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NONLINEAR SINGULAR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS WITH DISCONTINUOUS RIGHT-HAND SIDE*

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Abstract In this paper, an asymptotic method for nonlinear singular singularly perturbed boundary value problems with discontinuous right-hand side is investigated. We not only show existence of a solution with a step-like contrast structure, but also construct an asymptotic expansion of the solution. In addition, remainder estimation of the approximate solution is also given. Finally, an example is used to verify the correctness of the above theory.

Keywords Asymptotic method, singularly perturbed equations, discontinuous right-hand side.

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1. Introduction

Many phenomena in physics, biology, chemistry and so on can be described by singularly perturbed problems associated with various types of discontinuous equations. Thus the singularly perturbed problems with discontinuous right-hand side have been discussed by many scholars [1, 2, 5, 12–14]. Ding et al. [2] gave an asymptotic solution for a class of singularly perturbed semi-linear boundary value problems with discontinuous functions. Nefedov and Ni [5] developed asymptotic methods for a one-dimensional stationary reaction-diffusion equation in which the term describing reaction undergoes a discontinuity at some point.

The above researches usually require that there are only non-zero eigenvalues for the degenerate equations. To weaken this assumption, the singular singularly perturbed problems have been discussed by many experts [7, 8, 10, 11]. O'Malley and Flaherty [7] and Vasil'eva and Butuzov [10] investigated asymptotic methods for the nonlinear singular singularly perturbed initial value problem. Vasil'eva and Butuzov [10] also discussed the asymptotic solution for a class of singular singularly perturbed boundary value problem, and proved the consistent validity of the asymptotic expansion. Schmeiser and Weiss [8] constructed an asymptotic solution for nonlinear singular singularly perturbed boundary value problems, and estimated the remainder by the principle of squeezing mapping.

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The motivation of this paper is to study the asymptotic analysis of the nonlinear singular singularly perturbed problems with discontinuous right-hand side. Since the discontinuity of the right-hand side, solutions of the problems will change drastically in a short period of time. For simplicity, we assume that the problems have only one discontinuity point. The case of multiple discontinuity points can be analyzed by the similar method. We divide the problems into two parts at the discontinuity point, and study the asymptotic solutions of the two parts, which is inspired from the methods in [6]. Owing to the degenerate problems of the two parts have zero eigenvalue, some commonly used tools are not available, such as: the method of boundary layer functions [9] and the Fenichel's theorem [3], etc. In order to solve this problem, the method of boundary layer function [10] is used for the asymptotic solutions of the two parts.

The rest of the paper is structured as follows: In section 2, the nonlinear singular singularly perturbed problem with discontinuous right-hand side is formulated. In section 3, associated systems and assumptions are given to construct the asymptotic solution of the problem. In section 4, the asymptotic solution with a step-like contrast structure is constructed. In section 5, results for existence of the solution and estimations of remainders are presented. Finally, an example is given to verify the correctness of the above theory.

2. Statement of the Problem

Consider the nonlinear singular singularly perturbed problem

$$\begin{aligned} &\mu\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t,\mu), \\ &Ax(0,\mu) = Ax^0, \quad Bx(1,\mu) = Bx^1, \end{aligned} \tag{2.1}$$

where $0 < \mu \ll 1$ is a small parameter,

$$f(x,t,\mu) = \begin{cases} f^{(-)}(x,t,\mu), & 0 < t < t_0, \\ f^{(+)}(x,t,\mu), & t_0 \leqslant t < 1, \end{cases}$$
$$f^{(\mp)}: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n, \quad x \in \mathbb{R}^n$$

and

$$A = \mathrm{diag}(E_s, 0, E_c), \quad B = \mathrm{diag}(0, E_u, 0), \quad s+u+c = n$$

are $(n \times n)$-order matrices, E_a is the a-dimensional identity matrix. The functions $f^{(\mp)}(x,t,\mu)$ are sufficiently smooth in the domain $D = D^{(-)} \cup D^{(+)}$, where

$$\begin{aligned} D^{(-)} &= \left\{ \ (x,t) \ \middle| \ |x_i| \leqslant q,\ i = 1, \cdots, n,\ 0 \leqslant t \leqslant t_0 \ \right\}, \\ D^{(+)} &= \left\{ \ (x,t) \ \middle| \ |x_i| \leqslant q,\ i = 1, \cdots, n,\ t_0 \leqslant t \leqslant 1 \ \right\}, \end{aligned}$$

with q is some positive number. In addition, the functions $f^{(\mp)}(x,t,\mu)$ are satisfied

$$\lim_{t \to t_0} f^{(-)}(x,t,\mu) \neq f^{(+)}(x,t_0,\mu), \quad x \in D$$

at point t_0.

The solution of system (2.1) is a function

$$x(t,\mu) \in C[0,1] \cap C^1((0,t_0) \cup (t_0,1)).$$

The following assumption is given to construct asymptotic approximation for solution $u(t,\mu)$ of problem (2.1).

Assumption 2.1. *The degenerate problem*

$$f(x,t,0) = 0$$

has a family of solution of the form

$$x = \begin{cases} \varphi^{(-)}\left(\alpha^{(-)}(t),t\right) = \varphi^{(-)}\left(\alpha_1^{(-)}(t),\cdots,\alpha_c^{(-)}(t),t\right), & 0 \leqslant t < t_0, \\ \varphi^{(+)}\left(\alpha^{(+)}(t),t\right) = \varphi^{(+)}\left(\alpha_1^{(+)}(t),\cdots,\alpha_c^{(+)}(t),t\right), & t_0 \leqslant t \leqslant 1, \end{cases}$$

where $\varphi^{(\mp)}\left(\alpha^{(\mp)}(t),t\right)$ satisfy

(1) $\varphi^{(\mp)}\left(\alpha^{(\mp)}(t),t\right)$ *are sufficiently smooth in the domain D,*

(2) $\varphi_\alpha^{(\mp)}\left(\alpha^{(\mp)}(t),t\right)$ *have constant rank c.*

And the eigenvalues $\lambda_i^{(\mp)}$ of $f_x^{(\mp)}\left(\varphi^{(\mp)}(\alpha^{(\mp}(t),t),t,0\right)$ satisfy

$$\begin{aligned} \lambda_i^{(\mp)} &< 0, \quad i = 1,\cdots,s, \\ \lambda_i^{(\mp)} &> 0, \quad i = s+1,\cdots,s+u, \\ \lambda_i^{(\mp)} &= 0, \quad i = s+u+1,\cdots,n. \end{aligned}$$

The asymptotic solution to the problem (2.1) will be constructed by the above assumption.

