Journal of Applied Analysis and Computation
Volume 13, Number 2, April 2023, 758–781
Website:http://www.jaac-online.com
DOI:10.11948/20220110

BÄCKLUND TRANSFORMATION TO SOLVE THE GENERALIZED (3+1)-DIMENSIONAL KP-YTSF EQUATION AND KINKY PERIODIC-WAVE, WRONSKIAN AND GRAMMIAN SOLUTIONS

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Abstract The Kadomtsev-Petviashvili equation is considered to be a basic model describing nonlinear dispersive wave in fluids, which is an integrable equation with two spatial dimensions. The Yu-Toda-Sasa- Fukuyama equation plays a crucial role in fluid dynamics, plasma physics and weakly dispersive media. In this paper, we investigate a generalized (3+1)-dimensional Kadomtsev-Petviashvili-Yu-Toda-Sasa-Fukuyama equation, and multiple types of solutions are derived. With symbolic computation, a class of kinky periodic-wave solutions, determinant solutions and the bilinear Bäcklund transformation are constructed. We obtain two types of determinant solutions, that is, Wronskian and Grammian solutions. By choosing the appropriate matrix elements of determinants, many kinds of solutions are derived. In addition to the soliton solutions, the complexiton solutions and rational solutions are given. As illustrative examples, a few particular solutions are computed and plotted.

Keywords Kinky periodic-wave solutions, Bäcklund transformation, Wronskian solutions, Grammian solutions.

MSC(2010) 35A25, 37K10.

1. Introduction

The research of nonlinear science has attracted more and more attention [13, 16, 29, 30, 37]. Soliton theory is an important branch of nonlinear science. Solitons are considered to be an essential part of observing and understanding complex nonlinear systems [5, 10, 17, 19]. Soliton phenomena appear in many fields, such as shock wave at sea [5], biological system [17], plasma physics [10], light propagation in optical fiber [1, 19], laser propagation [34], hydrodynamics [41]. These nonlinear phenomena are often described with nonlinear evolution equations (NLEEs) [1, 4, 5, 10, 17, 19, 34, 41].

Many mathematicians and physicists have conducted a significant amount of research on solving the NLEEs [1, 4, 5, 10, 17, 19, 34, 41]. However, a large number of equations of practical value can not be solved accurately, nor a unified solution be

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given. NLEEs no longer satisfy the linear superposition principle, which makes the research of the exact solution more complex and difficult.

The emergence of soliton theory promotes the development of solving NLEEs, which leads to new solutions constantly [11,12,18,23,35,39,42,43]. For example, Hirota method [11,12], inverse scattering method [18,35], Bell polynomial method [39], Darboux transformation [23,42], and Bäcklund transformation (BT) [24,36,43] solve a series of significant equations in physics. In addition, Wronskian and Grammian techniques [2,38,44] are the effective approaches to construct N-soliton solutions to NLEEs in Hirota bilinear form. The basic idea is to select the appropriate elements and set up the determinant form of the solution, and then substituted into the bilinear equation to deduce the restrictive conditions [2, 6, 20, 38, 44]. Finally, the proof can be given by using the properties of determinant [12], Plücker relation [2,38] and Jacobi identity [6, 20, 44]. Through simplifying the soliton equation into a simple Maya chart [14], the properties of the solution are shown directly.

Recently, the N-soliton solutions are systematically studied for nonlocal integrable equations. A kind of Riemann-Hilbert problems are used to express the matrix spectral problems, and the soliton solutions to some physical equations are obtained [25–27]. The main idea is to apply Sokhotski-Plemelj formula to transform the relevant Riemann-Hilbert problems into Gelfand-Levitan-Marchenko type integral equations, and then solve the equations explicitly. This new formula provides a new way to construct soliton solutions to nonlocal real reverse-space time integra equations. Moreover, the relationship between Riemann-Hilbert problem and nonlocal equation is deeply studied [28, 31, 32]. The nonlocal integrable modified Korteweg-de Vries (mKdV) equations are reduced by two groups of reduction of the AKNS matrix spectral problems. One is local reduction, and the other is nonlocal reduction. According to the distribution of eigenvalues, the soliton solutions are constructed from the non reflectionless Riemann-Hilbert problems.

The Kadomtsev-Petviashvili I (KPI) equation [21] is given by

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0. \tag{1.1}$$

In Ref. [45], lump solutions to the KPI equation with a self-consistent source are constructed. A class of lump solutions to the KPI equation is obtained [33]. In addition, the parameters of the lump solutions [33] are specially selected to include the previous lump solutions obtained from the long wave limits of the soliton solution. The Wronskian and Grammian determinant solutions have been given to a variable-coefficient forced KP equation with inhomogeneous nonlinearity, dispersion, perturbed term and external force [44]. The N-soliton solutions to KPI can be expressed in Wronskian and Grammian determinant [12]. The bilinear form of KPI equation can be reduced to determinant identity [12]. Nevertheless, not all bilinear equations have the Wronskian and Grammian solutions, for example, B-type Kadomtsev-Petviashvili (BKP) equation only has the Pfaffian type solutions [12]. Through the Cole-Hopf transformation

$$u = 2(\ln f)_x, \tag{1.2}$$

the Hirota bilinear form of Eq. (1.1) is written as

$$\begin{aligned}&(D_xD_t + D_x^4 + D_y^2)f\cdot f\\ =&2(f_{xt}f - f_xf_t + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 + f_{yy}f - f_y^2) = 0,\end{aligned} \tag{1.3}$$

where the binary operator D is defined by [21]

$$\begin{aligned}&D_x^{\alpha}D_y^{\beta}D_t^{\gamma}(f\cdot g)\\=&\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\Big)^{\alpha}\Big(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\Big)^{\beta}\Big(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\Big)^{\gamma}f(x,y,t)g(x',y',t')\Big|_{x'=x,y'=y,t'=t}.\end{aligned}$$

The (3+1)-dimensional Yu-Toda-Sasa-Fukuyama (YTSF) equation [15, 46] is studied

$$[\Phi(w)w_z-4w_t]_x+3w_{yy}=0,\quad \Phi(w)=\partial_x^2+4w+2w_x\partial_x^{-1},\tag{1.4}$$

where w is a real funtion of the scaled space coordinates x,y,z and the scaled time coordinate t, and $\partial_x^{-1}=\int dx$. The (3+1)-dimensional potential YTSF equation plays a crucial role in the interfacial waves in a two-layer liquid or the elastic quasi-plane waves in a lattice [3]. Two types of interaction solutions to the potential YTSF equation have been shown by using Hirota bilinear method [3]. In Ref. [40], a new solution to the potential YTSF equation has been constructed by means of the sub-equation method based on variable coefficient Korteweg-de Vries (KdV) equation. The symmetry of the potential YTSF equation has been obtained by using the direct reduction method, and the corresponding reduction equation has been obtained in Ref. [47]. Further, the (3+1)-dimensional potential YTSF equation with $w=u_x$ have been introduced [9]

$$-4u_{xt}+u_{xxxz}+4u_xu_{xz}+2u_{xx}u_z+3u_{yy}=0.\tag{1.5}$$

Eq. (1.5) can be transformed into [8]

$$\begin{aligned}&(-4D_xD_t+D_x^3D_z+3D_y^2)f\cdot f\\=&-4(f_{xt}f-f_xf_t)+(f_{xxxz}f-3f_{xxz}f_x+3f_{xz}^2f_{xx}-f_{xxx}f_z)+3(f_{yy}f-f_y^2)=0,\end{aligned}\tag{1.6}$$

with the transformation $u=2(\ln f)_x$.

In this paper, we will investigate the following generalized (3+1)-dimensional Kadomtsev-Petviashvili-Yu-Toda-Sasa-Fukuyama (KP-YTSF) equation

$$u_{xt}+c_1u_{xxxz}+3c_1(u_xu_z)_x+c_2u_{xxxx}+6c_2u_xu_{xx}+c_3u_{yy}=0,\tag{1.7}$$

with $c_i\neq 0\,(i=1,2,3)$ are arbitrary real parameters. Under the dependent variable transformation

$$u=2(\ln f)_x,\tag{1.8}$$

the bilinear representation of Eq. (1.7) can be obtained

$$\begin{aligned}&(D_xD_t+c_1D_x^3D_z+c_2D_x^4+c_3D_y^2)f\cdot f\\=&(f_{xt}+c_1f_{xxxz}+c_2f_{xxxx}+c_3f_{yy})f-f_xf_t-3c_1f_{xxz}f_x\\&+3c_1f_{xz}f_{xx}-c_1f_{xxx}f_z-4c_2f_{xxx}f_x+3c_2f_{xx}^2-c_3f_y^2=0,\end{aligned}\tag{1.9}$$

which is a combination version of the bilinear KP Eq. (1.3) and the bilinear potential YTSF Eq. (1.6).

In this paper, we will study various types of solutions to Eq. (1.7) with different techniques, such as rational solutions, soliton solutions, kinky periodic-wave solutions and complexiton solutions. In Sec.2, we will investigate the interaction between soliton solutions and periodic solutions. In Sec.3, based on the bilinear form of Eq. (1.7), the Wronskian determinant solutions will be given and the soliton solutions and complexiton solutions will be derived. In Sec.4, we will investigate the Grammian solutions in a similar measure. In Sec.5, the bilinear BT will be constructed and a class of rational solutions will be derived from a particular solution. Finally, the conclusions will be summarized in Sec.6.

