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SHARP BOUNDS ON HANKEL DETERMINANTS FOR CERTAIN SUBCLASS OF STARLIKE FUNCTIONS*

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Abstract The main objective of this paper is to study coefficient problems for starlike functions located in the petal shaped domain. The bounds of the first three initial coefficients, bounds of Fekete-Szegö type inequality, estimates of the second and third Hankel determinants for the subclass of starlike functions are derived, all of these bounds are sharp.

Keywords Starlike function, petal shaped domain, Hankel determinant.

MSC(2010) 30C45, 30C50.

1. Introduction

Let the family of analytic functions in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be described by the symbol $\mathcal{H}(\mathcal{U})$, and let $\mathcal{A}$ be the subfamily of $\mathcal{H}(\mathcal{U})$ which is defined by

$$\mathcal{A} := \left\{ f \in \mathcal{H}(\mathcal{U}) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \right\}. \tag{1.1}$$

Furthermore, the set $\mathcal{S} \subset \mathcal{A}$ consists of all normalized univalent functions in $\mathcal{U}$. For two given functions $g_1, g_2 \in \mathcal{H}(\mathcal{U})$, we say that g_1 is subordinate to g_2, written by $g_1 \prec g_2$ or $g_1(z) \prec g_2(z)$, if a regular function v occurs in $\mathcal{U}$ with the restriction $v(0) = 0$ and $|v(z)| \leq 1$ such that $f(z) = g(v(z))$ for all $z \in \mathcal{U}$. Moreover, if g_2 is univalent in $\mathcal{U}$, then the following relationship holds:

$$g_1(z) \prec g_2(z) \ \ (z \in \mathcal{U}) \Longleftrightarrow g_1(0) = g_2(0) \text{ and } g_1(\mathcal{U}) \subset g_2(\mathcal{U}).$$

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Though the geometric function theory was started in 1851, due to the coefficient conjecture proposed by Bieberbach [6] in 1916, this field emerged as a hot research area. This conjecture was proved by de-Branges [10] in 1985. The families of starlike functions $\mathcal{S}^*$, convex functions $\mathcal{K}$ and close-to-convex functions $\mathcal{C}$ are the most basic subfamilies, which are defined as follows:

$$\mathcal{S}^* := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{U}) \right\},$$

$$\mathcal{K} := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{(zf'(z))'}{f'(z)}\right) > 0 \quad (z \in \mathcal{U}) \right\},$$

and

$$\mathcal{C} := \left\{ f \in \mathcal{S} : \exists\, g \in \mathcal{S}^* \text{ such that } \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad (z \in \mathcal{U}) \right\}.$$

The general form of the class $\mathcal{S}^*$ was studied in 1992 by Ma and Minda [24], which is given by

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \quad (z \in \mathcal{U}) \right\}, \tag{1.2}$$

where φ is a regular function with positive real part, $\varphi(0) = 1$ and $\varphi'(0) > 0$. Also, the function φ maps $\mathcal{U}$ onto a star-shaped region with respect to $\varphi(0) = 1$ and is symmetric with the real axis. They addressed some specific results such as distortion, growth, and covering theorems. In recent years, several subfamilies of the set $\mathcal{A}$ were studied as special cases of the class $\mathcal{S}^*(\varphi)$.

(i) If we take $\varphi(z) = \frac{1+Mz}{1+Nz}$ with $-1 \leq N < M \leq 1$, then the deduced family $\mathcal{S}^*[M,N] \equiv \mathcal{S}^*\left(\frac{1+Mz}{1+Nz}\right)$ is described by the functions of Janowski starlike family established in [12].

(ii) The family $\mathcal{S}^*_L \equiv \mathcal{S}^*(\varphi(z))$ with $\varphi(z) = \sqrt{1+z}$ was developed in [35] by Sokól and Stankiewicz. The image of the function $\varphi(z) = \sqrt{1+z}$ demonstrates that the image domain is bounded by the Bernoullis lemniscate right-half plane specified by $|w^2 - 1| < 1$.