3. Associated Systems

In order to construct the zeroth-order asymptotic solution, associated systems

$$\frac{\mathrm{d}\tilde{x}^{(\mp)}}{\mathrm{d}\tilde{\tau}} = f^{(\mp)}\left(\tilde{x}^{(\mp)},\tilde{t},0\right), \quad -\infty < \tilde{\tau} < \infty \tag{3.1}$$

of problem (2.1) are constructed, where $\tilde{t} \in [0,1]$ is temporarily fixed. Obviously, the equilibrium point of the associated system is $\tilde{x}^{(\mp)} = \varphi^{(\mp)}\left(\alpha^{(\mp)}\left(\tilde{t}\right),\tilde{t}\right)$ (denoted as $M^{(\mp)}\left(\tilde{t}\right)$). By Assumption 2.1 and [4, P548], through this equilibrium point $M^{(\mp)}\left(\tilde{t}\right)$ there passes a stable manifold $W^s\left(M^{(\mp)}\left(\tilde{t}\right)\right)$ of dimension s of the form

$$W^s\left(M^{(\mp)}\left(\tilde{t}\right)\right) = \left\{ \left(\tilde{x}_s^{(\mp)},\tilde{x}_u^{(\mp)},\tilde{x}_c^{(\mp)}\right) \;\middle|\; \begin{array}{l} \tilde{x}_u = \phi_u^{(\mp)}\left(\tilde{x}_s^{(\mp)},\alpha^{(\mp)}\right), \\ \qquad\qquad \tilde{x}_c^{(\mp)} = \phi_c^{(\mp)}\left(\tilde{x}_s,\alpha^{(\mp)}\right) \end{array} \right\}$$

and an unstable manifold $W^u\left(M^{(\mp)}\left(\tilde{t}\right)\right)$ of dimension u of the form

$$W^u\left(M^{(\mp)}\left(\tilde{t}\right)\right) = \left\{ \left(\tilde{x}_s^{(\mp)},\tilde{x}_u^{(\mp)},\tilde{x}_c^{(\mp)}\right) \;\middle|\; \begin{array}{l} \tilde{x}_s^{(\mp)} = \psi_s^{(\mp)}\left(\tilde{x}_u^{(\mp)},\alpha^{(\mp)}\right), \\ \qquad\qquad \tilde{x}_c^{(\mp)} = \psi_c^{(\mp)}\left(\tilde{x}_u,\alpha^{(\mp)}\right) \end{array} \right\},$$

where $\tilde{x}_s$ is the first s components of the vector $\tilde{x}$, $\tilde{x}_c$ is the middle c components of the vector $\tilde{x}$, $\tilde{x}_u$ is the last u components of the vector $\tilde{x}$, and $\phi_u^{(\mp)}$, $\phi_c^{(\mp)}$, $\psi_s^{(\mp)}$ and $\psi_c^{(\mp)}$ have the similar meaning. Moreover, $\phi_u^{(\mp)}$, $\phi_c^{(\mp)}$, $\psi_s^{(\mp)}$ and $\psi_c^{(\mp)}$ are sufficiently smooth functions. If $\tilde{x} \in W^s\left(M^{(\mp)}\left(\tilde{t}\right)\right)$ then

$$||\tilde{x}(\tau)|| \leq C\mathrm{e}^{-\kappa\tau}, \quad \tau \geqslant 0.$$

If $\tilde{x} \in W^u\left(M^{(\mp)}\left(\tilde{t}\right)\right)$ then

$$||\tilde{x}(\tau)|| \leq C\mathrm{e}^{\kappa\tau}, \quad \tau \leqslant 0.$$

Here and in the following, the parameters C and κ are some positive numbers, which can be different in different inequalities. Let

$$\tilde{x}^{(-)}(0) = \tilde{x}_0^{(-)}, \quad \tilde{x}^{(+)}(0) = \tilde{x}_0^{(+)}.$$

The following assumptions are made to construct zeroth-order asymptotic solution of (2.1) .

Assumption 3.1. *We assume*

$$A\left(\tilde{x}_0^{(-)} - \varphi^{(-)}\left(\alpha^{(-)}\left(0\right), 0\right)\right) \in W^s\left(M^{(\mp)}\left(0\right)\right),$$

$$B\left(\tilde{x}_0^{(+)} - \varphi^{(+)}\left(\alpha^{(+)}\left(1\right), 1\right)\right) \in W^u\left(M^{(\mp)}\left(1\right)\right).$$

Assumption 3.2. *We assume*

$$\Delta\tilde{x}^{(\mp)} = 0$$

have a solution $\alpha^{(-)}(0) = \alpha_0^{(-)}$ *and* $\alpha^{(+)}(t_0) = \alpha_0^{(+)}$*, respectively, where*

$$\Delta\tilde{x}^{(-)} = \tilde{x}_{0,c}^{(-)} - \varphi_c^{(-)}\left(\alpha^{(-)}(0), 0\right) - \phi_c^{(-)}\left(\tilde{x}_{0,s}^{(-)} - \varphi_s^{(-)}\left(\alpha^{(-)}(0), 0\right), \alpha^{(-)}\right),$$

$$\Delta\tilde{x}^{(+)} = \tilde{x}_{0,c}^{(+)} - \varphi_c^{(+)}\left(\alpha^{(+)}(t_0), t_0\right) - \phi_c^{(+)}\left(\tilde{x}_{0,s}^{(+)} - \varphi_s^{(+)}\left(\alpha^{(+)}(t_0), t_0\right), \alpha^{(+)}\right),$$

moreover,

$$\left.\frac{\mathrm{d}\Delta\tilde{x}^{(-)}}{\mathrm{d}\alpha^{(-)}(0)}\right|_{\alpha^{(-)}(0)=\alpha_0^{(-)}} \neq 0, \quad \left.\frac{\mathrm{d}\Delta\tilde{x}^{(+)}}{\mathrm{d}\alpha^{(+)}(0)}\right|_{\alpha^{(+)}(t_0)=\alpha_0^{(+)}} \neq 0.$$

Let $h^{(\mp)}\left(\alpha^{(\mp)}(t), t\right)$ are the $c \times n$ matrix whose rows are the eigenvectors $h_i^{(\mp)}\left(\alpha^{(\mp)}(t), t\right)$, $i = 1, \cdots, c$ of the adjoint matrix $\left[f_x^{(\mp)}\left(\varphi^{(\mp)}\left(\alpha^{(\mp)}(t), t\right), t, 0\right)\right]^*$ corresponding to the zero eigenvalue.

