,\ \textit{for}\ \ t\in[T,\infty)_{\mathbb{T}}. \tag{3.8}$$

Proof. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by $(H_1)-(H_4)$, there is a $T\in[t_0,\infty)_{\mathbb{T}}$ such that $y(t)>0$, $y(\tau(t))>0$ and $y(\theta(t))>0$ for all $t\in[T,\infty)_{\mathbb{T}}$. In view of $\tau(t)\leq t$ and $y(t)\leq L_0y(t)$, we derive from (3.3) that

$$L_0y(t)=y(t)+r(t)y(\tau(t))\leq y(t)+r(t)L_0y(\tau(t))\leq y(t)+r(t)L_0y(t)\ \text{ on }\ [T,\infty)_{\mathbb{T}},$$

which indicates $y(t)\geq[1-r(t)]L_0y(t)$ for $t\in[T,\infty)_{\mathbb{T}}$. By choosing T large enough, we have, for $t\in[T,\infty)_{\mathbb{T}}$,

$$y(\theta(t))\geq[1-r(\theta(t))]L_0y(\theta(t)). \tag{3.9}$$

As a consequence, (3.8) holds and the proof of this lemma is thereby complete with (1.1). □

3.1. The case when $\theta(t) > t$

In this subsection, we establish some sufficient oscillation conditions for (1.1) when $\theta(t) > t$. To formulate and prove our results, we use the following notations. Given $T_{n-1} \in (T_1, \infty)_{\mathbb{T}} \subseteq (t_0, \infty)_{\mathbb{T}}$ sufficiently large. For any $t \in [T_{n-1}, \infty)_{\mathbb{T}}$, we define

$$\beta(t,T_1) = \begin{cases} \left[\frac{R_1(t,T_1)}{R_1(\sigma(t),T_1)}\right]^{\alpha_{n-1}}, & \alpha_{n-1} \geq 1, \\ \frac{R_1(t,T_1)}{R_1(\sigma(t),T_1)}, & 0 < \alpha_{n-1} < 1, \end{cases} \qquad \gamma(t,T_1) = \frac{\alpha_{n-1}\delta(t)\beta(t,T_1)}{p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t)\delta^{1+\frac{1}{\alpha_{n-1}}}(\sigma(t))}, \tag{3.10}$$

$$\eta(t,T_1,T_{n-1}) = [1-r(\theta(t))]^{\prod_{i=1}^{n-1}\alpha_i} \frac{R_{n-1}^{\prod_{i=1}^{n-1}\alpha_i}(t,T_{n-1})}{R_1^{\alpha_{n-1}}(\sigma(t),T_1)}, \quad [g(t)]_+ = \max\{g(t),0\}, \tag{3.11}$$

and for any given function $\phi(t) > -1/p_{n-1}(t)R_1^{\alpha_{n-1}}(t,T_1)$ such that $p_{n-1}(t)\phi(t)$ is a Δ-differentiable function and a positive Δ-differentiable function $\delta(t)$, we assume

$$C(t,T_1) = \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))} + (1+\alpha_{n-1})(p_{n-1}\phi)^{\frac{1}{\alpha_{n-1}}}(\sigma(t))\frac{\beta(t,T_1)\delta(t)}{\delta(\sigma(t))p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t)},$$

and

$$\Psi(t,T_1,T_{n-1}) = \delta(t)\left[Mq(t)\eta(t,T_1,T_{n-1}) + \frac{\beta(t,T_1)}{p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t)}(p_{n-1}\phi)^{1+\frac{1}{\alpha_{n-1}}}(\sigma(t)) - [p_{n-1}(t)\phi(t)]^{\Delta}\right].$$

Now, we state and prove the first oscillation theorem in this subsection.

Theorem 3.1. *Let $\beta = \prod_{i=1}^{n-1}\alpha_i$. Assume that either* (3.1) *or* (3.2) *is satisfied. Furthermore, suppose that there exist a function $\phi(t)$ satisfying $\phi(t) = 0$ for $0 < \alpha_{n-1} < 1$ and a positive Δ-differentiable function $\delta(t)$ such that for a sufficiently large $T^* \in [T_{n-1}, \infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty}\int_{T^*}^{t}\left[\Psi(s,T_1,T_{n-1}) - \frac{\alpha_{n-1}^{\alpha_{n-1}}}{(1+\alpha_{n-1})^{1+\alpha_{n-1}}}\frac{([C(s,T_1)]_+)^{1+\alpha_{n-1}}}{[\gamma(s,T_1)]^{\alpha_{n-1}}}\right]\Delta s > A(T^*,T_1), \tag{3.12}$$

where $A(t,T_1) = \delta(t)\left[1/R_1(t,T_1)^{\alpha_{n-1}} + p_{n-1}(t)\phi(t)\right]$. Then,

(i) every solution $y(t)$ of Eq. (1.1) *is oscillatory when $n \in 2\mathbb{N}$;*

(ii) every solution $y(t)$ of Eq. (1.1) *is either oscillatory or* $\lim_{t\to\infty} y(t) = 0$ *when $n \in 2\mathbb{N}+1$.*

Proof. Assume by way of contradiction that Eq. (1.1) has a nonoscillatory solution $y(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, there is a sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$ such that for $t \in [T,\infty)_{\mathbb{T}}$, $y(t) > 0$, $y(\tau(t)) > 0$, $y(\theta(t)) > 0$, and Lemmas 3.1-3.3 hold.

When $n \in 2\mathbb{N}$, by Lemma 3.1, (3.3) holds. Define a generalized Riccati substitution:

$$w(t) = \delta(t)\left[\frac{p_{n-1}(t)L_{n-1}y(t)^{\alpha_{n-1}}}{(p_{n-2}(t)L_{n-2}y(t)^{\alpha_{n-2}})^{\alpha_{n-1}}} + p_{n-1}(t)\phi(t)\right]. \tag{3.13}$$

By the product rule, the sum rule and the quotient rule, we find

$$\begin{aligned}
w^{\Delta}(t) &= \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))} w(\sigma(t)) + \delta(t)\left[\frac{p_{n-1}(t)L_{n-1}y(t)^{\alpha_{n-1}}}{(p_{n-2}(t)L_{n-2}y(t)^{\alpha_{n-2}})^{\alpha_{n-1}}}\right]^{\Delta} + \delta(t)\left[p_{n-1}(t)\phi(t)\right]^{\Delta} \\
&= \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))} w(\sigma(t)) + \delta(t)\left[p_{n-1}(t)\phi(t)\right]^{\Delta} + \delta(t)\frac{L_n y(t)}{(p_{n-2}(\sigma(t))L_{n-2}y(\sigma(t))^{\alpha_{n-2}})^{\alpha_{n-1}}} \\
&\quad - \delta(t)\frac{p_{n-1}(t)L_{n-1}y(t)^{\alpha_{n-1}}\left[(p_{n-2}(t)L_{n-2}y(t)^{\alpha_{n-2}})^{\alpha_{n-1}}\right]^{\Delta}}{(p_{n-2}(t)L_{n-2}y(t)^{\alpha_{n-2}})^{\alpha_{n-1}}(p_{n-2}(\sigma(t))L_{n-2}y(\sigma(t))^{\alpha_{n-2}})^{\alpha_{n-1}}} \\
&= \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))} w(\sigma(t)) + \delta(t)\left[p_{n-1}(t)\phi(t)\right]^{\Delta} + \Lambda_1 - \Lambda_2. \qquad (3.14)
\end{aligned}$$

By Lemma 3.3, we have for $t \in [T,\infty)_{\mathbb{T}}$,

$$\begin{aligned}
\Lambda_1 \leq &- Mq(t)\delta(t)[1-r(\theta(t))]^{\prod_{i=1}^{n-1}\alpha_i}\left[\frac{L_0 y(\theta(t))}{L_0 y(\sigma(t))}\right]^{\prod_{i=1}^{n-1}\alpha_i} \\
&\times \frac{L_0 y(\sigma(t))^{\prod_{i=1}^{n-1}\alpha_i}}{(p_{n-2}(\sigma(t))L_{n-2}y(\sigma(t))^{\alpha_{n-2}})^{\alpha_{n-1}}}.
\end{aligned}$$

