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EXISTENCE AND GLOBAL ASYMPTOTIC BEHAVIOR OF MILD SOLUTIONS FOR DAMPED ELASTIC SYSTEMS WITH DELAY AND NONLOCAL CONDITIONS*

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Abstract In this paper, we are devoted to the study of a class of structural damped elastic systems with delay and nonlocal conditions in Banach space. Firstly, in the sense of compact semigroup, the existence of mild solutions is studied, where the nonlinearity f and nonlocal function g satisfy more general growth conditions rather than Lipschitz-type conditions. Secondly, based on a new Gronwall-Bellman type integral inequality with delay, the global asymptotic stability of the mild solution is discussed. At the end, a concrete example of nonlocal damped beam vibration equation is given to illustrate the feasibility and practical application value of our abstract results.

Keywords Elastic systems, structural damping, nonlocal problem, asymptotic stability, compact semigroup.

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1. Introduction

The aim of the paper is to discuss the existence and global asymptotic stability of mild solutions for the abstract model of the damped beam vibration equation with delay and nonlocal initial conditions in a Banach space E, i.e. the nonlocal problem of second order delayed evolution equation

$$\begin{cases} \ddot{u}(t)+\rho A\dot{u}(t)+A^2u(t)=f(t,u(t),u_t), & t\geq 0,\\ u(t)=\varphi(t), & t\in[-r,0],\\ \dot{u}(0)=g(u)+\psi, \end{cases} \tag{1.1}$$

where $\dot{u}$ and $\ddot{u}$ are the first and second order partial derivatives of u with respect to t, $\rho\geq 2$ is the damping coefficient, $A:\mathcal{D}(A)\subset E\to E$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t\geq 0)$ on E, $f:\mathbb{R}^+\times E\times\mathcal{B}\to E$ is a nonlinear mapping, $\mathcal{B}:=C([-r,0],E)$ is the Banach space of all continuous functions from $[-r,0]$ to E with the norm $\|\phi\|_{\mathcal{B}}=\sup\limits_{s\in[-r,0]}\|\phi(s)\|$. For $t\geq 0$, $u_t\in\mathcal{B}$ is the history

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state defined by $u_t(s) = u(t+s)$ for $s \in [-r, 0]$, $g : C(\mathbb{R}^+; E) \to E$, $\varphi \in \mathcal{B}$ and $\varphi(0) \in \mathcal{D}(A)$, $\psi \in E$, $r > 0$ is a constant.

As an important and independent branch of modern engineering research and applications, elastic beam is not only widely used in the fields of mechanics, material science, physics, geography and so on, but also plays an almost irreplaceable role in specific applications. Therefore, the study of beam vibration equations has gradually attracted high attention and great interest of scholars in various fields. In 1744, Leonhard Euler studied the transverse vibration of beams and gave the vibration functions and frequency equations under different boundary conditions. In 1751, on a similar issue, Daniel Bernoulli put forward the beam vibration equation

$$\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\Big(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\Big) = 0, \quad 0 < x < 1,\ t > 0, \tag{1.2}$$

where $\rho(x)$ is the mass density of the beam, E is the elastic modulus, and $I(x)$ is the moment of inertia of the beam cross section. Eq. (1.2) is the famous Euler-Bernoulli beam equation, which is the most basic beam vibration equation. During the early decades, this kind of equations have become the object of extensive research in several setting. In the specific mathematical research, the beam vibration equations are usually transformed into the corresponding abstract elastic systems, that is, the initial value problems of the second-order evolution equations, and then its properties are studied by various analysis methods.

Later, with the progress of science and technology and the rapid development of aerospace technology, the structural damping beam vibration equation, as an important mathematical model of the spacecraft, gradually came into people's vision. Since 1981, Chen and Russel [4] firstly proposed the damped elastic system

$$\begin{cases} \ddot{u}(t) + B\dot{u}(t) + Au(t) = 0, \\ u(0) = x_0,\ \dot{u}(0) = y_0, \end{cases} \tag{1.3}$$

such problems have successfully become one of the typical research objects in the field of evolution equations. Over the past few decades, the research on the well-posedness and asymptotic behavior of the solutions for linear and nonlinear elastic systems with structural damping have been relatively mature, and some rich and interesting results have been obtained, see [4, 10, 12–16, 18–20, 22, 29, 37] and their references. Especially, in [12–14], by means of the operator semigroups theory and various nonlinear analysis methods, Fan et al. studied the existence, uniqueness and asymptotic stability of solutions for the semilinear structural damped elastic system

$$\begin{cases} \ddot{u}(t) + \rho A\dot{u}(t) + A^2u(t) = f(t, u(t)), \quad t > 0, \\ u(0) = x_0,\ \dot{u}(0) = y_0, \end{cases} \tag{1.4}$$

and its corresponding linear elastic system.

Recently, in [30], by using the fixed point theorem of condensed mapping, Luong and Tung researched the existence and exponential decay of the mild solutions for the damped elastic system

$$\begin{cases} \ddot{u}(t) + \rho A\dot{u}(t) + A^2u(t) = f(t, u(t), u_t), & t \in [0, T], \\ u(s) = \varphi(s), & s \leq 0, \\ \dot{u}(0) + h(u) = \psi. \end{cases} \tag{1.5}$$

The characteristic of this equation is that the nonlinear function f contains delay and the initial value conditions are nonlocality. However, it is the effects of delay and nonlocal conditions that prevent such problems from being widely studied.

Actually, delay is a ubiquitous and extremely important phenomenon in natural and social. The delay phenomenon occurs not only in specific engineering systems, but also in many practical models. For example, when people study the problems of ecosystem, neural network, population reproduction, transportation and so on, they often come into contact with many systems involving delay, which prompts scholars to study various integral and differential equations with delay. Compared with the general evolution equations, the evolution equations with delay are usually a kind of mathematical models with broader practical application background. In recent decades, the research on the systems described by the evolution equations with delay have become a very active topic in the field of control theory and engineering, see [6, 8, 26–28, 30, 31, 36] and the references therein.

In addition, the theory of nonlocal evolution equations is an important research branch in the field of analysis, which can better describe some specific phenomena in physics, biology, aerospace, and medicine than the traditional Cauchy problems. In [9], Deng studied a kind of reaction-diffusion equations with nonlocal initial condition $u(0) + \sum_{k=1}^{n} c_k u(t_k) = \psi$, and used this problem to describe the gas diffusion phenomenon in transparent tubes. In [2], Byszewski studied the nonlocal problems of a class of abstract functional differential equations, and pointed out that the abstract conclusions obtained can be used to determine the position change of a physical object in kinematics and dynamics. Therefore, the nonlocal conditions are better than the classical initial conditions both in theory and practical applications. For more research on nonlocal evolution equations, see [1, 3, 7, 29, 30, 38, 39] etc.

What's more, we have noticed that in recent years, the research on the asymptotic behavior of solutions for evolution equations have been relatively active, see [5–7, 26, 27, 31, 34–36] and related references. In particular, Li [26] researched the asymptotic stability of the time periodic solutions for the evolution equation with multiple delays. Chen et al. [7] studied the global asymptotic stability of mild solutions for a class of nonlocal semilinear stochastic evolution equations. Li et al. [31] discussed the asymptotic stability of periodic mild solutions for the neutral evolution equation. Recently, Rubbioni [34] showed the asymptotic stability of solutions for some classes of impulsive differential equations with distributed delay. We know that in specific system applications, only the stable states or processes are meaningful, but various accidental disturbances will inevitably occur in the actual systems. Therefore, it is of great significance to study the stability of the solutions of evolution equations describing actual systems.