Assumption 3.3. *Let problems*

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha^{(\mp)}(t) = \left(h^{(\mp)}\left(\alpha^{(\mp)}(t), t\right)\varphi_\alpha^{(\mp)}(t)\right)^{-1} h^{(\mp)}\left(\alpha^{(\mp)}(t), t\right)\left(\bar{f}_\mu^{(\mp)}(t) - \varphi_t^{(\mp)}(t)\right)$$

with the initial conditions $\alpha^{(\mp)}(t_0) = \alpha_0^{(\mp)}$ *have unique solutions.*

The following assumptions are given to satisfy the continuity of the solution.

Assumption 3.4. *Let system* (3.1) *has* $n-1$ *independent first integrals*

$$\Phi_i\left(\tilde{x}_1^{(\mp)}, \tilde{x}_2^{(\mp)}, \cdots, \tilde{x}_n^{(\mp)}, t_0\right) = C_i, \quad i = 1, 2, \cdots, n-1.$$

Then

$$\Phi_i\left(\tilde{x}_1^{(\mp)}, \tilde{x}_2^{(\mp)}, \cdots, \tilde{x}_n^{(\mp)}, t_0\right) = \Phi_i\left(\varphi^{(\mp)}\left(\alpha^{(\mp)}(t_0), t_0\right), t_0\right), \quad i = 1, 2, \cdots, n-1$$

are the trajectory passing through the point $M^{(\mp)}(t_0)$*, and can be expressed as:*

$$\tilde{x}_i^{(\mp)} = X\left(\tilde{x}_1^{(\mp)}\right), \quad i = 2, \cdots, n.$$

Assumption 3.5.

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{x}_0^{(-)}}\left(\tilde{x}^{(-)}(0) - \tilde{x}^{(+)}(0)\right) \neq 0.$$

The asymptotic solution of (2.1) is constructed by the above assumptions in the next section.

4. Construction of Formal Asymptotic Representation

The main idea of the following asymptotic analysis is to divide the problem (2.1) at the discontinuity point $t = t_0$ into two parts. By the method of boundary layer function [10], asymptotic solutions on the interval $[0, t_0]$ and $[t_0, 1]$ are constructed, respectively. And the asymptotic solutions will be smoothly sewn together at the discontinuity point $t = t_0$ by the matching method. We consider the following left boundary value problem:

$$\begin{aligned} &\mu\frac{\mathrm{d}x^{(-)}}{\mathrm{d}t} = f^{(-)}\left(x^{(-)}, t, \mu\right), \quad 0 < t < t_0, \\ &Ax^{(-)}(0, \mu) = Ax^0, \quad Bx^{(-)}(t_0, \mu) = Bx^*, \end{aligned} \tag{4.1}$$

and right boundary value problem:

$$\begin{aligned} &\mu\frac{\mathrm{d}x^{(+)}}{\mathrm{d}t} = f^{(+)}\left(x^{(+)}, t, \mu\right), \quad t_0 < t < 1, \\ &Ax^{(+)}(t_0, \mu) = Ax^*, \quad Bx^{(+)}(1, \mu) = Bx^1, \end{aligned} \tag{4.2}$$

where the undetermined function x^* is expressed in explicit form:

$$x^* = x_0^* + \mu x_1^* + \cdots + \mu^k x_k^* + \cdots.$$

The formal asymptotic solutions of the problems (4.1) and (4.2) are constructed as follows:

$$\begin{aligned} x^{(-)}(t, \mu) &= Lx(\tau_0, \mu) + \bar{x}^{(-)}(t, \mu) + Q^{(-)}x(\tau, \mu), \\ x^{(+)}(t, \mu) &= Q^{(+)}x(\tau, \mu) + \bar{x}^{(+)}(t, \mu) + Rx(\tau_1, \mu), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
&\bar{x}^{(\mp)}(t,\mu) = \bar{x}_0^{(\mp)}(t) + \mu\bar{x}_1^{(\mp)}(t) + \cdots + \mu^k\bar{x}_k^{(\mp)}(t) + \cdots ,\\
&Lx(\tau_0,\mu) = L_0x(\tau_0) + \mu L_1x(\tau_0) + \cdots + \mu^k L_kx(\tau_0) + \cdots ,\\
&Q^{(\mp)}x(\tau,\mu) = Q_0^{(\mp)}x(\tau) + \mu Q_1^{(\mp)}x(\tau) + \cdots + \mu^k Q_k^{(\mp)}x(\tau) + \cdots ,\\
&Rx(\tau_1,\mu) = R_0x(\tau_1) + \mu R_1x(\tau_1) + \cdots + \mu^k R_kx(\tau_1) + \cdots ,\\
&\tau_0 = t\mu^{-1}, \quad \tau = (t-t_0)\mu^{-1}, \quad \tau_1 = (t-1)\mu^{-1}
\end{aligned} \tag{4.4}$$

with the functions $\bar{x}_k^{(\mp)}(t)$, $k \geqslant 0$ are regular series, $L_kx(\tau_0)\big(\text{resp. } R_kx(\tau_1)\big)$, $k \geqslant 0$ are boundary layer series in a neighborhood of $t = 0$(resp. $t = 1$), $Q_k^{(-)}x(\tau)\big(\text{resp. } Q_k^{(+)}x(\tau)\big)$, $k \geqslant 0$ are internal layer series in the left (resp. right) neighborhood the point $t = t_0$. At the same time, the boundary layer functions and internal layer functions satisfy $L_kx(+\infty) = 0$, $Q_k^{(\mp)}x(\mp\infty) = 0$, $R_kx(-\infty) = 0$, $k \geqslant 0$. Moreover, solutions $x^{(\mp)}(t,\mu)$ should satisfy the following continuity condition

$$x^{(-)}(t_0,\mu) = x^{(+)}(t_0,\mu) \tag{4.5}$$

at point t_0. According to the method of boundary layer function [10], we substitute (4.3) into (4.1) and (4.2) and separate them into three parts according to the scale variables t, τ_0, τ and τ_1 to get