(iii) By selecting $\varphi(z) = 1 + \sin z$, the class $\mathcal{S}^*(\varphi(z))$ leads to the family $\mathcal{S}^*_{\sin}$ which was explored in [9], while $\mathcal{S}^*_e \equiv \mathcal{S}^*(e^z)$ has been discussed in [25] and later studied in [32, 34].

(iv) The family $\mathcal{S}^*_c := \mathcal{S}^*(\varphi(z))$ with

$$\varphi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$$

was contributed by Sharma. In [31], it contains the function $f \in \mathcal{A}$ such that $\frac{zf'(z)}{f(z)}$ is located in the region bounded by the cardioid given by

$$(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0.$$

(v) The family $\mathcal{S}^*_R \equiv \mathcal{S}^*(\varphi(z))$ with

$$\varphi(z) = 1 + \frac{z}{\alpha} \cdot \frac{\alpha + z}{\alpha - z} \quad (\alpha = \sqrt{2} + 1)$$

was studied in [17], while $\mathcal{S}^*_{\cos} := \mathcal{S}^*(\cos(z))$ and $\mathcal{S}^*_{\cosh} := \mathcal{S}^*(\cosh(z))$ were recently discussed by Raza and Bano [5], and Abdullah *et al.* [1], respectively.

(vi) If we consider $\varphi(z) = 1 + \sinh^{-1} z$, the class

$$\mathcal{S}_\rho^* := \mathcal{S}^* \left(1 + \sinh^{-1} z\right)$$

was introduced by Arora and Kumar [3].

A function $f \in \mathcal{A}$ is in the family $\mathcal{S}_\rho^*$ if (1.2) holds for the function $\varphi(z) = \rho(z)$, where

$$\rho(z) = 1 + \sinh^{-1} z. \tag{1.3}$$

Clearly, the function ρ is a multivalued function and has the branch cuts about the line segments $(-i\infty, -i) \cup (i, i\infty)$, on the imaginary axis and hence it is holomorphic in $\mathcal{U}$. In a geometric point of view, the function ρ maps the unit disc $\mathcal{U}$ onto a petal shaped region Ω_ρ, where

$$\Omega_\rho = \{w \in \mathbb{C} : |\sinh (w - 1)| < 1\}.$$

From the above definition, we deduce that $f \in \mathcal{S}_\rho^*$ if and only if there exists a regular function $q(z) \prec \rho(z)$ such that

$$f(z) = z \exp\left(\int_0^z \frac{q(t) - 1}{t} dt\right). \tag{1.4}$$

If we set

$$q(z) = 1 + \sinh^{-1}(z) = 1 + z - \frac{1}{6}z^3 + \frac{3}{40}z^5 - \frac{5}{112}z^7 + \cdots,$$

it follows from (1.4) that

$$f_0(z) = z \exp\left(\int_0^z \frac{\sinh^{-1}(t)}{t} dt\right) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{9}z^4 - \frac{1}{72}z^5 - \frac{1}{225}z^6 + \cdots, \tag{1.5}$$

which will be played as an extremal function for the class $\mathcal{S}_\rho^*$.

The Hankel determinant $\mathcal{H}_{q,n}(f)$ $(q, n \in \mathbb{N} := \{1, 2, \ldots\})$ for a function $f \in \mathcal{S}$ of the series form (1.1) was given by Pommerenke [27,28] as follows:

$$\mathcal{H}_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In particular, the following determinants are known as the first-order, second-order and third-order Hankel determinants, respectively,

$$\mathcal{H}_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2,$$

$$\mathcal{H}_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2,$$

and

$$\mathcal{H}_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right). \quad (1.6)$$