Motivated by the above-mentioned aspects, this paper discusses the existence and global asymptotic stability of mild solutions for nonlocal problem (1.1). First of all, the equation (1.1) we studied contains both delay and nonlocal conditions, this will provide a very necessary theoretical support for some practical engineering projects involving the vibration of elastic beams but cannot vibrate simply and perfectly due to the interference of external factors. At the same time, this is also a great enrichment and development for the theory of evolution equations with delay. Secondly, we discuss the existence of global mild solutions for damped elastic systems with delay and nonlocal conditions on the infinite interval, which will have more practical application value than the existence of local mild solutions on a finite interval. Furthermore, in the establishment of the existence result, the conditions we put forward for the nonlinear function f and nonlocal function g are more general than the usual Lipschitz-type conditions and have better application value. Therefore, our results are novel and meaningful.

The main results of this paper are presented in Section 3 and 4. In Section 3, we establish the existence result of the mild solutions based on the generalized Arzela-Ascoli theorem and Schauder's fixed point theorem. Subsequently, by establishing a new Gronwall-Bellman type integral inequality with delay, the global asymptotic stability of the mild solution is discussed in Section 4. At the end of this article, a concrete beam vibration equation which can be transformed into the abstract problem (1.1) is discussed by using our abstract results. In order to facilitate the establishment and proof of our conclusions, we will review some necessary basic knowledge in the following section.

2. Preliminaries

Throughout this paper, we assume that $(E, \|\cdot\|)$ is a Banach space, $A : \mathcal{D}(A) \subset E \to E$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ on E. Now, we recall some basic concepts and properties of C_0-semigroup which are necessary for the subsequent discussion.

In accordance with the exponential boundedness of C_0-semigroup, there exist constants $M \geq 1$ and $\nu \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\nu t}, \qquad t \geq 0. \tag{2.1}$$

In particular, if $\nu = 0$, i.e., $T(t)(t \geq 0)$ satisfies $\|T(t)\| \leq M$ for all $t \geq 0$, then, $T(t)(t \geq 0)$ is said to be uniformly bounded.

Definition 2.1 ([11]). Let $T(t)(t \geq 0)$ be a C_0-semigroup on E, we say ν_0, the infimum of ν satisfying (2.1), is the growth exponent of $T(t)(t \geq 0)$ to mean

$$\nu_0 = \inf\{\nu \in \mathbb{R} | \textit{there exists } M \geq 1 \textit{ such that } \|T(t)\| \leq Me^{\nu t},\ \forall t \geq 0\}. \tag{2.2}$$

Moreover, if $\nu_0 < 0$, then C_0-semigroup $T(t)(t \geq 0)$ is said to be exponentially stable.

In view of the Definition 2.1, apparently, the exponentially stable C_0-semigroup $T(t)(t \geq 0)$ is uniformly bounded. If the C_0-semigroup $T(t)(t \geq 0)$ is continuous under the uniform operator topology in E for each $t > 0$, then ν_0 can also be expressed by the spectral set $\sigma(A)$ of A, i.e.,

$$\nu_0 = -\inf\{Re\lambda | \lambda \in \sigma(A)\}. \tag{2.3}$$

In fact, we have known from [35] that if C_0-semigroup $T(t)(t \geq 0)$ is compact, then it is continuous under the uniform operator topology for $t \geq 0$.

According to [30], the definition of the mild solution for the vibration equation of the damped beam, we can derive the following definition of the mild solution for nonlocal problem (1.1).

Definition 2.2. We say a function $u \in C\big([-r,+\infty),E\big)$ is a mild solution of nonlocal problen (1.1) provided $u(t)=\varphi(t)$ for $t\in[-r,0]$ and

$$\begin{aligned} u(t) =& T_2(t)\varphi(0)+\int_0^t T_2(t-s)T_1(s)(g(u)+\psi+\gamma_2 A\varphi(0))ds \\ &+\int_0^t\int_0^s T_2(t-s)T_1(s-\tau)f(\tau,u(\tau),u_\tau)d\tau ds,\ \text{for } t\geq 0, \end{aligned} \tag{2.4}$$

where $T_i(t)(t\geq 0)(i=1,2)$ are C_0-semigroups on E satisfing

$$T_i(t)=T(\gamma_i t)(i=1,2),\quad t\geq 0; \tag{2.5}$$

$$\gamma_1+\gamma_2=\rho,\ \gamma_1\gamma_2=1,\quad 0<\gamma_1<\gamma_2. \tag{2.6}$$

Lemma 2.1. *$T_i(t)(t\geq 0)(i=1,2)$ defined by (2.5) have the following properties:*

- (1°) *If $T(t)(t\geq 0)$ is compact on E, then $T_i(t)(t\geq 0)(i=1,2)$ are compact C_0-semigroups on E;*
- (2°) *If $T(t)(t\geq 0)$ is exponentially stable on E with growth exponent $\nu_0<0$, then $T_i(t)(t\geq 0)(i=1,2)$ are exponentially stable C_0-semigroups on E with growth exponent $\nu_i=\gamma_i\nu_0$ and $\nu_2<\nu_1<0$ (γ_i is defined in (2.6)).*

Proof. For (1°), the conclusion is obvious by the properties of compact semigroups, we can refer to [12,33]. Now, we verify that (2°) is valid.

(2°) Since $T(t)(t\geq 0)$ is exponentially stable, then according to [13] Theorem 3.1, we can easily obtained that $T_i(t)(t\geq 0)(i=1,2)$ are exponentially stable C_0-semigroups on E. In view of the Definition 2.1, we have

$$\|T(t)\|\leq Me^{\nu_0 t},\qquad t\geq 0. \tag{2.7}$$

Then, by (2.5), we can find

$$\|T_i(t)\|=\|T(\gamma_i t)\|\leq Me^{\nu_0\gamma_i t}:=Me^{\nu_i t},\qquad t\geq 0. \tag{2.8}$$

Since $\gamma_i>0$, from Definition 2.1, it follows that $\nu_i=\gamma_i\nu_0$ are the growth exponent of $T_i(t)(t\geq 0)(i=1,2)$. Since $\nu_0<0$ and $0<\gamma_1<\gamma_2$, thus, $\nu_2<\nu_1<0$.

This completes the proof. □

3. Existence Result

In this section, we firstly assume that $h\in C\big(\mathbb{R}^+,[1,+\infty)\big)$ is a nondecreasing function with $\lim\limits_{t\to+\infty}h(t)=+\infty$ and define the Banach space

$$C_h(E)=\Big\{u\in C\big(\mathbb{R}^+,E\big):\lim_{t\to+\infty}\frac{\|u(t)\|}{h(t)}=0\Big\}$$

with the norm $\|u\|_h=\sup\limits_{t\geq 0}\frac{\|u(t)\|}{h(t)}$.

The following compactness criterion is indispensable.

Lemma 3.1 ([21]). *A set $B \subset C_h(E)$ is relatively compact if and only if*

(a) B is equicontinuous;

(b) $B(t) = \{u(t) : u \in B\}$ is relatively compact in E for every $t \geq 0$;

(c) $\lim_{t\to+\infty} \frac{\|u(t)\|}{h(t)} = 0$, uniformly for $u \in B$.