2. Kinky periodic-wave solutions

In this section, we study the interaction solutions between soliton and periodic waves to the generalized (3+1)-dimensional KP-YTSF Eq. (1.7). We assume the solution f to Eq. (1.9) enjoys the form

$$f = e^{-\xi_1} + \delta_1 \sin(\xi_2) + \delta_2 e^{\xi_1}, \tag{2.1}$$

where $\xi_i = a_i x + b_i y + m_i z + d_i t$ $(i = 1, 2)$, and $a_i, b_i, m_i, d_i, \delta_i$ are all real parameters to be determined. With symbolic computation, we substitute Eq. (2.1) into Eq. (1.9), a set of algebraic equations about a_i, b_i, m_i, d_i and δ_i are obtained as

$$\begin{cases} \delta_1\delta_2(a_1^4c_2 + a_1^3c_1m_1 - 6a_1^2a_2^2c_2 - 3a_1^2a_2c_1m_2 - 3a_1a_2^2c_1m_1 \\ \quad + a_2^4c_2 + a_2^3c_1m_2 + b_1^2c_3 - b_2^2c_3 + a_1d_1 - a_2d_2) = 0, \\ \delta_1(a_1^4c_2 + a_1^3c_1m_1 - 6a_1^2a_2^2c_2 - 3a_1^2a_2c_1m_2 - 3a_1a_2^2c_1m_1 + a_2^4c_2 \\ \quad + a_2^3c_1m_2 + b_1^2c_3 - b_2^2c_3 + a_1d_1 - a_2d_2) = 0, \\ \delta_1\delta_2(4a_1^3a_2c_2 + a_1^3c_1m_2 + 3a_1^2a_2c_1m_1 - 4a_1a_2^3c_2 - 3a_1a_2^2c_1m_2 \\ \quad - a_2^3c_1m_1 + 2b_1b_2c_3 + a_1d_2 + a_2d_1) = 0, \\ -\delta_1(4a_1^3a_2c_2 + a_1^3c_1m_2 + 3a_1^2a_2c_1m_1 - 4a_1a_2^3c_2 - 3a_1a_2^2c_1m_2 \\ \quad - a_2^3c_1m_1 + 2b_1b_2c_3 + a_1d_2 + a_2d_1) = 0, \\ 4a_2^4c_2\delta_1^2 + 4a_2^3c_1\delta_1^2m_2 + 16a_1^4c_2\delta_2 + 16a_1^3c_1\delta_2m_1 - b_2^2c_3\delta_1^2 \\ \quad - a_2d_2\delta_1^2 + 4b_1^2c_3\delta_2 + 4a_1d_1\delta_2 = 0. \end{cases} \tag{2.2}$$

By solving the algebraic equations, the relations among these parameters a_i, b_i, m_i, d_i and δ_i can be derived, which are given in Appendix A. Taking the Case 3 in Appendix A as an example, we can obtain the solution f to Eq. (1.9)

$$\begin{aligned} f =& e^{-(b_1y+m_1z+\frac{a_2^3c_1m_1-2b_1b_2c_3}{a_2}t)} + \delta_1 \sin\left(a_2x + b_2y + \frac{-b_1^2c_3 - a_2^4c_2 + b_2^2c_3 + a_2d_2}{a_2^3c_1}t\right) \\ &+ \frac{4b_1^2\delta_1^2 - 3b_2^2\delta_1^2 - \frac{3}{c_3}a_2d_2\delta_1^2}{4b_1^2} e^{b_1y+m_1z+\frac{a_2^3c_1m_1-2b_1b_2c_3}{a_2}t} \end{aligned} \tag{2.3}$$

where $a_2 \neq 0, b_1 \neq 0, \delta_1 \neq 0$, and a_2, b_1, b_2, d_2, m_1 and δ_1 are free constants. The kinky periodic-wave solution to the Eq. (1.7) can be directly derived via transformation (1.8) as

$$u = \frac{2(-a_1e^{-\xi_1} + \delta_1\cos(\xi_2) + \delta_2a_1e^{\xi_1})}{e^{-\xi_1} + \delta_1\sin(\xi_2) + \delta_2e^{\xi_1}}. \tag{2.4}$$

If $\delta_2 > 0$, then

$$u = \frac{2(2a_1\sqrt{\delta_2}\sinh(\xi_1 - \theta) + a_2\delta_1\cos(\xi_2)}{2\sqrt{\delta_2}\cosh(\xi_1 - \theta) + \delta_1\sin(\xi_2)} = \frac{2a_2\delta_1\cos(\xi_2)}{2\sqrt{\delta_2}\cosh(\xi_1 - \theta) + \delta_1\sin(\xi_2)},$$

with

$$\theta = \frac{1}{2}\ln(\delta_2).$$

If $\delta_2 < 0$, then

$$u = \frac{2(2a_1\sqrt{-\delta_2}\cosh(\xi_1-\theta) + a_2\delta_1\cos(\xi_2)}{2\sqrt{-\delta_2}\sinh(\xi_1-\theta)+\delta_1\sin(\xi_2)} = \frac{2a_2\delta_1\cos(\xi_2)}{2\sqrt{-\delta_2}\sinh(\xi_1-\theta)+\delta_1\sin(\xi_2)},$$

with

$$\theta = \frac{1}{2}\ln(-\delta_2).$$

By selecting appropriate parameters, the three-dimensional plot and contour plot of the kinky periodic-wave solution are plotted in Fig. 1.

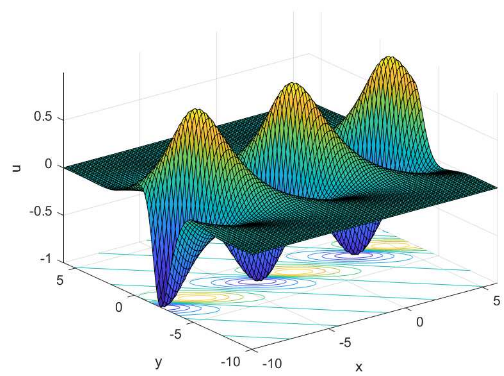


Figure 1. The kinky periodic-wave solution via Eq. (2.4) with $c_1 = -\frac{1}{4}, c_2 = 0, c_3 = -\frac{3}{4}$, $a_2 = b_1 = b_2 = d_2 = 1$, $\delta_1 = 2$, $t = 1$ and $z = 1$.

We get a class of kinky periodic-wave solutions which are different from those in Refs. [7, 22]. The form of the solution f to Eq. (1.9) is different. The relationship between the parameters obtained in Ref. [7] is a special case of this paper.

3. N-soliton solutions, complexiton solutions and rational solutions in the Wronskian form

3.1. Wronskian solutions

We firstly construct the Wronskian determinant solutions to Eq. (1.9), as follows

$$W(\phi_1,\phi_2,\cdots,\phi_N) = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix} = |\widehat{N-1}|, \tag{3.1}$$

where $\phi_i^{(m)} (i = 1, 2, \cdots, N,\ m = 0, 1, \cdots, N-1)$ are defined by

$$\phi_i^{(m)} = \frac{\partial^m \phi_i}{\partial x^m}. \tag{3.2}$$

The function $\phi_i's$ satisfy the linear partial differential conditions:

$$\phi_{i,y} = \gamma\phi_{i,xx}, \quad \phi_{i,z} = \frac{c_3\gamma^2 - 3c_2}{3c_1}\phi_{i,x}, \quad \phi_{i,t} = -\frac{4c_3\gamma^2}{3}\phi_{i,xxx}, \tag{3.3}$$

where $\gamma \neq 0$ is an arbitrary parameter.

Theorem 3.1. *If $\phi_i's$ satisfy the linear differential conditions in Eqs. (3.3), then the Wronskian determinant $f_N = |\widehat{N-1}|$ defined by Eq. (3.1) is the solution to the bilinear Eq. (1.9), and $u = 2(\ln f_N)_x$ leads to the solution to Eq. (1.7).*

Proof. According to the properties of the determinant, the derivative of f_N with respect to the independent variables x, y, z, and t can be easily calculated as

$$\begin{aligned}
f_{N,x} &= |\widehat{N-2}, N|,\\
f_{N,xx} &= |\widehat{N-2}, N+1| + |\widehat{N-3}, N-1, N|,\\
f_{N,xxx} &= |\widehat{N-2}, N+2| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-4}, N-2, N-1, N|,\\
f_{N,xxxx} &= |\widehat{N-2}, N+3| + 3|\widehat{N-3}, N-1, N+2| + 2|\widehat{N-3}, N, N+1|\\
&\quad + 3|\widehat{N-4}, N-2, N-1, N+1| + |\widehat{N-5}, N-3, N-2, N-1, N|,\\
f_{N,y} &= \gamma(|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N|),\\
f_{N,yy} &= \gamma^2(|\widehat{N-2}, N+3| - |\widehat{N-3}, N-1, N+2| + 2|\widehat{N-3}, N, N+1|\\
&\quad - |\widehat{N-4}, N-2, N-1, N+1| + |\widehat{N-5}, N-3, N-2, N-1, N|),\\
f_{N,z} &= \frac{c_3\gamma^2 - 3c_2}{3c_1}|\widehat{N-2}, N|,\\
f_{N,xz} &= \frac{c_3\gamma^2 - 3c_2}{3c_1}(|\widehat{N-2}, N+1| + |\widehat{N-3}, N-1, N|),\\
f_{N,xxz} &= \frac{c_3\gamma^2 - 3c_2}{3c_1}(|\widehat{N-2}, N+2| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-4}, N-2, N-1, N|),\\
f_{N,xxxz} &= \frac{c_3\gamma^2 - 3c_2}{3c_1}(|\widehat{N-2}, N+3| + 3|\widehat{N-3}, N-1, N+2| + 2|\widehat{N-3}, N, N+1|\\
&\quad + 3|\widehat{N-4}, N-2, N-1, N+1| + |\widehat{N-5}, N-3, N-2, N-1, N|),\\
f_{N,t} &= -\frac{4c_3\gamma^2}{3}(|\widehat{N-2}, N+2| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-4}, N-2, N-1, N|),\\
f_{N,xt} &= -\frac{4c_3\gamma^2}{3}(|\widehat{N-2}, N+3| - |\widehat{N-3}, N, N+1| + |\widehat{N-5}, N-3, N-2, N-1, N|).
\end{aligned}$$

Substituting the derivatives of f_N into Eq. (1.9), we have

$$\begin{aligned}
&(f_{xt} + c_1 f_{xxxz} + c_2 f_{xxxx} + c_3 f_{yy})f = 4c_3\gamma^2|\widehat{N-3}, N, N+1||\widehat{N-1}|,\\
&- f_x f_t - 3c_1 f_{xxz} f_x + 3c_1 f_{xz} f_{xx} - c_1 f_z f_{xxx} - 4c_2 f_{xxx} f_x + 3c_2 f_{xx}^2 - c_3 f_y^2\\
&= 4c_3\gamma^2(-|\widehat{N-2}, N||\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+1||\widehat{N-3}, N-1, N|)
\end{aligned} \tag{3.4}$$

and further get

$$(D_x D_t + c_1 D_x^3 D_z + c_2 D_x^4 + c_3 D_y^2) f \cdot f$$

$$=4c_3\gamma^2(|\widehat{N-1}||\widehat{N-3},N,N+1|-|\widehat{N-2},N||\widehat{N-3},N-1,N+1|$$
$$+|\widehat{N-2},N+1||\widehat{N-3},N-1,N|)=0, \tag{3.5}$$

which is exactly the Plücker relation of determinant [12], and the identity can be represented by Maya chart in Fig. 2.