$$\begin{aligned}
\mu\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^{(\mp)}(t,\mu) &= \bar{f}^{(\mp)}, \quad \frac{\mathrm{d}}{\mathrm{d}\tau_0}Lx(\tau_0,\mu) = Lf,\\
\frac{\mathrm{d}}{\mathrm{d}\tau}Q^{(\mp)}x(\tau,\mu) &= Q^{(\mp)}f, \quad \frac{\mathrm{d}}{\mathrm{d}\tau_1}Rx(\tau_1,\mu) = Rf,
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
\bar{f}^{(\mp)} &= f^{(\mp)}\left(\bar{x}^{(\mp)}(t,\mu), t, \mu\right),\\
Lf &= f^{(-)}\left(\bar{x}^{(-)}(\mu\tau_0,\mu) + Lx(\tau_0,\mu), \mu\tau_0, \mu\right)\\
&\quad - f^{(-)}\left(\bar{x}^{(-)}(\mu\tau_0,\mu), \mu\tau_0, \mu\right),\\
Q^{(\mp)}f &= f^{(\mp)}\left(\bar{x}^{(\mp)}(\mu\tau + t_0,\mu) + Q^{(\mp)}x(\tau,\mu), \mu\tau + t_0, \mu\right)\\
&\quad - f^{(\mp)}\left(\bar{x}^{(\mp)}(\mu\tau + t_0,\mu), \mu\tau + t_0, \mu\right),\\
Rf &= f^{(+)}\left(\bar{x}^{(+)}(\mu\tau_1 + 1,\mu) + Rx(\tau_1,\mu), \mu\tau_1 + 1, \mu\right)\\
&\quad - f^{(+)}\left(\bar{x}^{(+)}(\mu\tau_1 + 1,\mu), \mu\tau_1 + 1, \mu\right).
\end{aligned}$$

The functions $f^{(\mp)}$, Lf, $Q^{(\mp)}f$ and Rf can be written as power series of μ by substituting formal asymptotic series (4.4) into (4.6), and terms with different powers of μ should match, then the equations to determine $\bar{x}_k^{(\mp)}(t)$, $L_kx(\tau_0)$, $Q_k^{(\mp)}x(\tau)$ and $R_kx(\tau_1)$, $k \geqslant 0$ will be got. The following boundary conditions are required to determine functions $L_kx(\tau_0)$, $Q_k^{(\mp)}x(\tau)$ and $R_kx(\tau_1)$, $k \geqslant 0$,

$$\begin{aligned}
&A\bar{x}^{(-)}(0,\mu) + ALx(0,\mu) = Ax^0, \quad B\bar{x}^{(-)}(t_0,\mu) + BQ^{(-)}x(0,\mu) = Bx^*,\\
&A\bar{x}^{(+)}(t_0,\mu) + AQ^{(+)}x(0,\mu) = Ax^*, \quad B\bar{x}^{(+)}(1,\mu) + BRx(0,\mu) = Bx^1.
\end{aligned} \tag{4.7}$$

The boundary value conditions of functions $L_k x(\tau_0)$, $Q_k^{(\mp)} x(\tau)$ and $R_k x(\tau_1)$, $k \geqslant 0$ are derived by substituting (4.4) into (4.7) and comparing the same power of μ.

4.1. Zeroth-order Asymptotic Solution

Firstly, the equations for $\bar{x}_0^{(\mp)}(t)$ can be written out in the form

$$f^{(\mp)}\left(\bar{x}_0^{(\mp)}, t, \mu\right) = 0.$$

By Assumption 2.1, we get

$$\bar{x}_0^{(\mp)} = \varphi^{(\mp)}\left(\alpha^{(\mp)}(t), t\right),$$

where the functions $\alpha^{(\mp)}(t)$ are unknown now.

The problem of $L_0 x(\tau_0)$ is written in explicit form as

$$\begin{aligned} &\frac{\mathrm{d}L_0 x}{\mathrm{d}\tau_0} = f^{(-)}\left(\varphi^{(-)}\left(\alpha^{(-)}(0), 0\right) + L_0 x, 0, 0\right), \\ &A L_0 x(0) = A x^0 - A\varphi^{(-)}\left(\alpha^{(-)}(0), 0\right), \quad L_0 x(+\infty) = 0. \end{aligned} \tag{4.8}$$

The above problem can be turned into the associated system by scaling transformations $\tilde{x}(\tau_0) = \varphi^{(-)}\left(\alpha^{(-)}(0), 0\right) + L_0 x(\tau_0)$. The coefficient $L_0 x(\tau_0)$ can be determined by Assumption 3.1, and satisfy

$$||L_0 x(\tau_0)|| \leqslant C \mathrm{e}^{-\kappa \tau_0}, \quad \tau_0 \geqslant 0.$$

However, function $\alpha^{(-)}(0)$ in (4.8) is still an arbitrary vector. Condition $L_0 x(+\infty) = 0$ will is satisfied by the arbitrariness of $\alpha^{(-)}(0)$. From the discussion about the associated system in the previous section, it can be seen that

$$x_c^0 - \varphi_c^{(-)}\left(\alpha^{(-)}(0), 0\right) = \phi_c^{(-)}\left(x_s^0 - \varphi_s^{(-)}\left(\alpha^{(-)}(0), 0\right), \alpha^{(-)}\right). \tag{4.9}$$

We get $\alpha^{(-)}(0) = \alpha_0^{(-)}$ by Assumption 3.2, implicit function theorem and (4.9). The function $\alpha^{(-)}(t)$ will be determined by the solvability condition of $\bar{x}_1^{(-)}$. And the problem of $\bar{x}_1^{(-)}$ is written in explicit form as

$$\bar{f}_x^{(-)}(t)\bar{x}_1^{(-)} = \varphi_\alpha^{(-)}(t)\frac{\mathrm{d}\alpha^{(-)}(t)}{\mathrm{d}t} + \varphi_t^{(-)}(t) - \bar{f}_\mu^{(-)}(t), \tag{4.10}$$

where

$$\bar{f}_x^{(-)}(t) = f_x^{(-)}\left(\varphi^{(-)}\left(\alpha^{(-)}(t), t\right), t, 0\right), \quad \bar{f}_\mu^{(-)}(t) = f_\mu^{(-)}\left(\varphi^{(-)}\left(\alpha^{(-)}(t), t\right), t, 0\right).$$

