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A HYBRID SWITCHING S-DI-A EPIDEMIC MODEL WITH STANDARD INCIDENCE: PERSISTENCE, EXTINCTION AND POSITIVE RECURRENCE

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Abstract In this paper, a stochastic S-DI-A epidemic model with standard incidence under Markovian switching is investigated to study the spread of the HIV virus. For this purpose, we firstly obtain sufficient conditions for persistence in the mean of the disease. In addition, sufficient conditions for exponential extinction of the infectious disease is derived. Furthermore, by constructing a suitable stochastic Lyapunov function with regime switching, we establish sufficient conditions for the existence of positive recurrence of the solutions. Finally, numerical simulations are employed to demonstrate the analytical results.

Keywords S-DI-A model, Markovian switching, persistence in the mean, extinction, positive recurrence.

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1. Introduction

Human immunodeficiency virus (HIV) continues to be a major public health problem in the world. HIV attacks the body's immune system, specifically the CD4 cells (T cells), which help the immune system fight off infections. Untreated, HIV reduces the number of CD4 cells in the body, making the person more likely to get other infections or infection-related cancers. Over time, HIV can destroy so many of these cells that the body can't fight off infections and disease. These opportunistic infections or cancers take advantage of a very weak immune system and signal that the person has AIDS, the last stage of HIV infection [1].

Because of the long term predication and heavy expenses of clinical treatments, several mathematical models have been introduced to describe how the disease is distributed and to explain epidemic illnesses related to AIDS [2, 9–13, 20]. These Mathematical models are useful for understanding the spread of HIV/AIDS. Obviously, the most urgent public health problem globally is to devise effective strategies to minimize the destruction caused by the HIV/AIDS epidemic [24].

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The deterministic classical S-DI-A model employing with standard incidence is proposed by Hyman et al. [10], which has been widely used and studied, takes the form

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \mu S^0 - \mu S - \displaystyle\sum_{j=1}^{n} \beta_j \frac{I_j}{N} S, \\ \dfrac{\mathrm{d}I_i}{\mathrm{d}t} = p_i \displaystyle\sum_{j=1}^{n} \beta_j \frac{I_j}{N} S - (\mu + \gamma_i) I_i, \quad i = 1, 2, \ldots, n, \\ \dfrac{\mathrm{d}A}{\mathrm{d}t} = \displaystyle\sum_{i=1}^{n} \gamma_i I_i - \delta A. \end{cases} \tag{1.1}$$

Here they divided the total population N into the susceptible population S, the infected individuals I is subdivided into n subgroups $I_1, I_2, \ldots, I_n$, and the AIDS cases A. Therefore, $N(t) = S(t) + \sum_{j=1}^{n} I_j(t)$. S^0 is a positive constant steady state of the susceptible population S, if no virus is present in the population; μ denotes the rate of inflow and outflow; β_j denotes the transmission probability per partner of individuals in subgroup n; p_i denotes the probability of an infected individual entering the ith subgroup and $\sum_{i=1}^{n} p_i = 1$; γ_i is the rate of leaving the high-risk population because of behavior changes that are induced by either HIV-related illnesses or a positive HIV test and δ denotes the death rate of A which satisfies $\delta \geq \mu$.

Since the dynamics of AIDS cases A have no effects on those of susceptible population S and infected individuals I_i for system (1.1). Hence, we only need to consider the first two equations of system (1.1) as following reduced system

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \mu S^0 - \mu S - \displaystyle\sum_{j=1}^{n} \beta_j \frac{I_j}{N} S, \\ \dfrac{\mathrm{d}I_i}{\mathrm{d}t} = p_i \displaystyle\sum_{j=1}^{n} \beta_j \frac{I_j}{N} S - (\mu + \gamma_i) I_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{1.2}$$

According to the theory in [20], system (1.2) has the following properties

- System (1.2) always has the infection-free equilibrium $E_0 = (S^0, 0, 0, \ldots, 0)$, which is globally asymptotically stable in the region D if $R_0 = \sum_{i=1}^{n} \frac{\beta_i p_i}{\mu + \gamma_i} \leq 1$, where $D = \{(S, I_i) \in \mathbb{R}_+^{n+1} : 0 \leq N \leq S^0\}$.
- If $R_0 > 1$, the infection-free equilibrium E_0 is unstable and the endemic equilibrium $E^* = (S^*, I_1^*, I_2^*, \ldots, I_n^*)$ is globally asymptotically stable in the region D.

However, in many applications, it has been shown that modeling the behavior of dynamical systems by stochastic differential equations has more advantages than deterministic modeling [4, 6, 16, 18, 26, 30, 34]. Epidemic models are inevitably infected by various environmental noises such as telephone noise, which is important components in realism [19, 27, 28]. Therefore, in order to describe the interference of the external irresistible factors, many researchers tried to combine the Markov chain with the epidemic models [3, 5, 7, 15, 18, 25, 29, 32, 33]. By state switching of the Markov chain, the continuous-time Markov chain changes the main parameters of epidemic models. For example, Zhang et al. [33] considered stochastic SIS epidemic model with vaccination under regime switching. They establish sufficient conditions for the existence of a unique ergodic stationary distribution by constructing stochastic Lyapunov functions. Gary et al. [5] studied SIS epidemic model with noise introduced in the disease transmission term. They showed that the SDE has a unique positive global solution, and we established conditions for extinction and persistence of disease. Motivated by the abovementioned work, in this paper, we just make a first attempt to fill the gap and study the multigroup S-DI-A epidemic model with standard incidence and Markovian switching.

In the S-DI-A epidemic model, seasonality exerts a strong influence on the disease transmission coefficient β_j between compartments S and I_j. Moreover, β_j may be more sensitive to environmental fluctuations than other parameters of system (1.2) for human populations [14]. These changes usually cannot be described by the traditional deterministic or stochastic epidemic models. Hence, it is natural to consider the following S-DI-A epidemic model with standard incidence under a piecewise deterministic Markov process.

$$\begin{cases} \mathrm{d}S = (\mu S^0 - \mu S - \sum_{j=1}^{n} \beta_j(r(t)) \frac{I_j}{N} S)\mathrm{d}t, \\ \mathrm{d}I_i = (p_i \sum_{j=1}^{n} \beta_j(r(t)) \frac{I_j}{N} S - (\mu + \gamma_i) I_i)\mathrm{d}t, \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.3}$$

where the transmission coefficient β_j is obtained by a homogeneous continuous-time Markov chain $\{r(t), t > 0\}$ with values in finite state space $\mathcal{M} = \{1, 2, \ldots, \tilde{N}\}$ denoting different environments.

The paper is organized as follows. In Section 2, we introduce some results that will be used in our following analysis. In Section 3, the existence and uniqueness of the positive solution of system (1.3) are shown. In Section 4, we obtain sufficient conditions for persistence in the mean of the disease. Sufficient conditions for exponential extinction of the infectious disease are established in Section 5. In Section 6, we obtain sufficient conditions for positive recurrence of the solutions to system (1.3). Finally, in Section 7 and 8, we come to a conclusion and introduce numerical simulations to verify the theoretical results.