Moreover, for a given $\varphi \in \mathcal{B}$ and $x \in C_h(E)$, we define $x_\varphi(t) : [-r, +\infty) \to E$ by

$$x_\varphi(t) = \begin{cases} x(t), & t \geq 0, \\ \varphi(t), & t \in [-r, 0]. \end{cases} \tag{3.1}$$

And define a closed subspace of $C_h(E)$ by

$$C_{h,\varphi}(E) = \{u \in C_h(E) : \ u(0) = \varphi(0)\}$$

with the norm $\|u\|_{h,\varphi} = \max\{\|u\|_h, \ \|\varphi\|_{\mathcal{B}}\}$.

Now, we establish the existence result of mild solutions for equation (1.1).

Theorem 3.1. *Assume that $A : \mathcal{D}(A) \subset E \to E$ is a closed linear operator and $-A$ generates a compact and exponentially stable C_0-semigroup $T(t)(t \geq 0)$ on E, whose growth exponent $\nu_0 < 0$. Let $\varphi \in \mathcal{B}$, $\varphi(0) \in \mathcal{D}(A)$ and $\psi \in E$ be given. If $f : \mathbb{R}^+ \times E \times \mathcal{B} \to E$, $g : C(\mathbb{R}^+, E) \to E$ are continuous and satisfy the following conditions:*

(H1) For any $t \geq 0$, $x \in E$ and $\phi \in \mathcal{B}$, there exist functions $p_i : \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1, 2)$ and nondecreasing functions $\mathcal{F}_i \in C\big(\mathbb{R}^+, [0, +\infty)\mathbb{R}^+\big)$ $(i = 1, 2)$ as well as a positive constant K, such that

$$\|f(t, h(t)x, h(t)\phi)\| \leq p_1(t)\mathcal{F}_1(\|x\|) + p_2(t)\mathcal{F}_2(\|\phi\|_{\mathcal{B}}) + K,$$

where $\mathcal{F}_i$ and p_i satisfy

$$\liminf_{l\to+\infty} \frac{\mathcal{F}_i(l)}{l} := \zeta_i < +\infty, \quad i = 1, 2;$$

$$\sup_{t>0} \int_0^t e^{\nu_1(t-s)} p_i(s) ds := \delta_i < +\infty, \quad i = 1, 2.$$

(H2) For any $t \geq 0$, there exists a nondecreasing function $\mathcal{G} \in C\big(\mathbb{R}^+, [0, +\infty)\mathbb{R}^+\big)$ with

$$\liminf_{l\to+\infty} \frac{\mathcal{G}(l)}{l} := \eta < +\infty,$$

such that

$$\|g(h(t)v)\| \leq \mathcal{G}(\|v\|_C), \qquad v \in C\big(\mathbb{R}^+, E\big).$$

Then, the nonlocal problem (1.1) has at least one mild solution $u \in C\big([-r, +\infty), E\big)$ provided that

$$\frac{M^2}{\gamma_2 - \gamma_1}\eta + \frac{M^2}{\gamma_2}(\delta_1\zeta_1 + \delta_2\zeta_2) < |\nu_0|, \tag{3.2}$$

where γ_i and M are defined in (2.6) and (2.7).

Proof. Define an operator Q on $C_{h,\varphi}(E)$ by

$$\begin{aligned}Qu(t) =& T_2(t)\varphi(0) + \int_0^t T_2(t-s)T_1(s)(g(u)+\psi+\gamma_2 A\varphi(0))ds \\ &+ \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau,u(\tau),u_\tau)d\tau ds, \quad t \geq 0. \end{aligned} \tag{3.3}$$

Since $T(t)(t \geq 0)$ is a compact and exponentially stable C_0-semigroup, according to Lemma 2.3, $T_i(t)(t \geq 0)$ $(i = 1, 2)$ are compact, exponentially stable semigroups with corresponding growth exponent $\nu_2 < \nu_1 < 0$. It is clear that $Q : C_{h,\varphi}(E) \to C_{h,\varphi}(E)$ is well defined. In reality, for any $u \in C_{h,\varphi}(E)$ and $t \geq 0$, we know that $\|u(t)\| \leq h(t)\|u\|_h \leq h(t)\|u\|_{h,\varphi}$ and

$$\begin{aligned}\|u_t\|_{\mathcal{B}} = \sup_{s\in[-r,0]} \|u(t+s)\| &\leq \max\{\sup_{t\in[-r,0]} \|u(t)\|, \ \sup_{t\geq 0} \|u(t)\|\} \\ &\leq \max\{\|\varphi\|_{\mathcal{B}}, \ h(t)\|u\|_h\} \leq h(t)\|u\|_{h,\varphi}.\end{aligned}$$

Then, according to (3.3), for any $t \geq 0$, we have

$$\begin{aligned}\|Qu(t)\| \leq& \|T_2(t)\varphi(0)\| + \Big\| \int_0^t T_2(t-s)T_1(s)\Big(g(u)+\psi+\gamma_2 A\varphi(0)\Big)ds\Big\| \\ &+ \Big\| \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau,u(\tau),u_\tau)d\tau ds\Big\| \\ :=& I_1 + I_2 + I_3, \end{aligned} \tag{3.4}$$

where

$$I_1 := \|T_2(t)\varphi(0)\| \leq M\|\varphi\|_{\mathcal{B}}. \tag{3.5}$$

By $(H1)$ and $(H2)$, one can find

$$\begin{aligned}I_2 :=& \Big\| \int_0^t T_2(t-s)T_1(s)\Big(g(u)+\psi+\gamma_2 A\varphi(0)\Big)ds\Big\| \\ \leq& M^2 \int_0^t e^{\nu_2(t-s)}e^{\nu_1 s}\Big(\mathcal{G}(\frac{\|u\|_C}{h(t)}) + \|\psi\| + \gamma_2\|A\varphi(0)\|\Big)ds \\ \leq& \frac{M^2}{\nu_1-\nu_2}\Big(\mathcal{G}(\|u\|_{h,\varphi}) + \|\psi\| + \gamma_2\|A\varphi(0)\|\Big), \end{aligned} \tag{3.6}$$

$$\begin{aligned}I_3 :=& \Big\| \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau,u(\tau),u_\tau)d\tau ds\Big\| \\ \leq& M^2 \int_0^t \int_0^s e^{\nu_2(t-s)}e^{\nu_1(s-\tau)}\Big(p_1(\tau)\mathcal{F}_1(\frac{\|u(\tau)\|}{h(\tau)}) + p_2(\tau)\mathcal{F}_2(\frac{\|u_\tau\|_{\mathcal{B}}}{h(\tau)}) + K\Big)d\tau ds \\ \leq& M^2 \int_0^t e^{\nu_2(t-s)}\Big(\delta_1\mathcal{F}_1(\|u\|_{h,\varphi}) + \delta_2\mathcal{F}_2(\|u\|_{h,\varphi}) + \frac{K}{|\nu_1|}\Big)ds \\ \leq& \frac{M^2}{|\nu_2|}\Big(\delta_1\mathcal{F}_1(\|u\|_{h,\varphi}) + \delta_2\mathcal{F}_2(\|u\|_{h,\varphi}) + \frac{K}{|\nu_1|}\Big). \end{aligned} \tag{3.7}$$

Then, combining with $\lim_{t\to+\infty} h(t) = +\infty$, we can get that $\lim_{t\to+\infty} \frac{1}{h(t)}\|Qu(t)\| = 0$, which means that $Qu \in C_h(E)$ for $u \in C_{h,\varphi}(E)$. On the other hand, we know that

$(Qu)(0) = \varphi(0)$ from (3.3). Hence, it is not difficult to find that $Q : C_{h,\varphi}(E) \to C_{h,\varphi}(E)$ is well defined. According to Definition 2.2 and (3.1), if u is a fixed point of Q on $C_{h,\varphi}(E)$, then u_φ is undoubtedly a mild solution of nonlocal problem (1.1).