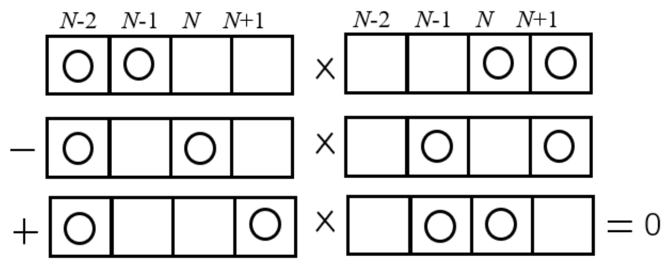


Figure 2. Maya chart of Eq. (3.5): the Plücker relation.

3.2. N-soliton solutions, complexiton solutions and rational solutions

Based on the linear partial differential system in Eqs. (3.3), we can obtain the N-soliton solutions to Eq. (1.7) in the form

$$u=2[\ln W(\phi_1,\phi_2,\cdots,\phi_N)]_x, \tag{3.6}$$

where $\phi_i=e^{\xi_i}+e^{\zeta_i}, \xi_i=l_ix+\gamma l_i^2y+\frac{c_3\gamma^2-3c_2}{3c_1}l_iz-\frac{4c_3\gamma^2}{3}l_i^3t+\xi_i^0, \zeta_i=k_ix+\gamma k_i^2y+\frac{c_3\gamma^2-3c_2}{3c_1}k_iz-\frac{4c_3\gamma^2}{3}k_i^3t+\zeta_i^0$, and $l_i,k_i,\xi_i^0,\zeta_i^0$ $(i=1,2,\cdots,N)$ are free parameters.

Taking $N=1$ in Eq. (3.6), we can derive the one-soliton solution. First of all, we have

$$f_1=W(\phi_1)=\phi_1=e^{\xi_1}+e^{\zeta_1},$$

and further substituting it into Eq. (3.6) to have

$$u=2(\ln f_1)_x=2\frac{l_1e^{\xi_1}+k_1e^{\zeta_1}}{e^{\xi_1}+e^{\zeta_1}}=(l_1+k_1)+(l_1-k_1)\tanh\frac{\xi_1-\zeta_1}{2}. \tag{3.7}$$

Taking $N=2$ in Eq. (3.6), we can obtain the two-soliton solution. Taking $\phi_1=e^{\xi_1}+e^{\zeta_1}$ and $\phi_2=e^{\xi_2}+e^{\zeta_2}$, we have

$$f_2=W(\phi_1,\phi_2)=\begin{vmatrix}\phi_1 & \phi_{1,x}\\ \phi_2 & \phi_{2,x}\end{vmatrix}=(e^{\xi_1}+e^{\zeta_1})(l_2e^{\xi_2}+k_2e^{\zeta_2})-(l_1e^{\xi_1}+k_1e^{\zeta_1})(e^{\xi_2}+e^{\zeta_2})$$
$$=(k_2-k_1)e^{\zeta_1+\zeta_2}\Big(1+\frac{k_2-l_1}{k_2-k_1}e^{\xi_1-\zeta_1}+\frac{l_2-k_1}{k_2-k_1}e^{\xi_2-\zeta_2}+\frac{l_2-l_1}{k_2-k_1}e^{\xi_1+\xi_2-\zeta_1-\zeta_2}\Big),$$

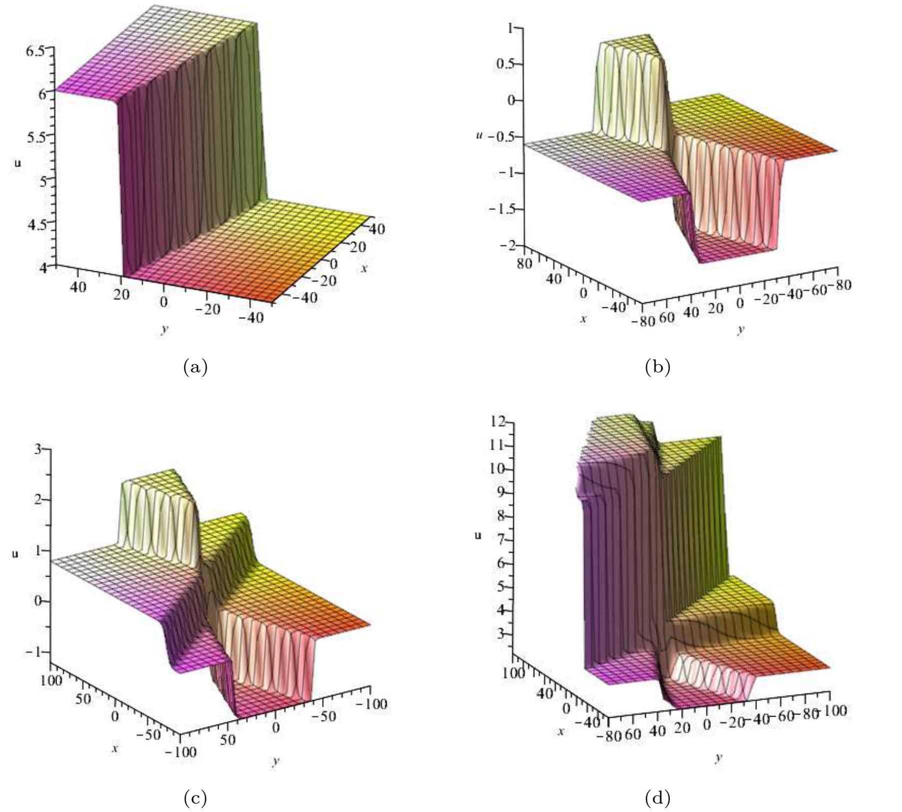
which leads to the two-soliton solution to Eq. (1.7) via $u=2(\ln f_2)_x$.

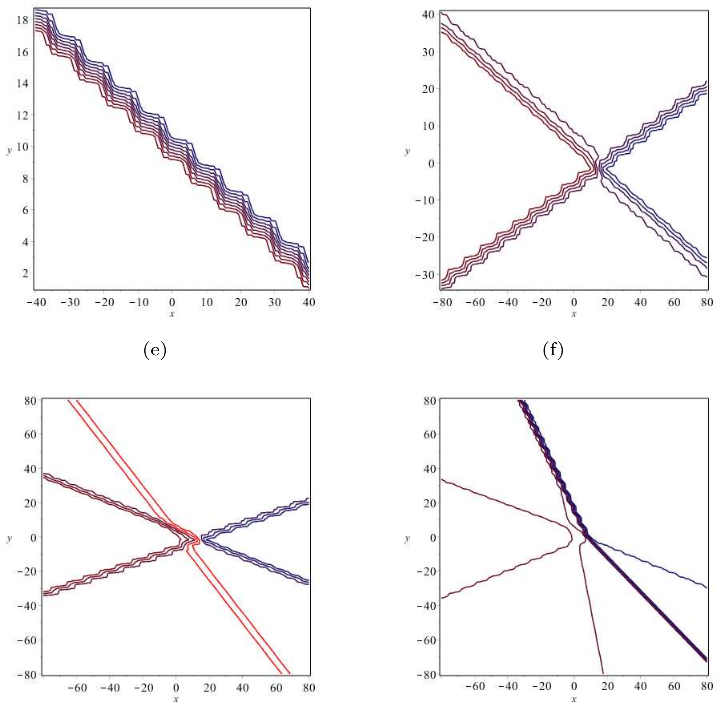
Taking $N = 3$ in Eq. (3.6), we can obtain the three-soliton solution. Let $\phi_1 = e^{\xi_1} + e^{\zeta_1}$, $\phi_2 = e^{\xi_2} + e^{\zeta_2}$ and $\phi_3 = e^{\xi_3} + e^{\zeta_3}$, then

$$
\begin{aligned}
f_3 = W(\phi_1, \phi_2, \phi_3) &= \begin{vmatrix} \phi_1 & \phi_{1,x} & \phi_{1,xx} \\ \phi_2 & \phi_{2,x} & \phi_{2,xx} \\ \phi_3 & \phi_{3,x} & \phi_{3,xx} \end{vmatrix} \\
&= (k_1 - k_2)(k_3 - k_2)(k_1 - k_3)e^{\zeta_1+\zeta_2+\zeta_3}\Big[1 + \frac{(k_2 - l_1)(k_3 - l_1)}{(k_1 - k_2)(k_1 - k_3)}e^{\xi_1-\zeta_1} \\
&\quad + \frac{(k_1 - l_2)(k_3 - l_2)}{(k_1 - k_2)(k_3 - k_2)}e^{\xi_2-\zeta_2} + \frac{(l_3 - k_2)(k_1 - l_3)}{(k_3 - k_2)(k_1 - k_3)}e^{\xi_3-\zeta_3} \\
&\quad + \frac{(l_1 - l_2)(k_3 - l_2)(l_1 - k_3)}{(k_1 - k_2)(k_3 - k_2)(k_1 - k_3)}e^{\xi_1+\xi_2-\zeta_1-\zeta_2} + \frac{(l_1 - k_2)(l_3 - k_2)(l_1 - l_3)}{(k_1 - k_2)(k_3 - k_2)(k_1 - k_3)}e^{\xi_1+\xi_3-\zeta_1-\zeta_3} \\
&\quad + \frac{(k_1 - l_2)(l_3 - l_2)(k_1 - l_3)}{(k_1 - k_2)(k_3 - k_2)(k_1 - k_3)}e^{\xi_2+\xi_3-\zeta_2-\zeta_3} \\
&\quad + \frac{(l_1 - l_2)(l_3 - l_2)(l_1 - l_3)}{(k_1 - k_2)(k_3 - k_2)(k_1 - k_3)}e^{\xi_1+\xi_2+\xi_3-\zeta_1-\zeta_2-\zeta_3}\Big],
\end{aligned}
$$

which gives rise to the three-soliton solution to Eq. (1.7) via $u = 2(\ln f_3)_x$.

