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ALMOST PERIODIC SYNCHRONIZATION FOR COMPLEX-VALUED NEURAL NETWORKS WITH TIME-VARYING DELAYS AND IMPULSIVE EFFECTS ON TIME SCALES*

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Abstract We propose a class of complex-valued neural networks with time-varying delays and impulsive effects on time scales. By employing the Banach fixed point theorem and differential inequality technique on time scales, we obtain the existence of almost periodic solutions for this networks. Then, by constructing a suitable Lyapunov function, we obtain that the drive-response structure of complex-valued neural networks with almost periodic coefficients can realize the global exponential synchronization. Our results are completely new. Finally, we give an example to illustrate the feasibility of our results.

Keywords Complex-valued neural networks, synchronization, impulsive effects, time scales.

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1. Introduction

In the past few years, complex-valued neural networks (CVNNs) have been extensively studied and analyzed in their dynamical behaviors, their application has been extended to optoelectronics, image, remote sensing, quantum neuron devices and systems, spatiotemporal analysis of physiological nervous system, and artificial neural information processing [1, 14]. As an extension of real-valued neural networks, CVNNs had more complicated dynamical properties. Hence, it is very important to study about the dynamical properties of CVNNs, such as the existence and stability of the equilibrium, periodic solutions, and almost periodic solutions, which have been studied by many scholars [6, 11, 12, 15, 32, 34, 47, 49].

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On the one hand, impulsive effects are used to describe some dynamical models in many fields, such as medicine and biology, economics and telecommunications. The theory of impulsive differential equations goes back to the works of Milman and Myshkis [31]. Later, the impulsive delay differential equations have been widely researched (see [23, 35, 36]). The stability problem of impulsive CVNNs with time delay is studied by many scholars [20, 33, 37, 38, 41, 44]. Therefore, impulsive effects play an important role in the dynamical behavior of CVNNs. Besides, the synchronization problem has played a significant role in nonlinear science since its potential applications in image processing, ecological systems, secure communication and harmonic oscillation generation, formation control and we can reference [9, 16, 17, 24, 45]. The main idea of the impulsive synchronization method is to adjust the state of the response system using synchronization pulses at discrete times so that the state of the response system is close to that of the drive system. These pulses are generated at discrete moments from a sample of state variables that drive the response system. Therefore, the impulsive synchronization scheme can be realized by digital technology. Moreover, due to the reduced amount of synchronization information transmitted from the drive system to the response system, impulsive synchronization lowers the communication cost and enhances the security of the chaotic crypto system. So far, few authors have considered the synchronization of complex-valued neural networks [22, 25, 26].

On the other hand, it is well known that the time scale theory was initiated by Hilger [18] in his Ph.D. thesis in 1988, which can unify the continuous and discrete cases. Many authors have studied the dynamic equations on time scales [3, 4, 21]. To study the almost periodic dynamic equation on time scales, the concept of almost periodic time scales was proposed by Li and Wang in [27]. Based on this concept, almost periodic functions on almost periodic time scales are defined [28], subsequently, many authors have studied the almost periodic solutions of neural networks on time scales [10, 13, 42, 46]. Subsequently, many scholars have considered CVNNs on time scales and established some sufficient conditions are obtained for the existence and global exponential stability of equilibrium point [5, 7, 39, 50]. Recently, some authors studied the dynamical behaviors, including synchronization and consensus on time scales in [2, 8, 19, 29, 30, 40]. But, few articles consider almost synchronization, let alone discuss the almost periodic synchronization of complex-valued neural networks with impulsive on time scales.

However, to the best of our knowledge, there is no published paper considering the almost periodic synchronization for impulsive complex-valued neural networks with leakage delays on time scales. Therefore, it is very important in theories and applications and also is a very challenging problem.

Inspired by the above analysis, in this paper, we consider the following impulsive complex-valued neural networks on time scales:

$$\begin{cases} x_l^{\Delta}(t) = -d_l(t)x_l(t-\eta_l(t)) + \displaystyle\sum_{j=1}^{n} a_{lj}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{lj}(t)g_j(x_j(t-\vartheta_{lj}(t))) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} c_{lj}(t)\int_{t-\delta_{lj}(t)}^{t} h_j(x_j(s))\Delta s + u_l(t), \quad t \neq t_k, \\ \tilde{\Delta} x_l(t_k) = x_l(t_k^+) - x_l(t_k^-) = I_{lk}(x_l(t_k)), \quad t = t_k,\ k \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where $t \in \mathbb{T}$, $l \in \{1, 2, \ldots, n\} := \mathcal{J}$, $x_l(t) \in \mathbb{C}$ is the state of the lth neuron at time

t; $d_l(t) > 0$ is the self-feedback connection weight; $a_{lj}(t)$, $b_{lj}(t)$ and $c_{lj}(t) \in \mathbb{C}$ are, respectively, the connection weight and the delay connection weight from neuron j to neuron l; $u_l(t)$ is an external input on the lth unit at time t; $\eta_l(t)$ is the leakage delays, $\vartheta_{lj}(t)$ and $\delta_{lj}(t)$ are the transmission delays, which satisfy that $t-\eta_l(t) \in \mathbb{T}$, $t-\vartheta_{lj}(t) \in \mathbb{T}$ and $t-\delta_{lj}(t) \in \mathbb{T}$ for all $t \in \mathbb{T}$; $x_l(t_k^-)$ and $x_l(t_k^+)$ are, respectively, the left and right limit at $t = t_k$; $\tilde{\Delta} x_l(t_k)$ is impulses at moments t_k, $x_l(t_k^-) = x_l(t_k)$ and $\{t_k\} \in \mathcal{B}$, $\mathcal{B} = \big\{\{t_k\} : t_k \in \mathbb{T}, t_k < t_{k+1}, k \in \mathbb{Z}, \lim\limits_{k\to\pm\infty} t_k = \pm\infty\big\}$; $I_{lk}(\cdot) \in \mathbb{C}$. Let i be the imaginary unit, i.e., $i = \sqrt{-1}$.

For every $x \in \mathbb{C}$, the norm of x is defined as $\|x\|_{\mathbb{C}} = \max\big\{|x^R|, |x^I|\big\}$, and for $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$, we define $\|x\|_0 = \max\limits_{l\in\mathcal{J}}\{\|x_l\|_{\mathbb{C}}\}$.

For convenience, we will adopt the following notations:

$$
\begin{aligned}
&d_l^- = \inf_{t\in\mathbb{T}} d_l(t), \quad a_{lj}^+ = \sup_{t\in\mathbb{T}} \|a_{lj}(t)\|_{\mathbb{C}}, \quad b_{lj}^+ = \sup_{t\in\mathbb{T}} \|b_{lj}(t)\|_{\mathbb{C}}, \quad c_{lj}^+ = \sup_{t\in\mathbb{T}} \|c_{lj}(t)\|_{\mathbb{C}},\\
&\eta = \max_{l\in\mathcal{J}} \big\{\sup_{t\in\mathbb{T}} \eta_l(t)\big\}, \quad \vartheta = \max_{l,j\in\mathcal{J}} \big\{\sup_{t\in\mathbb{T}} \vartheta_{lj}(t)\big\},\\
&\delta = \max_{l,j\in\mathcal{J}} \big\{\sup_{t\in\mathbb{T}} \delta_{lj}(t)\big\}, \quad \xi = \max\big\{\eta, \vartheta, \delta\big\}.
\end{aligned}
$$

The initial condition associated with (1.1) is of the form

$$
x_l(s) = \phi_l(s), \quad s \in [-\xi, 0]_{\mathbb{T}},\ l \in \mathcal{J},
$$

where $\phi_l(s) \in C([-\xi, 0]_{\mathbb{T}}, \mathbb{C})$.

The main purpose of this paper is to study the existence and global exponential synchronization of almost periodic solutions for system (1.1). The main contributions of this paper are listed as follows. Firstly, Our results include real-valued neural networks as its special cases. Secondly, Comparing the previous results, we studied the complex-valued neural networks via a direct method, and improved the norm. Thirdly, compared with other results, the results in this paper are the ones about complex-valued and with impulsive effects. Therefore, the results are less conservative and more general. Finally, our method of this paper can be used to study the the pseudo almost periodic synchronization for other types of impulsive-valued neural networks on time scales.