2. Preliminaries

In this section, we introduce the notations and two lemmas which will be used in the whole paper. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

Let $r(t)$ be a right-continuous Markov chain, taking values in a finite state space $\mathcal{M} = \{1, 2, \ldots, \tilde{N}\}$, with the generator $\tilde{\Gamma} = (\gamma_{ij})_{\tilde{N}\times\tilde{N}}$ given by [31]

$$P\{r(t+\delta) = v | r(t) = u\} = \begin{cases} \gamma_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta), & \text{if } i = j, \end{cases}$$

where $\delta > 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\sum_{j=1}^{\tilde{N}} \gamma_{ij} = 0$. Assume further that the Markov chain $r(t)$ is irreducible and has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_{\tilde{N}})$ which can be determined by equation

$$\pi\tilde{\Gamma} = 0, \tag{2.1}$$

subject to $\sum_{h=1}^{\tilde{N}} \pi_h = 1$, and $\pi_h > 0$, for any $h \in \mathcal{M}$. For any vector $g = (g(1), g(2), \ldots, g(\tilde{N}))$, define $\hat{g} = \min_{k\in\mathcal{M}} g(k)$, $\check{g} = \max_{k\in\mathcal{M}} g(k)$.

Let $(X(t), r(t))$ be the diffusion Markov process and satisfy the following equation

$$\mathrm{d}X(t) = b(X(t), r(t))\mathrm{d}t + \sigma(X(t), r(t))\mathrm{d}B(t), \quad X(0) = x_0, \quad r(0) = r, \tag{2.2}$$

where $b(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}^n$, $\sigma(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}^{n\times n}$, and $D(x, k) = \sigma(x, k)\sigma^T(x, k) = (d_{ij}(x, k))$. For each $k \in \mathcal{M}$, let $V(\cdot, k)$ be any twice continuously differentiable function, the operator $\mathcal{L}$ is defined by

$$\mathcal{L}V(x,k) = \sum_{i=1}^{n} b_i(x,k)\frac{\partial V(x,k)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} d_{ij}(x,k)\frac{\partial^2 V(x,k)}{\partial x_i \partial x_j} + \sum_{l=1}^{\tilde{N}} \gamma_{kl} V(x,l).$$

Then we can easily obtain the following results to prove the persistence in the mean from Liu et al. [17].

Lemma 2.1. $(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) \geq (\sum_{i=1}^{n} a_i b_i)^2$ *for any* $a_i, b_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$.

Lemma 2.2. $(\sum_{k=1}^{\tilde{N}} \pi_k x(k))^2 \leq \sum_{k=1}^{\tilde{N}} \pi_k x^2(k)$.

3. Existence and uniqueness of the positive solution

To study the dynamical behavior of stochastic S-DI-A epidemic model, the first concern is whether the solution is global and positive. In this section, we will show that system (1.3) has a unique global positive solution with any initial value.

Theorem 3.1. *For any initial value* $(S(0), I_1(0), I_2(0), \ldots, I_n(0), r(0)) \in \mathbb{R}_+^{n+1} \times \mathcal{M}$*, stochastic system (1.3) has a unique positive solution* $(S(t), I_1(t), I_2(t), \ldots, I_n(t), r(t))$ *on* $t \geq 0$*, and the solution will remain in* $\mathbb{R}_+^{n+1} \times \mathcal{M}$ *with probability one.*

Since the coefficients of model (1.3) satisfy the local Lipschitz condition, then for given initial value $(S(0), I_1(0), I_2(0), \ldots, I_n(0), r(0)) \in \mathbb{R}_+^{n+1} \times \mathcal{M}$, there exists a unique local solution $(S(t), I_1(t), I_2(t), \ldots, I_n(t), r(t))$ on $t \in (0, \rho)$, a.s., where ρ is the explosion time [23]. To show this solution is global, we only need to prove that $\rho = \infty$ a.s.. Let k_0 be sufficiently large for every component of $S(0)$ and $I_i(0)$ $(i = 1, 2, \ldots, n)$ all lie within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, define the stopping time as in [23]

$$\begin{aligned}\tau_k = \inf\{t \in (0,\tau) |\min\{S(t), I_1(t), I_2(t), \ldots, I_n(t)\} \leq \frac{1}{k} \\ \text{or } \max\{S(t), I_1(t), I_2(t), \ldots, I_n(t)\} \geq k\},\end{aligned}$$

where throughout this paper we set $\inf\emptyset = \infty$. Obviously, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, hence $\tau_\infty \leq \rho$ a.s.. If we show that $\tau_\infty = \infty$ a.s., then $\rho=\infty$ a.s.. This means that $(S(t),I_1(t),I_2(t),\ldots,I_n(t),r(t))\in\mathbb{R}_+^{n+1}\times\mathcal{M}$ a.s. for all $t\geq 0$. If $\tau_\infty<\infty$ a.s., then there is a pair of constants $T\geq 0$ and $\varepsilon\in(0,1)$ such that

$$\mathbb{P}\{\tau_\infty\leq T\}>\varepsilon. \tag{3.1}$$

Hence, there is an integer $k_1\geq k_0$ such that

$$\mathbb{P}\{\tau_k\leq T\}\geq\varepsilon,\qquad \text{for all } k\geq k_1.$$

Define a C^2-function V_0 on $\mathbb{R}_+^{n+1}\times\mathcal{M}$ to $\mathbb{R}_+$ as follows

$$V_0(S,I_1,I_2,\ldots,I_i,k)=(S-1-\log S)+\sum_{i=1}^{n}(I_i-1-\log I_i).$$

By using Itô's formula [21], we have

$$\mathrm{d}V(S,I_1,I_2,\ldots,I_i,k)=\mathcal{L}V(S,I_1,I_2,\ldots,I_i,k)\mathrm{d}t, \tag{3.2}$$

where $\mathcal{L}V_0:\mathbb{R}_+^{n+1}\times\mathcal{M}\to\mathbb{R}_+$ is defined by

$$\begin{aligned}
&\mathcal{L}V_0(S,I_1,I_2,\ldots,I_i,k)\\
=&\Big(1-\frac{1}{S}\Big)\Big(\mu S^0-\mu S-\sum_{j=1}^{n}\beta_j(k)\frac{I_j}{N}S\Big)+\sum_{i=1}^{n}(1-\frac{1}{I_i})\Big(\sum_{j=1}^{n}\beta_j(k)\frac{I_j}{N}S-(\mu+\gamma_i)I_i\Big)\\
=&\mu S^0-\mu S-\sum_{i=1}^{n}(\mu+\gamma_i)I_i-\frac{\mu S^0}{S}+\mu+\sum_{j=1}^{n}\beta_j(k)\frac{I_j}{N}-\sum_{i=1}^{n}\frac{1}{I_i}\sum_{j=1}^{n}\beta_j(k)\frac{I_j}{N}S+\sum_{i=1}^{n}(\mu+\gamma_i)\\
\leq&\mu S^0+\mu+\sum_{j=1}^{n}\check{\beta}_j+\sum_{i=1}^{n}(\mu+\gamma_i)\\
:=&Q,
\end{aligned} \tag{3.3}$$

where Q is a positive constant which is independent of S and I_i $(i=1,2,\ldots,n)$. The remained proof follows that in Mao et al. [22] and hence is omitted here. This completes the proof.