There are comparatively few observations in literature in relation to the Hankel determinant for the function f belongs to the general family $\mathcal{S}$. For the function $f \in \mathcal{S}$, the best established sharp inequality is $|\mathcal{H}_{2,n}(f)| \leq \lambda\sqrt{n}$, where λ is a constant, which is due to Hayman [11]. Further, for the same class $\mathcal{S}$, it was obtained in [26] that

$$|\mathcal{H}_{2,2}(f)| \leq \lambda \quad \left(1 \leq \lambda \leq \frac{11}{3}\right),$$

and

$$|\mathcal{H}_{3,1}(f)| \leq \mu \quad \left(\frac{4}{9} \leq \mu \leq \frac{32+\sqrt{285}}{15}\right).$$

In a given family of functions, the problem of calculating the bounds, probably sharp, of Hankel determinants attracted the interests of many researchers. For example, the sharp bound of $|\mathcal{H}_{2,2}(f)|$, for the subfamilies $\mathcal{K}$, $\mathcal{S}^*$ and $\mathcal{R}$ (family of bounded turning functions) of the set $\mathcal{S}$, were calculated by Janteng *et al.* [13, 14]. These estimates are

$$|\mathcal{H}_{2,2}(f)| \leq \begin{cases} 1/8, & \text{for } f \in \mathcal{K}, \\ 1, & \text{for } f \in \mathcal{S}^*, \\ 4/9, & \text{for } f \in \mathcal{R}. \end{cases}$$

For the families $\mathcal{S}^*(\beta)$ $(0 \leq \beta < 1)$ of starlike functions of order β and $\mathcal{SS}^*(\beta)$ $(0 < \beta \leq 1)$ of strongly starlike functions of order β, the authors [7, 8] showed that $|\mathcal{H}_{2,2}(f)|$ are bounded by $(1-\beta)^2$ and β^2, respectively. The exact bound for the family $\mathcal{S}^*(\varphi)$ of Ma-Minda starlike functions was investigated in [21]. For recent applications of Ma-Minda classes, we refer the reader to [39].

It is quite clear from the formulas given in (1.6) that the calculation of $|\mathcal{H}_{3,1}(f)|$ is far more challenging compared with finding the bound of $|\mathcal{H}_{2,2}(f)|$. Babalola [4] investigated the bounds of third order Hankel determinant for the families of $\mathcal{K}$, $\mathcal{S}^*$ and $\mathcal{R}$ in 2010. Moreover, Zaprawa [41] improved the results of Babalola by applying a new methodology. He obtained the following bounds:

$$|\mathcal{H}_{3,1}(f)| \leq \begin{cases} 49/540, & \text{for } f \in \mathcal{C}, \\ 1, & \text{for } f \in \mathcal{S}^*, \\ 41/60, & \text{for } f \in \mathcal{R}. \end{cases}$$

He also pointed out that such limits are indeed not the best one. In 2018, Kwon *et al.* [19] strengthened Zaprawa's result for $f \in \mathcal{S}^*$ and showed that $|\mathcal{H}_{3,1}(f)| \leq 8/9$, and this bound was further improved by Zaprawa *et al.* [40] in 2021. They obtained $|\mathcal{H}_{3,1}(f)| \leq 5/9$ for $f \in \mathcal{S}^*$. In 2018, Kowalczyk *et al.* [16] and Lecko *et al.* [20] succeeded in finding the sharp bounds of $|\mathcal{H}_{3,1}(f)|$ for the families $\mathcal{K}$ and $\mathcal{S}^*(1/2)$, respectively, where $\mathcal{S}^*(1/2)$ denote the starlike functions of order $1/2$. We note that Rath *et al.* [30] pointed out there was an error in the proof of the corresponding result obtained by Lecko *et al.* [20], they also gave a new corrected proof. These results are given as follows:

$$|\mathcal{H}_{3,1}(f)| \leq \begin{cases} 4/135, & \text{for } f \in \mathcal{K}, \\ 1/9, & \text{for } f \in \mathcal{S}^*(1/2). \end{cases}$$

Moreover, Wang *et al.* [38] determined the third and fourth-order Hankel determinants of a subclass of analytic functions.