In what follows, we will employ the fixed point theorem to verify in five steps that the operator Q has a fixed point on $C_{h,\varphi}(E)$.

I. *We claim that* $Q : C_{h,\varphi}(E) \to C_{h,\varphi}(E)$ *is continuous.*

To verify this assertion, let $\{u^{(n)}\} \subset C_{h,\varphi}(E)$ be a sequence such that $u^{(n)} \to u$ in $C_{h,\varphi}(E)$ as $n \to \infty$, then, $u^{(n)}(t) \to u(t)$ in E and $u_t^{(n)} \to u_t$ in $\mathcal{B}$ for every $t \geq 0$ as $n \to \infty$.

For $t \geq 0$, by the continuity of f and g, one can see that when $n \to \infty$,

$$f(t, u^{(n)}(t), u_t^{(n)}) \to f(t, u(t), u_t), \qquad g(u^{(n)}) \to g(u).$$

Hence, by the exponential boundedness of $T_i(t)(t \geq 0)$ $(i = 1, 2)$ and the Lebesgue dominated convergence theorem, we can deduce that

$$\begin{aligned}
&\|Qu^{(n)}(t) - Qu(t)\| \\
\leq &\int_0^t \|T_2(t-s)\| \cdot \|T_1(s)\| \cdot \|g(u^{(n)}) - g(u)\| ds \\
&+ \int_0^t \int_0^s \|T_2(t-s)\| \cdot \|T_1(s-\tau)\| \cdot \|f(\tau, u^{(n)}(\tau), u_\tau^{(n)}) - f(\tau, u(\tau), u_\tau)\| d\tau ds \\
\leq &M^2 \|g(u^{(n)}) - g(u)\| \cdot \int_0^t e^{\nu_2(t-s)} \cdot e^{\nu_1 s} ds \\
&+ M^2 \|f(\tau, u^{(n)}(\tau), u_\tau^{(n)}) - f(\tau, u(\tau), u_\tau)\| \cdot \int_0^t \int_0^s e^{\nu_2(t-s)} \cdot e^{\nu_1(s-\tau)} d\tau ds \\
\leq &\frac{M^2}{\nu_1 - \nu_2} \|g(u^{(n)}) - g(u)\| + \frac{M^2}{\nu_1 \nu_2} \|f(s, u^{(n)}(s), u_s^{(n)}) - f(s, u(s), u_s)\|.
\end{aligned}$$

As a result, we can get

$$\|Qu^{(n)} - Qu\|_{h,\varphi} = \sup_{t \geq 0} \frac{1}{h(t)} \|Qu^{(n)}(t) - Qu(t)\| \to 0 \ (n \to \infty),$$

which indicates that $Q : C_{h,\varphi}(E) \to C_{h,\varphi}(E)$ is continuous.

Now, for any $R > 0$, we define a set

$$\overline{\Omega}_R = \{u \in C_{h,\varphi}(E) : \|u\|_{h,\varphi} \leq R\}. \tag{3.8}$$

Apparently, $\overline{\Omega}_R$ is a closed ball in $C_{h,\varphi}(E)$.

II. *We check that there is a constant* $R_0 > 0$ *such that* $Q(\overline{\Omega}_{R_0}) \subset \overline{\Omega}_{R_0}$.

Indeed, if this were not so, it would follows that for any $R > 0$, there exists $u \in \overline{\Omega}_R$ such that $\|Qu\|_{h,\varphi} > R$. Thereupon, $\sup\limits_{t \geq 0} \frac{1}{h(t)} \|Qu(t)\| + \|\varphi\|_{\mathcal{B}} > R$.

Based on (3.4), we can get that for any $t \geq 0$,

$$\frac{1}{h(t)} \|Qu(t)\| \leq \|Qu(t)\| \leq I_1 + I_2 + I_3.$$

In view of (3.6) and (3.7), we can calculate that

$$\begin{aligned} I_2 &\leq \frac{M^2}{\nu_1-\nu_2}\Big(\mathcal{G}(\|u\|_{h,\varphi}) + \|\psi\| + \gamma_2\|A\varphi(0)\|\Big) \\ &\leq \frac{M^2}{\nu_1-\nu_2}\Big(\mathcal{G}(R) + \|\psi\| + \gamma_2\|A\varphi(0)\|\Big), \end{aligned}$$

and

$$\begin{aligned} I_3 &\leq \frac{M^2}{|\nu_2|}\Big(\delta_1\mathcal{F}_1(\|u\|_{h,\varphi}) + \delta_2\mathcal{F}_2(\|u\|_{h,\varphi}) + \frac{K}{|\nu_1|}\Big) \\ &\leq \frac{M^2}{|\nu_2|}\Big(\delta_1\mathcal{F}_1(R) + \delta_2\mathcal{F}_2(R) + \frac{K}{|\nu_1|}\Big). \end{aligned}$$

Hence, according to the above calculation and (3.5), we can see

$$\begin{aligned} R <&(M+1)\|\varphi\|_{\mathcal{B}} + \frac{M^2}{\nu_1-\nu_2}\Big(\mathcal{G}(R) + \|\psi\| + \gamma_2\|A\varphi(0)\|\Big) \\ &+ \frac{M^2}{|\nu_2|}\Big(\delta_1\mathcal{F}_1(R) + \delta_2\mathcal{F}_2(R) + \frac{K}{|\nu_1|}\Big). \end{aligned}$$

Dividing both sides by R and taking the lower limit as $R \to \infty$, we can get

$$\frac{M^2}{\nu_1-\nu_2}\eta + \frac{M^2}{|\nu_2|}(\delta_1\zeta_1 + \delta_2\zeta_2) > 1,$$

since $\nu_1 = \gamma_1\nu_0,\ \nu_2 = \gamma_2\nu_0$ and $\nu_0 < 0$. Hence, we have

$$\frac{M^2}{\gamma_2-\gamma_1}\eta + \frac{M^2}{\gamma_2}(\delta_1\zeta_1 + \delta_2\zeta_2) > |\nu_0|,$$

which is a contradiction with (3.2). Thus, there is a constant $R_0 > 0$ such that $Q(\overline{\Omega}_{R_0}) \subset \overline{\Omega}_{R_0}$.

III. *We further explain that* $\lim\limits_{t\to+\infty} \frac{1}{h(t)}\|Qu(t)\| = 0$, *uniformly for* $u \in \overline{\Omega}_{R_0}$.

By the proof of step II, we can easily get that for any $u \in \overline{\Omega}_{R_0}$,

$$\begin{aligned} \frac{1}{h(t)}\|Qu(t)\| \leq& \frac{1}{h(t)}\Big[M\|\varphi\|_{\mathcal{B}} + \frac{M^2}{\nu_1-\nu_2}\big(\mathcal{G}(R_0) + \|\psi\| + \gamma_2\|A\varphi(0)\|\big) \\ &+ \frac{M^2}{|\nu_2|}\big(\delta_1\mathcal{F}_1(R_0) + \delta_2\mathcal{F}_2(R_0) + \frac{K}{|\nu_1|}\big)\Big]. \end{aligned}$$

So, there is no doubt that $\lim\limits_{t\to+\infty} \frac{1}{h(t)}\|Qu(t)\| = 0$, uniformly for $u \in \overline{\Omega}_{R_0}$.