For the solvability of problem (4.10), the right-hand side is orthogonal to the eigenvectors $h_i^{(-)}\left(\alpha^{(-)}(t), t\right)$, $i = 1, \cdots, c$ of the adjoint matrix $\left(\bar{f}_x^{(-)}(t)\right)^*$ corresponding to the eigenvalue $\lambda = 0$. The orthogonality condition can be written as

$$\frac{\mathrm{d}\alpha^{(-)}(t)}{\mathrm{d}t} = \left(h^{(-)}\left(\alpha^{(-)}(t), t\right)\varphi_\alpha^{(-)}(t)\right)^{-1} h^{(-)}\left(\alpha^{(-)}(t), t\right)\left(\bar{f}_\mu^{(-)}(t) - \varphi_t^{(-)}(t)\right),$$

where $h^{(-)}\left(\alpha^{(-)}(t),t\right)$ are the $c\times n$ matrix whose rows are the eigenvectors $h_i^{(-)}\left(\alpha^{(-)}(t),t\right)$, $i=1,\cdots,c$. We get $\alpha^{(-)}(t)$ by the Assumption 3.3. Here $\bar{x}_0^{(-)}(t)$ and $L_0x(\tau_0)$ are fully determined.

The problems about internal layer functions $Q_0^{(\mp)}x(\tau)$ are represented as

$$\begin{aligned}&\frac{\mathrm{d}Q_0^{(\mp)}x}{\mathrm{d}\tau}=f^{(\mp)}\left(\varphi^{(\mp)}\left(\alpha^{(\mp)}(t_0),t_0\right)+Q_0^{(\mp)}x,t_0,0\right),\\&D^{(\mp)}Q_0^{(\mp)}x(0)=D^{(\mp)}x_0^*-D^{(\mp)}\varphi^{(\mp)}\left(\alpha^{(\mp)}(t_0),t_0\right),\quad Q_0^{(\mp)}x(\mp\infty)=0,\end{aligned}\tag{4.11}$$

where

$$D^{(-)}=B,\quad D^{(+)}=A.$$

By the continuity condition (4.5) and the following scaling transformation

$$\hat{x}^{(\mp)}(\tau)=Q_0^{(\mp)}x(\tau)+\varphi^{(\mp)}\left(\alpha^{(\mp)}(t_0),t_0\right),$$

(4.11) can be written as

$$\begin{aligned}&\frac{\mathrm{d}\hat{x}(\tau)}{\mathrm{d}\tau_0}=f\left(\hat{x}(\tau),t_0,0\right),\\&\hat{x}(\mp\infty)=\varphi^{(\mp)}\left(\alpha^{(\mp)}(t_0),t_0\right),\end{aligned}$$

where

$$\hat{x}(\tau)=\begin{cases}\hat{x}^{(-)}(\tau), & \tau<0,\\ \hat{x}^{(+)}(\tau), & \tau\geqslant 0.\end{cases}$$

From the Assumption 3.4, the problem

$$\hat{x}_i^{(-)}(0)=\hat{x}_i^{(+)}(0),\quad i=1,2,\cdots,n\tag{4.12}$$

containing the unknown quantities $\alpha^{(+)}(t_0)$ and x_0^* can be obtained. Using the Assumption 3.2 and the last c equations

$$\hat{x}_i^{(-)}(0)=\hat{x}_i^{(+)}(0),\quad i=n-c+1,n-c+2,\cdots,n,$$

we can find $\alpha^{(+)}\left(t_0,x_0^*\right)$. Substituting $\alpha^{(+)}\left(t_0,x_0^*\right)$ into (4.12), x_0^* can be found by Assumptions 3.5 and implicit function theorem.

Similarly, the problem about $\alpha^{(+)}(t)$ is represented as

$$\frac{\mathrm{d}\alpha^{(+)}(t)}{\mathrm{d}t}=\left(h^{(+)}\left(\alpha^{(+)}(t),t\right)\varphi_\alpha^{(+)}(t)\right)^{-1}h^{(+)}\left(\alpha^{(+)}(t),t\right)\left(\bar{f}_\mu^{(+)}(t)-\varphi_t^{(+)}(t)\right).$$

The function $\alpha^{(+)}(t)$ will be determined by the Assumption 3.3. Thus the functions $\bar{x}_0^{(\mp)}(t)$, $L_0x(\tau_0)$ and $Q_0^{(\mp)}x(\tau)$ are fully determined. The final study is the problem of $R_0x(\tau_1)$,

$$\begin{aligned}&\frac{\mathrm{d}R_0x}{\mathrm{d}\tau_1}=f^{(+)}\left(\varphi^{(+)}\left(\alpha^{(+)}(1),1\right)+R_0x,1,1\right),\\&BR_0x(1)=Bx^0-B\varphi^{(+)}\left(\alpha^{(+)}(1),1\right),\quad R_0x(-\infty)=0.\end{aligned}$$

The coefficient $R_0x(\tau_1)$ can be determined by the associated system and the Assumption 3.1, and satisfy

$$||R_0x(\tau_1)||\leqslant C\mathrm{e}^{\kappa\tau_1},\quad \tau_1\leqslant 0.$$

The zeroth-order terms in the asymptotic solution (4.3) are all determined so far.

4.2. First-order Approximate Solutions

Next, we will discuss the first-order term of the asymptotic solution (4.3). For functions $\bar{x}_1^{(\mp)}(t)$, we have

$$\bar{f}_x^{(\mp)}(t)\bar{x}_1^{(\mp)} = \varphi_\alpha^{(\mp)}(t)\frac{\mathrm{d}\alpha^{(\mp)}(t)}{\mathrm{d}t} + \varphi_t^{(\mp)}(t) - \bar{f}_\mu^{(\mp)}(t).$$

The solutions of the above problem are

$$\bar{x}_1^{(\mp)} = \varphi_\alpha^{(\mp)}(t)\beta^{(\mp)}(t) + \bar{x}_{1p}^{(\mp)},$$

where $\beta^{(\mp)}(t)$ are temporarily arbitrary and $\bar{x}_{1p}^{(\mp)}$ are special solutions of the above problems. Similar to the process of determining $\alpha^{(\mp)}(t)$, $\beta^{(\mp)}(t)$ can be determined by the solvability condition of $\bar{x}_2^{(\mp)}$.