This paper is organized as follows. In Section 2, we introduce some definitions, make some preparations for later sections and extend the piecewise almost periodic functions on time scales with the Δ-derivative. In Section 3, by utilizing the Banach fixed point theorem, a sufficient condition is derived for the existence of almost periodic solutions for (1.1). In section 4, based on the Lyapunov functional method and differential inequality technique on time scales, we obtain the global exponential synchronization of almost periodic solutions for the system (1.1). In Section 5, we give an illustrative example to demonstrate the feasibility of our results. Finally, we conclude Section 6.

2. Preliminaries

In this section, we shall recall some fundamental definitions and lemmas which are used in what follows.

Definition 2.1 ([18]). A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf \big\{ s \in \mathbb{T}, s > t \big\}, \quad \forall\, t \in \mathbb{T},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup \big\{ s \in \mathbb{T}, s < t \big\}, \quad \forall\, t \in \mathbb{T},$$

finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense or right-scattered if $\rho(t) = t$, $\rho(t) < t, \sigma(t) = t$ or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum m, define $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered maximum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$.

Definition 2.2 ([18]). Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^\Delta(t)$ to be the number(provided its exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t. Moreover, we say that f is delta (or Hilger) differentiable (or in short:differentiable) on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^\Delta : \mathbb{T}^k \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^k$.

Definition 2.3 ([18]). A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided

$$1 + \mu(t)p(t) \neq 0, \quad \forall\, t \in \mathbb{T}^\kappa.$$

The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ are denoted by

$$\begin{aligned} &\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}). \\ &\mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \quad \forall\, t \in \mathbb{T}\}. \end{aligned}$$

Definition 2.4 ([18]). If $p \in \mathcal{R}^+$, then we define the exponential function by

$$e_p(t, s) = \exp\bigg(\int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big)\Delta\tau \bigg), \quad \forall\, t, s \in \mathbb{T},$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\operatorname{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Definition 2.5. ([1]) Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu pq, \ \ominus p = -\frac{p}{1 + \mu p}, \ p \ominus q = p \oplus (\ominus q).$$

Lemma 2.1 ([18]). *Assume that $p:\mathbb{T}\to\mathbb{R}$ is regressive function, then*

(i) $e_0(t,s)\equiv 1$ *and* $e_p(t,t)\equiv 1$;

(ii) $e_p(t,s)=\frac{1}{e_p(s,t)}=e_{\ominus p}(s,t)$;

(iii) $e_p(t,s)e_p(s,r)=e_p(t,r)$;

(iv) $(e_{\ominus p}(t,s))^{\Delta}=(\ominus p)(t)e_{\ominus p}(t,s)$.

Lemma 2.2 ([18]). *Let f,g be Δ-differentiable functions on $\mathbb{T}$, then*

(i) $(v_1f+v_2g)^{\Delta}=v_1f^{\Delta}+v_2g^{\Delta}$, *for any constants* v_1,v_2;

(ii) $(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g(\sigma(t))$.

Lemma 2.3 ([18]). *Assume that $p(t)\geq 0$ for $t\geq s$, then $e_p(t,s)\geq 1$.*

Lemma 2.4 ([18]). *Suppose that $p\in\mathcal{R}^+$, then*

(i) $e_p(t,s)>0$, *for all* $t,s\in\mathbb{T}$;

(ii) *if $p(t)\leq q(t)$ for all $t\geq s$, $t,s\in\mathbb{T}$, then $e_p(t,s)\leq e_q(t,s)$ for all $t\geq s$.*

Lemma 2.5 ([18]). *If $p\in\mathcal{R}^+$ and $a,b,c\in\mathbb{T}$, then*

$$\big[e_p(c,\cdot)\big]^{\Delta}=-p\big[e_p(c,\cdot)\big]^{\sigma}\quad\text{and}\quad\int_a^b p(t)e_p\big(c,\sigma(t)\big)\Delta t=e_p(c,a)-e_p(c,b).$$

Definition 2.6 ([27,28]). A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi:=\{\tau\in\mathbb{R}:t\pm\tau\in\mathbb{T},\forall t\in\mathbb{T}\}\neq\{0\}.$$

$PC_{rd}(\mathbb{T},\mathbb{X})$ denotes the set of all rd-piecewise continuous and left continuous functions with points of discontinuity of first kind. For any integers k and j, consider the sequence $\{t_k^j\}$, where $t_k^j=t_{k+j}-t_k$, $k,j\in\mathbb{Z}$. From now on, $(\mathbb{X},\|\cdot\|)$ is a (real or complex) Banach space.

Definition 2.7 ([43]). The set of sequences $\{t_k^j\}$, $t_k^j=t_{k+j}-t_k$, $k,j\in\mathbb{Z}$ is said to be uniformly almost periodic, if for an arbitrary $\varepsilon>0$ there exists a relatively dense set of ε-almost periodic, common for all sequences $\{t_k^j\}$.

Definition 2.8 ([43]). Let $\mathbb{T}$ be an almost periodic time scale. The function $\varphi\in PC_{rd}(\mathbb{T},\mathbb{X})$ is said to be almost periodic, if the following holds:

(i) $\{t_k^j\}$, $t_k^j=t_{k+j}-t_k$, $k,j\in\mathbb{Z}$ is uniformly almost periodic;

(ii) for any $\varepsilon>0$, there is a positive number $\delta=\delta(\varepsilon)>0$ such that if the points t' and t'' belong to the same interval of continuity and $|t'-t''|<\delta$, then $\|\varphi(t')-\varphi(t'')\|<\varepsilon$;

(iii) for any $\varepsilon>0$, there is relative dense set $\Gamma_\varepsilon\subset\Pi$ such that if $\tau\in\Gamma_\varepsilon$, then $\|\varphi(t+\tau)-\varphi(t)\|<\varepsilon$ for all $t\in\mathbb{T}$, which satisfy the condition $|t-t_k|>\varepsilon$, $k\in\mathbb{Z}$.

We denote by $AP_T(\mathbb{T},\mathbb{X})$ the set of all piecewise almost periodic functions.

Throughout the rest of the paper, we assume that the following conditions hold:

(H_1) $d_l(t),\eta_l(t),\delta_{lj}(t):\mathbb{T}\to\mathbb{R}^+$ are all almost periodic functions, $\vartheta_{lj}(t)\in C^1(\mathbb{T},\Pi)$ with $\sup_{t\in\mathbb{R}}\vartheta_{lj}^{\Delta}(t)=\omega<1$ is almost periodic function, $a_{lj}(t),b_{lj}(t),c_{lj}(t)\in AP(\mathbb{T},\mathbb{C})$, $u_l\in AP(\mathbb{T},\mathbb{C})$, and there exists a constant λ such that $d_l(t)\geq\lambda>0$.

(H_2) $I_{lk} \in AP(\mathbb{R}, \mathbb{C})$ and there exist positive constants L_l and such that for $u, v \in \mathbb{C}$,

$$\|I_{lk}(u) - I_{lk}(v)\|_{\mathbb{C}} \leq L_l \|u - v\|_{\mathbb{C}}, \quad l \in \mathcal{J},\ k \in \mathbb{Z},$$

and $I_{lk}(\mathbf{0}) = \mathbf{0}$.

(H_3) There exist positive constants $\alpha_j, \beta_j, \zeta_j$ such that for any $x_j,\ y_j \in \mathbb{C}$,

$$\begin{aligned} \|f_j(x_j) - f_j(y_j)\|_{\mathbb{C}} &\leq \alpha_j \|x_j - y_j\|_{\mathbb{C}}, \\ \|g_j(x_j) - g_j(y_j)\|_{\mathbb{C}} &\leq \beta_j \|x_j - y_j\|_{\mathbb{C}}, \\ \|h_j(x_j) - h_j(y_j)\|_{\mathbb{C}} &\leq \zeta_j \|x_j - y_j\|_{\mathbb{C}}, \end{aligned}$$

and $f_j(\mathbf{0}) = g_j(\mathbf{0}) = h_j(\mathbf{0}) = \mathbf{0}$, where $j \in \mathcal{J}$.

(H_4) The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k, j \in \mathbb{Z}$ is uniformly almost periodic and $\inf_k t_k^1 = \theta > 0$.