IV. *It remains to verify that* $Q(\overline{\Omega}_{R_0})$ *is equicontinuous.*

For any $u \in \overline{\Omega}_{R_0}$ and $0 < t_1 < t_2$, by (3.3) and (2.8), one can deduce that

$$\begin{aligned} &\|Qu(t_2) - Qu(t_1)\| \\ \leq& \|T_2(t_2)\varphi(0) - T_2(t_1)\varphi(0)\| \\ &+ \Big\|\int_0^{t_1}\Big(T_2(t_2-s) - T_2(t_1-s)\Big)T_1(s)\Big(g(u) + \psi + \gamma_2A\varphi(0)\Big)ds\Big\| \end{aligned}$$

$$
\begin{aligned}
&+\Big\|\int_{t_1}^{t_2} T_2(t_2-s)T_1(s)\Big(g(u)+\psi+\gamma_2 A\varphi(0)\Big)ds\Big\| \\
&+\Big\|\int_0^{t_1}\int_0^s \Big(T_2(t_2-s)-T_2(t_1-s)\Big)T_1(s-\tau)f(\tau,u(\tau),u_\tau)d\tau ds\Big\| \\
&+\Big\|\int_{t_1}^{t_2}\int_0^s T_2(t_2-s)T_1(s-\tau)f(\tau,u(\tau),u_\tau)d\tau ds\Big\| \\
:=&J_1+J_2+J_3+J_4+J_5.
\end{aligned}
$$

Now, we verify that $J_i \to 0 (i=1,2,3,4,5)$ independently of $u \in \overline{\Omega}_{R_0}$ as $t_2-t_1 \to 0$. In light of the strong continuity of $T_2(t)(t \geq 0)$, it is easy to see that

$$
\begin{aligned}
J_1 =&\|T_2(t_2)\varphi(0)-T_2(t_1)\varphi(0)\| \\
\leq&\|T_2(t_2-t_1)-I\|\cdot\|T_2(t_1)\|\cdot\|\varphi\|_{\mathcal{B}} \to 0,\ \text{as } t_2-t_1\to 0.
\end{aligned}
$$

Taking a sufficiently small constant $\varepsilon \to 0^+$, then, we can get

$$
\begin{aligned}
&\int_0^{t_1}\|T_2(t_2-s)-T_2(t_1-s)\|ds \\
\leq&\int_0^{t_1-\varepsilon}\|T_2(t_2-s)-T_2(t_1-s)\|ds+\int_{t_1-\varepsilon}^{t_1}\|T_2(t_2-s)-T_2(t_1-s)\|ds \\
\leq&\|T_2(t_2-t_1+\varepsilon)-T_2(\varepsilon)\|\cdot\int_0^{t_1-\varepsilon}\|T_2(t_1-\varepsilon-s)\|ds \\
&+\int_{t_1-\varepsilon}^{t_1}\big(\|T_2(t_2-s)\|+\|T_2(t_1-s)\|\big)ds \\
\leq&\frac{M}{|\nu_2|}\|T_2(t_2-t_1+\varepsilon)-T_2(\varepsilon)\|+2M\varepsilon \to 0,\ \text{as } t_2-t_1\to 0.
\end{aligned}
$$

Hence, from the conditions $(H1)$ and $(H2)$, it follows that

$$
\begin{aligned}
J_2 \leq&\int_0^{t_1}\|T_2(t_2-s)-T_2(t_1-s)\|\cdot\|T_1(s)\|\cdot\|g(u)+\psi+\gamma_2 A\varphi(0))\|ds \\
\leq& M\big(\mathcal{G}(R_0)+\|\psi\|+\gamma_2\|A\varphi(0)\|\big)\cdot\int_0^{t_1}\|T_2(t_2-s)-T_2(t_1-s)\|ds \\
\to& 0,\ \text{as } t_2-t_1\to 0, \\
J_4 \leq&\int_0^{t_1}\int_0^s\|T_2(t_2-s)-T_2(t_1-s)\|\cdot\|T_1(s-\tau)\|\cdot\|f(\tau,u(\tau),u_\tau)\|d\tau ds \\
\leq& M\int_0^{t_1}\|T_2(t_2-s)-T_2(t_1-s)\| \\
&\times\int_0^s e^{\nu_1(s-\tau)}\big[p_1(\tau)\mathcal{F}_1(R_0)+p_2(\tau)\mathcal{F}_2(R_0)+K\big]d\tau ds \\
\leq& M\big(\delta_1\mathcal{F}_1(R_0)+\delta_2(\tau)\mathcal{F}_2(R_0)+\frac{K}{|\nu_1|}\big)\int_0^{t_1}\|T_2(t_2-s)-T_2(t_1-s)\|ds \\
\to& 0,\ \text{as } t_2-t_1\to 0.
\end{aligned}
$$

For J_3 and J_5, we have

$$
J_3 \leq \int_{t_1}^{t_2}\|T_2(t_2-s)\|\cdot\|T_1(s)\|\cdot\|g(u)+\psi+\gamma_2 A\varphi(0))\|ds
$$

$$
\begin{aligned}
&\leq M^2(\mathcal{G}(R_0)+\|\psi\|+\gamma_2\|A\varphi(0)\|)\cdot|t_2-t_1|,\\
&\to 0,\ \text{as } t_2-t_1\to 0,\\
J_5 &\leq \int_{t_1}^{t_2}\int_0^s \|T_2(t_2-s)\|\cdot\|T_1(s-\tau)\|\cdot\|f(\tau,u(\tau),u_\tau)\|d\tau ds\\
&\leq M^2\big(\delta_1\mathcal{F}_1(R_0)+\delta_2\mathcal{F}_2(R_0)+\frac{K}{|\nu_1|}\big)\cdot|t_2-t_1|\\
&\to 0,\ \text{as } t_2-t_1\to 0.
\end{aligned}
$$

Consequently, $\|Qu(t_2)-Qu(t_1)\|\to 0$ does not depend on $u\in\overline{\Omega}_{R_0}$ as $t_2-t_1\to 0$, which easily implies that $Q(\overline{\Omega}_{R_0})$ is equicontinuous.

V. *We finally show that* $\{Qu(t): u\in\overline{\Omega}_{R_0}\}$ *is precompact on* E *for* $t\geq 0$.

Let $t>0$ be given, then for any $\epsilon\in(0,t)$ and $u\in\overline{\Omega}_{R_0}$, we define a operator

$$
\begin{aligned}
Q_\epsilon u(t) =&T_2(t)\varphi(0)+\int_0^{t-\epsilon}T_2(t-s)T_1(s)\Big(g(u)+\psi+\gamma_2A\varphi(0)\Big)ds\\
&+\int_0^{t-\epsilon}\int_0^s T_2(t-s)T_1(s-\tau)f(\tau,u(\tau),u_\tau)d\tau ds\\
=&T_2(t)\varphi(0)+T_2(\epsilon)\int_0^{t-\epsilon}T_2(t-\epsilon-s)T_1(s)\Big(g(u)+\psi+\gamma_2A\varphi(0)\Big)ds\\
&+T_2(\epsilon)\int_0^{t-\epsilon}\int_0^s T_2(t-\epsilon-s)T_1(s-\tau)f(\tau,u(\tau),u_\tau)d\tau ds.
\end{aligned}
$$

According to the compactness of $T_1(t)(t\geq 0)$ and $T_2(t)(t\geq 0)$, the set $\{Q_\epsilon u(t): u\in\overline{\Omega}_{R_0}\}$ is precompact on E. Furthermore, for any $u\in\overline{\Omega}_{R_0}$ and $t>0$, we have

$$
\begin{aligned}
\|Qu(t)-Q_\epsilon u(t)\| =&\Big\|\int_{t-\epsilon}^{t}T_2(t-s)T_1(s)\Big(g(u)+\psi+\gamma_2A\varphi(0)\Big)ds\Big\|\\
&+\Big\|\int_{t-\epsilon}^{t}\int_0^s T_2(t-s)T_1(s-\tau)f(\tau,u(\tau),u_\tau)d\tau ds\Big\|\\
\leq&M^2\Big(\mathcal{G}(R_0)+\|\psi\|+\gamma_2\|A\varphi(0)\|\Big)\cdot\epsilon\\
&+M^2\Big(\delta_1\mathcal{F}_1(R_0)+\delta_2\mathcal{F}_2(R_0)+\frac{K}{|\nu_1|}\Big)\cdot\epsilon\\
\to&0,\ \text{as } \epsilon\to 0^+.
\end{aligned}
$$

It follows that $\{Qu(t): u\in\overline{\Omega}_{R_0}\}$ is precompact on E for $t>0$. Hence, it is precompact on E for every $t\geq 0$ by (3.3).