Because $L_0 y(t)/R_{n-1}(t,T_{n-1})$ is non-increasing on $[T_{n-1},\infty)_{\mathbb{T}}$, where T_{n-1} is given in Lemma 3.2, we have

$$\frac{L_0 y(t)}{L_0 y(\sigma(t))} \geq \frac{R_{n-1}(t,T_{n-1})}{R_{n-1}(\sigma(t),T_{n-1})} \quad \text{on} \quad [T_{n-1},\infty)_{\mathbb{T}}. \qquad (3.15)$$

As a consequence, for all $t \in [T_{n-1},\infty)_{\mathbb{T}}$,

$$\left[\frac{L_0 y(\theta(t))}{L_0 y(\sigma(t))}\right]^{\prod_{i=1}^{n-1}\alpha_i} = \left[\frac{L_0 y(\theta(t))}{L_0 y(t)}\frac{L_0 y(t)}{L_0 y(\sigma(t))}\right]^{\prod_{i=1}^{n-1}\alpha_i} \geq \left[\frac{R_{n-1}(t,T_{n-1})}{R_{n-1}(\sigma(t),T_{n-1})}\right]^{\prod_{i=1}^{n-1}\alpha_i}, \qquad (3.16)$$

since $L_1 y(t) = [L_0 y(t)]^{\Delta} > 0$ on $[T,\infty)_{\mathbb{T}}$. Using the induction method, we conclude from Lemma 3.2 that

$$\frac{L_0 y(t)}{R_{n-1}(t,T_{n-1})} \geq \left[\frac{p_1(t)L_1 y(t)^{\alpha_1}}{R_{n-2}(t,T_{n-2})}\right]^{\frac{1}{\alpha_1}} \geq \cdots \geq \left[\frac{p_{n-2}(t)L_{n-2}y(t)^{\alpha_{n-2}}}{R_1(t,T_1)}\right]^{\prod_{i=1}^{n-2}\frac{1}{\alpha_i}} \qquad (3.17)$$

on $[T_{n-1},\infty)_{\mathbb{T}}$. It then follows from (3.16) and (3.17) that for all $t \in [T_{n-1},\infty)_{\mathbb{T}}$,

$$\begin{aligned}
\Lambda_1 \leq &- Mq(t)\delta(t)[1-r(\theta(t))]^{\prod_{i=1}^{n-1}\alpha_i}\left[\frac{R_{n-1}(t,T_{n-1})}{R_{n-1}(\sigma(t),T_{n-1})}\right]^{\prod_{i=1}^{n-1}\alpha_i} \\
&\times \frac{\left\{R_{n-1}(\sigma(t),T_{n-1})\left[\frac{p_{n-2}(\sigma(t))L_{n-2}y(\sigma(t))^{\alpha_{n-2}}}{R_1(\sigma(t),T_1)}\right]^{\prod_{i=1}^{n-2}\frac{1}{\alpha_i}}\right\}^{\prod_{i=1}^{n-1}\alpha_i}}{(p_{n-2}(\sigma(t))L_{n-2}y(\sigma(t))^{\alpha_{n-2}})^{\alpha_{n-1}}} \\
= &- Mq(t)\delta(t)[1-r(\theta(t))]^{\prod_{i=1}^{n-1}\alpha_i}\frac{R_{n-1}^{\prod_{i=1}^{n-1}\alpha_i}(t,T_{n-1})}{R_1^{\alpha_{n-1}}(\sigma(t),T_1)}. \qquad (3.18)
\end{aligned}$$

By the Potzsche chain rule, Lemma 3.1 and Lemma 3.2, we have

$$\frac{\left[(p_{n-2}(t)L_{n-2}y(t)^{\alpha_{n-2}})^{\alpha_{n-1}}\right]^{\Delta}}{(p_{n-2}(\sigma(t))L_{n-2}y(\sigma(t))^{\alpha_{n-2}})^{\alpha_{n-1}}}$$

$$\geq \frac{\alpha_{n-1} L_{n-1} y(t) \left[p_{n-2}(t) L_{n-2} y(t)^{\alpha_{n-2}}\right]^{\alpha_{n-1}}}{p_{n-2}(t) L_{n-2} y(t)^{\alpha_{n-2}} \left[p_{n-2}(\sigma(t)) L_{n-2} y(\sigma(t))^{\alpha_{n-2}}\right]^{\alpha_{n-1}}}$$
$$\geq \frac{\alpha_{n-1} L_{n-1} y(t)}{p_{n-2}(t) L_{n-2} y(t)^{\alpha_{n-2}}} \left[\frac{R_1(t,T_1)}{R_1^\sigma(t,T_1)}\right]^{\alpha_{n-1}}, \ \alpha_{n-1} \geq 1,$$

and

$$\frac{\left[(p_{n-2}(t) L_{n-2} y(t)^{\alpha_{n-2}})^{\alpha_{n-1}}\right]^\Delta}{(p_{n-2}(\sigma(t)) L_{n-2} y(\sigma(t))^{\alpha_{n-2}})^{\alpha_{n-1}}}$$
$$\geq \frac{\alpha_{n-1} L_{n-1} y(t)}{p_{n-2}(\sigma(t)) L_{n-2} y(\sigma(t))^{\alpha_{n-2}}}$$
$$\geq \frac{\alpha_{n-1} L_{n-1} y(t)}{p_{n-2}(t) L_{n-2} y(t)^{\alpha_{n-2}}} \left[\frac{R_1(t,T_1)}{R_1^\sigma(t,T_1)}\right], \ 0 < \alpha_{n-1} < 1.$$

Noting that $L_n y(t) \leq 0$ and $L_{n-1} y(t) > 0$, combining with the definition of $\beta(t, T_1)$ and $w(t)$, we deduce that

$$\begin{aligned}
\Lambda_2 \geq& \frac{\alpha_{n-1}\delta(t)\beta(t,T_1) \left[p_{n-1}(t) L_{n-1} y(t)^{\alpha_{n-1}}\right]^{1+\frac{1}{\alpha_{n-1}}}}{p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t) (p_{n-2}(t) L_{n-2} y(t)^{\alpha_{n-2}})^{\alpha_{n-1}+1}} \\
\geq& \frac{\alpha_{n-1}\delta(t)\beta(t,T_1) \left[p_{n-1}(\sigma(t)) L_{n-1} y(\sigma(t))^{\alpha_{n-1}}\right]^{1+\frac{1}{\alpha_{n-1}}}}{p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t) (p_{n-2}(\sigma(t)) L_{n-2} y(\sigma(t))^{\alpha_{n-2}})^{\alpha_{n-1}+1}} \\
\geq& \frac{\alpha_{n-1}\delta(t)\beta(t,T_1)}{p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t)} \left[\frac{w(\sigma(t))}{\delta(\sigma(t))} - p_{n-1}(\sigma(t))\phi(\sigma(t))\right]^{1+\frac{1}{\alpha_{n-1}}}.
\end{aligned}$$

Define $E > 0$ and $F > 0$ by $E := w(\sigma(t))/\delta(\sigma(t))$ and $F := p_{n-1}(\sigma(t))\phi(\sigma(t))$ and using the inequality

$$E^{1+\frac{1}{\gamma}} - (E-F)^{1+\frac{1}{\gamma}} \leq F^{\frac{1}{\gamma}} \left[\left(1+\frac{1}{\gamma}\right) E - \frac{1}{\gamma} F\right], \ \gamma \geq 1,$$

we find, for $\alpha_{n-1} \geq 1$,

$$\begin{aligned}
\Lambda_2 \geq& \frac{\alpha_{n-1}\delta(t)\beta(t,T_1)}{p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t)} \Bigg\{ \left[\frac{w(\sigma(t))}{\delta(\sigma(t))}\right]^{1+\frac{1}{\alpha_{n-1}}} + \frac{1}{\alpha_{n-1}} (p_{n-1}\phi)^{1+\frac{1}{\alpha_{n-1}}}(\sigma(t)) \\
&- (1+\frac{1}{\alpha_{n-1}}) \frac{(p_{n-1}\phi)^{\frac{1}{\alpha_{n-1}}}(\sigma(t))}{\delta(\sigma(t))} w(\sigma(t)) \Bigg\}.
\end{aligned} \tag{3.19}$$