In the present paper, our objective is to calculate the sharp bounds of the coefficient inequalities, Fekete-Szegö type problem, Hankel determinants of order 2 and 3 for the class of starlike functions $\mathcal{S}^*_\rho$ connected with petal shaped domain.

2. Preliminary results

To prove our main results, we need the following definition and lemmas.

Definition 2.1. Let $\mathcal{P}$ denote the class of functions p which are analytic in $\mathcal{U}$ with $\mathrm{Re}\,(p(z)) > 0$ and has the series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathcal{U}). \tag{2.1}$$

Lemma 2.1. *Let $p \in \mathcal{P}$ has the series form* (2.1)*. Then for $x, \sigma, \rho \in \overline{\mathcal{U}} := \{z : |z| \leq 1\}$,*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{2.2}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\sigma, \tag{2.3}$$

and

$$\begin{aligned} 8c_4 =& c_1^4 + (4 - c_1^2)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] \\ &- 4(4 - c_1^2)(1 - |x|^2)\left[c(x-1)z + \overline{x}\sigma^2 - (1 - |\sigma|^2)\rho\right]. \end{aligned} \tag{2.4}$$

It contains the well-known formula for c_2 (see [29]), the formula for c_3 due to Libera and Zlotkiewicz [22], and the formula for c_4 was proved in [18].

Lemma 2.2. *If $p \in \mathcal{P}$ has the series form* (2.1)*, then*

$$|c_{n+k} - \mu c_n c_k| \leq 2\max\{1,\ |2\mu - 1|\}, \tag{2.5}$$

$$|c_n| \leq 2 \quad (n \geq 1), \tag{2.6}$$

and

$$\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| \leq 2|J| + |K - 2J| + 2|J - K + L|, \tag{2.7}$$

where $J, K, L, \mu \in \mathbb{C}$, and for $B \in [0,1]$ with $B(2B-1) \leq D \leq B$, we have

$$\left|c_3 - 2Bc_1c_2 + Dc_1^3\right| \leq 2. \tag{2.8}$$

The inequalities (2.5), (2.6), (2.7) and (2.8) in Lemma 2.2 are taken from [15], [29], [2] and [23], respectively.

3. Coefficient inequalities for the class $\mathcal{S}_{\rho}^{*}$

Firstly, we aim at finding the sharp bounds of the first three initial coefficients for functions in the class $\mathcal{S}_{\rho}^{*}$.

Theorem 3.1. *If $f \in \mathcal{S}_{\rho}^{*}$, then*

$$|a_2| \leq 1, \quad |a_3| \leq \frac{1}{2} \quad \text{and} \quad |a_4| \leq \frac{1}{3}. \tag{3.1}$$

These bounds are sharp.

Proof. From the definition of the class $\mathcal{S}_{\rho}^{*}$ along with the subordination principle, it follows that

$$\frac{zf'(z)}{f(z)} = 1 + \sinh^{-1}(w(z)).$$

If $p \in \mathcal{P}$, then in term of Schwarz function w, we have

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \tag{3.2}$$

or equivalently,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}.$$

In view of (1.1), we easily obtain

$$\begin{aligned} \frac{zf'(z)}{f(z)} =& 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2 a_3 + a_2^3)z^3 \\ &+ (4a_5 - 2a_3^2 - 4a_2 a_4 + 4a_2^2 a_3 - a_2^4)z^4 + \cdots. \end{aligned} \tag{3.3}$$

By the series expansion of $w(z)$, we get

$$\begin{aligned} 1 + \sinh^{-1}(w(z)) =& 1 + \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right) z^2 + \left(\frac{1}{2}c_3 + \frac{5}{48}c_1^3 - \frac{1}{2}c_1 c_2\right) z^3 \\ &+ \left(\frac{1}{2}c_4 - \frac{1}{4}c_2^2 - \frac{1}{32}c_1^4 + \frac{5}{16}c_1^2 c_2 - \frac{1}{2}c_1 c_3\right) z^4 + \cdots. \end{aligned} \tag{3.4}$$