As a result, according to Lemma 3.1, we can conclude that $Q:\overline{\Omega}_{R_0}\to\overline{\Omega}_{R_0}$ is relatively compact. Therefore, based on Schauder's fixed point theorem, there has at least one fixed point $u\in C_{h,\varphi}(E)$ of Q, which means u_φ defined by (3.1) is the mild solution of nonlocal problem (1.1).

The proof of the theorem is complete. □

4. Asymptotic Stability

In this section, we first show the following Gronwall-Bellman type inequality involving delay to facilitate the establishment of the result of asymptotic stability.

Lemma 4.1 ([34]). *Assume that there exist a function $w \in C\big([-r,+\infty),\mathbb{R}^+\big)$ and constants $\beta_1, \beta_2 \geq 0$ satisfying*

$$w(t) \leq w(0) + Aw(0) + \int_0^t \big[\beta_1 w(s) + \beta_2 \sup_{\tau\in[-r,0]} w(s+\tau)\big]ds,$$

for $t \geq 0$ and $w(t) = \chi(t)$ for $t \in [-r,0]$, where $\chi \in C\big([-r,0],\mathbb{R}^+\big)$, $r > 0$ is a constant. Then, for any $t \geq 0$,

$$w(t) \leq \Big(\|\chi\|_{C[-r,0]} + \|A\chi(0)\|\Big)e^{(\beta_1+\beta_2)t}.$$

Theorem 4.1. *Assume that $A : \mathcal{D}(A) \subset E \to E$ is a closed linear operator and $-A$ generates a exponentially stable C_0-semigroup $T(t)(t \geq 0)$ on E, whose growth exponent denotes $\nu_0 < 0$, $\varphi \in \mathcal{B}$ and $\varphi(0) \in \mathcal{D}(A)$, $\psi \in E$, $f : \mathbb{R}^+ \times E \times \mathcal{B} \to E$, $g : C(\mathbb{R}^+; E) \to E$ are continuous with $\sup_{t\geq 0} \|f(t,\theta,\theta_{\mathcal{B}})\| < \infty$, where θ and $\theta_{\mathcal{B}}$ stand for the zero elements of the space E and $\mathcal{B}$, respectively. If the following conditions are established:*

(H3) There exist nonnegative constants C_{f1}, C_{f2}, such that

$$\|f(t,x_1,\phi_1) - f(t,x_2,\phi_2)\| \leq C_{f1}\|x_1 - x_2\| + C_{f2}\|\phi_1 - \phi_2\|_{\mathcal{B}}$$

for all $t \geq 0$ and $x_1, x_2 \in E$, $\phi_1, \phi_2 \in \mathcal{B}$.

(H4) There exists a nonnegative constant C_g such that

$$\|g(v_1) - g(v_2)\| \leq C_g\|v_1(t) - v_2(t)\|$$

for all $v_1, v_2 \in C\big(\mathbb{R}^+, E\big)$ and $t \geq 0$.

Then, the nonlocal problem (1.1) has a unique mild solution $u \in C\big([-r,+\infty), E\big)$ provided that

$$\frac{C_{f1} + C_{f2}}{\gamma_1\gamma_2} + \frac{|\nu_0|C_g}{\gamma_1} < \frac{\nu_0^2}{M^2}, \tag{4.1}$$

where γ_i and M are defined in (2.6) and (2.7).

Moreover, if

$$\frac{C_{f1} + C_{f2}}{\gamma_1(\gamma_2 - \gamma_1)} + \frac{|\nu_0|C_g}{\gamma_1} < \frac{\nu_0^2}{M^2}, \tag{4.2}$$

then, the unique mild solution $u \in C\big([-r,+\infty), E\big)$ is globally asymptotic stable.

Proof. Let $C_b\big(\mathbb{R}^+, E\big)$ denote the Banach space of all bounded and continuous functions from $\mathbb{R}^+$ to E with the norm $\|u\|_b = \sup_{t\geq 0}\|u(t)\|$, and for given $\varphi \in \mathcal{B}$, define

$$C_{b,\varphi}(E) := \Big\{u \in C_b\big(\mathbb{R}^+, E\big) : \ u(0) = \varphi(0)\Big\},$$

then, it is obvious that $C_{b,\varphi}(E)$ is a Banach space in the sense of the norm $\|u\|_{b,\varphi} = \max\{\|\varphi\|_{\mathcal{B}},\ \|u\|_b\}$. In addition, for any $t \geq 0$, we can easily deduce that

$$\|u(t)\| \leq \|u\|_b \leq \|u\|_{b,\varphi}, \quad \|u_t\|_{\mathcal{B}} \leq \|u\|_{b,\varphi}.$$

Then, for any $t \geq 0$ and $u \in C_{b,\varphi}(E)$, by $(H3)$ and $(H4)$, we have

$$\|g(u)\| \leq C_g\|u\|_{b,\varphi} + \|g(\theta)\| := C_1,$$
$$\|f(t,u(t),u_t)\| \leq (C_{f1}+C_{f2})\|u\|_{b,\varphi} + \|f(t,\theta,\theta_{\mathcal{B}})\| := C_2.$$

Let Q be the operator defined in (3.3), then, based on (3.4), we can easily derive that for any $t \geq 0$,

$$\|Qu(t)\| \leq M\|u\|_{b,\varphi} + \frac{M^2}{\nu_1-\nu_2}(C_1+\|\psi\|+\gamma_2\|A\varphi(0)\|) + \frac{M^2C_2}{\nu_1\nu_2} := \widetilde{C},$$

which means that $(Qu)|_{t\geq 0} \in C_b([0,+\infty),E)$. Hence, according to (3.3), $Q : C_{b,\varphi}(E) \to C_{b,\varphi}(E)$ is well defined and continuous. According to Definition 2.2 and (3.1), if u is a fixed point of Q on $C_{h,\varphi}(E)$, then u_φ is undoubtedly a mild solution of nonlocal problem (1.1).

The proof will be completed in two steps.