Plugging (3.18) and (3.19) into (3.14) and using the definition of $\Psi(t, T_1, T_{n-1})$, $C(t, T_1)$ and $\gamma(t, T_1)$, we obtain

$$\begin{aligned}
w^\Delta(t) \leq& -\Psi(t,T_1,T_{n-1}) + C(t,T_1) w(\sigma(t)) - \gamma(t,T_1) w^{1+\frac{1}{\alpha_{n-1}}}(\sigma(t)) \\
\leq& -\Psi(t,T_1,T_{n-1}) + [C(t,T_1)]_+ w(\sigma(t)) - \gamma(t,T_1) w^{1+\frac{1}{\alpha_{n-1}}}(\sigma(t)).
\end{aligned} \tag{3.20}$$

For $0 < \alpha_{n-1} < 1$ and $\phi(t) = 0$, it is obvious that (3.20) also holds. Applying the inequality

$$Bw - Aw^{1+\frac{1}{\gamma}} \leq \frac{\gamma^\gamma}{(1+\gamma)^{1+\gamma}} \frac{B^{1+\gamma}}{A^\gamma}, \quad A, B > 0, \tag{3.21}$$

to the inequality (3.20), we have

$$w^{\Delta}(t) \le -\Psi(t, T_1, T_{n-1}) + \frac{\alpha_{n-1}^{\alpha_{n-1}}}{(1+\alpha_{n-1})^{1+\alpha_{n-1}}} \frac{([C(t, T_1)]_+)^{1+\alpha_{n-1}}}{[\gamma(t, T_1)]^{\alpha_{n-1}}}. \tag{3.22}$$

Integrating (3.22) with respect to t from T^* to t, where $t \in (T^*, \infty)_{\mathbb{T}}$ with $T^* \in (\max\{T, T_{n-1}\}, \infty)_{\mathbb{T}}$, we get

$$\begin{aligned}
&\int_{T^*}^{t} \left[\Psi(s, T_1, T_{n-1}) - \frac{\alpha_{n-1}^{\alpha_{n-1}}}{(1+\alpha_{n-1})^{1+\alpha_{n-1}}} \frac{([C(s, T_1)]_+)^{1+\alpha_{n-1}}}{[\gamma(s, T_1)]^{\alpha_{n-1}}} \right] \Delta s \\
\le & w(T^*) - w(t) \le \delta(T^*) \left[\frac{1}{(R_1(T^*, T_1))^{\alpha_{n-1}}} + p_{n-1}(T^*)\phi(T^*) \right],
\end{aligned}$$

which contradicts (3.12). Thus, every solution $y(t)$ of (1.1) is oscillatory.

When $n \in 2\mathbb{N}+1$, we infer from Lemma 3.1 that (3.3) holds or $\lim_{t\to\infty} y(t) = 0$. If (3.3) holds, similarly, then we see that Eq. (1.1) is oscillatory. Thus, we omit the details. The proof of this theorem is completed. □

In what follows, we will use the function class $\mathcal{X}$ to study oscillation of (1.1). We say that a function $\Xi := \Xi(t, s, \ell)$ belongs to the function class $\mathcal{X}$, denoted by $\Xi \in \mathcal{X}$, if $\Xi \in \mathrm{C_{rd}}(\Gamma, \mathbb{R})$, where $\Gamma := \{(t, s, \ell) : \infty > t \ge s \ge \ell \ge t_0, t, s, \ell \in [t_0, \infty)_{\mathbb{T}}\}$, which satisfies $\Xi(t, t, \ell) = 0$, $\Xi(t, \ell, \ell) = 0$ and $\Xi(t, s, \ell) \ne 0$ for $t > s > \ell$, and has the partial derivative Ξ^{Δ_s} on Γ such that Ξ^{Δ_s} is Δ-integrable with respect to s in Γ. We define the operator $\mathcal{B}[\cdot; \ell, t]$ by

$$\mathcal{B}[g; \ell, t] := \int_{\ell}^{t} \Xi^2(t, s, \ell) g(s) \Delta s \text{ for } t \ge s \ge \ell \ge t_0 \text{ and } g \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}), \tag{3.23}$$

and the function $\xi(t, s, \ell)$ is defined by

$$\Xi^{\Delta_s}(t, s, \ell) = \xi(t, s, \ell)\Xi(t, s, \ell). \tag{3.24}$$

It is not difficult to verify that $\mathcal{B}[\cdot; \ell, t]$ is a linear operator and satisfies

$$\mathcal{B}[g^{\Delta}; \ell, t] = -\mathcal{B}[g^{\sigma}(2\xi + \mu\xi^2); \ell, t] \text{ for } g \in \mathrm{C^1_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}). \tag{3.25}$$

Theorem 3.2. *Let $\beta = \prod_{i=1}^{n-1} \alpha_i$. Assume that either* (3.1) *or* (3.2) *is satisfied. Furthermore, suppose that for each $T \in [t_0, \infty)_{\mathbb{T}}$, there exists a function $\Xi \in \mathcal{X}$ such that*

$$\limsup_{t\to\infty} \mathcal{B}\left[\Psi(s, T_1, T_{n-1}) - \frac{\alpha_{n-1}^{\alpha_{n-1}}([2\xi(s)+\mu(s)\xi^2(s)+C(s, T_1)]_+)^{1+\alpha_{n-1}}}{(1+\alpha_{n-1})^{\alpha_{n-1}}\gamma^{\alpha_{n-1}}(s, T_1)}; T, t\right] > 0, \tag{3.26}$$

where the operator $\mathcal{B}$ is defined by (3.23) *and the function ξ is defined by* (3.24)*. Then,*

- *(i) every solution $y(t)$ of Eq.* (1.1) *is oscillatory when $n \in 2\mathbb{N}$;*
- *(ii) every solution $y(t)$ of Eq.* (1.1) *is either oscillatory or $\lim_{t\to\infty} y(t) = 0$ when $n \in 2\mathbb{N}+1$.*

Proof. On the contrary, assume that Eq. (1.1) has a nonoscillatory solution $y(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, there is a sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ such that for $t \in [T, \infty)_{\mathbb{T}}$, $y(t) > 0$, $y(\tau(t)) > 0$, $y(\theta(t)) > 0$, and Lemmas 3.1-3.3 hold.

When $n \in 2\mathbb{N}$, we proceed as in the proof of Theorem 3.1 to obtain (3.20). It then follows that

$$\Psi(s, T_1, T_{n-1}) \leq -w^{\Delta}(s) + C(s, T_1)w(\sigma(s)) - \gamma(s, T_1)w^{1+\frac{1}{\alpha_{n-1}}}(\sigma(s))$$

for all $s \in [\max\{T, T_{n-1}\}, \infty)_{\mathbb{T}}$. Applying the operator $\mathcal{B}[\cdot; T^*, t]$, where $t \in [T^*, \infty)_{\mathbb{T}}$ with $T^* \in (\max\{T, T_{n-1}\}, \infty)_{\mathbb{T}}$, and utilizing the property (3.25), we have

$$\begin{aligned}
&\mathcal{B}[\Psi(s, T_1, T_{n-1}); T^*, t] \\
\leq& \mathcal{B}\left[-w^{\Delta}(s) + C(s, T_1)w(\sigma(s)) - \gamma(s, T_1)w^{1+\frac{1}{\alpha_{n-1}}}(\sigma(s)); T^*, t\right] \\
\leq& \mathcal{B}\left[(2\xi(s) + \mu(s)\xi^2(s) + C(s, T_1))w(\sigma(s)) - \gamma(s, T_1)w^{1+\frac{1}{\alpha_{n-1}}}(\sigma(s)); T^*, t\right] \\
\leq& \mathcal{B}\left[[2\xi(s) + \mu(s)\xi^2(s) + C(s, T_1)]_+ w(\sigma(s)) - \gamma(s, T_1)w^{1+\frac{1}{\alpha_{n-1}}}(\sigma(s)); T^*, t\right].
\end{aligned}$$

It then follows from (3.21) that

$$\mathcal{B}[\Psi(s, T_1, T_{n-1}); T^*, t] \leq \mathcal{B}\left[\frac{\alpha_{n-1}^{\alpha_{n-1}}([2\xi(s) + \mu(s)\xi^2(s) + C(s, T_1)]_+)^{1+\alpha_{n-1}}}{(1+\alpha_{n-1})^{1+\alpha_{n-1}}\gamma^{\alpha_{n-1}}(s, T_1)}; T^*, t\right].$$

Taking the super limit in the above inequality, we obtain that

$$\limsup_{t\to\infty}\mathcal{B}\left[\Psi(s, T_1, T_{n-1}) - \frac{\alpha_{n-1}^{\alpha_{n-1}}([2\xi(s)+\mu(s)\xi^2(s)+C(s, T_1)]_+)^{1+\alpha_{n-1}}}{(1+\alpha_{n-1})^{1+\alpha_{n-1}}\gamma^{\alpha_{n-1}}(s, T_1)}; T^*, t\right] \leq 0.$$

This is a contradiction to (3.26). Therefore, every solution $y(t)$ of (1.1) is oscillatory.