By virtue of (3.3) and (3.4), we know that

$$a_2 = \frac{1}{2}c_1, \tag{3.5}$$

$$a_3 = \frac{1}{4}c_2, \tag{3.6}$$

and

$$a_4 = \frac{1}{6}c_3 - \frac{1}{144}c_1^3 - \frac{1}{24}c_1 c_2. \tag{3.7}$$

For a_2 and a_3, by using (2.6) in (3.5) and (3.6), we easily obtain

$$|a_2| \leq 1 \text{ and } |a_3| \leq \frac{1}{2}.$$

The above two inequalities are sharp for the extremal function given by (1.5).

For a_4, we rewrite (3.7) as

$$|a_4| = \frac{1}{6}\left|c_3 - \frac{1}{4}c_1c_2 - \frac{1}{24}c_1^3\right|. \tag{3.8}$$

Then by using (2.7) and (3.8) , we obtain

$$0 \leq B = \frac{1}{8} < 1, \quad B = \frac{1}{8} > D = -\frac{1}{48},$$

and

$$B(2B-1) = -\frac{3}{32} < D = -\frac{1}{48}.$$

Thus, all the constraints of Lemma 2.2 are satisfied, by virtue of (2.8), we obtain the required bound

$$|a_4| \leq \frac{1}{3}.$$

The sharpness can be found from the following extremal function

$$f_1(z) = z\exp\left(\int_0^z \frac{\sinh^{-1}(t^3)}{t}dt\right) = z + \frac{1}{3}z^4 + \frac{1}{18}z^7 - \frac{1}{81}z^{10} + \cdots . \tag{3.9}$$

□

Now, we study the Fekete-Szegö type problem for the family $\mathcal{S}_\rho^*$.

Theorem 3.2. *If $f \in \mathcal{S}_\rho^*$, then*

$$\left|a_3 - \gamma a_2^2\right| \leq \max\left\{\frac{1}{2},\ \frac{1}{2}\left|2\gamma - 1\right|\right\}.$$

This inequality is sharp.

Proof. From (3.5) and (3.6) , we may write

$$\left|a_3 - \gamma a_2^2\right| = \frac{1}{4}\left|\left(c_2 - \gamma c_1^2\right)\right|.$$

An application of (2.5) , we obtain

$$\left|a_3 - \gamma a_2^2\right| \leq \frac{1}{2}\max\left\{1,\ |2\gamma - 1|\right\}.$$

□

By putting $\gamma = 1$, we obtain the following result.

Corollary 3.1. *If $f \in \mathcal{S}_\rho^*$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{2}.$$

This inequality is sharp with the extremal function

$$f_2(z) = z\exp\left(\int_0^z \frac{\sinh^{-1}(t^2)}{t}dt\right) = z + \frac{1}{2}z^3 + \frac{1}{8}z^5 - \frac{1}{144}z^7 + \cdots . \tag{3.10}$$

Theorem 3.3. *If $f \in \mathcal{S}_\rho^*$, then*

$$|a_2a_3 - a_4| \leq \frac{1}{3}.$$

This inequality is sharp.

Proof. From (3.5), (3.6) and (3.7), we obtain

$$|a_2a_3 - a_4| = \frac{1}{144}\left|c_1^3 + 24c_1c_2 - 24c_3\right|.$$

In view of (2.2) and (2.3) along with $c_1 = c \in [0, 2]$, we have

$$|a_2a_3 - a_4| = \frac{1}{144}\left|7c^3 + 6\left(4 - c^2\right)cx^2 - 12\left(4 - c^2\right)\left(1 - |x|^2\right)z\right|.$$