Fig.3 show three-dimensional plots and contour plots of the one-, two-, three- and four-soliton solutions by selecting the appropriate parameters.





(e) (f)

(g) (h)

Figure 3. Soliton solutions to Eq. (1.7) with $c_1 = -1, c_2 = 1, c_3 = 2$, $z = 1$, $t = 1$ and $\gamma = 1$ (a) one-soliton solution via Eq. (3.7): $l_1 = 2$, $k_1 = 3$, $\xi_1^0 = \zeta_1^0 = 0$; (b) two-soliton solution: $l_1 = 1$, $l_2 = -1.2$, $k_1 = 1.5$, $k_2 = -1.8$, $\xi_1^0 = \xi_2^0 = \zeta_1^0 = \zeta_2^0 = 0$; (c) three-soliton solution: $l_1 = 1$, $l_2 = -1.2$, $l_3 = 0.2$, $k_1 = 1.6$, $k_2 = -1.8$, $k_3 = 0.6$, $\xi_1^0 = \xi_2^0 = \xi_3^0 = \zeta_1^0 = \zeta_2^0 = \zeta_3^0 = 0$; (d) four-soliton solution: $l_1 = 1$, $l_2 = 0.4$, $l_3 = 1.8$, $l_4 = -1$, $k_1 = 1.5$, $k_2 = -0.2$, $k_3 = 2$, $k_4 = -1.5$, $\xi_1^0 = \xi_2^0 = \xi_3^0 = \xi_4^0 = \zeta_1^0 = \zeta_2^0 = \zeta_3^0 = \zeta_4^0 = 0$, and (e), (f), (g), (h) are the corresponding contour plots.

In addition to the soliton solutions, we will construct some new type of solutions. In terms to the following expressions

$$l_i = \alpha_{1i} + \beta_{1i} I, \quad k_i = \alpha_{2i} + \beta_{2i} I, \tag{3.8}$$

where α_{1i}, α_{2i}, β_{1i}, β_{2i} are arbitrary real constants and $I = \sqrt{-1}$ is an imaginary unit, ξ_i and ζ_i can be rewritten as

$$\xi_i = \hat{\xi_{1i}} + \hat{\xi_{2i}} I, \quad \zeta_i = \hat{\zeta_{1i}} + \hat{\zeta_{2i}} I, \tag{3.9}$$

where

$$\hat{\xi_{1i}} = \alpha_{1i} x + \gamma(\alpha_{1i}^2 - \beta_{1i}^2) y + \frac{c_3\gamma^2 - 3c_2}{3c_1}\alpha_{1i} z - \frac{4c_3\gamma^2}{3}(\alpha_{1i}^3 - 3\alpha_{1i}\beta_{1i}^2)t,$$

$$\hat{\xi_{2i}} = \beta_{1i} x + 2\gamma\alpha_{1i}\beta_{1i} y + \frac{c_3\gamma^2 - 3c_2}{3c_1}\beta_{1i} z - \frac{4c_3\gamma^2}{3}(3\alpha_{1i}^2\beta_{1i} - \beta_{1i}^3)t,$$

$$\hat{\zeta_{1i}} = \alpha_{2i} x + \gamma(\alpha_{2i}^2 - \beta_{2i}^2) y + \frac{c_3\gamma^2 - 3c_2}{3c_1}\alpha_{2i} z - \frac{4c_3\gamma^2}{3}(\alpha_{2i}^3 - 3\alpha_{2i}\beta_{2i}^2)t,$$

$$\hat{\zeta_{2i}} = \beta_{2i} x + 2\gamma\alpha_{2i}\beta_{2i} y + \frac{c_3\gamma^2 - 3c_2}{3c_1}\beta_{2i} z - \frac{4c_3\gamma^2}{3}(3\alpha_{2i}^2\beta_{2i} - \beta_{2i}^3)t.$$

Therefore, ϕ_i is sorted out as follows

$$\phi_i = e^{\hat{\xi_{1i}}}(\cos\hat{\xi_{2i}} + I\sin\hat{\xi_{2i}}) + e^{\hat{\zeta_{1i}}}(\cos\hat{\zeta_{2i}} + I\sin\hat{\zeta_{2i}}), \tag{3.10}$$

which satisfies the linear partial differential conditions. If we take $\alpha_{1i} = \alpha_{2i}, \beta_{1i} = -\beta_{2i}$, then $\zeta_i = \xi_i^*$ and ξ_i^* represents the conjugate complex number of ξ_i. So Eq. (3.10) can be reduced to

$$\phi_i = 2e^{\hat{\xi_{1i}}}\cos\hat{\xi_{2i}}, \tag{3.11}$$

which results in the complexiton solution to Eq. (1.7) via $u = 2[\ln W(\phi_1, \phi_2, \cdots, \phi_N)]_x$.

When $N = 1$ in Eq. (3.6), the complexiton solution to Eq. (1.7) is given by

$$\begin{aligned} u &= 2(\ln f_1)_x = 2\alpha_{11} - 2\beta_{11}\tan\hat{\xi_{21}}, \\ f_1 &= W(\phi_1) = 2e^{\hat{\xi_{11}}}\cos\hat{\xi_{21}}. \end{aligned} \tag{3.12}$$

In addition, when $N = 2$ in Eq. (3.6), we take $\phi_1 = 2e^{\hat{\xi_{11}}}\cos\hat{\xi_{21}}$, $\phi_2 = 2e^{\hat{\xi_{12}}}\cos\hat{\xi_{22}}$, then we get the solution to Eq. (1.9)

$$\begin{aligned} f_2 = W(\phi_1, \phi_2) &= \begin{vmatrix} \phi_1 & \phi_{1,x} \\ \phi_2 & \phi_{2,x} \end{vmatrix} \\ &= 2e^{\hat{\xi_{11}}}\cos\hat{\xi_{21}}\cdot[2e^{\hat{\xi_{11}}}(\alpha_{11}\cos\hat{\xi_{21}} - \beta_{11}\sin\hat{\xi_{21}})] \\ &\quad - 2e^{\hat{\xi_{12}}}\cos\hat{\xi_{22}}\cdot[2e^{\hat{\xi_{12}}}(\alpha_{12}\cos\hat{\xi_{22}} - \beta_{12}\sin\hat{\xi_{22}})] \\ &= -4[(\alpha_{11} - \alpha_{12})\cos\hat{\xi_{21}}\cos\hat{\xi_{22}} + \beta_{12}\cos\hat{\xi_{21}}\sin\hat{\xi_{22}} - \beta_{11}\cos\hat{\xi_{22}}\sin\hat{\xi_{21}}]e^{\hat{\xi_{11}}+\hat{\xi_{12}}}, \end{aligned}$$

which generates the complexiton solution to Eq. (1.7) via $u = 2(\ln f_2)_x$.

Fig. 4 show the three-dimensional plot of the complexiton solution to Eq. (1.7) by taking some special values of parameters.

Moreover, a kind of rational solutions to Eq. (1.7) can be given based on the Wronskian form. We introduce a new type solution to the linear partial differential system (3.3) as follows

$$\begin{aligned} \phi_j &= \sum_{s=1}^{j}\sum_{l=0}^{[\frac{s}{3}]}\sum_{n=0}^{[\frac{s-3l}{2}]} \frac{\gamma^n\left(-\frac{4c_3\gamma^2}{3}\right)^l\left(x + \left(\frac{c_3\gamma^2 - 3c_2}{3c_1}\right)z\right)^{s-3l-2n} y^n t^l}{n!l!\,(s-3l-2n)!} m_s \\ &(j = 1, 2, \cdots, N), \end{aligned} \tag{3.13}$$

where $[\frac{s}{3}]$ represents the largest integer that does not exceed the real number $\frac{s}{3}$ and m_s is an arbitrary constant. In particular, if $m_s = 0$ $(s = 1, 2, \cdots, j-1), m_j = 1$, Eq. (3.13) is reduced as

$$\phi_j = \sum_{l=0}^{[\frac{j}{3}]}\sum_{n=0}^{[\frac{j-3l}{2}]} \frac{\gamma^n\left(-\frac{4c_3\gamma^2}{3}\right)^l\left(x + \left(\frac{c_3\gamma^2 - 3c_2}{3c_1}\right)z\right)^{j-3l-2n} y^n t^l}{n!l!\,(j-3l-2n)!}. \tag{3.14}$$

According to the above formula, we can calculate $\phi_j's$ as follows

$$\phi_1 = x + \frac{c_3\gamma^2 - 3c_2}{3c_1}z,$$

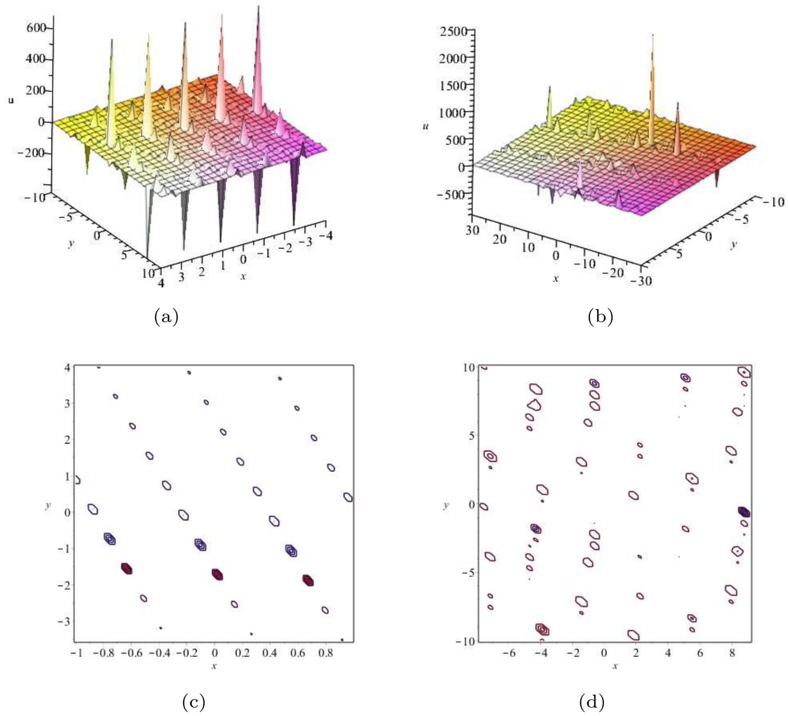


Figure 4. The complexiton solution to Eq. (1.7) with $c_1 = 1, c_2 = 2, c_3 = 3$, $t = 1$, $z = 1$ and $\gamma = 1$ (a)$N = 1$ via Eq. (3.12): $\alpha_{11} = 2$, $\beta_{11} = 1$;(b)$N = 2$: $\alpha_{11} = 2, \alpha_{12} = 1$, $\beta_{11} = 1, \beta_{12} = 2$, and (c), (d) are the corresponding contour plots.

$$\phi_2 = \frac{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^2}{2!} + \gamma y, \tag{3.15}$$

$$\phi_3 = \frac{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^3}{3!} + \gamma \left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right) y - \frac{4c_3\gamma^2}{3} t,$$

$$\phi_4 = \frac{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^4}{4!} + \frac{\gamma\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^2 y}{2!} + \frac{\gamma^2 y^2}{2!} - \frac{4c_3\gamma^2}{3}\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right) t,$$

$$\vdots$$

Based on the polynomial solutions to Eq. (1.9), we can construct the rational solutions to Eq. (1.7).