When $n \in 2\mathbb{N}+1$, we derive from Lemma 3.1 that (3.3) holds or $\lim_{t\to\infty} y(t) = 0$. If (3.3) holds, similar to the proof of the case (i), then we can show that Eq. (1.1) is oscillatory and hence omit its proof. The proof is thereby complete. □

If we choose $\Xi(t, s, \ell) = \varphi(s)(t - s)(s - \ell)$, where $\varphi(s) \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, then obviously, $\Xi \in \mathcal{X}$ and

$$\xi(t, s, \ell) = \frac{\varphi^{\Delta}(s)}{\varphi(s)} + \frac{\varphi^{\sigma}(s)(t - \sigma(s))}{\varphi(s)(t - s)(s - \ell)} - \frac{\varphi^{\sigma}(s)}{\varphi(s)(t - s)}. \tag{3.27}$$

For an application of Theorem 3.2, we obtain the following corollary.

Corollary 3.1. *Let* $\beta = \prod_{i=1}^{n-1} \alpha_i$. *Assume that either* (3.1) *or* (3.2) *is satisfied. Furthermore, suppose that for each* $T \in [t_0, \infty)_{\mathbb{T}}$, *there exists a function* $\varphi \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *such that*

$$\begin{aligned}
\limsup_{t\to\infty} \int_T^t \varphi^2(s)(t - s)^2(s - T)^2 \Bigg[&\Psi(s, T_1, T_{n-1}) \\
- &\frac{\alpha_{n-1}^{\alpha_{n-1}}([2\xi(s) + \mu(s)\xi^2(s) + C(s, T_1)]_+)^{1+\alpha_{n-1}}}{(1+\alpha_{n-1})^{\alpha_{n-1}}\gamma^{\alpha_{n-1}}(s, T_1)}\Bigg]\Delta s > 0,
\end{aligned}$$

where the function ξ *is defined by* (3.27). *Then,*

(i) every solution $y(t)$ *of Eq.* (1.1) *is oscillatory when* $n \in 2\mathbb{N}$;

(ii) every solution $y(t)$ of Eq. (1.1) *is either oscillatory or* $\lim_{t\to\infty} y(t) = 0$ *when* $n \in 2\mathbb{N}+1$.

Remark 3.1. Theorem 3.2 is in a form with a high degree of generality. When $\mathbb{T} = \mathbb{R}$, we can see that Xu and Meng [29, Theorem 2.1] is a special case of Theorem 3.2.

If the assumptions (3.1) and (3.2) do not hold, we give the following more general theorem.

Theorem 3.3. *Assume that* $(H_1) - (H_4)$ *hold and* $\beta = \Pi_{i=1}^{n-1}\alpha_i$.

(i) Suppose that $n \in 2\mathbb{N}$ *and for each odd integer* $l \in \{1, 3, \cdots, n\}$ *and a sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$, *there holds*

$$\limsup_{t\to\infty} M^{\Pi_{i=l+1}^{n-1}\frac{1}{\alpha_i}} H_l^{\prod_{i=1}^{l}\alpha_i}(t,T)\int_t^\infty \chi_{n-l-1}(s)\Delta s > 1, \tag{3.28}$$

where

$$\chi_0(t) = q(t)[1 - r(\theta(t))]^{\Pi_{i=1}^{n-1}\alpha_i},$$
$$\chi_k(t) = \left[\frac{\int_t^\infty \chi_{k-1}(s)\Delta s}{p_{n-k}(t)}\right]^{\frac{1}{\alpha_{n-k}}}, \ k = 1, 2, \cdots, n-l-1, \tag{3.29}$$

and

$$H_1(t,T) = \int_T^t p_l(s)^{-\frac{1}{\alpha_l}}\Delta s, \ H_j(t,T) = \int_T^t \left[\frac{H_{j-1}(s,T)}{p_{l-j+1}(s)}\right]^{\frac{1}{\alpha_{l-j+1}}}\Delta s, \ j = 2, \cdots, l. \tag{3.30}$$

Then, every solution $y(t)$ of Eq. (1.1) *is oscillatory;*

(ii) Suppose that $n \in 2\mathbb{N}+1$,

$$\int_{t_0}^\infty \rho_{n-1}(s)\Delta s = \infty, \tag{3.31}$$

and for each even integer $l \in \{2, 4, \cdots, n\}$, (3.28) *holds, where* $\rho_i(t)$ *are defined as in Lemma 3.1. Then, every solution* $y(t)$ *of Eq.* (1.1) *is either oscillatory or* $\lim_{t\to\infty} y(t) = 0$.

Proof. Suppose that Eq.(1.1) has a nonoscillatory solution $y(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by $(H_1) - (H_4)$, there is a $T \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\theta(t)) > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. Moreover, by Theorem 2.1, we have for $t \in [T, \infty)_{\mathbb{T}}$, $L_j y(t) > 0$, $j = 1, 2, \cdots, l$, and $(-1)^{n+j} y(t) L_j y(t) < 0$ for $j = l+1, l+2, \cdots, n$.

When $n \in 2\mathbb{N}$, since $L_n y(t) = [p_{n-1}(t)L_{n-1}y(t)^{\alpha_{n-1}}]^\Delta \le 0$ and $L_{n-1}y(t) > 0$, then

$$\lim_{t\to\infty} p_{n-1}(t)L_{n-1}y(t)^{\alpha_{n-1}} = \zeta \ge 0.$$

Integrating both sides of (3.8) from t to ∞, we have

$$\zeta - p_{n-1}(t)L_{n-1}y(t)^{\alpha_{n-1}} + M\int_t^\infty q(s)[1 - r(\theta(s))]^{\prod_{i=1}^{n-1}\alpha_i} L_0 y(\theta(s))^{\prod_{i=1}^{n-1}\alpha_i}\Delta s \le 0. \tag{3.32}$$

Since $n+l$ is odd, we know that l must be an odd integer and $L_1y(t) = [L_0y(t)]^\Delta > 0$ for $t \in [T,\infty)_{\mathbb{T}}$. It then follows from (3.32) that

$$p_{n-1}(t)L_{n-1}y(t)^{\alpha_{n-1}} \geq ML_0y(t)^{\Pi_{i=1}^{n-1}\alpha_i}\int_t^\infty q(s)[1-r(\theta(s))]^{\Pi_{i=1}^{n-1}\alpha_i}\Delta s, \quad (3.33)$$

which implies that

$$L_{n-1}y(t) \geq L_0y(t)^{\Pi_{i=1}^{n-2}\alpha_i}\left[\frac{M}{p_{n-1}(t)}\int_t^\infty q(s)[(1-r(\theta(s)))]^{\Pi_{i=1}^{n-1}\alpha_i}\Delta s\right]^{\frac{1}{\alpha_{n-1}}}.$$

Next, integrating $L_{n-1}y(t) = [-p_{n-2}(t)|L_{n-2}y(t)|^{\alpha_{n-2}}]^\Delta$ over $[t,\infty)_{\mathbb{T}}$, where $t \in [T,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned}&p_{n-2}(t)|L_{n-2}y(t)|^{\alpha_{n-2}}\\ \geq &M^{\frac{1}{\alpha_{n-1}}}L_0y(t)^{\Pi_{i=1}^{n-2}\alpha_i}\int_t^\infty\left[\frac{\int_s^\infty q(u)[(1-r(\theta(u)))]^{\Pi_{i=1}^{n-1}\alpha_i}\Delta u}{p_{n-1}(s)}\right]^{\frac{1}{\alpha_{n-1}}}\Delta s.\end{aligned}$$