Remark 3.1. Theorem 3.1 shows that for any initial value $(S(0), I_1(0), I_2(0),\ldots, I_n(0), r(0))\in\mathbb{R}_+^{n+1}\times\mathcal{M}$, system (1.3) has a unique positive solution $(S(t),I_1(t), I_2(t),\ldots, I_n(t),r(t))\in\mathbb{R}_+^{n+1}\times\mathcal{M}$ a.s.. Therefore,

$$\begin{aligned}
&\left[\mu S^0-(\mu+\sum_{i=1}^{n}\gamma_i)(S+\sum_{i=1}^{n}I_i)\right]\mathrm{d}t\\
&<\mathrm{d}(S+\sum_{i=1}^{n}I_i)\\
&=\mu S^0-\mu S-\sum_{i=1}^{n}(\mu+\gamma_i)I_i<\left[\mu S^0-\mu(S+\sum_{i=1}^{n}I_i)\right]\mathrm{d}t,
\end{aligned}$$

thus

$$\begin{aligned}
&\frac{\mu S^0}{\mu+\sum_{i=1}^{n}\gamma_i}+e^{-(\mu+\sum_{i=1}^{n}\gamma_i)t}\Big(S(0)+\sum_{i=1}^{n}I_i(0)-\frac{\mu S^0}{\mu+\sum_{i=1}^{n}\gamma_i}\Big)\\
<&S(t)+\sum_{i=1}^{n}I_i(t)<S^0+e^{-\mu t}\Big(S(0)+\sum_{i=1}^{n}I_i(0)-S^0\Big).
\end{aligned}$$

If

$$\frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i} < S(0) + \sum_{i=1}^n I_i(0) < S^0,$$

then we arrive at

$$\frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i} < S(t) + \sum_{i=1}^n I_i(t) < S^0, \quad a.s..$$

Hence, system (1.3) has a positively invariant set

$$\bar{\Gamma} = \Big\{(S, I_1, I_2, \ldots, I_n) \in \mathbb{R}_+^{n+1} : \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i} < S + \sum_{i=1}^n I_i < S^0\Big\}.$$

From now on, we always assume that the initial value $(S(0), I_1(0), I_2(0), \ldots, I_n(0), r(0)) \in \bar{\Gamma} \times \mathcal{M}$.

4. Persistence of the diseases

When considering HIV epidemic models, we are mainly interested in when the disease will prevail in the population. In this section, we shall study the persistence of the diseases. First of all, we present a definition of the persistence in the mean as follows.

Definition 4.1. System (1.3) is said to be persistent in the mean if

$$\liminf_{t \to +\infty} \frac{1}{t} \int_0^t I_i(s) > 0, \quad a.s. \quad i = 1, 2, \ldots, n.$$

Theorem 4.1. *Assume that* $R_0^s = \sum_{i=1}^n \frac{p_i(\sum_{k=1}^{\tilde{N}} \pi_k \sqrt{\beta_i(k)})^2}{(\mu+\gamma_i)^2} > 1$. *Then for any initial value* $(S(0), I_1(0), I_2(0), \ldots, I_n(0), r(0)) \in \mathbb{R}_+^{n+1} \times \mathcal{M}$, *the solution* $(S(t), I_1(t), I_2(t), \ldots, I_n(t), r(t))$ *of system (1.3) has the following property*

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t I_i(s) \mathrm{d}s \geq \frac{R_0^s(R_0^s - 1)}{[\frac{\mu + \sum_{i=1}^n \gamma_i}{\mu S^0}\left(\sum_{i=1}^n \sqrt{c_i p_i \breve{\beta}_i}\right)^2](\frac{\mu+\gamma_1}{p_1 \mu})}, \ a.s. \ i = 1, 2, \ldots, n,$$

which implies if $R_0^s > 1$ *the disease will spread in the world.*

Proof. First we define

$$V_1 = -\sum_{i=1}^n c_i \log I_i,$$

where c_i $(i = 1, 2, \ldots, n)$ are positive constants will be determined later.

Applying the generalized Itô's formula, we obtain

$$
\begin{aligned}
\mathcal{L}V_1 &= -\frac{S}{N}\sum_{i,j=1}^{n} c_i p_i \beta_j(r(t))\frac{I_j}{I_i} + \sum_{i=1}^{n} c_i(\mu+\gamma_i) \\
&= -\frac{S}{N}\sum_{i=1}^{n}\Big(\sqrt{\frac{c_i p_i}{I_i}}\Big)^2 \sum_{j=1}^{n}\Big(\sqrt{\beta_j(r(t))I_j}\Big)^2 + \sum_{i=1}^{n} c_i(\mu+\gamma_i) \\
&\le -\frac{S}{N}\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(r(t))}\Big)^2 + \sum_{i=1}^{n} c_i(\mu+\gamma_i) \\
&= -\frac{N-\sum_{i=1}^{n} I_i}{N}\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(r(t))}\Big)^2 + \sum_{i=1}^{n} c_i(\mu+\gamma_i) \\
&= -\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(r(t))}\Big)^2 + \frac{\sum_{i=1}^{n} I_i}{N}\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(r(t))}\Big)^2 + \sum_{i=1}^{n} c_i(\mu+\gamma_i) \\
&\le -\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(r(t))}\Big)^2 + \frac{\mu+\sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2 \sum_{i=1}^{n} I_i + \sum_{i=1}^{n} c_i(\mu+\gamma_i) \\
&= -\sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(k)}\Big)^2 + \frac{\mu+\sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2 \sum_{i=1}^{n} I_i + \sum_{i=1}^{n} c_i(\mu+\gamma_i) \\
&\quad -\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(r(t))}\Big)^2 + \sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(k)}\Big)^2.
\end{aligned}
\tag{4.1}
$$

Using the inequality in Lemma 2.2 into (4.1) leads to

$$
\begin{aligned}
\mathcal{L}V_1 &\le -\Big(\sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(k)}\Big)\Big)^2 + \sum_{i=1}^{n} c_i(\mu+\gamma_i) + \frac{\mu+\sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2 \sum_{i=1}^{n} I_i \\
&\quad -\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(r(t))}\Big)^2 + \sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(k)}\Big)^2 \\
&= -\Big(\sum_{i=1}^{n}\sqrt{c_i p_i}\sum_{k=1}^{\tilde{N}}\pi_k\sqrt{\beta_i(k)}\Big)^2 + \sum_{i=1}^{n} c_i(\mu+\gamma_i) + \frac{\mu+\sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2 \sum_{i=1}^{n} I_i \\
&\quad -\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(r(t))}\Big)^2 + \sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(k)}\Big)^2 \\
&= -\Big(\sum_{i=1}^{n}\sqrt{c_i p_i\Big(\sum_{k=1}^{\tilde{N}}\pi_k\sqrt{\beta_i(k)}\Big)^2}\Big)^2 + \sum_{i=1}^{n} c_i(\mu+\gamma_i) + \frac{\mu+\sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2 \sum_{i=1}^{n} I_i \\
&\quad -\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(r(t))}\Big)^2 + \sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^{n}\sqrt{c_i p_i \beta_i(k)}\Big)^2.
\end{aligned}
$$