For the functions $L_1x(\tau_0)$, $Q_1^{(\mp)}x(\tau)$ and $R_1x(\tau_1)$, we have the following problems:

$$\begin{aligned}
&\frac{\mathrm{d}L_1x}{\mathrm{d}\tau_0} = f_x^{(-)}\left(\varphi^{(-)}\left(\alpha^{(-)}(0),0\right) + L_0x,0,0\right)L_1x + g_1^L(\tau_0),\\
&AL_1x(0) = -A\bar{x}_1^{(-)}(0), \quad L_1x(+\infty) = 0,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&\frac{\mathrm{d}Q_1^{(-)}x}{\mathrm{d}\tau} = f_x^{(-)}\left(\varphi^{(-)}\left(\alpha^{(-)}(t_0),t_0\right) + Q_0^{(-)}x,t_0,0\right)Q_1^{(-)}x + g_1^{(-)}(\tau),\\
&BQ_1^{(-)}x(0) = Bx_1^* - B\bar{x}_1^{(-)}(t_0), \quad Q_1^{(-)}x(-\infty) = 0,
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\frac{\mathrm{d}Q_1^{(+)}x}{\mathrm{d}\tau} = f_x^{(+)}\left(\varphi^{(+)}\left(\alpha^{(+)}(t_0),t_0\right) + Q_0^{(+)}x,t_0,0\right)Q_1^{(+)}x + g_1^{(+)}(\tau),\\
&AQ_1^{(+)}x(0) = Ax_1^* - A\bar{x}_1^{(-)}(t_0), \quad Q_1^{(+)}x(+\infty) = 0,
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}R_1x}{\mathrm{d}\tau_1} = f_x^{(+)}\left(\varphi^{(+)}\left(\alpha^{(+)}(1),1\right) + R_0x,1,0\right)R_1x + g_1^R(\tau_1),\\
&BR_1x(0) = -B\bar{x}_1^{(+)}(1), \quad R_1x(-\infty) = 0,
\end{aligned} \tag{4.16}$$

where g_1^j, $j = L, (\mp), R$ are known exponential decaying functions, for example,

$$\begin{aligned}
g_1^L(\tau_0) = &\left(f_x(\tau_0) - \bar{f}_x(0)\right)\left(\bar{x}_1^{(-)}(0) + \tau_0\left(\bar{x}_0^{(-)}(0)\right)'\right)\\
&+ \left(f_t(\tau_0) - \bar{f}_t(0)\right)\tau_0 + \left(f_\mu(\tau_0) - \bar{f}_\mu(0)\right),
\end{aligned}$$

where

$$\begin{aligned}
&f_i(\tau_0) = f_i^{(-)}\left(\varphi^{(-)}\left(\alpha^{(-)}(0),0\right) + L_0x,0,0\right),\\
&\bar{f}_i(0) = f_i^{(-)}\left(\varphi^{(-)}\left(\alpha^{(-)}(0),0\right),0,0\right), \quad i = x,t,\mu.
\end{aligned}$$

The solutions of these linear problems (4.13)–(4.16) can be found without any additional conditions [8], for example, in problem (4.13), we firstly rewrite the expression $f_x(\tau_0)$ in the form

$$f_x(\tau_0) = \bar{f}_x(0) + f_x(\tau_0) - \bar{f}_x(0) \triangleq \bar{f}_x(0) - f^L(\tau_0),$$

where $f^L(\tau_0)$ satisfy

$$\left|\left|f^L(\tau_0)\right|\right| \leqslant C\mathrm{e}^{-\kappa\tau_0}, \quad \tau_0 \geqslant 0.$$

There is an invertible matrix T that reduces the matrix $\bar{f}_x(0)$ to the block-diagonal form

$$T^{-1}f_x(0)T = \Lambda = \mathrm{diag}(\Lambda_-, \Lambda_+, 0),$$

where the $s \times s$(resp. $u \times u$) matrix Λ_-(resp. Λ_+) has eigenvalues with negative (resp. positive) real part. Let

$$T = (T_-, T_+, T_0), \quad T^{-1} = \left(T_-^{-1}, T_+^{-1}, T_0^{-1}\right)^T,$$

where T_-, T_+, and T_0 are matrices (blocks) of sizes $n \times s$, $n \times u$, and $n \times c$, respectively. And T_-^{-1}, T_+^{-1}, and T_0^{-1} are matrices (blocks) of the sizes $s \times n$, $u \times n$, and $c \times n$, respectively. Then the solution of problem (4.13) can be represented in the form

$$L_1x(\tau_0) = -\Phi_1^L(\tau_0)A\bar{x}_1^{(-)}(0) + \tilde{H}^L g_1^L(\tau_0),$$

where

$$\Phi_1^L(\tau_0) = \left(E - H^\delta f^L\right)^{-1} T_-\mathrm{e}^{\Lambda_-\tau_0}, \quad \tilde{H}^L g_1^L(\tau_0) = \left(E - H^\delta f^L\right)^{-1} H^\delta g_1^L(\tau_0),$$

$$H^\delta g_1^L(\tau_0) = T \begin{pmatrix} \int_\delta^\tau \mathrm{e}^{\Lambda_-(\tau_0 - s)}T_-^{-1}g_1^L(s)\,\mathrm{d}s \\ \\ \int_{+\infty}^\tau \mathrm{e}^{\Lambda_+(\tau_0 - s)}T_+^{-1}g_1^L(s)\,\mathrm{d}s \\ \\ \int_{+\infty}^\tau T_0^{-1}g_1L(s)\,\mathrm{d}s \end{pmatrix},$$

here δ can be continuously extended to $+\infty$.