Thus, we have that for $t \in [T,\infty)_{\mathbb{T}}$,

$$\begin{aligned}&-L_{n-2}y(t)\\ \geq &L_0y(t)^{\Pi_{i=1}^{n-3}\alpha_i}\left\{\frac{M^{\frac{1}{\alpha_{n-1}}}}{p_{n-2}(t)}\int_t^\infty\left[\frac{\int_s^\infty q(u)[(1-r(\theta(u)))]^{\Pi_{i=1}^{n-1}\alpha_i}\Delta u}{p_{n-1}(s)}\right]^{\frac{1}{\alpha_{n-1}}}\Delta s\right\}^{\frac{1}{\alpha_{n-2}}}.\end{aligned}$$

Using the definition of $\chi_i(t)$, we continue in this fashion to get

$$p_l(t)L_ly(t)^{\alpha_l} \geq M^{\Pi_{i=l+1}^{n-1}\frac{1}{\alpha_i}}L_0y(t)^{\Pi_{i=1}^{l}\alpha_i}\int_t^\infty \chi_{n-l-1}(s)\Delta s \quad (3.34)$$

on $[T,\infty)_{\mathbb{T}}$. Noting that $L_{l+1}y(t) = [p_l(t)L_ly(t)^{\alpha_l}]^\Delta < 0$, we derive that $p_l(t)L_ly(t)^{\alpha_l}$ is decreasing on $[T,\infty)_{\mathbb{T}}$. Consequently,

$$\begin{aligned}p_{l-1}(t)L_{l-1}y(t)^{\alpha_{l-1}} &\geq \int_T^t [p_l(s)L_ly(s)^{\alpha_l}]^{\frac{1}{\alpha_l}}p_l(s)^{-\frac{1}{\alpha_l}}\Delta s\\ &\geq [p_l(t)L_ly(t)^{\alpha_l}]^{\frac{1}{\alpha_l}}\int_T^t p_l(s)^{-\frac{1}{\alpha_l}}\Delta s.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}p_{l-2}(t)L_{l-2}y(t)^{\alpha_{l-2}} &\geq \int_T^t [p_{l-1}(s)L_{l-1}y(s)^{\alpha_{l-1}}]^{\frac{1}{\alpha_{l-1}}}p_{l-1}(s)^{-\frac{1}{\alpha_{l-1}}}\Delta s\\ &\geq \int_T^t [p_l(s)L_ly(s)^{\alpha_l}]^{\frac{1}{\alpha_l\alpha_{l-1}}}H_1^{\frac{1}{\alpha_{l-1}}}(s,T)p_{l-1}(s)^{-\frac{1}{\alpha_{l-1}}}\Delta s\\ &\geq [p_l(t)L_ly(t)^{\alpha_l}]^{\frac{1}{\alpha_l\alpha_{l-1}}}H_2(t,T).\end{aligned}$$

Continuing in this way, we obtain

$$L_0y(t) \geq L_0y(t) - L_0y(T) \geq [p_l(t)L_ly(t)^{\alpha_l}]^{\Pi_{i=1}^{l}\frac{1}{\alpha_i}}H_l(t,T), \quad t \in [T,\infty)_{\mathbb{T}}. \quad (3.35)$$

Plugging (3.35) into (3.34) gives

$$M^{\prod_{i=l+1}^{n-1}\frac{1}{\alpha_i}} H_l^{\prod_{i=1}^{l}\alpha_i}(t,T)\int_t^\infty \chi_{n-l-1}(s)\Delta s \le 1, \quad t\in[T,\infty)_{\mathbb{T}}.$$

Taking the limsup on both sides of the above inequality as $t\to\infty$, we obtain a contradiction to (3.28). Therefore, every solution $y(t)$ of (1.1) is oscillatory.

Now we consider the case $n\in 2\mathbb{N}+1$. First, we assume $l=0$. Then, $L_1y(t) = [L_0y(t)]^\Delta < 0$. Consequently, $\lim_{t\to\infty} L_0y(t)=\nu\ge 0$. When $\lim_{t\to\infty} L_0y(t)=\nu > 0$, it follows from $0\le r(t)<1$ and $\lim_{t\to\infty}\theta(t)=\infty$ that there exists $T^{**}\in[T,\infty)_{\mathbb{T}}$ such that $y(\theta(t))\ge\hat{\mu}$ for some $\hat{\mu}\in[0,\nu)$ on $[T^{**},\infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 3.1, we obtain

$$-p_{n-1}(t)L_{n-1}y(t)^{\alpha_{n-1}} \le -M\hat{\mu}^\beta\int_t^\infty q(s)\Delta s.$$

Integrating $L_{n-1}y(t)=[p_{n-2}(t)|L_{n-2}y(t)|^{\alpha_{n-2}}sgn(L_{n-3}y(t))]^\Delta$ over $[t,\infty)_{\mathbb{T}}$, we infer that for $t\in[T^{**},\infty)_{\mathbb{T}}$,

$$p_{n-2}(t)[-L_{n-2}y(t)]^{\alpha_{n-1}} \ge \left(\hat{\mu}M^{\frac{1}{\beta}}\right)^{\frac{\beta}{\alpha_{n-1}}}\int_t^\infty \rho_1(s)\Delta s.$$

Continuing in the fashion, we have

$$|L_iy(t)| \ge \left(\hat{\mu}M^{\frac{1}{\beta}}\right)^{\prod_{k=i}^{n-1}\frac{\beta}{\alpha_k}}\int_t^\infty \rho_{n-i}(s)\Delta s,\ i=1,2,\cdots,n-3,$$

and so

$$L_1y(t) \le -\left(\hat{\mu}M^{\frac{1}{\beta}}\right)^{\prod_{i=1}^{n-1}\frac{\beta}{\alpha_i}}\int_t^\infty \rho_{n-2}(s)\Delta s \quad \text{on} \quad [T^{**},\infty)_{\mathbb{T}}.$$

Integrating the above inequality from T^{**} to $t\in[T^{**},\infty)_{\mathbb{T}}$, we have

$$L_0y(t)-L_0y(T^{**}) \le -\left(\hat{\mu}M)^{\frac{1}{\beta}}\right)^{\prod_{i=1}^{n-1}\frac{\beta}{\alpha_i}}\int_{T^{**}}^t \rho_{n-1}(s)\Delta s,$$

which yields $\lim_{t\to\infty} L_0y(t)=-\infty$ by (3.31). This is clearly impossible since $y(t)\le L_0y(t)$ and $y(t)>0$ on $[T,\infty)_{\mathbb{T}}$. From the above discussion, we conclude that $\lim_{t\to\infty} y(t)=0$ if $l=0$.

If $l\ge 2$, then $L_1y(t)=[L_0y(t)]^\Delta>0$. As in the proof of the case $n\in 2\mathbb{N}$, we deduce that every solution $y(t)$ of (1.1) is oscillatory and the proof of this theorem is thereby complete. □

If $l=n-1$, then we have the following result.

Corollary 3.2. *Let $\beta=\prod_{i=1}^{n-1}\alpha_i$. Assume that $(H_1)-(H_4)$ hold and either* (3.1) *or* (3.2) *is satisfied. If*

$$\limsup_{t\to\infty} MR_{n-1}^{\prod_{i=1}^{n-1}\alpha_i}(t,T_{n-1})\int_t^\infty q(s)[1-r(\theta(s))]^{\prod_{i=1}^{n-1}\alpha_i}\Delta s > 1 \tag{3.36}$$

for sufficiently large $T_{n-1}\in[T_0,\infty)_{\mathbb{T}}$. Then,

(i) every solution $y(t)$ of Eq. (1.1) *is oscillatory when $n\in 2\mathbb{N}$;*

(ii) every solution $y(t)$ of Eq. (1.1) *is either oscillatory or $\lim_{t\to\infty} y(t)=0$ when $n\in 2\mathbb{N}+1$.*

3.2. The case when $\theta(t) \le t$

In this subsection, we establish some sufficient oscillation conditions for (1.1) when $\theta(t) \le t$. We first introduce the following notations. Given $T_{n-1} > T_1 > t_0$ sufficiently large. For any $t \in [T_{n-1}, \infty)_{\mathbb{T}}$, we define

$$\widetilde{\eta}(t, T_1, T_{n-1}) = [1 - r(\theta(t))]^{\prod_{i=1}^{n-1} \alpha_i} \frac{R_{n-1}^{\prod_{i=1}^{n-1} \alpha_i}(\theta(t), T_{n-1})}{R_1^{\alpha_{n-1}}(\sigma(t), T_1)}, \tag{3.37}$$

and for any given function $\phi(t) > -1/p_{n-1}(t) R_1^{\alpha_{n-1}}(t, T_1)$ such that $p_{n-1}(t)\phi(t)$ is a Δ-differentiable function and a positive Δ-differentiable function $\delta(t)$, and we assume

$$\begin{aligned}\widetilde{\Psi}(t, T_1, T_{n-1}) =&\delta(t)\Bigg[Mq(t)\widetilde{\eta}(t, T_1, T_{n-1}) + \frac{\beta(t, T_1)}{p_{n-1}^{\frac{1}{\alpha_{n-1}}}(t)} (p_{n-1}\phi)^{1+\frac{1}{\alpha_{n-1}}}(\sigma(t)) \\ &- [p_{n-1}(t)\phi(t)]^{\Delta}\Bigg].\end{aligned}$$

Now, we state and prove the parallel oscillation theorems in this subsection.