For $N = 1$ in Eq. (3.6), the polynomial solutions to Eq. (1.9) are presented

$$f_1 = W(\phi_1) = \phi_1 = x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z.$$

Therefore, the rational solutions to Eq. (1.7) can be derived as

$$u = 2(\ln f_1)_x = \frac{2}{x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z}.$$

For $N = 2$ in Eq. (3.6), the polynomial solutions to Eq. (1.9) are given

$$f_2 = W(\phi_1, \phi_2) = \begin{vmatrix} x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z & 1 \\ \frac{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^2}{2!} + \gamma y & x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z \end{vmatrix}$$
$$= \frac{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^2}{2} - \gamma y,$$

which leads to the rational solutions to Eq. (1.7) as

$$u = 2(\ln f_2)_x = 4 \frac{x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z}{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^2 - 2\gamma y}. \tag{3.16}$$

For $N = 3$ in Eq. (3.6), the polynomial solutions to Eq. (1.9) are shown

$$f_3 = W(\phi_1, \phi_2, \phi_3)$$
$$= \begin{vmatrix} x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z & 1 & 0 \\ \frac{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^2}{2!} + \gamma y & x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z & 1 \\ \frac{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^3}{3!} + \gamma\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right) y - \frac{4c_3\gamma^2}{3} t & \frac{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^2}{2!} + \gamma y & x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z \end{vmatrix}$$
$$= \frac{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^3}{3!} - \gamma\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right) y - \frac{4c_3\gamma^2}{3} t,$$

which gives rise to the rational solutions to Eq. (1.7) as

$$u = 2(\ln f_3)_x = 6 \frac{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^2 - 2\gamma y}{\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right)^3 - 6\gamma\left(x + \frac{c_3\gamma^2 - 3c_2}{3c_1} z\right) y - 8c_3\gamma^2 t}. \tag{3.17}$$

From the above process, we can see that if we choose different special forms for the unknown element ϕ_i in the determinant, which satisfies the linear differential conditions in Eqs. (3.3), different types of exact solutions to Eq. (1.7) can be derived.

4. N-soliton solutions and complexiton solutions in the Grammian form

4.1. Grammian solutions

In this part, we consider another form solutions to Eq. (1.9) by using Grammian determinant to express f as follows

$$f_N = \det(a_{ij})_{1\le i,j\le N}, \quad a_{ij} = c_{ij} + \int^x \phi_i \psi_j \, dx, \tag{4.1}$$

where c_{ij} is constant, and $\phi_i's$ and $\psi_j's$ are the functions of the scaled space coordinates x, y, z and the time coordinate t satisfying the following conditions:

$$\begin{aligned}
&\phi_{i,y} = \gamma\phi_{i,xx}, \quad \phi_{i,z} = \frac{c_3\gamma^2 - 3c_2}{3c_1}\,\phi_{i,x}, \quad \phi_{i,t} = -\frac{4c_3\gamma^2}{3}\,\phi_{i,xxx},\\
&\psi_{i,y} = -\gamma\psi_{i,xx}, \quad \psi_{i,z} = \frac{c_3\gamma^2 - 3c_2}{3c_1}\,\psi_{i,x}, \quad \psi_{i,t} = -\frac{4c_3\gamma^2}{3}\,\psi_{i,xxx}.
\end{aligned} \tag{4.2}$$

In fact, the coefficients in this partial differential system are the same as those in the above conditions (3.3).

In order to prove that f_N satisfies Eq. (1.9) more simply, Pfaffian of the N-soliton is introduced

$$(1, 2, \cdots, 2N), \tag{4.3}$$

and its expansion with the first character "1" as the datum element is

$$\begin{aligned}
(1, 2, \cdots, 2N) =& (1, 2)(3, 4, \cdots, 2N) - (1, 3)(2, 4, \cdots, 2N)\\
&+ (1, 4)(2, 3, \cdots, 2N) - \cdots + (1, 2N)(2, 3, \cdots, 2N - 1)\\
=& \sum_{j=2}^{2n} (-1)^j (1, j)(2, 3, \cdots, \hat{j}, \cdots, 2N),
\end{aligned}$$

where $\hat{j}$ means to remove the character j and the 1-order Pfaffian (i, j) is called an element of Pfaffian, satisfying $(i, j) = -(j, i)$. By making use of Pfaffian, the N-order Grammian determinant f_N can be rewritten as

$$f_N = (1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*) = (\bullet), \tag{4.4}$$

where $(i, j^*) = a_{ij}$ and $(i, j) = (i^*, j^*) = 0$. To express the derivatives of the entries (i, j^*) with Pfaffian, we define

$$(d_n, i^*) = \frac{\partial^n}{\partial x^n}\phi_i, \quad (d_n^*, j) = \frac{\partial^n}{\partial x^n}\psi_j, \quad (d_m, d_n^*) = (d_m^*, j^*) = (d_n, i) = 0, \tag{4.5}$$

where $m, n = 0, 1, 2, 3, 4$.

Theorem 4.1. *If $\phi_i's$ and $\psi_j's$ satisfy the linear differential conditions in Eqs. (4.2), then the Grammian determinant $f_N = det(a_{ij})_{1\le i,j\le N}$ defined by Eq. (4.4) solves the bilinear Eq. (1.9), and $u = 2(\ln f_N)_x$ leads to the solution to Eq. (1.7).*

Proof. According to the definition of Pfaffian entries $(i, j^*) = a_{ij}$ and the linear partial differential conditions in Eqs. (4.2), the derivatives of a_{ij} can be easily computed

$$\begin{aligned}
\frac{\partial}{\partial x}a_{ij} &= \phi_i\psi_j = (d_0, d_0^*, i, j^*),\\
\frac{\partial}{\partial y}a_{ij} &= \int^x \phi_{i,y}\psi_j + \phi_i\psi_{j,y}\,dx\\
&= \gamma(\phi_{i,x}\psi_j - \phi_i\psi_{j,x}) = \gamma[(d_0, d_1^*, i, j^*) - (d_1, d_0^*, i, j^*)],\\
\frac{\partial}{\partial z}a_{ij} &= \int^x \phi_{i,z}\psi_j + \phi_i\psi_{j,z}\,dx\\
&= \frac{c_3\gamma^2 - 3c_2}{3c_1}\phi_i\psi_j = \frac{c_3\gamma^2 - 3c_2}{3c_1}(d_0, d_0^*, i, j^*),
\end{aligned}$$

$$\frac{\partial}{\partial t}a_{ij} = \int^x \phi_{i,t}\psi_j + \phi_i\psi_{j,t}\,dx = -\frac{4c_3\gamma^2}{3}(\phi_{i,xx}\psi_j - \phi_{i,x}\psi_{j,x} + \phi_i\psi_{j,xx})$$
$$= -\frac{4c_3\gamma^2}{3}[(d_0, d_2^*, i, j^*) - (d_1, d_1^*, i, j^*) + (d_2, d_0^*, i, j^*)].$$

Through using the properties of the Pfaffian, the derivatives of the f_N with respect to independent variables x, y, z and t can be not difficult to calculated as

$$\begin{aligned}
&f_{N,x} = (d_0, d_0^*, \bullet),\\
&f_{N,xx} = (d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet),\\
&f_{N,xxx} = (d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet),\\
&f_{N,xxxx} = (d_3, d_0^*, \bullet) + 3(d_2, d_1^*, \bullet) + 3(d_1, d_2^*, \bullet) + (d_0, d_3^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet),\\
&f_{N,y} = \gamma[(d_0, d_1^*, \bullet) - (d_1, d_0^*, \bullet)],\\
&f_{N,yy} = \gamma^2[-(d_2, d_1^*, \bullet) + (d_0, d_3^*, \bullet) + (d_3, d_0^*, \bullet) - (d_1, d_2^*, \bullet)] + 2(d_0, d_0^*, d_1, d_1^*, \bullet),\\
&f_{N,z} = \frac{c_3\gamma^2 - 3c_2}{3c_1}(d_0, d_0^*, \bullet),\\
&f_{N,xz} = \frac{c_3\gamma^2 - 3c_2}{3c_1}[(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)],\\
&f_{N,xxz} = \frac{c_3\gamma^2 - 3c_2}{3c_1}[(d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)],\\
&f_{N,xxxz} = \frac{c_3\gamma^2 - 3c_2}{3c_1}[(d_3, d_0^*, \bullet) + 3(d_2, d_1^*, \bullet) + 3(d_1, d_2^*, \bullet) + (d_0, d_3^*, \bullet)\\
&\qquad + 2(d_0, d_0^*, d_1, d_1^*, \bullet)],\\
&f_{N,t} = -\frac{4c_3\gamma^2}{3}[(d_0, d_2^*, \bullet) - (d_1, d_1^*, \bullet) + (d_2, d_0^*, \bullet)],\\
&f_{N,xt} = -\frac{4c_3\gamma^2}{3}[(d_0, d_3^*, \bullet) + (d_3, d_0^*, \bullet) - (d_0, d_0^*, d_1, d_1^*, \bullet)].
\end{aligned} \tag{4.6}$$

Substituting the above results into Eq. (1.9), we have

$$(f_{xt} + c_1 f_{xxxz} + c_2 f_{xxxx} + c_3 f_{yy})f = 4c_3\gamma^2(d_0, d_0^*, d_1, d_1^*, \bullet)(\bullet),$$
$$- f_x f_t - 3c_1 f_{xxz} f_x + 3c_1 f_{xz} f_{xx} - c_1 f_z f_{xxx} - 4c_2 f_{xxx} f_x + 3c_2 f_{xx}^2 - c_3 f_y^2$$
$$= 4c_3\gamma^2[-(d_0, d_0^*, \bullet)(d_1, d_1^*, \bullet) + (d_0, d_1^*, \bullet)(d_1, d_0^*, \bullet)]$$

and further derive

$$(D_x D_t + c_1 D_x^3 D_z + c_2 D_x^4 + c_3 D_y^2) f \cdot f$$
$$= 4c_3\gamma^2[(d_0, d_0^*, d_1, d_1^*, \bullet)(\bullet) - (d_0, d_0^*, \bullet)(d_1, d_1^*, \bullet) + (d_0, d_1^*, \bullet)(d_1, d_0^*, \bullet)] = 0, \tag{4.7}$$

which is exactly the Jacobi identity of determinant [12], and the identity can be represented by Maya chart in Fig. 5.