Step 1. *Existence and uniqueness of global mild solutions.*

For any $u,v \in C_{b,\varphi}(E), t \geq 0$, by (3.3) and $(H3)$, $(H4)$,we have

$$\begin{aligned}
&\|Qu(t)-Qv(t)\| \\
\leq&\Big\|\int_0^t T_2(t-s)T_1(s)\Big(g(u)-g(v)\Big)ds\Big\| \\
&+\Big\|\int_0^t\int_0^s T_2(t-s)T_1(s-\tau)\Big(f(\tau,u(\tau),u_\tau)-f(\tau,v(\tau),v_\tau)\Big)d\tau ds\Big\| \\
\leq&M^2C_g\|u-v\|_{b,\varphi}\cdot\int_0^t e^{\nu_2(t-s)}e^{\nu_1 s}ds \\
&+M^2(C_{f1}+C_{f2})\|u-v\|_{b,\varphi}\cdot\int_0^t\int_0^s e^{\nu_2(t-s)}e^{\nu_1(s-\tau)}d\tau ds \\
\leq&\frac{M^2}{|\nu_1|}C_g\|u-v\|_{b,\varphi}+\frac{M^2}{\nu_1\nu_2}(C_{f1}+C_{f2})\|u-v\|_{b,\varphi} \\
\leq&\Big(\frac{M^2}{|\nu_1|}C_g+\frac{M^2}{\nu_1\nu_2}(C_{f1}+C_{f2})\Big)\cdot\|u-v\|_{b,\varphi}.
\end{aligned}$$

Then, combining (4.1) and $\nu_i = \gamma_i\nu_0$ $(i=1,2)$, we can deduce that

$$\frac{M^2}{|\nu_1|}C_g+\frac{M^2}{\nu_1\nu_2}(C_{f1}+C_{f2}) < 1,$$

which indicates that

$$\|Qu-Qv\|_{b,\varphi} \leq \|u-v\|_{b,\varphi},$$

that is, $Q : C_{b,\varphi}(E) \to C_{b,\varphi}(E)$ is a contraction mapping. Consequently, according to the Banach fixed point theorem, Q has a unique fixed point $u^* \in C_{b,\varphi}(E)$, then, $u^*_\varphi \in C\big([-r,+\infty),E\big)$ defined by (3.1) is the unique mild solution of nonlocal problem (1.1).

Step 2. *Asymptotic stability of global mild solution.*

Obviously, from (4.2), we can deduce that (4.1) is tenable. Hence, the nonlocal problem (1.1) has a unique mild solution $u^*_\varphi = u^*(t,\varphi) \in C\big([-r,+\infty),E\big)$. For any

$\phi \in \mathcal{B}$, let $u_\phi = u(t,\phi) \in C\big([-r,+\infty), E\big)$ be the unique mild solution of nonlocal problem (1.1) corresponding to the initial value $u_\phi(t) = \phi(t)$ for $t \in [-r,0]$, then, in view of the Definition 2.2 and (3.1), it can be expressed as

$$u_\phi(t) = \begin{cases} u(t), & t \geq 0, \\ \phi(t), & t \in [-r,0], \end{cases} \tag{4.3}$$

where

$$\begin{aligned} u(t) =& T_2(t)\phi(0) + \int_0^t T_2(t-s)T_1(s)(g(u) + \psi + \gamma_2 A\phi(0))ds \\ &+ \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau, u(\tau), u_\tau)d\tau ds, \quad t \geq 0, \end{aligned}$$

we can easily obtained that for any $t \geq 0$,

$$\begin{aligned} &\|u^*(t) - u(t)\| \\ \leq& \|T_2(t)(u^*(0) - u(0))\| \\ &+ \Big\| \int_0^t T_2(t-s)T_1(s)\Big[\Big(g(u^*) - g(u)\Big) + \gamma_2\Big(Au^*(0) - Au(0)\Big)\Big]ds\Big\| \\ &+ \Big\| \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)\Big(f(\tau, u^*(\tau), u^*_\tau) - f(\tau, u(\tau), u_\tau)\Big)d\tau ds\Big\| \\ :=& \Lambda_1 + \Lambda_2 + \Lambda_3, \end{aligned}$$

where

$$\Lambda_1 \leq Me^{\nu_2 t}\|u^*(0) - u(0)\| \leq Me^{\nu_1 t}\|u^*(0) - u(0)\|.$$

According to $(H4)$, we can get

$$\begin{aligned} \Lambda_2 \leq& \Big\| \int_0^t T_2(t-s)T_1(s)\Big(g(u^*) - g(u)\Big)ds\Big\| \\ &+ \Big\| \int_0^t T_2(t-s)T_1(s)\gamma_2\Big(Au^*(0) - Au(0)\Big)ds\Big\| \\ \leq& M^2 e^{\nu_1 t} C_g \int_0^t \|u^*(s) - u(s)\|ds + \frac{M^2\gamma_2}{\nu_1 - \nu_2} e^{\nu_1 t}\|Au^*(0) - Au(0)\| \\ \leq& M^2 e^{\nu_1 t} C_g \int_0^t e^{-\nu_1 s}\|u^*(s) - u(s)\|ds + \frac{M^2\gamma_2}{\nu_1 - \nu_2} e^{\nu_1 t}\|Au^*(0) - Au(0)\|. \end{aligned}$$

On the basis of $(H3)$, and combined with the integration by parts, we have

$$\begin{aligned} \Lambda_3 \leq& M^2 e^{\nu_2 t} \int_0^t e^{(\nu_1 - \nu_2)s} \int_0^s e^{-\nu_1 \tau}\|f(\tau, u^*(\tau), u^*_\tau) - f(\tau, u(\tau), u_\tau)\|d\tau ds \\ \leq& \frac{M^2}{\nu_1 - \nu_2} e^{\nu_1 t} \int_0^t e^{-\nu_1 s}\|f(s, u^*(s), u^*_s) - f(s, u(s), u_s\|ds \\ \leq& \frac{M^2}{\nu_1 - \nu_2} e^{\nu_1 t} \int_0^t e^{-\nu_1 s}\Big(C_{f1}\|u^*(s) - u(s)\| \\ &+ C_{f2} \sup_{\tau \in [-r,0]} \|u^*(s+\tau) - u(s+\tau)\|\Big)ds. \end{aligned}$$

Wrap this all together, therefore, we have

$$\begin{aligned}&\|u^*(t)-u(t)\|\\ \le& Me^{\nu_1 t}\|u^*(0)-u(0)\|+\frac{M^2\gamma_2}{\nu_1-\nu_2}e^{\nu_1 t}\|Au^*(0)-Au(0)\|\\ &+M^2e^{\nu_1 t}\int_0^t e^{-\nu_1 s}\Big[(C_g+\frac{C_{f1}}{\nu_1-\nu_2})\|u^*(s)-u(s)\|\\ &+\frac{C_{f2}}{\nu_1-\nu_2}\sup_{\tau\in[-r,0]}\|u^*(s+\tau)-u(s+\tau)\|\Big]ds.\end{aligned}$$

Let $\omega(t)=e^{-\nu_1 t}\|u^*(t)-u(t)\|$ for $t\ge 0$, then we can find that

$$\begin{aligned}\omega(t)\le& M\omega(0)+\frac{M^2\gamma_2}{\nu_1-\nu_2}A\omega(0)\\ &+\int_0^t\Big[M^2(C_g+\frac{C_{f1}}{\nu_1-\nu_2})\omega(s)+\frac{M^2C_{f2}}{\nu_1-\nu_2}\sup_{\tau\in[-r,0]}\omega(s+\tau)\Big]ds.\end{aligned}$$

It follows from Lemma 4.1 that

$$\begin{aligned}\omega(t)=&e^{-\nu_1 t}\|u^*(t)-u(t)\|\\ \le&\Big(M\omega(0)+\frac{M^2\gamma_2}{\nu_1-\nu_2}A\omega(0)\Big)\cdot e^{M^2(C_g+\frac{C_{f1}+C_{f2}}{\nu_1-\nu_2})t}.\end{aligned}$$