Theorem 3.4. *Let $\beta = \prod_{i=1}^{n-1} \alpha_i$ and $\theta^{\Delta}(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. Assume that either* (3.1) *or* (3.2) *is satisfied. Furthermore, assume that there exist a function $\phi(t)$ satisfying $\phi(t) = 0$ for $0 < \alpha_{n-1} < 1$ and a positive Δ-differentiable function $\delta(t)$ such that for a sufficiently large $T^* \in [T_{n-1}, \infty)_{\mathbb{T}}$,*

$$\begin{aligned}&\limsup_{t\to\infty} \int_{T^*}^{t} \left[\widetilde{\Psi}(s, T_1, T_{n-1}) - \frac{\alpha_{n-1}^{\alpha_{n-1}}}{(1+\alpha_{n-1})^{1+\alpha_{n-1}}} \frac{([C(s, T_1)]_+)^{1+\alpha_{n-1}}}{[\gamma(s, T_1)]^{\alpha_{n-1}}}\right] \Delta s \\ &> A(T^*, T_1),\end{aligned} \tag{3.38}$$

where $A(t, T_1) = \delta(t)\,[1/R_1(t, T_1)^{\alpha_{n-1}} + p_{n-1}(t)\phi(t)]$. Then,

(i) every solution $y(t)$ of Eq. (1.1) *is oscillatory when $n \in 2\mathbb{N}$;*

(ii) every solution $y(t)$ of Eq. (1.1) *is either oscillatory or $\lim_{t\to\infty} y(t) = 0$ when $n \in 2\mathbb{N}+1$.*

Proof. Assume by way of contradiction that Eq. (1.1) has a nonoscillatory solution $y(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, there is a sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ such that for $t \in [T, \infty)_{\mathbb{T}}$, $y(t) > 0$, $y(\tau(t)) > 0$, $y(\theta(t)) > 0$, and Lemmas 3.1-3.3 hold.

When $n \in 2\mathbb{N}$, by Lemma 3.1, (3.3) holds. Define the function $w(t)$ by (3.13). In view of (3.17) and

$$\left[\frac{L_0 y(\theta(t))}{L_0 y(\sigma(t))}\right]^{\prod_{i=1}^{n-1} \alpha_i} \ge \left[\frac{R_{n-1}(\theta(t), T_{n-1})}{R_{n-1}(\sigma(t), T_{n-1})}\right]^{\prod_{i=1}^{n-1} \alpha_i}, \tag{3.39}$$

we find

$$\Lambda_1 \le -Mq(t)\delta(t)[1 - r(\theta(t))]^{\prod_{i=1}^{n-1} \alpha_i} \frac{R_{n-1}^{\prod_{i=1}^{n-1} \alpha_i}(\theta(t), T_{n-1})}{R_1^{\alpha_{n-1}}(\sigma(t), T_1)}. \tag{3.40}$$

Substituting (3.19) and (3.40) into (3.14) and using the definition of $\widetilde{\Psi}(t, T_1, T_{n-1})$, $C(t, T_1)$ and $\gamma(t, T_1)$, we infer that

$$w^{\Delta}(t) \leq -\widetilde{\Psi}(t, T_1, T_{n-1}) + C(t, T_1)w(\sigma(t)) - \gamma(t, T_1)w^{1+\frac{1}{\alpha_{n-1}}}(\sigma(t)). \tag{3.41}$$

For $0 < \alpha_{n-1} < 1$ and $\phi(t) = 0$, obviously, (3.41) also holds. Proceeding as in the proof of Theorem 3.1, we have

$$\begin{aligned}&\int_{T^*}^{t} \left[\widetilde{\Psi}(s, T_1, T_{n-1}) - \frac{\alpha_{n-1}^{\alpha_{n-1}}}{(1+\alpha_{n-1})^{1+\alpha_{n-1}}} \frac{([C(s, T_1)]_+)^{1+\alpha_{n-1}}}{[\gamma(s, T_1)]^{\alpha_{n-1}}}\right] \Delta s \\ \leq & w(T^*) - w(t) \leq \delta(T^*) \left[\frac{1}{(R_1(T^*, T_1))^{\alpha_{n-1}}} + p_{n-1}(T^*)\phi(T^*)\right],\end{aligned}$$

which contradicts the assumption (3.38). Therefore, every solution $y(t)$ of (1.1) is oscillatory.

When $n \in 2\mathbb{N}+1$, we deduce from Lemma 3.1 that (3.3) holds or $\lim_{t\to\infty} y(t) = 0$. If (3.3) holds, then as shown in the proof of the case when n is even, we see that Eq. (1.1) is oscillatory. This ends the proof of this theorem. □

Theorem 3.5. *Let $\beta = \prod_{i=1}^{n-1} \alpha_i$ and $\theta^{\Delta}(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. Assume that either* (3.1) *or* (3.2) *is satisfied. Furthermore, assume that for each $T \in [t_0, \infty)_{\mathbb{T}}$, there exists a function $\Xi \in \mathcal{X}$ such that*

$$\limsup_{t\to\infty} \mathcal{B}\left[\widetilde{\Psi}(s, T_1, T_{n-1}) - \frac{\alpha_{n-1}^{\alpha_{n-1}}([2\xi(s) + \mu(s)\xi^2(s) + C(s, T_1)]_+)^{1+\alpha_{n-1}}}{(1+\alpha_{n-1})^{\alpha_{n-1}}\gamma^{\alpha_{n-1}}(s, T_1)}; T, t\right] > 0, \tag{3.42}$$

where the operator $\mathcal{B}$ and the function ξ are defined as in (3.23) *and* (3.24)*. Then,*

- *(i) every solution $y(t)$ of Eq.* (1.1) *is oscillatory when $n \in 2\mathbb{N}$;*
- *(ii) every solution $y(t)$ of Eq.* (1.1) *is either oscillatory or $\lim_{t\to\infty} y(t) = 0$ when $n \in 2\mathbb{N}+1$.*

Proof. Based on (3.41), the proof is similar to Theorem 3.2 and hence is omitted. □

Corollary 3.3. *Let $\beta = \prod_{i=1}^{n-1} \alpha_i$ and $\theta^{\Delta}(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that either* (3.1) *or* (3.2) *is satisfied. Furthermore, assume that for each $T \in [t_0, \infty)_{\mathbb{T}}$, there exists a function $\varphi \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ such that*

$$\begin{aligned}\limsup_{t\to\infty} \int_{T}^{t} \varphi^2(s)(t-s)^2(s-T)^2 &\left[\widetilde{\Psi}(s, T_1, T_{n-1})\right. \\ &\left.- \frac{\alpha_{n-1}^{\alpha_{n-1}}([2\xi(s) + \mu(s)\xi^2(s) + C(s, T_1)]_+)^{1+\alpha_{n-1}}}{(1+\alpha_{n-1})^{\alpha_{n-1}}\gamma^{\alpha_{n-1}}(s, T_1)}\right] \Delta s > 0,\end{aligned}$$

where the function ξ is given by (3.27)*. Then,*

- *(i) every solution $y(t)$ of Eq.* (1.1) *is oscillatory when $n \in 2\mathbb{N}$;*
- *(ii) every solution $y(t)$ of Eq.* (1.1) *is either oscillatory or $\lim_{t\to\infty} y(t) = 0$ when $n \in 2\mathbb{N}+1$.*