Let

$$
c_i = \frac{p_i\big(\sum_{k=1}^{\tilde{N}}\pi_k\sqrt{\beta_i(k)}\big)^2}{(\mu+\gamma_i)^2}, \quad i = 1, 2, \ldots, n.
$$

Then we obtain

$$
\begin{aligned}
\mathcal{L}V_1 \leq & -\sum_{i=1}^{n}\left(\frac{p_i\big(\sum_{k=1}^{\tilde{N}}\sqrt{\beta_i(k)}\big)^2}{\mu+\gamma_i}\right)^2+\sum_{i=1}^{n}\frac{p_i\big(\sum_{k=1}^{\tilde{N}}\sqrt{\beta_i(k)}\big)^2}{\mu+\gamma_i} \\
& +\frac{\mu+\sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_ip_i\check{\beta}_i}\right)^2\sum_{i=1}^{n}I_i \\
& -(\sum_{i=1}^{n}\sqrt{c_ip_i\beta_i(r(t))})^2+\sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^{n}\sqrt{c_ip_i\beta_i(k)}\Big)^2 \\
\leq & -R_0^s(R_0^s-1)+\frac{\mu+\sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_ip_i\check{\beta}_i}\right)^2\sum_{i=1}^{n}I_i \\
& -(\sum_{i=1}^{n}\sqrt{c_ip_i\beta_i(r(t))})^2+\sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^{n}\sqrt{c_ip_i\beta_i(k)}\Big)^2,
\end{aligned}
\tag{4.2}
$$

where $R_0^s=\sum_{i=1}^{n}\frac{p_i\big(\sum_{k=1}^{\tilde{N}}\pi_k\sqrt{\beta_i(k)}\big)^2}{\mu+\gamma_i}$.

Next, we consider

$$
V_2=\sum_{i=1}^{n}\frac{p_i}{\mu+\gamma_i}(\frac{I_i}{p_i}-\frac{I_1}{p_1}).
$$

Applying the generalized Itô's formula, one arrives at

$$
\mathcal{L}V_2\leq-\sum_{i=1}^{n}I_i+\sum_{i=1}^{n}\frac{p_i(\mu+\gamma_1)}{p_1(\mu+\gamma_i)}I_1\leq-\sum_{i=1}^{n}I_i+\sum_{i=1}^{n}p_i\frac{\mu+\gamma_1}{p_1\mu}I_1=-\sum_{i=1}^{n}I_i+\frac{\mu+\gamma_1}{p_1\mu}I_1.
\tag{4.3}
$$

Therefore, we define

$$
V_3=V_1+\left[\frac{\mu+\sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_ip_i\check{\beta}_i}\right)^2\right]V_2.
$$

From (4.2) and (4.3), it follows that

$$
\begin{aligned}
\mathcal{L}V_3\leq & -R_0^s(R_0^s-1)+[\frac{\mu+\sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_ip_i\check{\beta}_i}\right)^2](\frac{\mu+\gamma_1}{p_1\mu})I_1 \\
& -(\sum_{i=1}^{n}\sqrt{c_ip_i\beta_i(r(t))})^2+\sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^{n}\sqrt{c_ip_i\beta_i(k)}\Big)^2.
\end{aligned}
\tag{4.4}
$$

Integrating (4.4) from 0 to t and then dividing by t on both sides, we obtain

$$
\begin{aligned}
\frac{V_3(t)-V_3(0)}{t}\leq & -R_0^s(R_0^s-1)+[\frac{\mu+\sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_ip_i\check{\beta}_i}\right)^2](\frac{\mu+\gamma_1}{p_1\mu})\frac{1}{t}\int_0^t I_1(s)\mathrm{d}s \\
& -\frac{1}{t}\int_0^t(\sum_{i=1}^{n}\sqrt{c_ip_i\beta_i(r(s))})^2\mathrm{d}s+\sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^{n}\sqrt{c_ip_i\beta_i(k)}\Big)^2.
\end{aligned}
\tag{4.5}
$$

By the ergodic property of the Markov chain $r(t)$, one gets

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t(\sum_{i=1}^n\sqrt{c_ip_i\beta_i(r(s))})^2\mathrm{d}s=\sum_{k=1}^{\tilde{N}}\pi_k\Big(\sum_{i=1}^n\sqrt{c_ip_i\beta_i(k)}\Big)^2,\quad a.s.. \tag{4.6}$$

Since $\frac{\mu S^0}{\mu+\sum_{i=1}^n\gamma_i}<S+\sum_{i=1}^n I_i<S^0$, we obtain

$$V_3(I_1,\ldots,I_n)>-\sum_{i=1}^n\Big(c_i+\frac{p_i}{p_1(\mu+\gamma_i)}\Big)S^0:=\text{constant}.$$

Therefore

$$\liminf_{t\to\infty}\frac{V_3(t)}{t}\geq 0. \tag{4.7}$$

Taking the inferior limit on both sides of (4.5) and combining with (4.6) and (4.7), we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I_1(s)\mathrm{d}s\geq\frac{R_0^s(R_0^s-1)}{[\frac{\mu+\sum_{i=1}^n\gamma_i}{\mu S^0}\Big(\sum_{i=1}^n\sqrt{c_ip_i\check{\beta}_i}\Big)^2](\frac{\mu+\gamma_1}{p_1\mu})},\quad a.s..$$

Using the same approach as above, we can also obtain that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I_i(s)\mathrm{d}s\geq\frac{R_0^s(R_0^s-1)}{[\frac{\mu+\sum_{i=1}^n\gamma_i}{\mu S^0}\Big(\sum_{i=1}^n\sqrt{c_ip_i\check{\beta}_i}\Big)^2](\frac{\mu+\gamma_1}{p_1\mu})},\ i=2,3,\ldots,n,\ a.s..$$

Therefore, we derive the desired statement in Theorem 4.1. This completes the proof. □

5. Extinction of system (1.3)

The other of the main concerns in epidemiology is finding the condition that the disease will be eradicated in a long term. In this section, we shall establish sufficient conditions for exponential extinction of system (1.3).

Theorem 5.1. *Let $(S(t),I_1(t),I_2(t),\ldots,I_n(t),r(t))$ be the positive solution of system (1.3) with any initial value $(S(0),I_1(0),I_2(0),\ldots,I_n(0),r(0))\in\mathbb{R}_+^{n+1}\times\mathcal{M}$. If*

$$R_0^*=\sum_{i=1}^n\frac{\check{\beta}_ip_i}{\mu+\gamma_i}<1,$$

then

$$\limsup_{t\to\infty}\frac{\log I_i(t)}{t}<0,\quad a.s.\quad i=1,2,\ldots,n,$$

which implies the disease will tend to zero exponentially with probability one.