The Jacobi identity is a special case of Pfaffian identity [12]. Because the number of indexes with $*$ and without $*$ in the two Pfaffian $(d_0, d_1, \bullet)$, $(d_0^*, d_1^*, \bullet)$ is not equal, the two Pfaffian are all zero. □

4.2. N-soliton solutions and complexiton solutions

If we select the appropriate elements in the determinant and satisfy the linear partial differential conditions, we can get different forms of solutions to Eq. (1.9).

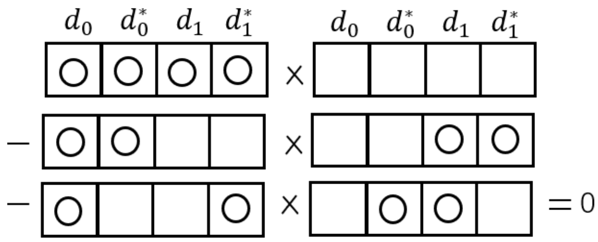


Figure 5. Maya chart of Eq. (4.7): the Jacobi identity.

In particular, the linear partial differential system in Eqs. (4.2) have the following form of solution

$$\begin{aligned}
&\phi_i = e^{\xi_i}, \quad \psi_i = e^{\zeta_i}, \\
&\xi_i = l_i x + \gamma l_i^2 y + \frac{c_3\gamma^2 - 3c_2}{3c_1} l_i z - \frac{4c_3\gamma^2}{3} l_i^3 t + \xi_i^0, \\
&\zeta_i = k_i x - \gamma k_i^2 y + \frac{c_3\gamma^2 - 3c_2}{3c_1} k_i z - \frac{4c_3\gamma^2}{3} k_i^3 t + \zeta_i^0,
\end{aligned} \tag{4.8}$$

where l_i, k_i, ξ_i^0 and ζ_i^0 $(i = 1, 2, \cdots, N)$ are arbitrary parameters.

When $N = 1$ in Eq. (4.4), the one-soliton solution to Eq. (1.7) can be given. Taking $c_{11} = 1$, $\phi_1 = e^{\xi_1}$ and $\psi_1 = e^{\zeta_1}$, we have

$$f_1 = (1, 1^*) = 1 + \frac{1}{l_1 + k_1} e^{\xi_1 + \zeta_1}, \tag{4.9}$$

which leads to the one-soliton solution to Eq. (1.7) as

$$u = 2(\ln f_1)_x = \frac{2e^{\xi_1+\zeta_1}}{1 + \frac{1}{l_1+k_1} e^{\xi_1+\zeta_1}} = (l_1+k_1) + (l_1+k_1)\tanh[\frac{1}{2}(\xi_1+\zeta_1+\ln\frac{1}{l_1+k_1})]. \tag{4.10}$$

When $N = 2$ in Eq. (4.4), the two-soliton solution to Eq. (1.7) can be given. Taking $c_{11} = c_{22} = 1$, $c_{12} = c_{21} = 0$, $\phi_j = e^{\xi_j}, \psi_j = e^{\zeta_j} (j = 1, 2)$, we have

$$\begin{aligned}
f_2 = (1, 2, 2^*, 1^*) &= \begin{vmatrix} 1 + \frac{1}{l_1+k_1} e^{\xi_1+\zeta_1} & \frac{1}{l_1+k_2} e^{\xi_1+\zeta_2} \\ \frac{1}{l_2+k_1} e^{\xi_2+\zeta_1} & 1 + \frac{1}{l_2+k_2} e^{\xi_2+\zeta_2} \end{vmatrix} \\
&= 1 + \frac{1}{l_1 + k_1} e^{\xi_1+\zeta_1} + \frac{1}{l_2 + k_2} e^{\xi_2+\zeta_2} \\
&\quad + \frac{(l_1 - l_2)(k_1 - k_2)}{(l_1 + k_1)(l_2 + k_2)(l_1 + k_2)(l_1 + k_2)} e^{\xi_1+\zeta_1+\xi_2+\zeta_2},
\end{aligned} \tag{4.11}$$

which results in the two-soliton solution to Eq. (1.7) via $u = 2(\ln f_2)_x$.

When $N = 3$ in Eq. (4.4), the three-soliton solution to Eq. (1.7) can be given. Taking $c_{11} = c_{22} = c_{33} = 1$, the rest of $c_{ij} = 0 (i, j = 1, 2, 3)$, and $\phi_j = e^{\xi_j}, \psi_j =$

$e^{\zeta_j}(j=1,2,3)$, we have

$$f_3=(1,2,3,3^*,2^*,1^*)=\begin{vmatrix} 1+\frac{1}{l_1+k_1}e^{\xi_1+\zeta_1} & \frac{1}{l_1+k_2}e^{\xi_1+\zeta_2} & \frac{1}{l_1+k_3}e^{\xi_1+\zeta_3} \\ \frac{1}{l_2+k_1}e^{\xi_2+\zeta_1} & 1+\frac{1}{l_2+k_2}e^{\xi_2+\zeta_2} & \frac{1}{l_2+k_3}e^{\xi_2+\zeta_3} \\ \frac{1}{l_3+k_1}e^{\xi_3+\zeta_1} & +\frac{1}{l_3+k_2}e^{\xi_3+\zeta_2} & 1+\frac{1}{l_3+k_3}e^{\xi_3+\zeta_3} \end{vmatrix}$$
$$=1+e^{\eta_1}+e^{\eta_2}+e^{\eta_3}+b_{12}e^{\eta_1+\eta_2}+b_{13}e^{\eta_1+\eta_3}+b_{23}e^{\eta_2+\eta_3}+b_{12}b_{13}b_{23}e^{\eta_1+\eta_2+\eta_3},$$

where

$$\eta_1=\xi_1+\zeta_1+\delta_1,\quad \eta_2=\xi_2+\zeta_2+\delta_2,\quad \eta_3=\xi_3+\zeta_3+\delta_3,\quad e^{\delta_i}=\frac{1}{l_i+k_i},$$

and

$$b_{12}=\frac{(l_1-l_2)(k_1-k_2)}{(l_1+k_2)(l_2+k_1)},\quad b_{13}=\frac{(l_1-l_3)(k_1-k_3)}{(l_1+k_3)(l_3+k_1)},\quad b_{23}=\frac{(l_2-l_3)(k_2-k_3)}{(l_2+k_3)(l_3+k_2)},$$

which gives rise to the three-soliton solution to Eq. (1.7) via $u=2(\ln f_3)_x$.

Similarly, we can derive the four-soliton solution to Eq. (1.7). In order to illustrate the soliton solutions more vividly, the one-, two-, three- and four-soliton solutions and the contours are plotted in Fig. 6 by choosing the appropriate parameters. As an example, if $c_1=-\frac{1}{4}, c_2=0, c_3=-\frac{3}{4}$, Eq. (1.9) is reduced to the (3+1)-dimensional YTSF (1.6).

5. Bilinear BT and rational solutions

5.1. Bilinear BT

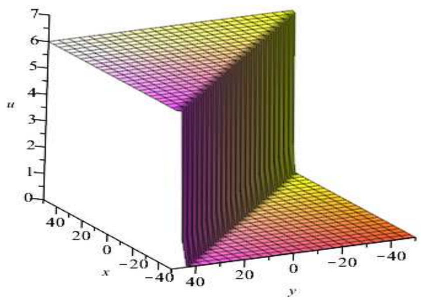
BT is an effective tool to solve the exact solution to the NLEEs [24, 36, 43]. It establishes the relationship between the solution to one NLEE and another known NLEE or the relationship between two different solutions to the same NLEE, and then derives the new solution from the known solution. Here we construct the bilinear BT between one solution f and another solution f' to the bilinear KP-YTSF Eq. (1.9). Fist of all, we consider

$$\begin{aligned}\mathrm{P} :=&[(D_xD_t+c_1D_x^3D_z+c_2D_x^4+c_3D_y^2)f'\cdot f']f^2\\ &-f'^2[(D_xD_t+c_1D_x^3D_z+c_2D_x^4+c_3D_y^2)f\cdot f].\end{aligned}\tag{5.1}$$

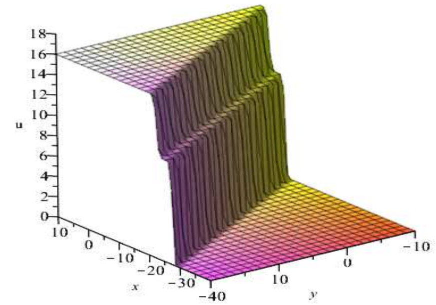
It can be observed from the above equation that if $\mathrm{P}=0$, then f satisfies Eq. (1.9) if and only if f' satisfies Eq. (1.9). By means of the bilinear operation identities, a series of bilinear equation of the interaction between the variables f and f' can be derived from $\mathrm{P}=0$.