Then, one can obtained that

$$\|u^*(t)-u(t)\|\le \widetilde{M}e^{\beta t}, \tag{4.4}$$

where

$$\begin{aligned}&\widetilde{M}:=M\|\varphi(0)-\phi(0)\|+\frac{M^2\gamma_2}{\nu_1-\nu_2}\|A\varphi(0)-A\phi(0)\|,\\ &\beta:=M^2(C_g+\frac{C_{f1}+C_{f2}}{\nu_1-\nu_2})+\nu_1.\end{aligned}$$

In view of (4.2), we can easily obtained that,

$$\beta=\frac{M^2\gamma_1}{|\nu_0|}\Big(\frac{C_g|\nu_0|}{\gamma_1}+\frac{C_{f1}+C_{f2}}{\gamma_1(\gamma_2-\gamma_1)}-\frac{\nu_0^2}{M^2}\Big)<0.$$

Then, from (4.4), it follows that

$$\|u^*(t)-u(t)\|\le\widetilde{M}e^{\beta t}\to 0,\quad \text{as } t\to+\infty.$$

Thanks to (4.3), one can easily see that

$$\|u^*_\varphi(t)-u_\phi(t)\|\to 0,\quad \text{as } t\to+\infty.$$

Therefore, the unique mild solution u^*_φ of nonlocal problem (1.1) is globally asymptotically stable.

This completes the proof of Theorem 4.2 □

5. Application

We consider the following damped beam vibration equation with delay and nonlocal initial conditions

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} - 4\dfrac{\partial^3 u(x,t)}{\partial x^2 \partial t} + \dfrac{\partial^4 u(x,t)}{\partial x^4} \\ = \dfrac{a\sin t}{e^{2t}}\sqrt{u^2(x,t)+1} + \dfrac{b\cos t}{e^{2t}} u(x,t+s) + 1, & (x,t) \in [0,1]\times\mathbb{R}^+, \\ u(0,t) = u(1,t) = 0, \ \dfrac{\partial^2 u(0,t)}{\partial x^2} = \dfrac{\partial^2 u(1,t)}{\partial x^2} = 0, & t \geq 0, \\ u(x,t) = \varphi(x,t), & (x,t) \in [0,1]\times[-r,0], \\ \dfrac{\partial u(x,0)}{\partial t} = \dfrac{c}{|u(x,t)|+1} + \psi(x), & x \in [0,1], \end{cases} \tag{5.1}$$

where $r, a, b, c > 0$ are constants, $\varphi \in C\big([0,1]\times[-r,0], \mathbb{R}^+\big), \psi \in E$.

We write $I = [0,1]$ and choose the Banach space $E = L^p(I)$ $(2 \leq p < +\infty)$ with L^p-norm $\|\cdot\|_p$. Define the linear operator $A : D(A) \subset E \to E$ by

$$D(A) = W^{2,p}(I) \cap W_0^{1,p}(I), \quad Au = -\frac{\partial^2 u(x,t)}{\partial x^2}.$$

Let $\rho = 4,\ \varphi(t) = \varphi(\cdot,t),\ \psi = \psi(\cdot),\ u(t) = u(\cdot,t),\ u_t(s) = u(\cdot,t+s)$ and

$$f(t,u(t),u_t) = \frac{a\sin t}{e^{2t}}\sqrt{u^2(\cdot,t)+1} + \frac{b\cos t}{e^{2t}}u(\cdot,t+s) + 1; \tag{5.2}$$

$$g(u) = \frac{c}{|u(\cdot,t)|+1}. \tag{5.3}$$

It is easy to see that $f : \mathbb{R}^+ \times E \times \mathcal{B} \to E$ and $g : C\big(\mathbb{R}^+, E\big) \to E$ are continuous. Meanwhile, (5.1) can be transformed into the form of nonlocal problem (1.1) in $L^p(I)$.

By [17,33], it is well known that $-A$ generates an exponentially stable compact semigroup $T(t)(t \geq 0)$ satisfying $\|T(t)\| \leq e^{-\pi^2 t}$ for $t \geq 0$, that is, $M = 1, \nu_0 = -\pi^2$. On the other hand, according to (2.6), $\gamma_1 = 2-\sqrt{3},\ \gamma_2 = 2+\sqrt{3}$. Hence, by Lemma 2.3, we can easily obtain that $T_i(t)(t \geq 0), (i = 1,2)$ are exponentially stable compact semigroup with growth exponent $\nu_1 = -(2-\sqrt{3})\pi^2,\ \nu_2 = -(2+\sqrt{3})\pi^2$, respectively.

Theorem 5.1. *Let $a + b < (1+\sqrt{3})\pi^2$, then, for any $\varphi \in C\big([0,1]\times[-r,0],\mathbb{R}^+\big)$, $\varphi(\cdot,0) \in W^{2,p}(I) \cap W_0^{1,p}(I)$ and $\psi \in E$, the problem (5.1) has at least one mild solutions.*

Proof. Let $h(t) = e^t$ $(t \in \mathbb{R}^+)$, then, according to (5.2) and (5.3), we can obtain

$$\begin{aligned} \|f(t, e^t u(t), e^t u_t)\| &\leq \left\| \frac{a\sin t}{e^{2t}}\sqrt{e^{2t}u^2(t)+1} + \frac{b\cos t}{e^t}\cdot u_t + 1 \right\| \\ &\leq \frac{a}{e^t}\sqrt{\|u(t)\|^2+1} + \frac{b}{e^t}\|u_t\|_{\mathcal{B}} + 1; \\ g(e^t u) = \frac{c}{e^t|u(t)|+1} &\leq \frac{c}{\|u\|_C + 1}, \end{aligned}$$

which follows that the conditions $(H1)$ and $(H2)$ hold with $\mathcal{F}_1(l) = \sqrt{l^2+1}$, $\mathcal{F}_2(l) = l$; $p_1(t) = \frac{a}{e^t}$, $p_2(t) = \frac{b}{e^t}$, $\zeta_1 = \zeta_2 = 1, \delta_1 = \frac{a}{\sqrt{3}-1}$, $\delta_2 = \frac{b}{\sqrt{3}-1}$, $K = 1$ and $\mathcal{G}(l) = \frac{c}{l+1}$, $\eta = 0$. By combining these with $a + b < 1 + \sqrt{3}$, we know that (3.1) holds. Consequently, all the conditions of Theorem 3.2 are established, which shows that the problem (5.1) has at least one mild solutions. □

Theorem 5.2. *Let* $\frac{a+b}{2\sqrt{3}} + \pi^2 c < (2-\sqrt{3})\pi^4$*, then, for any* $\varphi \in C\big([0,1]\times[-r,0], \mathbb{R}^+\big)$, $\varphi(\cdot,0) \in W^{2,p}(I) \cap W_0^{1,p}(I)$ *and* $\psi \in E$*, the problem* (5.1) *has a unique globally asymptotic mild solution.*

Proof. In view of (5.2) and (5.3), we can get that for any $t \geq 0$ and $u, v \in E$,

$$\|f(t,u(t),u_t) - f(t,v(t),v_t)\| \leq a\|u(t)-v(t)\| + b\|u_t - v_t\|_{\mathcal{B}};$$
$$\|g(u(t)) - g(v(t))\| \leq c\|u(t)-v(t)\|,$$

which follows that the conditions $(H3)$ and $(H4)$ hold with $C_{f1} = a$, $C_{f1} = b$, $C_g = c$. By combining these with $\frac{a+b}{2\sqrt{3}} + \pi^2 c < (2-\sqrt{3})\pi^4$, we know that (4.2) holds. Consequently, the conditions of Theorem 4.2 are established, which shows that the problem (5.1) has a unique globally asymptotic mild solution. □

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