3. Existence of piecewise almost periodic solution

Let $\mathbb{X} = \big\{ f \mid f, f^{\Delta} \in AP_T(\mathbb{T}, \mathbb{C}^n) \big\}$ with the norm

$$\|f\|_{\mathbb{X}} = \max \big\{ \sup_{t \in \mathbb{T}} \|f(t)\|_0 \,,\, \sup_{t \in \mathbb{T}} \|f^{\Delta}(t)\|_0 \big\},$$

then $\mathbb{X}$ is a Banach space.

Theorem 3.1. *Assume that conditions (H_1)-(H_4) hold and the following conditions are satisfied:*

(H_5) *There exists positive constants r and θ such that*

$$\max_{l \in \mathcal{J}} \left\{ \frac{\Theta_l r + u_l^+}{d_l^-} + \frac{L_l r}{1 - e^{-\theta d_l^-}}, \left(1 + \frac{d_l^+}{d_l^-}\right)(\Theta_l r + u_l^+) + \frac{d_l^+ L_l r}{1 - e^{-\theta d_l^-}} \right\} \leq r,$$
$$\max_{l \in \mathcal{J}} \left\{ \frac{\Theta_l}{d_l^-} + \frac{L_l}{1 - e^{-\theta d_l^-}}\,, \left(1 + \frac{d_l^+}{d_l^-}\right)\Theta_l + \frac{d_l^+ L_l}{1 - e^{-\theta d_l^-}} \right\} := \rho < 1,$$

where

$$\Theta_l = d_l^+ \eta_l^+ + \sum_{j=1}^{n} a_{lj}^+ \alpha_j + \sum_{j=1}^{n} b_{lj}^+ \beta_j + \sum_{j=1}^{n} \delta_{lj}^+ c_{lj}^+ \zeta_j, \quad l \in \mathcal{J}.$$

Then, the system (1.1) *has a unique piecewise almost periodic solution in the region* $\mathbb{X}^* = \{\varphi \mid \varphi \in \mathbb{X}, \|\varphi\|_{\mathbb{X}} \leq r\}$.

Proof. Firstly, it is easy to see that if $x = (x_1, x_2, \ldots, x_n)^T \in AP_T(\mathbb{T}, \mathbb{C}^n)$ is a solution of the following system

$$x_l(t) = \int_{-\infty}^{t} e_{-d_l}(t, \sigma(s)) \big(\mathcal{K}_l(s, x) + u_l(s)\big) \Delta s + \sum_{t_k < t} e_{-d_l}(t, t_k) I_{lk}(x_l(t_k)),$$

where

$$\mathcal{K}_l(s, x) = d_l(s) \int_{s - \eta_l(s)}^{t} x_l^{\Delta}(u) \Delta u + \sum_{j=1}^{n} a_{lj}(s) f_j(x_j(s))$$

$$+\sum_{j=1}^{n} b_{lj}(s)g_j(x_j(s-\vartheta_{lj}(s)))+\sum_{j=1}^{n} c_{lj}(s)\int_{s-\delta_{lj}(s)}^{s} h_j(x_j(u))\Delta u,$$

then $x(t)$ is a solution of (1.1). Define the nonlinear operator Φ as follows, for each $\varphi, \varphi^{\Delta} \in AP_T(\mathbb{T}, \mathbb{C}^n)$ and $l \in \mathcal{J}$,

$$(\Phi\varphi)_l(t)=\int_{-\infty}^{t} e_{-d_l}(t,\sigma(s))\big(\mathcal{K}_l(s,\varphi)+u_l(s)\big)\Delta s+\sum_{t_k<t} e_{-d_l}(t,t_k)I_{lk}(\varphi_l(t_k)).$$

Now, we show that the mapping Φ is a self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$. Note that

$$\begin{aligned}
\big|\mathcal{K}_l(s,\varphi)\big| &\le d_l^+\int_{s-\eta_l(s)}^{s}\big\|\varphi_l^{\Delta}(u)\big\|_{\mathbb{C}}\Delta u+\sum_{j=1}^{n}\big\|a_{lj}(t)\big\|_{\mathbb{C}}\big\|f_j(\varphi_j(s))\big\|_{\mathbb{C}}\\
&\quad+\sum_{j=1}^{n}\big\|b_{lj}(t)\big\|_{\mathbb{C}}\big\|g_j(\varphi_j(s-\vartheta_{lj}(s)))\big\|_{\mathbb{C}}\\
&\quad+\sum_{j=1}^{n}\big\|c_{lj}(t)\big\|_{\mathbb{C}}\int_{s-\delta_{lj}(s)}^{s}\big\|h_j(\varphi_j(u))\big\|_{\mathbb{C}}\Delta u\\
&\le d_l^+\eta_l^+\big\|\varphi_l^{\Delta}(s)\big\|_{\mathbb{C}}+\sum_{j=1}^{n}a_{lj}^+\alpha_j\big\|\varphi_j(s)\big\|_{\mathbb{C}}\\
&\quad+\sum_{j=1}^{n}b_{lj}^+\beta_j\big\|\varphi_j(s-\vartheta_{lj}(s))\big\|_{\mathbb{C}}+\sum_{j=1}^{n}c_{lj}^+\delta_{lj}^+\zeta_j\big\|\varphi_j(s)\big\|_{\mathbb{C}}\\
&\le\Big[d_l^+\eta_l^++\sum_{j=1}^{n}a_{lj}^+\alpha_j+\sum_{j=1}^{n}b_{lj}^+\beta_j+\sum_{j=1}^{n}\delta_{lj}^+c_{lj}^+\zeta_j\Big]\|\varphi\|_{\mathbb{X}}\\
&\le\Theta_l r,
\end{aligned}$$

and

$$\begin{aligned}
\Big\|\sum_{t_k<t} e_{-d_l}(t,t_k)I_{lk}(\varphi_l(t_k))\Big\|_{\mathbb{C}} &\le\sum_{t_k<t}\big|e_{-d_l}(t,t_k)\big|L_l\big\|\varphi_l(t_k)\big\|_{\mathbb{C}}\\
&\le\frac{L_l r}{1-e^{-\theta d_l^-}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\big\|(\Phi\varphi)_l(t)\big\|_{\mathbb{C}} &\le\int_{-\infty}^{t} e_{-d_l}(t,\sigma(s))\big\|\mathcal{K}_l(s,\varphi)+u_l(s)\big\|_{\mathbb{C}}\Delta s\\
&\quad+\sum_{t_k<t}\Big\|e_{-d_l}(t,t_k)I_{lk}(\varphi_l(t_k))\Big\|_{\mathbb{C}}\\
&\le\frac{\Theta_l r+u_l^+}{d_l^-}+\frac{L_l r}{1-e^{-\theta d_l^-}},\quad l\in\mathcal{J}.
\end{aligned}$$

On the other hand, we have

$$\Big\|\big((\Phi\varphi)^{\Delta}\big)_l(t)\Big\|_{\mathbb{C}}=\Big\|\mathcal{K}_l(t,\varphi)+u_l(t)-d_l(t)\int_{-\infty}^{t} e_{-d_l}(t,\sigma(s))\big(\mathcal{K}_l(s,\varphi)+u_l(s)\big)\Delta s$$

$$-d_l(t)\sum_{t_k<t} e_{-d_l}(t,t_k)I_{lk}(\varphi_l(t_k))\Big\|_{\mathbb{C}}$$
$$\le \left(1+\frac{d_l^+}{d_l^-}\right)\left(\Theta_l r+u_l^+\right)+\frac{d_l^+L_l r}{1-e^{-\theta d_l^-}},\quad l\in\mathcal{J}.$$

Hence, by (H_5), we obtain $\|\Phi\varphi\|_{\mathbb{X}}\le r$, which implies that $\Phi\varphi\in\mathbb{X}^*$.