Proof. Define

$$P=\log\left(\sum_{i=1}^n d_iI_i\right),$$

where $d_i = \frac{\breve{\beta}_i}{\mu+\gamma_i}$. Applying the generalized Itô's formula to U, one can get that

$$\begin{aligned}
\mathrm{d}P =&\frac{1}{\sum_{i=1}^n d_i I_i}\left[\sum_{i=1}^n p_i d_i \sum_{j=1}^n \beta_j(k)\frac{I_j S}{N} - \sum_{i=1}^n(\mu+\gamma_i)\right]\mathrm{d}t \\
\leq&\frac{1}{\sum_{i=1}^n d_i I_i}\left[\sum_{i=1}^n p_i d_i \sum_{j=1}^n \breve{\beta}_j I_j - \sum_{i=1}^n(\mu+\gamma_i)\right]\mathrm{d}t \\
=&\frac{1}{\sum_{i=1}^n d_i I_i}(R_0^*-1)\sum_{i=1}^n(\mu+\gamma_i)d_i I_i \mathrm{d}t \\
\leq&(R_0^*-1)(\mu+\max_{1\leq i\leq n}\gamma_i),
\end{aligned} \tag{5.1}$$

where $R_0^* = \sum_{i=1}^n \frac{\breve{\beta}_i p_i}{\mu+\gamma_i}$.

Integrating (5.1) from 0 to t and then dividing by t on both sides lead to that

$$\frac{\log P(t) - \log P(0)}{t} \leq (R_0^*-1)(\mu+\max_{1\leq i\leq n}\gamma_i), \quad a.s.. \tag{5.2}$$

Taking the superior limit on both sides of (7.2), if $R_0^* < 1$, we obtain

$$\limsup_{t\to\infty}\frac{\log P(t)}{t} \leq (R_0^*-1)(\mu+\max_{1\leq i\leq n}\gamma_i) < 0, \quad a.s.,$$

which implies that

$$\limsup_{t\to\infty}\frac{\log I_i(t)}{t} < 0, \quad a.s., \quad i=1,2,\ldots,n.$$

In other words, the disease I_i $(i=1,2,\ldots,n)$ will tend to zero exponentially with probability one. This completes the proof. □

6. Positive recurrence

In the previous section, we have verified the persistence in the mean of the diseases. In this section, in the case of persistence, we will find a domain $U \in \bar{\Gamma}$ which is positive recurrence for the process $(S(t), I_1(t), I_2(t), \ldots, I_n(t))$.

Definition 6.1. The process X_t^x with $X_0 = x$ is recurrent with respect to U, if for any $x \notin U$, $\mathbb{P}(\tau_U < \infty) = 1$, where τ_U is the hitting time of U for the process X_t^x, i.e.,

$$\tau_U = \inf\{t > 0, X_t^x \in U\}.$$

The process X_t^x is said to be positive recurrent with respect to U, if $\mathbb{E}(\tau_U) < \infty$ for any $x \notin U$.

Theorem 6.1. *Let $(S(t), I_1(t), I_2(t), \ldots, I_n(t), r(t))$ be the positive solution of system (1.3) with any initial value $(S(0), I_1(0), I_2(0), \ldots, I_n(0), r(0)) \in \mathbb{R}_+^{n+1} \times \mathcal{M}$. Assume that $R_0^s > 1$, where R_0^s is defined in Theorem 4.1. Then there exists a* *positive solution* $(S(t), I_1(t), I_2(t), \ldots, I_n(t), r(t))$ *of system (1.3) which is positive recurrence with respect to the domain* $U \times \mathcal{M}$, *where*

$$U = \{(S, I_1, I_2, \ldots, I_n) \in \bar{\Gamma} : \varepsilon \le S \le 1/\varepsilon, \varepsilon \le I_1 \le 1/\varepsilon, \varepsilon^3 \le I_i \le 1/\varepsilon^3,$$
$$\frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i} < S + \sum_{i=1}^n I_i < S^0\},$$

where ε *is a sufficiently small positive constant and* $i = 2, 3, \ldots, n$.

Proof. According to Theorem 3.1, we can derive that for any initial value $(S(0), I_1(0), I_2(0), \ldots, I_n(0), r(0)) \in \mathbb{R}_+^{n+1} \times \mathcal{M}$, the solution to system (1.3) is regular. Define a C^2-function

$$h(S, I_1, I_2, \ldots, I_n, k) = M(V_3 - \omega(k)) - \log S - \sum_{i=2}^n \log I_i - \log\left(S^0 - S - \sum_{i=1}^n I_i\right)$$
$$- \log\left(S + \sum_{i=1}^n I_i - \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i}\right),$$

where M satisfies

$$M_0 - M R_0^s (R_0^s - 1) \le -2, \tag{6.1}$$

$\omega(k) = (\omega(1), \omega(2), \ldots, \omega(\tilde{N}))^T$ and constant M_0 will be determined later.

By the monotonicity of $h(S, I_1, I_2, \ldots, I_n, k)$, it is easy to check that $h(S, I_1, I_2, \ldots, I_n, k)$ has the minimum point $h(\bar{S}, \bar{I}_1, \bar{I}_2, \ldots, \bar{I}_n, \bar{k})$.
Then we construct a non-negative Lyapunov function defined by

$$V(S, I_1, I_2, \ldots, I_n, k) = M(V_3 - \omega(k)) - \log S - \sum_{i=2}^n \log I_i - \log\left(S^0 - S - \sum_{i=1}^n I_i\right)$$
$$- \log\left(S + \sum_{i=1}^n I_i - \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i}\right) - h(\bar{S}, \bar{I}_1, \bar{I}_2, \ldots, \bar{I}_n, \bar{k})$$
$$= M\bar{V} + V_4 + V_5 + V_6 + V_7.$$

where $\bar{V} = V_3 - \omega(k)$, $V_4 = -\log S$, $V_5 = -\sum_{i=2}^n \log I_i$, $V_6 = -\log(S^0 - S - \sum_{i=1}^n I_i)$, $V_7 = -\log(S + \sum_{i=1}^n I_i - \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i}) - h(\bar{S}, \bar{I}_1, \bar{I}_2, \ldots, \bar{I}_n, \bar{k})$.

The differential operator $\mathcal{L}$ acting on the function $\bar{V}$ along the solutions, we have

$$\begin{aligned}
\mathcal{L}\bar{V} \le& -(\sum_{i=1}^n \sqrt{c_i p_i \beta_i(k)})^2 - \sum_{l=1}^{\tilde{N}} \gamma_{kl}\omega(l) + \sum_{i=1}^n c_i(\mu + \gamma_i) \\
&+ \left[\frac{\mu + \sum_{i=1}^n \gamma_i}{\mu S^0}\left(\sum_{i=1}^n \sqrt{c_i p_i \breve{\beta}_i}\right)^2\right]\left(\frac{\mu + \gamma_1}{p_1 \mu}\right) I_1 \\
=& -\tilde{R}_0(k) - \sum_{l=1}^{\tilde{N}} \gamma_{kl}\omega(l) + \sum_{i=1}^n c_i(\mu + \gamma_i) \\
&+ \left[\frac{\mu + \sum_{i=1}^n \gamma_i}{\mu S^0}\left(\sum_{i=1}^n \sqrt{c_i p_i \breve{\beta}_i}\right)^2\right]\left(\frac{\mu + \gamma_1}{p_1 \mu}\right) I_1,
\end{aligned} \tag{6.2}$$

where $\tilde{R}_0(k) = (\sum_{i=1}^{n} \sqrt{c_i p_i \beta_i(k)})^2$.