By applying triangle inequality and replacing $|z| \leq 1$, $|x| = b$ with $b \leq 1$, which leads to

$$|a_2a_3 - a_4| \leq \frac{1}{144}\left[7c^3 + 6\left(4 - c^2\right)(c - 2)b^2 + 12\left(4 - c^2\right)\right] = F(c, b).$$

Now, we differentiate $F(c, b)$ with respect to b, it is easy to show that $F'(c, b) \leq 0$ on rectangle $[0, 2] \times [0, 1]$. If we put $b = 0$, then

$$\max\{F(c, b)\} = F(c, 0).$$

Thus, we have

$$|a_2a_3 - a_4| \leq \frac{1}{144}\left[7c^3 + 12\left(4 - c^2\right)\right] =: G(c).$$

By taking $G'(c) = 0$, we obtain $c = 0$ or $8/7$, and $G''(c) < 0$ at $c = 0$. Thus, $G(c)$ arrives at its maximum value at $c = 0$, it follows that

$$|a_2a_3 - a_4| \leq |G(0)| = \frac{1}{3}.$$

Equality holds for the function given by (3.9). □

Next, we will determine the second-order Hankel determinant for f belongs to $\mathcal{S}_\rho^*$.

Theorem 3.4. *If $f \in \mathcal{S}_\rho^*$, then*

$$|\mathcal{H}_{2,2}(f)| \leq \frac{1}{4}.$$

The inequality is sharp.

Proof. We can write $\mathcal{H}_{2,2}(f)$ as

$$\mathcal{H}_{2,2}(f) = a_2a_4 - a_3^2.$$

From (3.5), (3.6) and (3.7), we have

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{288}\left|-c_1^4 - 6c_1^2c_2 + 24c_1c_3 - 18c_2^2\right|.$$

Without loss of generality, we can write $c_1 = c$ with $0 \leq c \leq 2$. In view of (2.2) and (2.3), we obtain

$$\left|a_2a_4-a_3^2\right|=\frac{1}{288}\left|-\frac{5}{2}c^4-6\left(4-c^2\right)c^2x^2+12\left(4-c^2\right)\left(1-|x|^2\right)cz-\frac{9}{2}\left(4-c^2\right)^2x^2\right|.$$

By applying triangle inequality and replacing $|z| \leq 1$, $|x| = b$ with $b \leq 1$, we get

$$\begin{aligned}&\left|a_2a_4-a_3^2\right|\\ \leq&\frac{1}{288}\left[\frac{5}{2}c^4+6\left(4-c^2\right)c^2b^2+12\left(4-c^2\right)\left(1-b^2\right)c+\frac{9}{2}\left(4-c^2\right)^2b^2\right]=:\phi\left(c,b\right).\end{aligned}$$

By differentiating $\phi\left(c,b\right)$ with respect to b, we have

$$\frac{\partial\phi\left(c,b\right)}{\partial b}=\frac{1}{288}\left(c^2-8c+12\right)\left(4-c^2\right)b.$$

It easily to show that $\phi'\left(c,b\right) \geq 0$ on $[0,1]$, thus, we see that $\phi\left(c,b\right) \leq \phi\left(c,1\right)$.

If we put $b = 1$, then

$$\left|a_2a_4-a_3^2\right| \leq \frac{1}{288}\left[\frac{5}{2}c^4+6c^2\left(4-c^2\right)+\frac{9}{2}\left(4-c^2\right)^2\right]=:G_1\left(c\right).$$

Since $G_1'\left(c\right) \leq 0$, we know that $G_1\left(c\right)$ is a decreasing function, it attains its maximum value at $c = 0$. Therefore, we conclude that

$$\left|\mathcal{H}_{2,2}\left(f\right)\right| \leq \left|G_1(0)\right| = \frac{1}{4}.$$

The required second Hankel determinant is sharp and it can be obtained by (3.10). □

4. Third-order Hankel determinant for the class $\mathcal{S}_\rho^*$

In this section, we will determine the bounds of third-order Hankel determinant for f belongs to $\mathcal{S}_\rho^*$.