Theorem 3.6. *Let $\theta^{\Delta}(t) > 0$ and $\beta = \Pi_{i=1}^{n-1}\alpha_i$.*

(i) *Suppose that $n \in 2\mathbb{N}$ and for each odd integer $l \in \{1, 3, \cdots, n\}$ and a sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, there holds*

$$\limsup_{t\to\infty} M^{\Pi_{i=l+1}^{n-1}\frac{1}{\alpha_i}} H_l^{\prod_{i=1}^{l}\alpha_i}(\theta(t), T)\int_t^\infty \chi_{n-l-1}(s)\Delta s > 1, \tag{3.43}$$

where $\chi_i(t)(i = 0, \cdots, n-l-1)$ and $H_i(t, T)(i = 1, \cdots, l)$ are defined as in (3.29) *and* (3.30)*. Then, every solution $y(t)$ of Eq.* (1.1) *is oscillatory;*

(ii) *Suppose that $n \in 2\mathbb{N}+1$,* (3.31) *and* (3.43) *hold for each even integer $l \in \{2, 4, \cdots, n\}$, where $\rho_i(t)(i = 1, \cdots, n)$ are defined as in Lemma 3.1. Then, every solution $y(t)$ of Eq.* (1.1) *is either oscillatory or $\lim_{t\to\infty} y(t) = 0$.*

Proof. Suppose that Eq.(1.1) has a nonoscillatory solution $y(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by $(H_1) - (H_4)$, there is a $T \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\theta(t)) > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. Moreover, by Theorem 2.1, we have for $t \in [T, \infty)_{\mathbb{T}}$, $L_j y(t) > 0$, $j = 1, 2, \cdots, l$, and $(-1)^{n+j} y(t) L_j y(t) < 0$ for $j = l+1, l+2, \cdots, n$.

When $n \in 2\mathbb{N}$, l must be an odd integer and $L_1 y(t) = [L_0 y(t)]^\Delta > 0$ for $t \in [T, \infty)_{\mathbb{T}}$. Thus, (3.32) holds. Then, it follows from $\theta^\Delta(t) > 0$ that

$$p_{n-1}(t) L_{n-1} y(t)^{\alpha_{n-1}} \geq M L_0 y(\theta(t))^{\Pi_{i=1}^{n-1}\alpha_i} \int_t^\infty q(s)[(1 - r(\theta(s)))]^{\Pi_{i=1}^{n-1}\alpha_i} \Delta s.$$

Proceeding as in the proof of (3.34), we have

$$p_l(\theta(t)) L_l y(\theta(t))^{\alpha_l} \geq M^{\Pi_{i=l+1}^{n-1}\frac{1}{\alpha_i}} L_0 y(\theta(t))^{\Pi_{i=1}^{l}\alpha_i} \int_t^\infty \chi_{n-l-1}(s)\Delta s. \tag{3.44}$$

On the other hand, (3.35) leads to

$$L_0 y(\theta(t)) \geq [p_l(\theta(t)) L_l y(\theta(t))^{\alpha_l}]^{\Pi_{i=1}^{l}\frac{1}{\alpha_i}} H_l(\theta(t), T), \quad t \in [T, \infty)_{\mathbb{T}}. \tag{3.45}$$

Substituting (3.45) into (3.44) gives

$$M^{\Pi_{i=l+1}^{n-1}\frac{1}{\alpha_i}} H_l^{\prod_{i=1}^{l}\alpha_i}(\theta(t), T)\int_t^\infty \chi_{n-l-1}(s)\Delta s \leq 1.$$

This contradicts (3.43). Therefore, every solution $y(t)$ of (1.1) is oscillatory.

When $n \in 2\mathbb{N}+1$, proceeding as in the proof of Theorem 3.3, we can conclude that $\lim_{t\to\infty} y(t) = 0$ if $l = 0$. Similar to the proof of the case when $n \in 2\mathbb{N}$, we can infer that every solution $y(t)$ of (1.1) is oscillatory. Hence, the proof of this theorem is completed. □

Corollary 3.4. *Let $\beta = \prod_{i=1}^{n-1}\alpha_i$ and $\theta^\Delta(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that $(H_1) - (H_4)$ hold and either* (3.1) *or* (3.2) *is satisfied. If*

$$\limsup_{t\to\infty} M R_{n-1}^{\prod_{i=1}^{n-1}\alpha_i}(\theta(t), T_{n-1}) \int_t^\infty q(s)[1 - r(\theta(s))]^{\prod_{i=1}^{n-1}\alpha_i}\Delta s > 1, \tag{3.46}$$

for a sufficiently large $T_{n-1} \in [T_0, \infty)_{\mathbb{T}}$, then,

(i) *every solution $y(t)$ of Eq.* (1.1) *is oscillatory when $n \in 2\mathbb{N}$;*

(ii) *every solution $y(t)$ of Eq.* (1.1) *is either oscillatory or $\lim_{t\to\infty} y(t) = 0$ when $n \in 2\mathbb{N}+1$.*

4. Examples

Example 4.1. Let h, M be positive constants, $\alpha_{n-1} \geq 1$, and the time scale $\mathbb{T} = h\mathbb{Z}$. Consider the dynamic equation

$$L_n y(t) + q(t) f(|y(\theta(t))|^{\beta} sgn(y(\theta(t)))) = 0, \quad t \in [2h, \infty)_{\mathbb{T}}, \tag{4.1}$$

where $p_{n-1}(t) = 1$, $p_i(t) = t, \alpha_i = 1 (1 \leq i \leq n-2)$, $\tau(t) = \theta(t) = t - h$ and $q(t) = \frac{1}{t\sigma(t)}$.

It is clear that conditions (H_2) and (3.2) hold, since

$$\int_{t_0}^{\infty} \left(\frac{1}{p_{n-1}(s)}\right)^{\frac{1}{\alpha_{n-1}}} \Delta s = \int_{t_0}^{\infty} \Delta s = \infty,$$
$$\int_{t_0}^{\infty} \left(\frac{1}{p_i(s)}\right)^{\frac{1}{\alpha_i}} \Delta s = \int_{t_0}^{\infty} \frac{1}{s} \Delta s = \infty, \quad 1 \leq i \leq n-2$$

from [4, Theorem 1.75] and

$$\int_{t_0}^{\infty} \rho_2(t) \Delta t = \int_{t_0}^{\infty} \frac{1}{s} \left\{ \int_{s}^{\infty} \left[\int_{u}^{\infty} \frac{1}{v\sigma(v)} \Delta v \right]^{\frac{1}{\alpha_{n-1}}} \Delta u \right\} \Delta s$$
$$\geq \int_{t_0}^{t^*} \frac{1}{s} \Delta s \int_{t^*}^{\infty} \frac{1}{u^{1/\alpha_{n-1}}} \Delta u = \infty$$

for some constant $t^* \in [t_0, \infty)_{\mathbb{T}}$. Note that

$$\frac{t}{2} < R_1(t, T_1) = \int_{T_1}^{t} \left[\frac{R_0(s, T_0)}{p_{n-1}(s)}\right]^{\frac{1}{\alpha_{n-1}}} \Delta s = t - T_1 < t$$

for $t > T_2 := 2T_1 = 2^2 T_0$, then

$$\frac{t}{2^2} < \frac{1}{2} \int_{T_2}^{t} \Delta s < R_2(t, T_2) = \int_{T_2}^{t} \left[\frac{R_1(s, T_0)}{p_{n-2}(s)}\right]^{\frac{1}{\alpha_{n-2}}} \Delta s$$
$$= \int_{T_2}^{t} \frac{R_1(s, T_1)}{s} \Delta s < \int_{T_2}^{t} \Delta s < t$$

for $t > T_3 := 2^3 T_0$. By using the induction method, we can derive that

$$\frac{t}{2^{n-1}} < R_{n-1}(t, T_{n-1}) < t, \quad \text{for} \quad t > T_n := 2^n T_0.$$