Let $\omega(k) = (\omega(1), \omega(2), \ldots, \omega(\tilde{N}))^T$ be the solution of the following Poisson system

$$\Gamma\omega = \sum_{h=1}^{\tilde{N}} \pi_h \tilde{R}_0(h) - \tilde{R}_0,$$

where $\tilde{R}_0 = (\tilde{R}_0(1), \tilde{R}_0(2), \ldots, \tilde{R}_0(\tilde{N}))^T$. This implies that

$$-\tilde{R}_0(k) - \sum_{l=1}^{\tilde{N}} \gamma_{kl}\omega(l) = -\sum_{k=1}^{\tilde{N}} \pi_k \tilde{R}_0(k). \tag{6.3}$$

Substituting (6.3) into (6.2) and using $c_i = \frac{p_i(\sum_{k=1}^{\tilde{N}} \pi_k \sqrt{\beta_i(k)})^2}{(\mu+\gamma_i)^2}$ in Theorem 4.1, one gets

$$\begin{aligned}
\mathcal{L}\bar{V} \leq & -\Big(\sum_{k=1}^{\tilde{N}} \pi_k (\sum_{i=1}^{n} \sqrt{c_i p_i \beta_i(k)})\Big)^2 + \sum_{i=1}^{n} c_i(\mu+\gamma_i) \\
& + \left[\frac{\mu + \sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2\right]\left(\frac{\mu+\gamma_1}{p_1\mu}\right) I_1 \\
\leq & -\Big(\sum_{i=1}^{n} \sqrt{c_i p_i (\sum_{k=1}^{\tilde{N}} \pi_k \sqrt{\beta_i(k)})^2}\Big)^2 + \sum_{i=1}^{n} c_i(\mu+\gamma_i) \\
& + \left[\frac{\mu + \sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2\right]\left(\frac{\mu+\gamma_1}{p_1\mu}\right) I_1 \\
\leq & -(\sum_{i=1}^{n} \frac{p_i(\sum_{k=1}^{\tilde{N}} \pi_k \sqrt{\beta_i(k)})^2}{(\mu+\gamma_i)^2})^2 + \sum_{i=1}^{n} \frac{p_i(\sum_{k=1}^{\tilde{N}} \pi_k \sqrt{\beta_i(k)})^2}{(\mu+\gamma_i)^2} \\
& + \left[\frac{\mu + \sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2\right]\left(\frac{\mu+\gamma_1}{p_1\mu}\right) I_1 \\
\leq & -R_0^s(R_0^s - 1) + \left[\frac{\mu + \sum_{i=1}^{n}\gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2\right]\left(\frac{\mu+\gamma_1}{p_1\mu}\right) I_1,
\end{aligned}$$

where $R_0^s = \sum_{i=1}^{n} \frac{p_i(\sum_{k=1}^{\tilde{N}} \pi_k \sqrt{\beta_i(k)})^2}{(\mu+\gamma_i)^2}$ is defined in Theorem 4.1.

The differential operator $\mathcal{L}$ acting on the function V_4, V_5, V_6 and V_7 along the solutions, we have

$$\mathcal{L}V_4 = -\frac{\mu S^0}{S} + \mu + \sum_{j=1}^{n} \beta_j(k)\frac{I_j}{N} \leq -\frac{\mu S^0}{S} + \mu + \sum_{j=1}^{n} \check{\beta}_j.$$

$$\begin{aligned}
\mathcal{L}V_5 = & -\sum_{i=2}^{n} p_i \sum_{j=1}^{n} \beta_j(k)\frac{I_j S}{I_i N} + \sum_{i=2}^{n}(\mu+\gamma_i) \\
\leq & -\sum_{i=2}^{n} p_i \beta_1(k)\frac{I_1 S}{I_i S^0} + \sum_{i=2}^{n}(\mu+\gamma_i)
\end{aligned}$$

$$\le -\frac{\hat{\beta}_1}{S^0}\sum_{i=2}^{n} p_i \frac{I_1 S}{I_i} + \sum_{i=2}^{n}(\mu+\gamma_i).$$

$$\begin{aligned}
\mathcal{L}V_6 &= \frac{1}{S^0 - S - \sum_{i=1}^n I_i}(\mu S^0 - \mu(S + \sum_{i=1}^{n} I_i) - \sum_{i=1}^{n}\gamma_i I_i) \\
&\le \mu - \frac{\sum_{i=1}^n \gamma_i I_i}{S^0 - S - \sum_{i=1}^n I_i}. \\
\mathcal{L}V_7 &= -\frac{1}{S + \sum_{i=1}^n I_i - \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i}}(\mu S^0 - \mu(S + \sum_{i=1}^{n} I_i) - \sum_{i=1}^{n}\gamma_i I_i) \\
&= -\frac{1}{S + \sum_{i=1}^n I_i - \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i}}(\mu S^0 - (\mu + \sum_{i=1}^{n}\gamma_i)(S + \sum_{i=1}^{n} I_i) + (\sum_{i=1}^{n}\gamma_i)S) \\
&\le \mu + \sum_{i=1}^{n}\gamma_i - \frac{(\sum_{i=1}^n \gamma_i)S}{S + \sum_{i=1}^n I_i - \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i}}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathcal{L}V =& \mathcal{L}(M\bar{V} + V_4 + V_5 + V_6 + V_7) \\
\le& - MR_0^s(R_0^s - 1) + M\left[\frac{\mu + \sum_{i=1}^n \gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2\right]\left(1 + \frac{\mu+\gamma_1}{p_1\mu}\right) I_1 \\
& - \frac{\mu S^0}{S} + \mu + \sum_{j=1}^{n}\check{\beta}_j - \frac{\hat{\beta}_1}{S^0}\sum_{i=2}^{n} p_i \frac{I_1 S}{I_i} + \sum_{i=2}^{n}(\mu+\gamma_i) + \mu - \frac{\sum_{i=1}^n \gamma_i I_i}{S^0 - S - \sum_{i=1}^n I_i} \\
& + \mu + \sum_{i=1}^{n}\gamma_i - \frac{(\sum_{i=1}^n \gamma_i)S}{S + \sum_{i=1}^n I_i - \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i}} \\
\le& M_0 - MR_0^s(R_0^s - 1) + f(I_1) - \frac{\mu S^0}{S} - \frac{\hat{\beta}_1}{S^0}\sum_{i=2}^{n} p_i \frac{I_1 S}{I_i} - \frac{\sum_{i=1}^n \gamma_i I_i}{S^0 - S - \sum_{i=1}^n I_i} \\
& - \frac{(\sum_{i=1}^n \gamma_i)S}{S + \sum_{i=1}^n I_i - \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i}},
\end{aligned}$$

where

$$M_0 = (n+2)\mu + \sum_{j=1}^{n}\check{\beta}_j + \gamma_1 + 2\sum_{i=2}^{n}\gamma_i,$$

and

$$f(I_1) = M[\frac{\mu + \sum_{i=1}^n \gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2]\left(\frac{\mu+\gamma_1}{p_1\mu}\right) I_1$$

has the upper bound $f^u = M[\frac{\mu + \sum_{i=1}^n \gamma_i}{\mu S^0}\left(\sum_{i=1}^{n}\sqrt{c_i p_i \check{\beta}_i}\right)^2]\left(\frac{\mu+\gamma_1}{p_1\mu}\right) S^0$.