Next, we show that Φ is a contraction operator. In fact, for any $\varphi,\psi\in\mathbb{X}^*$, we denote

$$\begin{aligned}\mathcal{W}_l(s,\varphi,\psi)=d_l(s)\int_{s-\eta_l(s)}^{s}\left(\varphi_l^\Delta(u)-\psi_l^\Delta(u)\right)\Delta u+\sum_{j=1}^{n}a_{lj}(s)\left[f_j(\varphi_j(s))-f_j(\psi_j(s))\right]\\+\sum_{j=1}^{n}b_{lj}(s)\left[g_j(\varphi_j(s-\vartheta_{lj}(s)))-g_j(\psi_j(s-\vartheta_{lj}(s)))\right]\\+\sum_{j=1}^{n}c_{lj}(s)\int_{s-\delta_{lj}(s)}^{s}\left[h_j(\varphi_j(u))-h_j(\psi_j(u))\right]\Delta u.\end{aligned}$$

Then we have

$$\begin{aligned}\left\|\mathcal{W}_l(s,\varphi,\psi)\right\|_{\mathbb{C}}&\le\left(d_l^+\eta_l^+ +\sum_{j=1}^{n}a_{lj}^+\alpha_j+\sum_{j=1}^{n}b_{lj}^+\beta_j+\sum_{j=1}^{n}\delta_{lj}^+c_{lj}^+\zeta_j\right)\|\varphi-\psi\|_{\mathbb{X}}\\&=\Theta_l\|\varphi-\psi\|_{\mathbb{X}},\quad l\in\mathcal{J},\end{aligned}$$

and

$$\begin{aligned}\Big\|\sum_{t_k<t}e_{-d_l}(t,t_k)\left[I_{lk}(\varphi_l(t_k))-I_{lk}(\psi_l(t_k))\right]\Big\|_{\mathbb{C}}&\le\sum_{t_k<t}\left|e_{-d_l}(t,t_k)\right|L_l\|\varphi-\psi\|_{\mathbb{X}}\\&\le\frac{L_l}{1-e^{-\theta d_l^-}}\|\varphi-\psi\|_{\mathbb{X}}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\left\|(\Phi\varphi)_l(t)-(\Phi\psi)_l(t)\right\|_{\mathbb{C}}&=\Big\|\int_{-\infty}^{t}e_{-d_l}(t,\sigma(s))\mathcal{W}_l(s,\varphi,\psi)\Delta s\\&\quad+\sum_{t_k<t}e_{-d_l}(t,t_k)\left[I_{lk}(\varphi_l(t_k))-I_{lk}(\psi_l(t_k))\right]\Big\|_{\mathbb{C}}\\&\le\left(\frac{\Theta_l}{d_l^-}+\frac{L_l}{1-e^{-\theta d_l^-}}\right)\|\varphi-\psi\|_{\mathbb{X}},\quad l\in\mathcal{J}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\Big\|\left((\Phi\varphi)^\Delta\right)_l(t)-\left((\Phi\psi)^\Delta\right)_l(t)\Big\|_{\mathbb{C}}\\&=\Big\|\left(\int_{-\infty}^{t}e_{-d_l}(t,\sigma(s))\mathcal{W}_l(s,\varphi,\psi)\Delta s+\sum_{t_k<t}e_{-d_l}(t,t_k)\left[I_{lk}(\varphi_l(t_k))-I_{lk}(\psi_l(t_k))\right]\right)^\Delta\Big\|_{\mathbb{C}}\\&\le\left\|\mathcal{W}_l(t,\varphi,\psi)\right\|_{\mathbb{C}}+d_l^+\sup_{t\in\mathbb{T}}\Big\|\int_{-\infty}^{t}e_{-d_l}(t,\sigma(s))\mathcal{W}_l(s,\varphi,\psi)\Delta s\Big\|_{\mathbb{C}}\end{aligned}$$

$$
\begin{aligned}
&+d_l^+ \sum_{t_k<t} e_{-d_l}(t,t_k)\big\|I_{lk}(\varphi_l(t_k)) - I_{lk}(\psi_l(t_k))\big\|_{\mathbb{C}} \\
&\leq \left\{\left(1+\frac{d_l^+}{d_l^-}\right)\Theta_l + \frac{d_l^+ L_l}{1-e^{-\theta d_l^-}}\right\}\|\varphi-\psi\|_{\mathbb{X}}, \quad l \in \mathcal{J}.
\end{aligned}
$$

From (H_5), we have $\|\Phi(\varphi) - \Phi(\psi)\|_{\mathbb{X}} \leq \rho\|\varphi - \psi\|_{\mathbb{X}}$. Hence, Φ is a contraction mapping. By Banach fixed point theorem, Φ has a unique fixed point in $\mathbb{X}^*$, that is, (1.1) has a unique piecewise almost periodic solution in $\mathbb{X}^*$. The proof is complete. □

4. Almost periodic synchronization

In this section, by designing a controller, utilizing some analytic techniques and constructing a suitable Lyapunov function, we consider the exponential synchronization problem of CVNNs with time-varying delays and impulsive effects on time scales and almost periodic coefficients. Thus, we consider the system (1.1) as a drive system, and a response system is designed as

$$
\begin{cases}
y_l^\Delta(t) = -d_l(t)y_l(t) + d_l(t)\displaystyle\int_{t-\eta_l(t)}^{t} y_l^\Delta(s)\Delta s \\
\qquad + \displaystyle\sum_{j=1}^{n} a_{lj}(t)f_j(y_j(t)) + \sum_{j=1}^{n} b_{lj}(t)g_j(y_j(t-\vartheta_{lj}(t))) \\
\qquad + \displaystyle\sum_{j=1}^{n} c_{lj}(t)\int_{t-\delta_{lj}(t)}^{t} h_j(y_j(s))\Delta s + u_l(t) + \mathcal{U}_l(t), \quad t \neq t_k, \\
\tilde{\Delta} y_l(t_k) = I_{lk}(y_l(t_k)), \quad t = t_k,\ k \in \mathbb{Z},
\end{cases}
\tag{4.1}
$$

where $l \in \mathcal{J}$, $\mathcal{U}_l(t)$ is a controlled input.

Let signals $z_l(t) = y_l(t) - x_l(t)$, $z_l(t) = \left(z_l^R(t), z_l^I(t)\right)^T$. Then, we can obtain the following error system:

$$
\begin{cases}
z_l^\Delta(t) = -d_l(t)z_l(t) + d_l(t)\displaystyle\int_{t-\eta_l(t)}^{t} z_l^\Delta(s)\Delta s + \sum_{j=1}^{n} a_{lj}(t)\big[f_j(y_j(t)) - f_j(x_j(t))\big] \\
\qquad + \displaystyle\sum_{j=1}^{n} b_{lj}(t)\big[g_j(y_j(t-\vartheta_{lj}(t))) - g_j(x_j(t-\vartheta_{lj}(t)))\big] \\
\qquad + \displaystyle\sum_{j=1}^{n} c_{lj}(t)\int_{t-\delta_{lj}(t)}^{t} \big[h_j(y_j(s)) - h_j(x_j(s))\big]\Delta s + \mathcal{U}_l(t), \quad t \neq t_k, \\
\tilde{\Delta} z_l(t_k) = I_{lk}(z_l(t_k)), \quad t = t_k,\ k \in \mathbb{Z}.
\end{cases}
\tag{4.2}
$$

In order to realize the synchronization between (1.1) and (4.1), we design the following state-feedback controller:

$$
\mathcal{U}_l(t) = -\kappa_l z_l(t), \tag{4.3}
$$

where $\kappa_l \in \mathbb{R}^+$ is the control input.

Definition 4.1. The response system (4.1) and the drive system (1.1) can be globally exponentially synchronized, if there exist positive constant $M \geq 1$, $\lambda > 0$ such that

$$\|z(t)\|_0 \leq \|\psi - \varphi\|_1 M e_{\ominus\lambda}(t,0), \quad t \in [0,+\infty)_{\mathbb{T}}$$

where

$$\|z(t)\|_0 = \max_{l\in\mathcal{J}} \Big\{ \|z_l(t)\|_{\mathbb{C}}, \|z_l^{\Delta}(t)\|_{\mathbb{C}} \Big\},$$
$$\|\psi - \phi\|_1 = \max_{l\in\mathcal{J}} \Big\{ \sup_{s\in[-\xi,0]_{\mathbb{T}}} \|\psi_l(s) - \varphi_l(s)\|_{\mathbb{C}}, \sup_{s\in[-\xi,0]_{\mathbb{T}}} \|\psi_l^{\Delta}(s) - \varphi_l^{\Delta}(s)\|_{\mathbb{C}} \Big\}.$$

Theorem 4.1. *Assume that conditions* (H_1)-(H_5) *hold and the following conditions are satisfied:*

(H_6) $I_{lk}(x_l(t_k)) = -\gamma_{lk} x_l(t_k)$ *with* $0 \leq \gamma_{lk} \leq 2$, $l \in \mathcal{J}$, $k \in \mathbb{Z}$.