Based on the exchange formula, the following bilinear operation identities are obtain

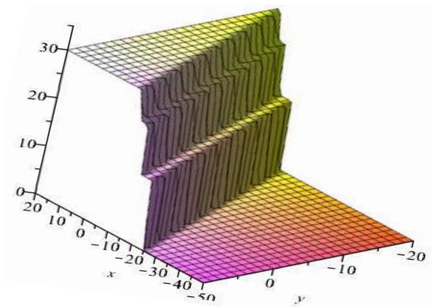
$$\begin{aligned}&(D_y^2f'\cdot f')f^2-(D_y^2f\cdot f)f'^2=2D_y(D_yf'\cdot f)\cdot ff',\\ &(D_xD_tf'\cdot f')f^2-(D_xD_tf\cdot f)f'^2=2D_x(D_tf'\cdot f)\cdot ff',\\ &D_y(D_x^2f'\cdot f)\cdot ff'=D_x[(D_xD_yf'\cdot f)\cdot ff'+(D_yf'\cdot f)\cdot(D_xf'\cdot f)],\end{aligned}$$



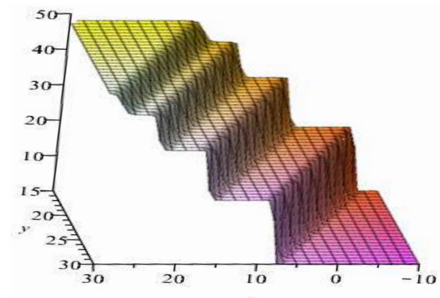
(a) one-soliton solution via Eq. (4.10): $l_1 = 2$, $k_1 = 1$, $\xi_1^0 = \zeta_1^0 = 0$



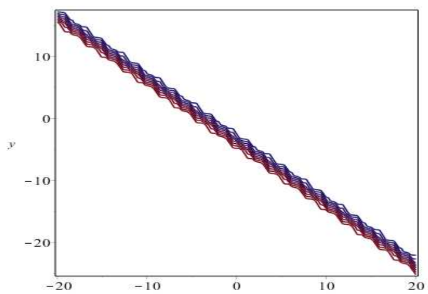
(b) two-soliton solution: $l_1=2$, $l_2=3$, $k_1=1$, $k_2=2$, $\xi_1^0=\xi_2^0=\zeta_1^0=\zeta_2^0=0$



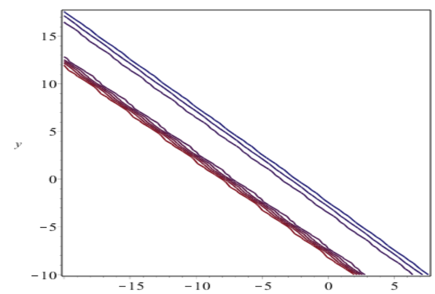
(c) three-soliton solution: $l_1=2$, $l_2=3$, $l_3=4$, $k_1=1$, $k_2=2$, $k_3=3$, $\xi_1^0=\xi_2^0=\xi_3^0=\zeta_1^0=\zeta_2^0=\zeta_3^0=0$



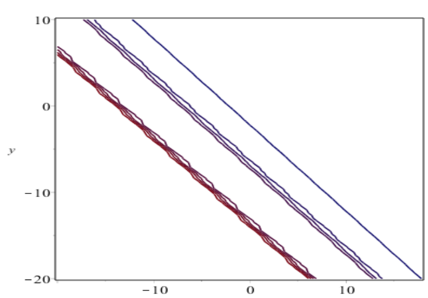
(d) four-soliton solution: $l_1 = 1$, $l_2 = 2$, $l_3 = 0.5$, $l_4 = 1.6$, $k_1 = 2$, $k_2 = 3$, $k_3 = 1.5$, $k_4 = 2.6$, $\xi_1^0 = \xi_2^0 = \xi_3^0 = \xi_4^0 = \zeta_1^0 = \zeta_2^0 = \zeta_3^0 = \zeta_4^0 = 0$



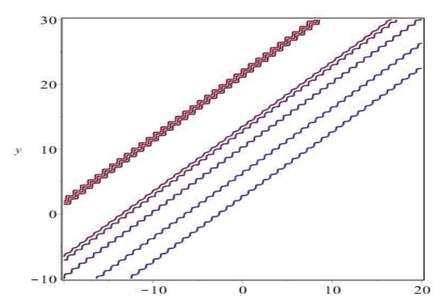
(e) the corresponding contour plots



(f) the corresponding contour plots



(g) the corresponding contour plots



(h) the corresponding contour plots

Figure 6. Soliton solutions to Eq. (1.7) with $z = 1$, $t = 1$ and $\gamma = 1$.

$$
\begin{aligned}
&(D_x^4 f'\cdot f')f^2-(D_x^4 f\cdot f)f'^2=2D_x(D_x^3 f'\cdot f)\cdot ff'-6D_x(D_x^2 f'\cdot f)\cdot(D_x f'\cdot f),\\
&\quad 2(D_x^3D_z f'\cdot f')f^2-2(D_x^3D_y f\cdot f)f'^2\\
&=3D_x(D_x^2D_z f'\cdot f)\cdot ff'-6D_x(D_xD_z f'\cdot f)\cdot(D_x f'\cdot f)\\
&\quad -3D_x(D_x^2 f'\cdot f)\cdot(D_z f'\cdot f)+D_z(D_x^3 f'\cdot f)\cdot ff'-3D_z(D_x^2 f'\cdot f)\cdot(D_x f'\cdot f).
\end{aligned}
\tag{5.2}
$$

The above five identities are substituted into Eq. (5.1), which can also be rewritten as

$$
\begin{aligned}
2\text{P}=&4D_x(D_t f'\cdot f)\cdot ff'+3c_1D_x(D_x^2D_z f'\cdot f)\cdot ff'-6c_1D_x(D_xD_z f'\cdot f)\cdot(D_x f'\cdot f)\\
&-3c_1D_x(D_x^2 f'\cdot f)\cdot(D_z f'\cdot f)+c_1D_z(D_x^3 f'\cdot f)\cdot ff'-3c_1D_z(D_x^2 f'\cdot f)\cdot(D_x f'\cdot f)\\
&+4c_2D_x(D_x^3 f'\cdot f)\cdot ff'-12c_2D_x(D_x^2 f'\cdot f)\cdot(D_x f'\cdot f)+4c_3D_y(D_y f'\cdot f)\cdot ff'\\
&+4c_3D_x[(D_xD_y f'\cdot f)\cdot ff'+(D_y f'\cdot f)\cdot(D_x f'\cdot f)]-4c_3D_y(D_x^2 f'\cdot f)\cdot ff'\\
=&D_x[(4D_t+3c_1D_x^2D_z+4c_2D_x^3+4c_3D_xD_y+4c_3\lambda_1D_y+c_1\lambda_2D_z+\lambda_3)f'\cdot f]\cdot ff'\\
&-2D_x[(3c_1D_xD_z+6c_2D_x^2-2c_3D_y+\lambda_4D_x)f'\cdot f]\cdot(D_x f'\cdot f)\\
&+4c_3D_y[(D_y-D_x^2-\lambda_1D_x+c_1\lambda_5D_z+\lambda_6)f'\cdot f]\cdot ff'\\
&-3c_1D_x[(D_x^2+\lambda_7+\lambda_8D_z)f'\cdot f]\cdot(D_z f'\cdot f)\\
&-3c_1D_z[(D_x^2-\lambda_7+\lambda_9D_x)f'\cdot f]\cdot(D_x f'\cdot f)\\
&+c_1D_z[(D_x^3-\lambda_2D_x-4c_3\lambda_5D_y+\lambda_{10})f'\cdot f]\cdot ff',
\end{aligned}
\tag{5.3}
$$

where $\lambda_i(i=1,2,\cdots,10)$ are all arbitrary parameters. At present, the bilinear BT for Eq. (1.9) is constructed as follows

$$
\begin{cases}
B_1 f'\cdot f=(4D_t+3c_1D_x^2D_z+4c_2D_x^3+4c_3D_xD_y+4c_3\lambda_1D_y+c_1\lambda_2D_z+\lambda_3)f'\cdot f=0,\\
B_2 f'\cdot f=(3c_1D_xD_z+6c_2D_x^2-2c_3D_y+\lambda_4D_x)f'\cdot f=0,\\
B_3 f'\cdot f=(D_y-D_x^2-\lambda_1D_x+c_1\lambda_5D_z+\lambda_6)f'\cdot f=0,\\
B_4 f'\cdot f=(D_x^2+\lambda_7+\lambda_8D_z)f'\cdot f=0,\\
B_5 f'\cdot f=(D_x^2-\lambda_7+\lambda_9D_x)f'\cdot f=0,\\
B_6 f'\cdot f=(D_x^3-\lambda_2D_x-4c_3\lambda_5D_y+\lambda_{10})f'\cdot f=0,
\end{cases}
\tag{5.4}
$$

which is composed of six bilinear equations about f and f' and includes ten arbitrary parameters. In the above calculation, the parameters $\lambda_i(i=3,4,6,8,9,10)$ are equal to 0 via $D_\gamma f\cdot f=0$ and the parameters $\lambda_j(j=1,2,5,7)$ are equal to 0 via $D_x(D_y f'\cdot f)\cdot ff'=D_y(D_x f'\cdot f)\cdot ff'$.

5.2. Rational solutions

If a simple function $f=1$ is taken into the bilinear BT of Eq. (1.9), then Eq. (5.4) can be reduced to a linear partial differential system

$$
\begin{cases}
4f'_t+3c_1f'_{xxz}+4c_2f'_{xxx}+4c_3f'_{xy}+4c_3\lambda_1f'_y+c_1\lambda_2f'_z+\lambda_3f'=0,\\
3c_1f'_{xz}+6c_2f'_{xx}-2c_3f'_y+\lambda_4f'_x=0,\\
f'_y-f'_{xx}-\lambda_1f'_x+c_1\lambda_5f'_z+\lambda_6f'=0,\\
f'_{xx}+\lambda_7f'+\lambda_8f'_z=0,\ f'_{xx}-\lambda_7f'+\lambda_9f'_x=0,\\
f'_{xxx}-\lambda_2f'_x-4c_3\lambda_5f'_y+\lambda_{10}f'=0,
\end{cases}
\tag{5.5}
$$

which makes it easier to get a new solution f'.