Next we consider the bounded open set

$$\begin{aligned}
U =& \{(S, I_1, I_2, \ldots, I_n) \in \bar{\Gamma} : \varepsilon \le S \le 1/\varepsilon, \varepsilon \le I_1 \le 1/\varepsilon, \varepsilon^3 \le I_i \le 1/\varepsilon^3, i = 2, 3, \ldots, n, \\
& \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i} < S + \sum_{i=1}^{n} I_i < S^0\},
\end{aligned}$$

where ε is a sufficiently small positive constant satisfying the following conditions

$$f^u - \frac{\mu S^0}{\varepsilon} \le -1, \tag{6.4}$$

$$M\Big[\frac{\mu + \sum_{i=1}^n \gamma_i}{\mu S^0}\left(\sum_{i=1}^n \sqrt{c_i p_i \check{\beta}_i}\right)^2 \left(\frac{\mu+\gamma_1}{p_1 \mu}\right)\Big]\varepsilon \le 1, \tag{6.5}$$

$$f^u - \frac{\hat{\beta}_1 \sum_{i=2}^n p_i}{\varepsilon S^0} \le -1, \tag{6.6}$$

$$f^u - \frac{\sum_{i=1}^n \gamma_i}{\varepsilon} \le -1. \tag{6.7}$$

Then we divide $\bar{\Gamma} \setminus U$ into the following five domains

$$\bar{\Gamma} \setminus U = U_1^c \cup U_2^c \cup U_3^c \cup U_4^c \cup U_5^c,$$

and

$$U_1^c = \{(S, I_1, I_2, \dots, I_n) \in \bar{\Gamma} : 0 < S < \varepsilon\},\ U_2^c = \{(S, I_1, I_2, \dots, I_n) \in \bar{\Gamma} : 0 < I_1 < \varepsilon\},$$
$$U_3^c = \{(S, I_1, I_2, \dots, I_n) \in \bar{\Gamma} : S \ge \varepsilon,\ I_1 \ge \varepsilon,\ 0 < I_i < \varepsilon^3,\ i = 2, 3, \dots, n\},$$
$$U_4^c = \{(S, I_1, I_2, \dots, I_n) \in \bar{\Gamma} : I_i > \varepsilon^3,\ S^0 - \varepsilon^4 < S + \sum_{i=1}^n I_i < S^0,\ i = 2, 3, \dots, n\},$$
$$U_5^c = \{(S, I_1, I_2, \dots, I_n) \in \bar{\Gamma} : S > \varepsilon,\ \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i} < S + \sum_{i=1}^n I_i < \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i} + \varepsilon^2\}.$$

Now we verify the negativity of $\mathcal{L}V(S, I_1, I_2, \dots, I_n, k)$ on $(\mathbb{R}_+^{n+1} \setminus U) \times \mathcal{M}$, which is equivalent to verifying it on the above five domains, respectively.
Case 1. If$(S, I_1, I_2, \dots, I_n, k) \in U_1^c \times \mathcal{M}$, from (6.1) and (6.4) we know

$$\mathcal{L}V \le M_0 - MR_0^s(R_0^s - 1) + f(I_1) - \frac{\mu S^0}{S} \le f^u - \frac{\mu S^0}{\varepsilon} \le -1.$$

Case 2. If$(S, I_1, I_2, \dots, I_n, k) \in U_2^c \times \mathcal{M}$, from (6.1) and (6.5) we get

$$\begin{aligned}\mathcal{L}V \le& M_0 - MR_0^s(R_0^s - 1) + M\Big[\frac{\mu + \sum_{i=1}^n \gamma_i}{\mu S^0}\left(\sum_{i=1}^n \sqrt{c_i p_i \check{\beta}_i}\right)^2 \left(\frac{\mu+\gamma_1}{p_1 \mu}\right)\Big] I_1 \\ \le& M_0 - MR_0^s(R_0^s - 1) + M\Big[\frac{\mu + \sum_{i=1}^n \gamma_i}{\mu S^0}\left(\sum_{i=1}^n \sqrt{c_i p_i \check{\beta}_i}\right)^2 \left(\frac{\mu+\gamma_1}{p_1 \mu}\right)\Big]\varepsilon \le -1.\end{aligned}$$

Case 3. If$(S, I_1, I_2, \dots, I_n, k) \in U_3^c \times \mathcal{M}$, (6.1) and from (6.6) we obtain

$$\mathcal{L}V \le M_0 - MR_0^s(R_0^s - 1) + f(I_1) - \frac{\hat{\beta}_1}{S^0}\sum_{i=2}^n p_i \frac{I_1 S}{I_i} \le f^u - \frac{\hat{\beta}_1 \sum_{i=2}^n p_i}{\varepsilon S^0} \le -1,$$

Case 4. If$(S, I_1, I_2, \dots, I_n, k) \in U_4^c \times \mathcal{M}$, from (6.1) and (6.7) we have

$$\mathcal{L}V \le M_0 - MR_0^s(R_0^s - 1) + f(I_1) - \frac{\sum_{i=1}^n \gamma_i I_i}{S^0 - S - \sum_{i=1}^n I_i}$$

$$< f^u - \frac{\sum_{i=1}^n \gamma_i \varepsilon^3}{\varepsilon^4} = f^u - \frac{\sum_{i=1}^n \gamma_i}{\varepsilon} \leq -1.$$

Case 5. If$(S, I_1, I_2, \ldots, I_n, k) \in U_5^c \times \mathcal{M}$, from (6.1) and (6.7) we know

$$\begin{aligned}\mathcal{L}V &\leq M_0 - MR_0^s(R_0^s - 1) + f(I_1) - \frac{(\sum_{i=1}^n \gamma_i)S}{S + \sum_{i=1}^n I_i - \frac{\mu S^0}{\mu + \sum_{i=1}^n \gamma_i}} \\ &< f^u - \frac{\sum_{i=1}^n \gamma_i \varepsilon}{\varepsilon^2} = f^u - \frac{\sum_{i=1}^n \gamma_i}{\varepsilon} \leq -1.\end{aligned}$$

From the above discussions, one can see that for a sufficiently small ε_0,

$$\mathcal{L}V \leq -1, \quad \text{for any } (S, I_1, I_2, \ldots, I_n, k) \in (\bar{\Gamma} \setminus U) \times \mathcal{M}. \tag{6.8}$$

Let $(S(0), I_1(0), I_2(0), \ldots, I_n(0), r(0)) \in (\bar{\Gamma} \setminus U) \times \mathcal{M}$, according to the generalized Itô's formula and using (6.8), we obtain

$$\begin{aligned}&\mathbb{E}[V(S(\tau_U), I_1(\tau_U), I_2(\tau_U), \ldots, I_n(\tau_U))] - V(S(0), I_1(0), I_2(0), \ldots, I_n(0), r(0)) \\ =&\mathbb{E}\int_0^{\tau_U} \mathcal{L}V(S(t), I_1(t), I_2(t), \ldots, I_n(t), r(t))\mathrm{d}t \leq -\mathbb{E}(\tau_U).\end{aligned}$$