(H_7) *There exists a positive constant* λ *such that*

$$\max_{l\in\mathcal{J}} \Bigg\{ \Big(\big(-\kappa_l - d_l^- \big) + \lambda + 2\bar{\mu}\big(\kappa_l + d_l^+\big)^2 \Big) + \big(2\bar{\mu}\big(\kappa_l + d_l^+\big) + 1\big)$$
$$\times \Bigg(d_l^+ \eta_l^+ + \sum_{j=1}^n a_{lj}\alpha_j + \frac{e_\lambda(\vartheta,0)}{1-\omega} \sum_{j=1}^n b_{lj}^+ \beta_j + \sum_{j=1}^n \delta_{lj}^+ c_{lj}^+ \zeta_j \Bigg) \Bigg\} \big(1 + \bar{\mu}\lambda\big) < 0.$$

Then the drive system (1.1) *and the response system* (4.1) *are globally exponentially synchronized based on the controller* (4.3).

Proof. From (4.2), for $l \in \mathcal{J}$, $k \in \mathbb{Z}$, we have

$$\begin{aligned} z_l(t_k^+) = y_l(t_k^+) - x_l(t_k^+) &= y_l(t_k) - \gamma_{lk} y_l(t_k) - \big(x_l(t_k) - \gamma_{lk} x_l(t_k)\big) \\ &= (1 - \gamma_{lk})\big(y_l(t_k) - x_l(t_k)\big) \\ &\leq y_l(t_k) - x_l(t_k) = z_l(t_k), \end{aligned}$$

so, we have $z_l(t_k^+) \leq z_l(t_k)$. For $t \in [0,+\infty)_{\mathbb{T}}$, by (H_4), one can easily to see that

$$\begin{aligned} \big\|(z_l)^{\Delta}(t)\big\|_{\mathbb{C}} \leq (\kappa_l + d_l^+)\big\|z_l(t)\big\|_{\mathbb{C}} + d_l^+\eta_l^+\big\|z_l^{\Delta}(t)\big\|_{\mathbb{C}} + \sum_{j=1}^n a_{lj}\alpha_j \big\|z_j(t)\big\|_{\mathbb{C}} \\ + \sum_{j=1}^n b_{lj}^+\beta_j \big\|z_j(t - \vartheta_{lj}(t))\big\|_{\mathbb{C}} + \sum_{j=1}^n \delta_{lj}^+ c_{lj}^+ \zeta_j \big\|z_j(t)\big\|_{\mathbb{C}}. \end{aligned}$$

We consider the Lyapunov function as follows:

$$V(t) = \max_{l\in\mathcal{J}} \Big(\big\|z_l(t)\big\|_{\mathbb{C}} e_\lambda(t,0) + \mathcal{P}_l \Big)$$

where

$$\mathcal{P}_l = \frac{\big(2\bar{\mu}(\kappa_l + d_l^+) + 1\big)(1 + \bar{\mu}\lambda)}{1-\omega} \sum_{j=1}^n \Bigg(b_{lj}^+ \int_{t-\vartheta_{lj}(t)}^{t} \beta_j \big\|z_j(s)\big\|_{\mathbb{C}} e_\lambda(s+\tau,0)\Delta s \Bigg).$$

It follows from the definition of delta derivatives and properties of exponential function, we denote $\bar{\mu} = \sup\limits_{t\in\mathbb{T}} |\mu(t)|$, then we have

$$
\begin{aligned}
&\Big[\big\|z_l(t)\big\|_{\mathbb{C}} e_\lambda(t,0)\Big]^{\Delta} \\
&\le \operatorname{sign}\big(z_l(\sigma(t))\big)\big(z_l\big)^{\Delta}(t) e_\lambda(\sigma(t),0) + \lambda \big\|z_l(t)\big\|_{\mathbb{C}} e_\lambda(t,0) \\
&= \operatorname{sign}\big(z_l(\sigma(t))\big)\bigg[\big(-\kappa_l - d_l(t)\big) z_l(t) + d_l(t)\int_{t-\eta_l(t)}^{t} (z_l)^{\Delta}(s)\Delta s \\
&\quad + \sum_{j=1}^{n} a_{lj}(t)\big[f_j(y_j(t)) - f_j(x_j(t))\big] + \sum_{j=1}^{n} b_{lj}(t)\big[g_j(y_j(t-\vartheta_{lj}(t))) \\
&\quad - g_j(x_j(t-\vartheta_{lj}(t)))\big] + \sum_{j=1}^{n} c_{lj}(t)\int_{t-\delta_{lj}(t)}^{t}\big[h_j(y_j(s)) \\
&\quad - h_j(x_j(s))\big]\Delta s + \lambda\big\|z_l(t)\big\|_{\mathbb{C}} e_\lambda(t,0)\bigg]\big(1+\mu(t)\lambda\big)e_\lambda(t,0) \\
&\le \operatorname{sign}\big(z_l(\sigma(t))\big)\bigg[\big(-\kappa_l - d_l(t)\big)\big[z_l(\sigma(t)) - \mu(t)\big(z_l\big)^{\Delta}(t)\big] \\
&\quad + d_l^{+}\int_{t-\eta_l(t)}^{t} \big|(z_l)^{\Delta}(s)\big|\Delta s + \sum_{j=1}^{n} a_{lj}^{+}\alpha_j\big\|z_j(t)\big\|_{\mathbb{C}} \\
&\quad + \sum_{j=1}^{n} b_{lj}^{+}\beta_j\big\|z_j(t-\vartheta_{lj}(t))\big\|_{\mathbb{C}} + \lambda\big\|z_l(t)\big\|_{\mathbb{C}} e_\lambda(t,0) \\
&\quad + \sum_{j=1}^{n} \delta_{lj}^{+} c_{lj}^{+}\zeta_j\big\|z_j(t)\big\|_{\mathbb{C}}\bigg]\big(1+\mu(t)\lambda\big)e_\lambda(t,0) \\
&\le \bigg[\big(-\kappa_l - d_l^{-}\big)\big\|z_l(t) - \mu(t)\big(z_l\big)^{\Delta}(t)\big\|_{\mathbb{C}} + \big(\kappa_l + d_l^{+}\big)\bar{\mu}\big\|\big(z_l\big)^{\Delta}(t)\big\|_{\mathbb{C}} \\
&\quad + \lambda\big\|z_l(t)\big\|_{\mathbb{C}} e_\lambda(t,0) + d_l^{+}\int_{t-\eta_l(t)}^{t}\big\|z_l^{\Delta}(s)\big\|_{\mathbb{C}}\Delta s + \sum_{j=1}^{n} a_{lj}^{+}\alpha_j\big\|z_j(t)\big\|_{\mathbb{C}} \\
&\quad + \sum_{j=1}^{n} b_{lj}^{+}\beta_j\big\|z_j(t-\vartheta_{lj}(t))\big\|_{\mathbb{C}} + \sum_{j=1}^{n} \delta_{lj}^{+} c_{lj}^{+}\zeta_j\big\|z_j(t)\big\|_{\mathbb{C}}\bigg]\big(1+\mu(t)\lambda\big)e_\lambda(t,0) \\
&\le \bigg[\big(-\kappa_l - d_l^{-}\big)\big\|z_l(t)\big\|_{\mathbb{C}} + 2\big(\kappa_l + d_l^{+}\big)\bar{\mu}\big\|\big(z_l\big)^{\Delta}(t)\big\|_{\mathbb{C}} \\
&\quad + \lambda\big\|z_l(t)\big\|_{\mathbb{C}} e_\lambda(t,0) + d_l^{+}\int_{t-\eta_l(t)}^{t}\big\|(z_l)^{\Delta}(s)\big\|_{\mathbb{C}}\Delta s + \sum_{j=1}^{n} a_{lj}^{+}\alpha_j\big\|z_j(t)\big\|_{\mathbb{C}} \\
&\quad + \sum_{j=1}^{n} b_{lj}^{+}\beta_j\big\|z_j(t-\vartheta_{lj}(t))\big\|_{\mathbb{C}} + \sum_{j=1}^{n} \delta_{lj}^{+} c_{lj}^{+}\zeta_j\big\|z_j(t)\big\|_{\mathbb{C}}\bigg]\big(1+\mu(t)\lambda\big)e_\lambda(t,0) \\
&\le \bigg[\big(-\kappa_l - d_l^{-}\big)\big\|z_l(t)\big\|_{\mathbb{C}} + 2\big(\kappa_l + d_l^{+}\big)\bar{\mu}\bigg(\big(\kappa_l + d_l^{+}\big)\big\|z_l(t)\big\|_{\mathbb{C}} \\
&\quad + d_l^{+}\int_{t-\eta_l(t)}^{t}\big\|z_l^{\Delta}(s)\big\|_{\mathbb{C}}\Delta s + \sum_{j=1}^{n} a_{lj}^{+}\alpha_j\big\|z_j(t)\big\|_{\mathbb{C}}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} b_{lj}^{+}\beta_j\big\|z_j(t-\vartheta_{lj}(t))\big\|_{\mathbb{C}}+\sum_{j=1}^{n}\delta_{lj}^{+}c_{lj}^{+}\zeta_j\big\|z_j(t)\big\|_{\mathbb{C}}\Big)\\
&+\lambda\big\|z_l(t)\big\|_{\mathbb{C}}e_\lambda(t,0)+d_l^{+}\int_{t-\eta_l(t)}^{t}\big\|z_l^{\Delta}(s)\big\|_{\mathbb{C}}\Delta s+\sum_{j=1}^{n}a_{lj}^{+}\alpha_j\big\|z_j(t)\big\|_{\mathbb{C}}\\
&+\sum_{j=1}^{n} b_{lj}^{+}\beta_j\big\|z_j(t-\vartheta_{lj}(t))\big\|_{\mathbb{C}}+\sum_{j=1}^{n}\delta_{lj}^{+}c_{lj}^{+}\zeta_j\big\|z_j(t)\big\|_{\mathbb{C}}\Big]\big(1+\mu(t)\lambda\big)e_\lambda(t,0)\\
\le&\Big[\Big(\big(-\kappa_l-d_l^{-}\big)+\lambda+2\bar\mu\big(\kappa_l+d_l^{+}\big)^2\Big)\big\|z_l(t)\big\|_{\mathbb{C}}+\big(2\bar\mu\big(\kappa_l+d_l^{+}\big)+1\big)\\
&\times\Big(d_l^{+}\int_{t-\eta_l(t)}^{t}\big\|z_l^{\Delta}(s)\big\|_{\mathbb{C}}\Delta s+\sum_{j=1}^{n}a_{lj}\alpha_j\big\|z_j(t)\big\|_{\mathbb{C}}\\
&+\sum_{j=1}^{n} b_{lj}^{+}\beta_j\big\|z_j(t-\vartheta_{lj}(t))\big\|_{\mathbb{C}}+\sum_{j=1}^{n}\delta_{lj}^{+}c_{lj}^{+}\zeta_j\big\|z_j(t)\big\|_{\mathbb{C}}\Big)\Big]\big(1+\bar\mu\lambda\big)e_\lambda(t,0),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{P}_l^{\Delta}(t)\le e_\lambda(t,0)\sum_{l=1}^{n}\Big\{&\frac{\big(2\bar\mu(\kappa_l+d_l^{+})+1\big)(1+\bar\mu\lambda)}{1-\omega}\sum_{j=1}^{n}b_{lj}^{+}\beta_j\big\|z_j(t)\big\|_{\mathbb{C}}e_\lambda(\vartheta,0)\\
&-\big(2\bar\mu(\kappa_l+d_l^{+})+1\big)(1+\bar\mu\lambda)\sum_{j=1}^{n}b_{lj}^{+}\beta_j\big\|z_j(t-\vartheta_{lj}(t))\big\|_{\mathbb{C}}\Big\},
\end{aligned}
$$