Picking up $\delta(t) = 1 + \frac{1}{t}$ and $\phi(t) = 0$ in Theorem 3.4, then we have $\delta^{\Delta}(t) = -\frac{1}{t\sigma(t)} < 0$ and $[C(t, T_1)]_+ = \max\{-\frac{t+h}{(t+h+1)t\sigma(t)}, 0\} = 0$. Furthermore, pick $r(t) = \frac{1}{2} - \frac{1}{t}$ and choose $T^* = \max\{T_n + h, T + 3h\}$, then for $t > T^*$,

$$\frac{1}{2^{n\alpha_{n-1}+1}} \leq \frac{1}{2^{n\alpha_{n-1}}} \frac{(t-h)^{\alpha_{n-1}}}{(t+h)^{\alpha_{n-1}}} \leq \widetilde{\eta}(t, T_1, T_{n-1})$$
$$\leq \left(\frac{1}{2} + \frac{1}{h}\right)^{\alpha_{n-1}} \frac{[2(t-h)]^{\alpha_{n-1}}}{(t+h)^{\alpha_{n-1}}} < \left(1 + \frac{2}{h}\right)^{\alpha_{n-1}},$$

$$\frac{M}{2^{n\alpha_{n-1}+1}}\frac{1}{t\sigma(t)} \le \widetilde{\Psi}(t,T_1,T_{n-1}) = \left(1+\frac{1}{t}\right)M\frac{1}{t\sigma(t)}\widetilde{\eta}(t,T_1,T_{n-1})$$
$$\le 2M\left(1+\frac{2}{h}\right)^{\alpha_{n-1}}\frac{1}{t\sigma(t)}.$$

Consequently, for $M > 2^{(n+1)\alpha_{n-1}+2}/(T^*)^{\alpha_{n-1}-1}$, we conclude that

$$\begin{aligned}
\infty >&2\left(1+\frac{2}{h}\right)^{\alpha_{n-1}}\frac{M}{T^*}\\
>&2M\left(1+\frac{2}{h}\right)^{\alpha_{n-1}}\limsup_{t\to\infty}\int_{T^*}^{t}\frac{1}{s\sigma(s)}\Delta s\\
>&\limsup_{t\to\infty}\int_{T^*}^{t}\left[\widetilde{\Psi}(s,T_1,T_{n-1})-\frac{\alpha_{n-1}^{\alpha_{n-1}}}{(1+\alpha_{n-1})^{1+\alpha_{n-1}}}\frac{([C(s,T_1)]_+)^{1+\alpha_{n-1}}}{[\gamma(s,T_1)]^{\alpha_{n-1}}}\right]\Delta s\\
>&\frac{M}{2^{n\alpha_{n-1}+1}}\limsup_{t\to\infty}\int_{T^*}^{t}\frac{1}{s\sigma(s)}\Delta s\\
=&\frac{M}{2^{n\alpha_{n-1}+1}T^*} > \frac{2^{\alpha_{n-1}+1}}{(T^*)^{\alpha_{n-1}}} > \delta(T^*)\left[\frac{1}{R_1^{\alpha_{n-1}}(T^*,T_1)}+p_{n-1}(T^*)\phi(T^*)\right].
\end{aligned}$$

By Theorem 3.4, every solution $y(t)$ of Eq. (4.1) is oscillatory when $n \in 2\mathbb{N}$, and every solution $y(t)$ of Eq. (4.1) is either oscillatory or $\lim_{t\to\infty} y(t) = 0$ when $n \in 2\mathbb{N}+1$.

Example 4.2. Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $M > 0$, $q > 1$, $\alpha_i (1 \le i \le n-1)$ be constants, $\alpha_{n-1} \ge 2$, and $M(q-1)q^{(n-2)\alpha_{n-1}/2} > q^{\alpha_{n-1}/2} - 1$. Consider the delay q-difference equation

$$L_n y(t) + q(t) f(|y(\theta(t))|^{\beta} sgn(y(\theta(t)))) = 0, \quad t \in \overline{q^{\mathbb{Z}}}, \quad t \ge q, \tag{4.2}$$

where $\alpha_i = 1 (1 \le i \le n-2)$, $\tau(t) \le t$ and $\theta(t) = qt$. Here,

$$q(t) = t^{-1-\frac{\alpha_{n-1}}{2}}, \quad p_{n-1}(t) = t^{\frac{\alpha_{n-1}}{2}} \quad \text{and} \quad p_i(t) = t, \ i = 1, 2, \cdots, n-2.$$

The conditions (H_2) and (3.2) hold, since

$$\int_{t_0}^{\infty}\left(\frac{1}{p_{n-1}(s)}\right)^{\frac{1}{\alpha_{n-1}}}\Delta s = \int_{t_0}^{\infty}\frac{1}{s^{\frac{1}{2}}}\Delta s = \infty,$$
$$\int_{t_0}^{\infty}\left(\frac{1}{p_i(s)}\right)^{\frac{1}{\alpha_i}}\Delta s = \int_{t_0}^{\infty}\frac{1}{s}\Delta s = \infty, \quad 1 \le i \le n-2$$

from [4, Theorem 1.75] and

$$\int_{t_0}^{\infty}\rho_2(t)\Delta t = \int_{t_0}^{\infty}\frac{1}{s}\left\{\int_{s}^{\infty}\left[\frac{1}{u^{\frac{\alpha_{n-1}}{2}}}\int_{u}^{\infty}\frac{1}{v^{1+\frac{\alpha_{n-1}}{2}}}\Delta v\right]^{\frac{1}{\alpha_{n-1}}}\Delta u\right\}\Delta s$$
$$\ge \left[\frac{q-1}{q^{\frac{\alpha_{n-1}}{2}}-1}\right]^{\frac{1}{\alpha_{n-1}}}\int_{t_0}^{t^*}\frac{1}{s}\Delta s\int_{t^*}^{\infty}\frac{1}{u}\Delta u = \infty$$

for some constant $t^* \in [t_0, \infty)_{\mathbb{T}}$.

Note that

$$\sqrt{q}\sqrt{t} < R_1(t,T_1) = \int_{T_1}^{t} \left[\frac{R_0(s,T_0)}{p_{n-1}(s)}\right]^{\frac{1}{\alpha_{n-1}}} \Delta s$$
$$= \int_{T_1}^{t} \frac{1}{s^{\frac{1}{2}}} \Delta s = (\sqrt{q}+1)(\sqrt{t}-\sqrt{T_1}) < (\sqrt{q}+1)\sqrt{t},$$

for $t > T_2 := (\sqrt{q}+1)^2 T_1$, then

$$q\sqrt{t} < \sqrt{q}\int_{T_2}^{t} \frac{1}{s^{\frac{1}{2}}}\Delta s < R_2(t,T_2) = \int_{T_2}^{t} \frac{R_1(s,T_1)}{s}\Delta s$$
$$< (\sqrt{q}+1)\int_{T_2}^{t} \frac{1}{s^{\frac{1}{2}}}\Delta s < (\sqrt{q}+1)^2\sqrt{t}$$

for $t > T_3 := (\sqrt{q}+1)^4 T_1$. By using the induction method, we can derive that

$$(\sqrt{q})^{n-1}\sqrt{t} < R_{n-1}(t,T_{n-1}) < (\sqrt{q}+1)^{n-1}\sqrt{t},$$
$$\text{for} \quad t > T_n := (\sqrt{q}+1)^{2(n-1)}T_1 = (\sqrt{q}+1)^{2n}T_1.$$

Pick up $r(t) = 1 - \frac{1}{q} - \frac{1}{t}$. Thus, for $M > (q^{\alpha_{n-1}/2}-1)/(q-1)q^{(n-2)\alpha_{n-1}/2}$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} M R_{n-1}^{\prod_{i=1}^{n-1}\alpha_i}(t,T_{n-1}) \int_t^\infty q(s)[1-r(\theta(s))]^{\prod_{i=1}^{n-1}\alpha_i}\Delta s \\
\geq& \frac{Mq^{\frac{(n-1)\alpha_{n-1}}{2}}}{2^{\alpha_{n-1}}} \limsup_{t\to\infty} t^{\frac{\alpha_{n-1}}{2}} \int_t^\infty \frac{1}{s^{1+\frac{\alpha_{n-1}}{2}}}\Delta s \\
\geq& \frac{M(q-1)q^{\frac{(n-2)\alpha_{n-1}}{2}}}{q^{\frac{\alpha_{n-1}}{2}}-1} \\
>& 1.
\end{aligned}$$

By Corollary 3.2, every solution $y(t)$ of Eq. (4.2) is oscillatory when $n \in 2\mathbb{N}$, and every solution $y(t)$ of Eq. (4.2) is either oscillatory or $\lim_{t\to\infty} y(t) = 0$ when $n \in 2\mathbb{N}+1$.

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