Similarly, the solutions of problems (4.14)–(4.16) can be write out in the form

$$\begin{aligned} Q_1^{(-)}x(\tau) &= \Phi_1^{(-)}(\tau)\left(Bx_1^* - B\bar{x}_1^{(-)}(t_0)\right) + \tilde{H}^{(-)}g_1^{(-)}(\tau), \\ Q_1^{(+)}x(\tau) &= \Phi_1^{(+)}(\tau)\left(Ax_1^* - A\bar{x}_1^{(+)}(t_0)\right) + \tilde{H}^{(+)}g_1^{(+)}(\tau), \\ R_1x(\tau_1) &= -\Phi_1^R(\tau_1)B\bar{x}_1(1) + \tilde{H}^R g_1^R(\tau_1). \end{aligned}$$

And the solutions of problems (4.13)–(4.16) satisfy the inequalities

$$\begin{aligned} &||L_1x(\tau_0)|| \leqslant C\mathrm{e}^{-\kappa\tau_0}, \quad \tau_0 \geqslant 0, \quad \left|\left|Q_1^{(-)}x(\tau)\right|\right| \leqslant C\mathrm{e}^{\kappa\tau}, \quad \tau \leqslant 0, \\ &\left|\left|Q_1^{(+)}x(\tau)\right|\right| \leqslant C\mathrm{e}^{-\kappa\tau}, \quad \tau \geqslant 0, \quad ||R_1x(\tau_1)|| \leqslant C\mathrm{e}^{\kappa\tau_1}, \quad \tau_1 \leqslant 0. \end{aligned}$$

According to the continuity condition (4.5), we have

$$x_1^* = \bar{\Phi}_1^{-1}\left[\left(E - \Phi_1^{(+)}(0)A\right)\bar{x}_1^{(+)}(t_0) - \left(E - \Phi_1^{(-)}(0)B\right)\bar{x}_1^{(-)}(t_0) + \Delta\tilde{H}g_1(0)\right],$$

where

$$\bar{\Phi}_1 = \Phi_1^{(-)}(0)A - \Phi_1^{(+)}(0)B, \quad \Delta\tilde{H}g_1(0) = \tilde{H}^{(+)}g_1^{(+)}(0) - \tilde{H}^{(-)}g_1^{(-)}(0).$$

The first-order terms in the asymptotic solution (4.3) are all determined so far. The functions $L_kx(\tau_0)$, $Q_k^{(\mp)}x(\tau)$ and $R_kx(\tau_1)$, $k \geqslant 2$ can be determined in a similar way. Therefore, all coefficients of the series (4.3) can be determined.

5. Main Result

After determining the first $(k+1)$-order terms of the series (4.3), denote the kth partial sum of the series (4.3) by $X_k^{(\mp)}(t,\mu)$ respectively, i.e.

$$X_k^{(-)}(t,\mu) = \sum_{i=0}^{k} \mu^i \left(\bar{x}_i^{(-)}(t) + L_i x(\tau_0) + Q_i^{(-)} x(\tau)\right), \quad 0 \leqslant t < t_0,$$

$$X_k^{(+)}(t,\mu) = \sum_{i=0}^{k} \mu^i \left(\bar{x}_i^{(+)}(t) + Q_i^{(+)} x(\tau) + R_i x(\tau_1)\right), \quad t_0 \leqslant t \leqslant 1,$$

where

$$\tau_0 = t\mu^{-1}, \quad \tau = (t-t_0)\mu^{-1}, \quad \tau_1 = (t-1)\mu^{-1}.$$

Lemma 5.1. *Under the Assumptions 2.1–3.5, there exist a positive constant μ_0, such that for $0 < \mu \leqslant \mu_0$, problems (4.1) and (4.2) have unique solutions $u^{(-)}$ and $u^{(+)}$, respectively, and solutions $x^{(\mp)}(t,\mu)$ satisfy*

$$\left|\left|x^{(-)}(t,\mu) - X_k^{(-)}(t,\mu)\right|\right| \leqslant C\mu^{k+1}, \quad 0 \leqslant t \leqslant t_0,$$

$$\left|\left|x^{(+)}(t,\mu) - X_k^{(+)}(t,\mu)\right|\right| \leqslant C\mu^{k+1}, \quad t_0 \leqslant t \leqslant 1.$$

The proof of the above Lemma 5.1 is similar to [8], which can be proved by the principle of compression mapping, and will not be repeated here. From Lemma 5.1, one can obtain

$$\left|\left|x^{(\mp)}(t_0,\mu) - X_k^{(\mp)}(t_0,\mu)\right|\right| \leqslant C\mu^{k+1}.$$

So

$$\left|\left|x^* - \sum_{i=0}^{k} \mu^i x_i^*\right|\right| \leqslant C\mu^{k+1}.$$

We summarize the above discussion as the following main theorem of this paper.

Theorem 5.1. *Suppose the Assumptions 2.1–3.5 hold. Then problem (2.1) has a solution with a step-like contrast structure, and its asymptotic representation can be represented in the form:*

$$x(t,\mu) = \begin{cases} \sum\limits_{i=0}^{k} \mu^i \left(\bar{x}_i^{(-)}(t) + L_i x(\tau_0) + Q_i^{(-)} x(\tau)\right) + \mathrm{O}(\mu^{k+1}), & 0 \leqslant t < t_0, \\ \sum\limits_{i=0}^{k} \mu^i \left(\bar{x}_i^{(+)}(t) + Q_i^{(+)} x(\tau) + R_i x(\tau_1)\right) + \mathrm{O}(\mu^{k+1}), & t_0 \leqslant t \leqslant 1. \end{cases}$$

6. Example

Consider the following problem

$$\mu\frac{\mathrm{d}x}{\mathrm{d}t}=\begin{cases} y, & 0\leqslant t<\frac{1}{2},\\ y+2, & \frac{1}{2}\leqslant t\leqslant 1,\end{cases}$$

$$\mu\frac{\mathrm{d}y}{\mathrm{d}t}=\begin{cases} x, & 0\leqslant t<\frac{1}{2},\\ x+1, & \frac{1}{2}\leqslant t\leqslant 1,\end{cases} \tag{6.1}$$

$$\mu\frac{\mathrm{d}z}{\mathrm{d}t}=\begin{cases} y, & 0\leqslant t<\frac{1}{2},\\ y+2, & \frac{1}{2}\leqslant t\leqslant 1,\end{cases}$$

$$x(0,\mu)=1,\quad x(1,\mu)=1,\quad z(0,\mu)=1.$$

According to the asymptotic method proposed in this paper, we will construct the zeroth-order expression of the solution with a step-like contrast structure. First, divide (6.1) into the left boundary value problem

$$\mu\frac{\mathrm{d}x^{(-)}}{\mathrm{d}t}=y^{(-)},$$

$$\mu\frac{\mathrm{d}y^{(-)}}{\mathrm{d}t}=x^{(-)},$$

$$\mu\frac{\mathrm{d}z^{(-)}}{\mathrm{d}t}=y^{(-)},$$

$$x^{(-)}(0,\mu)=1,\quad y^{(-)}\left(\frac{1}{2},\mu\right)=y^*,\quad z^{(-)}(0,\mu)=1$$

and right boundary value problem

$$\mu\frac{\mathrm{d}x^{(+)}}{\mathrm{d}t}=y^{(+)}+2,$$

$$\mu\frac{\mathrm{d}y^{(+)}}{\mathrm{d}t}=x^{(+)}+1,$$

$$\mu\frac{\mathrm{d}z^{(+)}}{\mathrm{d}t}=y^{(+)}+2,$$

$$x^{(+)}\left(\frac{1}{2},\mu\right)=x^*,\quad x^{(+)}(1,\mu)=1,\quad z^{(+)}\left(\frac{1}{2},\mu\right)=z^*.$$