according to the above two inequality, we have

$$
\begin{aligned}
V^{\Delta}(t)\le\max_{l\in\mathcal{J}}\Big\{&\Big[\Big(\big(-\kappa_l-d_l^{-}\big)+\lambda+2\bar\mu\big(\kappa_l+d_l^{+}\big)^2\Big)+\big(2\bar\mu\big(\kappa_l+d_l^{+}\big)+1\big)\\
&\times\Big(d_l^{+}\eta_l^{+}+\sum_{j=1}^{n}a_{lj}\alpha_j+\frac{e_\lambda(\vartheta,0)}{1-\omega}\sum_{j=1}^{n}b_{lj}^{+}\beta_j+\sum_{j=1}^{n}\delta_{lj}^{+}c_{lj}^{+}\zeta_j\Big)\Big]\\
&\times\big(1+\bar\mu\lambda\big)\Big\}e_\lambda(t,0)\|z(t)\|_0.
\end{aligned}
$$

Then, we can obtain

$$
V^{\Delta}(t)\le 0,\quad t\in[0,+\infty)_{\mathbb{T}},
$$

and

$$
\begin{aligned}
V(t_k^{+})-V(t_k)&\le\max_{l\in\mathcal{J}}\|z_l(t_k^{+})\|_{\mathbb{C}}e_\lambda(t_k^{+},0)-\max_{l\in\mathcal{J}}\|z_l(t_k)\|_{\mathbb{C}}e_\lambda(t_k,0)\\
&\le\max_{l\in\mathcal{J}}\big\|(1-\gamma_{lk})z_l(t_k)-z_l(t_k)\big\|_{\mathbb{C}}e_\lambda(t_k^{+},0)\le 0,\quad k\in\mathbb{Z}.
\end{aligned}
$$

Thus, for $t\in[0,+\infty)_{\mathbb{T}}$, we can obtain $V(t)\le V(0)$. On the other hand,

$$
\begin{aligned}
V(0)=\max_{l\in\mathcal{J}}\Big\{&\big\|z_l(0)\big\|_{\mathbb{C}}e_\lambda(t,0)+\frac{\big(2\bar\mu(\kappa_l+d_l^{+})+1\big)(1+\bar\mu\lambda)}{1-\omega}\sum_{j=1}^{n}b_{lj}^{+}\\
&\times\int_{0-\vartheta_{lj}(0)}^{0}\beta_j\big\|z_j(s)\big\|_{\mathbb{C}}e_\lambda(s+\tau,0)\Delta s
\end{aligned}
$$

$$\leq \sum_{l=1}^{n}\left\{1+\frac{\big(2\bar{\mu}(\kappa_l+d_l^+)+1\big)(1+\bar{\mu}\lambda)}{1-\omega}\sum_{j=1}^{n}b_{lj}^+\beta_j\vartheta_{lj}^+e_\lambda(\vartheta,0)\right\}e_\lambda(t,0)\|\psi-\phi\|_1.$$

Thus, for $t\neq t_k$, we have

$$\|z(t)\|_0\leq\|\phi-\psi\|_1Me_{\ominus\lambda}(t,0),$$

where

$$M=\max_{l\in\mathcal{J}}\left\{1+\frac{\big(2\bar{\mu}(\kappa_l+d_l^+)+1\big)(1+\bar{\mu}\lambda)}{1-\omega}\sum_{j=1}^{n}b_{lj}^+\beta_j\vartheta_{lj}^+e_\lambda(\vartheta,0)\right\}>1.$$

Furthermore, for $t=t_k$,

$$\|z(t_k)\|_0\leq\|\phi-\psi\|_1Me_{\ominus\lambda}(t,0).$$

Therefore, the drive system (1.1) and the response system (4.1) are globally exponential synchronized based on the controller (4.3). The proof is complete. □

5. Numerical simulations

In this section, we give an example to illustrate the feasibility and effectiveness of our results obtained in Sections 3 and 4.