Theorem 4.1. *If $f \in \mathcal{S}_\rho^*$, then*

$$\left|\mathcal{H}_{3,1}\left(f\right)\right| \leq \frac{1}{9}.$$

Equality can be obtained by (3.9).

Proof. The third Hankel determinant can be written as follows:

$$\mathcal{H}_{3,1}\left(f\right) = 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5.$$

From (3.3) and (3.4), we know that

$$a_5 = \frac{1}{4}\left(\frac{1}{2}c_4 - \frac{1}{8}c_2^2 + \frac{5}{288}c_1^4 - \frac{1}{48}c_1^2c_2 - \frac{1}{6}c_1c_3\right). \tag{4.1}$$

In view of (3.5), (3.6), (3.7) and (4.1) along with $c_1 = c \in [0,2]$, we get

$$\begin{aligned}\mathcal{H}_{3,1}(f) =&\frac{1}{41472}\Big(-47c^6 + 3c^4c_2 + 528c^3c_3 - 234c^2c_2^2 - 1296c^2c_4 \\ &+ 1872cc_2c_3 - 972c_2^3 + 1296c_2c_4 - 1152c_3^2\Big).\end{aligned} \tag{4.2}$$

To simplify the computation, we assume that $t = 4 - c^2$ in (2.2), (2.3) and (2.4), by using (2.2), (2.3) and (2.4) along with straightforward algebraic computation, we find that

$$\begin{aligned}
2c^4c_2 &= c^6 + c^4tx,\\
4c^3c_3 &= c^6 + 2c^4tx - c^4tx^2 + 2c^3t\left(1-|x|^2\right)z,\\
4c^2c_2^2 &= c^6 + 2c^4tx + c^2t^2x^2,\\
8c^2c_4 =& c^4tx^3 - 4c^3tx\left(1-|x|^2\right)z - 4c^2t\overline{x}\left(1-|x|^2\right)z^2 - 3c^4tx^2\\
&+4c^2t\left(1-|x|^2\right)\left(1-|z|^2\right)\rho + 4c^3t\left(1-|x|^2\right)z + 3c^4tx + c^6 + 4c^2tx^2 + \frac{104}{9}cc_2c_3\\
=& -c^2t^2x^3 - c^4tx^2 + 2ct^2x\left(1-|x|^2\right)z + 2c^2t^2x^2 + 2c^3t\left(1-|x|^2\right)z + 3c^4tx + c^6,\\
8c_2^3 &= t^3x^3 + 3c^2t^2x^2 + 3c^4tx + c^6,\\
16c_2c_4 =& 4c^2tx^2 + 3t^2x^3 + c^6 + 4c^4tx + 4c^3t\left(1-|x|^2\right)z\\
&+ 4c^2t\left(1-|x|^2\right)\left(1-|z|^2\right)\rho + 3c^2t^2x^2 + 4ct^2x\left(1-|x|^2\right)z\\
&+ 4t^2x\left(1-|x|^2\right)\left(1-|z|^2\right)\rho - 3c^4tx^2 - 4c^2t\overline{x}\left(1-|x|^2\right)z^2\\
&- 4c^3tx\left(1-|x|^2\right)z - 3c^2t^2x^3 - 4t^2x\overline{x}\left(1-|x|^2\right)z^2 + c^4tx^3\\
&+ c^2t^2x^4 - 4ct^2x^2\left(1-|x|^2\right)z,
\end{aligned}$$

and

$$\begin{aligned}16c_3^2 =& c^2t^2x^4 - 4ct^2x^2\left(1-|x|^2\right)z - 4c^2t^2x^3 - 2c^4tx^2 + 4t^2\left(1-|x|^2\right)^2z^2\\ &+ 8ct^2x\left(1-|x|^2\right)z + 4c^2t^2x^2 + 4c^3t\left(1-|x|^2\right)z + 4c^4tx + c^6.\end{aligned}$$