For $\mu=0$, we obtain the degenerate solution

$$\bar{x}^{(-)}=0,\quad \bar{x}^{(+)}=-1,\quad \bar{y}^{(-)}=0,\quad \bar{y}^{(+)}=-2,\quad \bar{z}^{(\mp)}=\alpha^{(\mp)}.$$

The following problem

$$\frac{\mathrm{d}L_0x(\tau_0)}{\mathrm{d}\tau_0} = L_0y(\tau_0), \quad \tau_0 \leqslant 0,$$

$$\frac{\mathrm{d}L_0y(\tau_0)}{\mathrm{d}\tau_0} = L_0x(\tau_0), \quad \tau_0 \leqslant 0,$$

$$\frac{\mathrm{d}L_0z(\tau_0)}{\mathrm{d}\tau_0} = L_0y(\tau_0), \quad \tau_0 \leqslant 0,$$

$$L_0x(0) = 1, \quad L_0z(0) = 1 - \alpha^{(-)}(0), \quad L_0u(+\infty) = 0$$

has the first integrals

$$\alpha^{(-)}(0) + L_0z = L_0x + C_1, \tag{6.2}$$
$$L_0y = -L_0x + C_2. \tag{6.3}$$

Substituting $L_0x(0) = 1$ and $L_0z(0) + \alpha^{(-)}(0) = 1$ into (6.2), one can see $C_1 = 0$. Let $\tau_0 \to +\infty$ in (6.2), then $\alpha^{(-)}(0) = 0$. From (4.10), the problem to determine $\alpha^{(-)}(t)$ has the form

$$\frac{\mathrm{d}\alpha^{(-)}(t)}{\mathrm{d}t} = 0.$$

So $\alpha^{(-)}(t) = 0$. Let $\tau_0 \to +\infty$ in (6.3), we get $C_2 = 0$. Then

$$L_0y = -L_0x.$$

Combining $L_0x(0) = 1$, one can obtain

$$L_0x(\tau_0) = \exp(-\tau_0), \quad L_0y(\tau_0) = -\exp(-\tau_0), \quad L_0z(\tau_0) = \exp(-\tau_0).$$

In a similar way, we can see

$$Q_0^{(-)}x(\tau) = y_0^*\exp(\tau), \quad Q_0^{(-)}y(\tau) = y_0^*\exp(\tau), \quad Q_0^{(-)}z(\tau) = y_0^*\exp(\tau),$$
$$Q_0^{(+)}x(\tau) = x_0^*\exp(-\tau), \quad Q_0^{(+)}y(\tau) = -x_0^*\exp(-\tau), \quad Q_0^{(+)}z(\tau) = x_0^*\exp(-\tau),$$
$$R_0x(\tau_1) = 2\exp(\tau_1), \quad R_0y(\tau_1) = 2\exp(\tau_1), \quad R_0z(\tau_1) = 2\exp(\tau_1).$$

From the continuity condition (4.5), we get

$$x_0^* = -\frac{1}{2}, \quad y_0^* = -\frac{3}{2}, \quad z_0^* = -\frac{3}{2}.$$

The specific expression of contrast structure solution for problem (6.1) is as follows:

$$x(t,\mu)=\begin{cases}\exp(-\tau_0)-\dfrac{3}{2}\exp(\tau)+O(\mu), & 0\leqslant t<t_0,\\ -\dfrac{1}{2}\exp(-\tau)-1+2\exp(\tau_1)+O(\mu), & t_0\leqslant t\leqslant 1,\end{cases}$$

$$y(t,\mu)=\begin{cases}-\exp(-\tau_0)-\dfrac{3}{2}\exp(\tau)+O(\mu), & 0\leqslant t<t_0,\\ \dfrac{1}{2}\exp(-\tau)-2+2\exp(\tau_1)+O(\mu), & t_0\leqslant t\leqslant 1,\end{cases}$$

$$z(t,\mu)=\begin{cases}\exp(-\tau_0)-\dfrac{3}{2}\exp(\tau)+O(\mu), & 0\leqslant t<t_0,\\ -\dfrac{1}{2}\exp(-\tau)-1+2\exp(\tau_1)+O(\mu), & t_0\leqslant t\leqslant 1,\end{cases}$$

where

$$\tau_0=t\mu^{-1},\quad \tau=\left(t-\frac{1}{2}\right)\mu^{-1},\quad \tau_1=(t-1)\mu^{-1}.$$

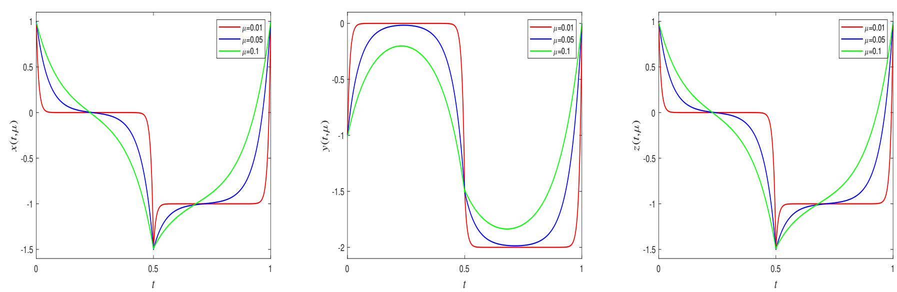

Figure 1. The asymptotic solutions $x(t,\mu)$, $y(t,\mu)$ and $z(t,\mu)$ of (6.1) under different parameters μ.

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