Example 5.1. Consider a two-neuron impulsive complex-valued neural networks on time scales

$$\begin{cases} x_l^\Delta(t)=-d_l(t)x_l(t-\eta_l(t))+\displaystyle\sum_{j=1}^{2}a_{lj}(t)f_j(x_j(t))+\sum_{j=1}^{2}b_{lj}(t)g_j(x_j(t-\vartheta_{lj}(t)))\\ \qquad+\displaystyle\sum_{j=1}^{2}c_{lj}(t)\int_{t-\delta_{lj}(t)}^{t}h_j(x_j(s))\Delta s+u_l(t),\quad t\neq t_k,\\ \tilde{\Delta}x_l(t_k)=x_l(t_k^+)-x_l(t_k^-)=I_{lk}(x_l(t_k)),\quad t=t_k,\ k\in\mathbb{Z},\end{cases}\tag{5.1}$$

where $l=1,2$, $t_k=2k$, $t\in\mathbb{T}$ and the coefficients are follows:

$f_j(x_j)=\dfrac{1}{65}|x_j^R+x_j^I|+i\dfrac{1}{50}\sin^2x_j^I,\quad g_j(x_j)=\dfrac{1}{40}\tanh x_j^R+i\dfrac{1}{40}|x_j^I|,$

$h_j(x_j)=\dfrac{1}{60}(|x_j^R+1|+|x_j^I|-1)+i\dfrac{1}{45}|x_j^R+x_j^I|,\quad d_1(t)=0.2-0.02|\sin t|,$

$d_2(t)=0.2+0.05|\cos t|,\quad u_1(t)=0.04+0.05i\sin t,\quad u_2(t)=0.08+0.06i\cos t,$

$a_{11}(t)=a_{12}(t)=0.1+0.2i|\sin(\sqrt{3}t)|,\quad a_{21}(t)=a_{22}(t)=0.3+0.1i|\cos\sqrt{2}t|,$

$b_{11}(t)=b_{12}(t)=0.15+0.05i|\sin t|,\quad b_{21}(t)=b_{22}(t)=0.05+0.02i|\sin(\sqrt{2}t)|,$

$c_{11}(t)=c_{12}(t)=0.015+0.015i\cos^2t,\quad c_{21}(t)=c_{22}(t)=0.01+0.015i|\cos(\sqrt{2}t)|,$

$\eta_1(t)=0.01|\cos(\pi t)|,\eta_2(t)=0.03|\cos(2\pi t)|,\vartheta_{lj}(t)=\dfrac{4}{5}|\cos(2\pi t)|,\ \delta_{lj}(t)=2|\cos(2\pi t)|,$

$\tilde{\Delta}x_1(2k)=-\dfrac{1}{15}x_1^R(2k)+\dfrac{1}{15}\sin(x_1^R(2k))+i\big(-\dfrac{1}{15}x_1^I(2k)+\dfrac{1}{15}\sin(x_1^I(2k))\big),$

$$\tilde{\Delta}x_2(2k) = -\frac{1}{15}x_2^R(2k) + \frac{1}{15}\cos(x_2^R(2k)) + i\big(-\frac{1}{15}x_2^I(2k) + \frac{1}{15}\cos(x_2^I(2k))\big).$$

By a simple calculation, we have $\alpha_j = \frac{1}{50}$, $\beta_j = \frac{1}{40}$, $\zeta_j = \frac{1}{45}$, $L_l = \frac{2}{15}$ and

$$\begin{aligned}
&d_1^- = 0.18, \quad d_2^- = 0.2, \quad d_1^+ = 0.2, \quad d_2^+ = 0.25, \quad a_{11}^+ = a_{12}^+ = 0.2,\\
&a_{21}^+ = a_{22}^+ = 0.3, \quad b_{11}^+ = b_{12}^+ = 0.15, \quad b_{21}^+ = b_{22}^+ = 0.05,\\
&c_{11}^+ = c_{12}^+ = 0.015, \quad c_{21}^+ = c_{22}^+ = 0.015, \quad u_1^+ = 0.05, \quad u_2^+ = 0.08,\\
&\eta_1^+ = 0.01, \quad \eta_2^+ = 0.03, \quad \vartheta_{lj}^+ = \frac{4}{5}, \quad \delta_{lj}^+ = 2, \quad l, j = 1, 2,\\
&\Theta_1 = d_1^+\eta_1^+ + \sum_{j=1}^{2} a_{1j}^+\alpha_j + \sum_{j=1}^{2} b_{1j}^+\beta_j + \sum_{j=1}^{2} \delta_{1j}^+ c_{1j}^+\zeta_j \approx 0.0188,\\
&\Theta_2 = d_2^+\eta_2^+ + \sum_{j=1}^{2} a_{2j}^+\alpha_j + \sum_{j=1}^{2} b_{2j}^+\beta_j + \sum_{j=1}^{2} \delta_{2j}^+ c_{1j}^+\zeta_j \approx 0.0233.
\end{aligned}$$

And we take $r = 0.8$, $\theta = 4$, then we obtain

$$\begin{aligned}
&\max_{1\le l\le 2}\left\{\frac{\Theta_l r + u_l^+}{d_l^-} + \frac{L_l}{1-e^{-\theta d_l^-}}r, \left(1+\frac{d_l^+}{d_l^-}\right)(\Theta_l r + u_l^+) + \frac{d_l^+ L_l}{1-e^{-\theta d_l^-}}r\right\}\\
\approx{} &\max\{0.5692, 0.1789, 0.6869, 0.2704\} = 0.6869 < r = 0.8,\\
&\max_{1\le l\le 2}\left\{\frac{\Theta_l}{d_l^-} + \frac{L_l}{1-e^{-\theta d_l^-}}, \left(1+\frac{d_l^+}{d_l^-}\right)\Theta_l + \frac{d_l^+ L_l}{1-e^{-\theta d_l^-}}\right\}\\
\approx{} &\max\{0.3642, 0.3586, 0.0916, 0.1130\} = 0.3642 = \rho < 1.
\end{aligned}$$

Moreover, take $\lambda = 0.1$, $\kappa_l = 0.1$, $\omega = \frac{4}{5}$, if $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$, $e_\lambda(\vartheta, 0) = e^{\lambda\vartheta} = e^{\frac{2}{25}}$. We obtain

$$\begin{aligned}
&\max_{1\le l\le 2}\Bigg\{\Big(\big(-\kappa_l - d_l^-\big) + \lambda + 2\bar{\mu}\big(\kappa_l + d_l^+\big)^2\Big) + \big(2\bar{\mu}\big(\kappa_l + d_l^+\big) + 1\big)\\
&\times\left(d_l^+\eta_l^+ + \sum_{j=1}^{2} a_{lj}^+\alpha_j + \frac{e_\lambda(\vartheta,0)}{1-\omega}\sum_{j=1}^{2} b_{lj}^+\beta_j + \sum_{j=1}^{2}\delta_{2j}^+ c_{lj}^+\zeta_j\right)\Bigg\}\big(1+\bar{\mu}\lambda\big)\\
\approx{} &-0.1281 < 0,
\end{aligned}$$

if $\mathbb{T} = \frac{\mathbb{Z}}{5}$, $\mu(t) = \frac{1}{5}$, $e_\lambda(\vartheta, 0) = (1+\lambda)^\vartheta = 1.1^{\frac{4}{5}}$. We obtain

$$\begin{aligned}
&\max_{1\le l\le 2}\Bigg\{\Big(\big(-\kappa_l - d_l^-\big) + \lambda + 2\bar{\mu}\big(\kappa_l + d_l^+\big)^2\Big) + \big(2\bar{\mu}\big(\kappa_l + d_l^+\big) + 1\big)\\
&\times\left(d_l^+\eta_l^+ + \sum_{j=1}^{2} a_{lj}^+\alpha_j + \frac{e_\lambda(\vartheta,0)}{1-\omega}\sum_{j=1}^{2} b_{lj}^+\beta_j + \sum_{j=1}^{2}\delta_{2j}^+ c_{lj}^+\zeta_j\right)\Bigg\}\big(1+\bar{\mu}\lambda\big)\\
\approx{} &-0.0877 < 0.
\end{aligned}$$

Take the time scale $\mathbb{T} = \mathbb{R}$ as a special case, represents a continuous case. We can verify that all assumptions in Theorems 3.1 and 4.1 are satisfied. Therefore,the the system (5.1) has a unique piecewise almost periodic solution in the region $\mathbb{X}^* = \{\varphi \mid \varphi \in \mathbb{X}, \|\varphi\|_{\mathbb{X}} \le r\}$ and the drive system (5.1) and its response system are globally exponential synchronized. By using the Simulink toolbox in MATLAB, Figures 1 depict the time revolution of real parts x_1 and imaginary parts x_2 of system (5.1) without control. Figures 2 depict the time revolution of real parts y_1 and imaginary parts y_2 of system (5.1) with control (4.3). Figure 3 shows state response curve of the real and imaginary parts of synchronization error. From simulation results in Figures 1-3, it is clearly seen that the drive system (5.1) and its response system achieve synchronization based on the controller (4.3). Similarly, take the time scale $\mathbb{T} = \frac{\mathbb{Z}}{5}$ as a special case, represents a discrete case, we can get similar results. (see Figures 3-6).