Therefore, by virtue of the positivity of V, we conclude

$$\mathbb{E}(\tau_U) \leq V(S(0), I_1(0), I_2(0), \ldots, I_n(0), r(0)).$$

This proof is completed. □

7. Numerical simulations

In this section, we shall use the Milstein's Higher Order Method developed in [8] to illustrate our theoretical analysis. Assume that the Markov chain $r(t)$ is on the state space $\mathcal{M} = \{1, 2, 3\}$ with the generator as follows

$$\tilde{\Gamma} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix}.$$

Then, we solve the linear equation (2.1) and obtain that the Markov chain $r(t)$ has a unique stationary distribution

$$\pi = (\pi_1, \pi_2, \pi_3) = \left(\frac{3}{7}, \frac{2}{7}, \frac{2}{7}\right).$$

Example 7.1. The parameter values in system (1.3) are fixed

$$\begin{aligned}&S_0 = 2,\ \mu = 0.2,\ p_1 = 0.2,\ p_2 = 0.3,\ p_3 = 0.5,\ \gamma_1 = 0.05,\ \gamma_2 = 0.06,\ \gamma_3 = 0.07, \\ &(\beta_1(1),\ \beta_1(2),\ \beta_1(3)) = (0.2, 0.25, 0.3),\ (\beta_2(1), \beta_2(2),\ \beta_2(3)) = (0.3, 0.35, 0.4), \\ &(\beta_3(1),\ \beta_3(2),\ \beta_3(3)) = (0.4, 0.45, 0.5).\end{aligned}$$

Let initial value be $(S(0), I_1(0), I_2(0), I_3(0)) = (1.5, 1, 0.9, 0.8)$. Simple computations result

$$R_0^s = \sum_{i=1}^{n} \frac{p_i(\sum_{k=1}^{\tilde{N}} \pi_k \sqrt{\beta_i(k)})^2}{\mu + \gamma_i} \approx 1.4053 > 1.$$

Then according to Theorem 4.1, we can obtain the disease is persistent in the mean a.s., and from Theorem 6.1, the solution $(S(t), I_1(t), I_2(t), \dots I_n(t), r(t))$ of system (1.3) is positive recurrence. Fig.1 confirms this.

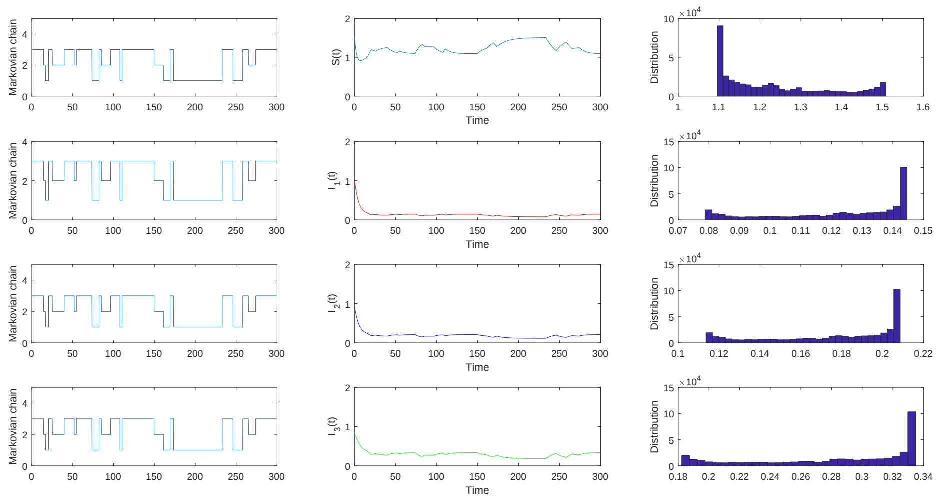

Figure 1. The solution $(S(t), I_1(t), I_2(t), I_3(t), r(t))$ of system (1.3) is positive recurrence. The picture on the left and right are the Markovian chain and the density distributions of system (1.3), respectively.

Example 7.2. Reselect the parameter values in system (1.3) as follows, other parameters are the same as earlier:

$$\mu = 0.3, \quad \gamma_1 = 0.2, \quad \gamma_2 = 0.25, \quad \gamma_3 = 0.3.$$

Let $(S(0), I_1(0), I_2(0), I_3(0)) = (1.5, 3, 2, 1)$. Then direct calculation leads to

$$R_0^* = \sum_{i=1}^{n} \frac{\check{\beta}_i p_i}{\mu + \gamma_i} \approx 0.7548 < 1.$$

Therefore, by the condition of Theorem 5.1, we can obtain that $I_i(t)$ of system (1.3) will tend to zero exponentially with probability one, where $i = 1, 2, 3$.

8. Conclusion

In this paper, considering that the transmission rate may change over time, we have studied a stochastic S-DI-A epidemic model with standard incidence under Markovian switching. Firstly, we obtain the existence of the unique global positive solution. Then By using stochastic Lyapunov functions with regime switching, the critical condition R_0^s for persistence in the mean of the disease, which is also shown to determine the existence of positive recurrence of the solutions to the model (1.3). Furthermore, we establish sufficient conditions R_0^* for the exponential extinction of the disease.

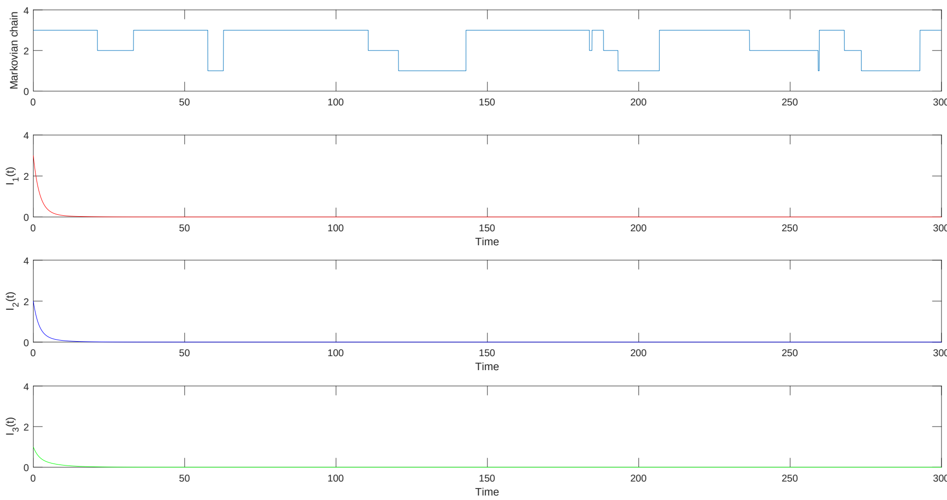

Figure 2. $I_1(t)$, $I_2(t)$ and $I_3(t)$ of stochastic model (1.3) will go to extinction exponentially with probability one.

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