By substituting these expressions into (4.2), we deduce that

$$\begin{aligned}\mathcal{H}_3(1) =& \frac{1}{41472}\Big[-\frac{25}{2}c^6 + 324t^2x^3 - \frac{243}{2}t^3x^3 - 324c^2tx^2 - 81c^4tx^3 + 21c^4tx^2\\
&+36c^4tx + 9c^2t^2x^4 - 189c^2t^2x^3 - 288t^2\left(1-|x|^2\right)^2z^2 + 120c^3t\left(1-|x|^2\right)z\\
&+ 324c^3tx\left(1-|x|^2\right)z + 324c^2t\overline{x}\left(1-|x|^2\right)z^2 - 324c^2t\left(1-|x|^2\right)\left(1-|z|^2\right)\rho\\
&- 36ct^2x^2\left(1-|x|^2\right)z - 324t^2x\overline{x}\left(1-|x|^2\right)z^2 + 216ct^2x\left(1-|x|^2\right)z\\
&+ 324t^2x\left(1-|x|^2\right)\left(1-|z|^2\right)\rho\Big].\end{aligned}$$

By noting that $t = 4 - c^2$, we get

$$\mathcal{H}_{3,1}(f) = \frac{1}{41472}\left[v_1(c,x) + v_2(c,x)z + v_3(c,x)z^2 + \Psi(c,x,z)\rho\right],$$

where $\rho, x, z \in \overline{\mathcal{U}}$, and

$$\begin{aligned} v_1(c,x) =& -\frac{25}{2}c^6 + \left(4-c^2\right)\Bigg[\left(4-c^2\right)\left(-162x^3 - \frac{135}{2}c^2x^3 + 9c^2x^4\right) \\ & - 324c^2x^2 - 81c^4x^3 + 21c^4x^2 + 36c^4x\Bigg], \\ v_2(c,x) =& \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(216cx - 36cx^2\right) + 324c^3x + 120c^3\right], \\ v_3(c,x) =& \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-36x^2 - 288\right) + 324c^2\overline{x}\right], \end{aligned}$$

and

$$\Psi(c,x,z) = \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|z|^2\right)\left[-324c^2 + 324x\left(4-c^2\right)\right].$$

Now, by using $|x| = x, |z| = y$ and utilizing the fact $|\rho| \le 1$, we get

$$\begin{aligned} |\mathcal{H}_{3,1}(f)| \le& \frac{1}{41472}\left(|v_1(c,x)| + |v_2(c,x)|\, y + |v_3(c,x)|\, y^2 + |\Psi(c,x,z)|\right) \\ \le& \frac{1}{41472}|G(c,x,y)|, \end{aligned} \tag{4.3}$$

where

$$G(c,x,y) = g_1(c,x) + g_4(c,x) + g_2(c,x)\, y^2 + \left(g_3(c,x) - g_4(c,x)\right) y^2$$

with

$$\begin{aligned} g_1(c,x) =& \frac{25}{2}c^6 + \left(4-c^2\right)\Bigg[\left(4-c^2\right)\left(162x^3 + \frac{135}{2}c^2x^3 + 9c^2x^4\right) \\ & + 324c^2x^2 + 81c^4x^3 + 21c^4x^2 + 36c^4x\Bigg], \\ g_2(c,x) =& \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(216cx + 36cx^2\right) + 324c^3x + 120c^3\right], \\ g_3(c,x) =& \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(36x^2 + 288\right) + 324c^2x\right], \end{aligned}$$

and

$$g_4(c,x) = \left(4-c^2\right)\left(1-x^2\right)\left[324c^2 + 324x\left(4-c^2\right)\right].$$

It is shown in [33] that $|G(c,x,y)|$ achieves its maximum value 4608 on the closed cuboid $[0,2] \times [0,1] \times [0,1]$. Thus, we deduce that

$$|\mathcal{H}_{3,1}(f)| \le \frac{1}{41472}|G(c,x,y)| \le \frac{1}{9}.$$

□

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