Next, we take into account of a class of rational solutions to Eq. (1.9), so let f be a first-order polynomial function

$$f' = a_1x + a_2y + a_3z - a_4t, \tag{5.6}$$

where a_1, a_2, a_3, a_4 are all arbitrary constants. By substituting f' into Eq. (5.5) and taking $\lambda_3 = \lambda_6 = \lambda_7 = \lambda_{10} = 0$ in Eq. (5.5), we get

$$a_2 = -a_3c_1\lambda_5 + a_1\lambda_1, \quad a_4 = \frac{(-a_3c_1\lambda_5 + a_1\lambda_1)^2c_3}{a_1}, \tag{5.7}$$

$$\lambda_2 = -\frac{4(-a_3c_1\lambda_5 + a_1\lambda_1)c_3\lambda_5}{a_1}, \quad \lambda_4 = \frac{2(-a_3c_1\lambda_5 + a_1\lambda_1)c_3}{a_1}, \tag{5.8}$$

and

$$\lambda_8 = 0, \quad \lambda_9 = 0. \tag{5.9}$$

Hence, the first-order polynomial solution to Eq. (1.9) is derived

$$f' = a_1x + (-a_3c_1\lambda_5 + a_1\lambda_1)y + a_3z - \frac{(-a_3c_1\lambda_5 + a_1\lambda_1)^2c_3}{a_1}t. \tag{5.10}$$

Based on the dependent variable transformation (1.8), the following rational solution to Eq. (1.7) is given

$$u = 2(\ln f')_x = \frac{2a_1}{f'}, \tag{5.11}$$

where $a_1, a_3, \lambda_1, \lambda_5$ are arbitrary constants. By selecting $a_1 = 1.8, a_3 = 4, \lambda_1 = 2$ and $\lambda_5 = 1.5$, this rational solution and the contour are plotted in Fig. 7.

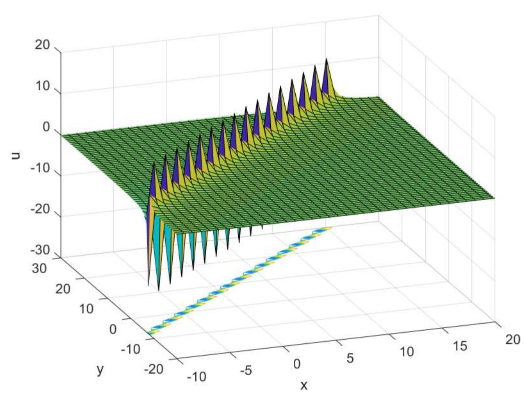


Figure 7. Plot of the rational solution and the contour via Eq. (5.11) with $c_1 = 1, c_2 = -2$, $z = 1$ and $t = 3$.

6. Conclusions

In this paper, we have used four methods to investigate exact solutions to Eq. (1.7), and obtained different types of solutions, such as rational solutions, soliton solutions, kinky periodic-wave solutions and complexiton solutions. Fist of all, we have investigated the interaction between soliton and periodic solution, and a class of kinky periodic-wave solutions has been obtained. Based on the bilinear form of Eq. (1.7), the Wronskian determinant solutions have been given. With symbolic computation, the soliton solutions, complexiton solutions and rational solutions have been derived. We have also investigated the Grammian solutions to Eq. (1.9) in a similar measure. By selecting appropriate parameters, we have plotted three-dimensional plot of a few particular solutions and corresponding contour maps with Maple. The results show that determinant technique is an important means to solve NLEEs. Furthermore, the bilinear BT of Eq. (1.9) has been constructed, which consists of six bilinear equations and ten free parameters. Based on the bilinear BT, a class of rational solutions have been derived from a particular solution. It is proved that we can get new solutions from some known solutions to Eq. (1.9) according to BT.

Declaration of Competing Interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Compliance with ethical standards.

Conflict of interest. The authors declare that they have no conflict of interest.

Data availability statement. The authors declare that the manuscript has no associated data.

Acknowledgements. Authors would like to thank Prof. Bo-Ling Guo for his valuable discussion. This work is supported by Beijing Laboratory of National Economic Security Early-warning Engineering, Beijing Jiaotong University.

Appendix A

Using Maple, we get three groups of solutions about the relations among parameters a_i, b_i, m_i, d_i and δ_i .

Case 1

$$
\begin{aligned}
d_1 = {} & \frac{1}{2a_1(a_2^2\delta_1^2+6a_1^2\delta_2+2a_2^2\delta_2)(a_1^2+a_2^2)}(a_1^6a_2^2c_2\delta_1^2+7a_1^4a_2^4c_2\delta_1^2+3a_1^2a_2^6c_2\delta_1^2-3a_2^8c_2\delta_1^2 \\
& +32a_1^6a_2^2c_2\delta_2+32a_1^4a_2^4c_2\delta_2-a_1^4b_2^2c_3\delta_1^2+2a_1^3a_2b_1b_2c_3\delta_1^2-3a_1^2a_2^2b_1^2c_3\delta_1^2 \\
& +3a_1^2a_2^2b_2^2c_3\delta_1^2-6a_1a_2^3b_1b_2c_3\delta_1^2+a_2^4b_1^2c_3\delta_1^2-12a_1^4b_1^2c_3\delta_2+16a_1^4b_2^2c_3\delta_2 \\
& -32a_1^3a_2b_1b_2c_3\delta_2-4a_1^4b_1^2c_3\delta_2), \\
d_2 = {} & \frac{1}{a_2^2\delta_1^2+6a_1^2\delta_2+2a_2^2\delta_2}(-4a_2^5c_2\delta_1^2+6a_1^4a_2c_2\delta_2-12a_1^2a_2^3c_2\delta_2-2a_2^5c_2\delta_2+a_2b_2^2c_3\delta_1^2 \\
& +12a_1b_1b_2c_3\delta_2-6a_2b_1^2c_3\delta_2+2a_2b_2^2c_3\delta_2), \\
m_1 = {} & \frac{-1}{2a_1c_1(a_2^2\delta_1^2+6a_1^2\delta_2+2a_2^2\delta_2)(a_1^2+a_2^2)}(3a_1^4a_2^2c_2\delta_1^2+6a_1^2a_2^4c_2\delta_1^2+3a_2^6c_2\delta_1^2 \\
& +12a_1^6c_2\delta_2+24a_1^4a_2^2c_2\delta_2+12a_1^2a_2^4c_2\delta_2-a_1^2b_2^2c_3\delta_1^2+2a_1a_2b_1b_2c_3\delta_1^2 \\
& -a_2^2b_1^2c_3\delta_1^2+4a_1^2b_2^2c_3\delta_2-8a_1a_2b_1b_2c_3\delta_2+4a_2^2b_1^2c_3\delta_2), \\
m_2 = {} & 0,
\end{aligned}
$$

with

$$a_1 \neq 0, a_2 \neq 0, a_2^2\delta_1^2 + 6a_1^2\delta_2 + 2a_2^2\delta_2 \neq 0,$$

where $a_1, a_2, b_1, b_2, \delta_1$ and δ_2 are free constants.

Case 2

$$\begin{aligned}
d_1 = {} & \frac{1}{2a_1a_2^2(a_1^2+a_2^2)^2}(a_1^8a_2^2c_2 + a_1^8a_2c_1m_2 + 4a_1^6a_2^4c_2 + 4a_1^6a_2^3c_1m_2 + 6a_1^4a_2^6c_2 \\
& + 6a_1^4a_2^5c_1m_2 + 4a_1^2a_2^8c_2 + 4a_1^2a_2^7c_1m_2 + a_2^{10}c_2 + a_2^9c_1m_2 - a_1^6b_2^2c_1 + 2a_1^5a_2b_1b_2c_3 \\
& - 3a_1^4a_2^2b_1^2c_3 + 6a_1^4a_2^2b_2^2c_3 - 12a_1^3a_2^3b_1b_2c_3 + 2a_1^2a_2^4b_1^2c_3 - a_1^2a_2^4b_2^2c_3 \\
& + 2a_1a_2^5b_1b_2c_3 - 3a_2^6b_1^2c_3), \\
d_2 = {} & \frac{-c_3(a_1^4b_2^2 - 2a_1^2a_2^2b_2^2 + 8a_1a_2^3b_1b_2 - 4a_2^4b_1^2 + a_2^4b_2^2)}{a_2(a_1^2+a_2^2)^2}, \\
m_1 = {} & \frac{-1}{2a_1a_2^2c_1(a_1^2+a_2^2)^2}(3a_1^6a_2^2c_2 + a_1^6a_2c_1m_2 + 5a_1^4a_2^4c_2 + 4a_1^4a_2^3c_1m_2 + a_1^2a_2^6c_2 \\
& - a_1^2a_2^5c_1m_2 - a_2^8c_2 - a_2^7c_1m_2 - a_1^4b_2^2c_3 + 2a_1^3a_2b_1b_2c_3 - a_1^2a_2^2b_1^2c_3 \\
& + 3a_1^2a_2^2b_2^2c_3 - 6a_1a_2^3b_1b_2c_3 + 3a_2^4b_1^2c_3), \\
\delta_2 = {} & \frac{2a_2^4\delta_1^2}{3a_1^4 - 6a_1^2a_2^2 - a_2^4},
\end{aligned}$$

with

$$a_1 \neq 0, \quad a_2 \neq 0, \quad 3a_1^4 - 6a_1^2a_2^2 - a_2^4 \neq 0,$$

where a_1, a_2, b_1, b_2, m_2 and δ_1 are free constants.

Case 3

$$\begin{aligned}
& a_1 = 0, \quad d_1 = \frac{a_2^3c_1m_1 - 2b_1b_2c_3}{a_2}, \\
& m_2 = \frac{-b_1^2c_3 - a_2^4c_2 + b_2^2c_3 + a_2d_2}{a_2^3c_1}, \quad \delta_2 = \frac{4b_1^2\delta_1^2 - 3b_2^2\delta_1^2 - \frac{3}{c_3}a_2d_2\delta_1^2}{4b_1^2},
\end{aligned}$$

with

$$a_2 \neq 0, \quad b_1 \neq 0, \quad \delta_1 \neq 0,$$

where a_2, b_1, b_2, d_2, m_1 and δ_1 are free constants.

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