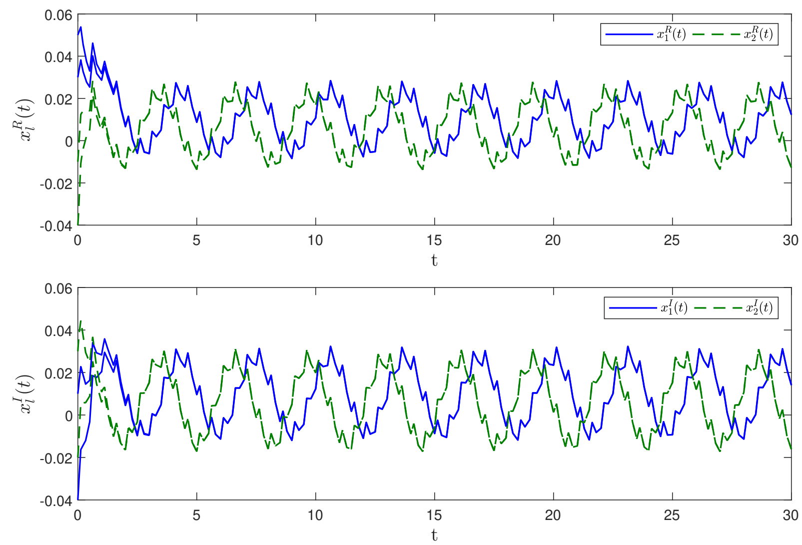

Figure 1. $\mathbb{T} = \mathbb{R}$, the states of two parts of $x_1(t)$ and $x_2(t)$.

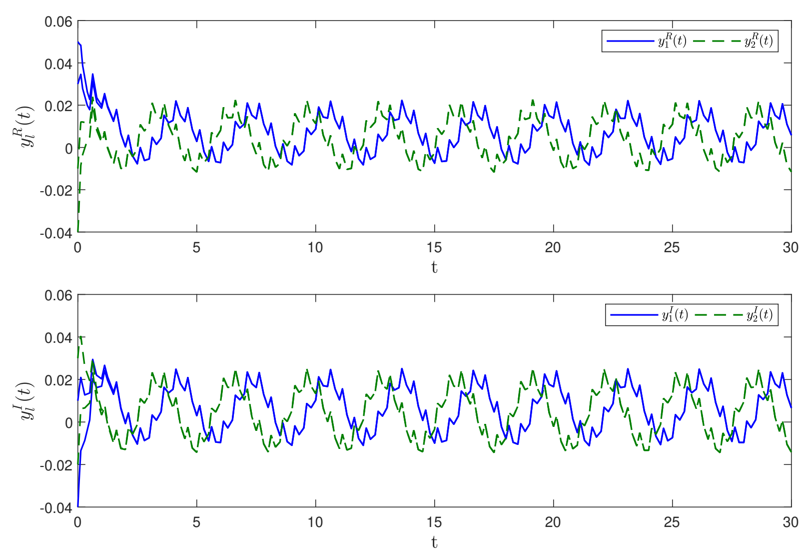

Figure 2. $\mathbb{T} = \mathbb{R}$, the states of two parts of $y_1(t)$ and $y_2(t)$.

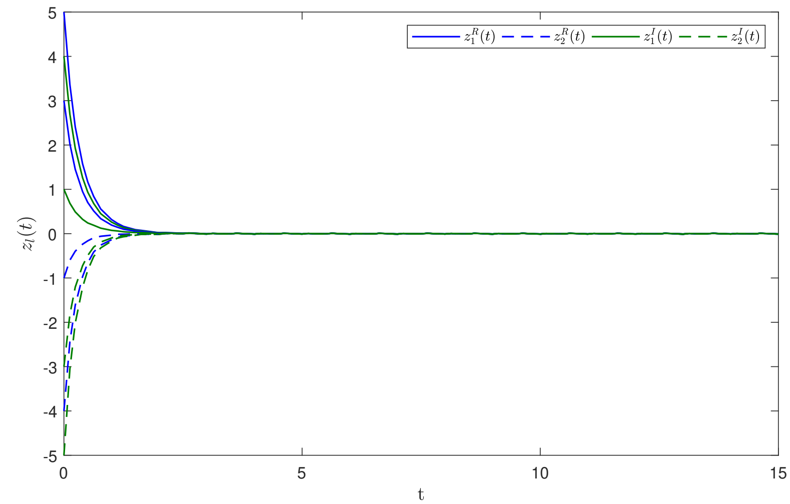

Figure 3. $\mathbb{T} = \mathbb{R}$. Synchronization errors $z(t) = y(t) - x(t)$.

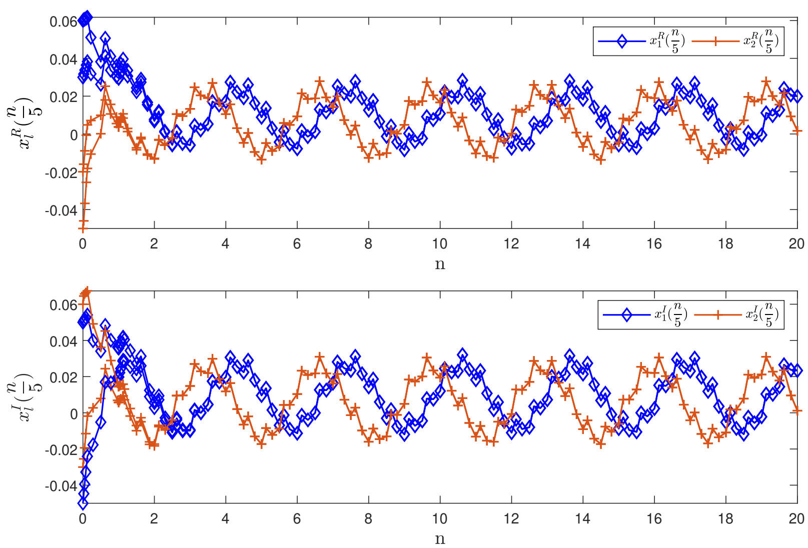

Figure 4. $\mathbb{T} = \frac{\mathbb{Z}}{5}$, the states of two parts of $x_1(\frac{n}{5})$ and $x_2(\frac{n}{5})$.

6. Conclusion

In this paper, we have investigated the impulsive complex-valued neural networks with leakage delays on time scales. Base on the Banach fixed point theorem, Lyapunov functional method and differential inequality technique on time scales, we obtain the existence and global exponential synchronization of almost periodic solutions for impulsive complex-valued neural networks. An example has been given

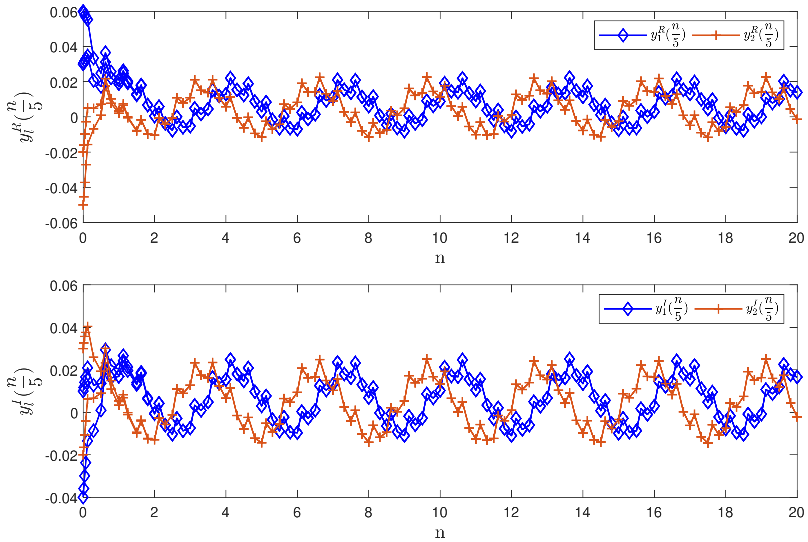

Figure 5. $\mathbb{T} = \frac{\mathbb{Z}}{5}$, the states of two parts of $y_1(\frac{n}{5})$ and $y_2(\frac{n}{5})$.

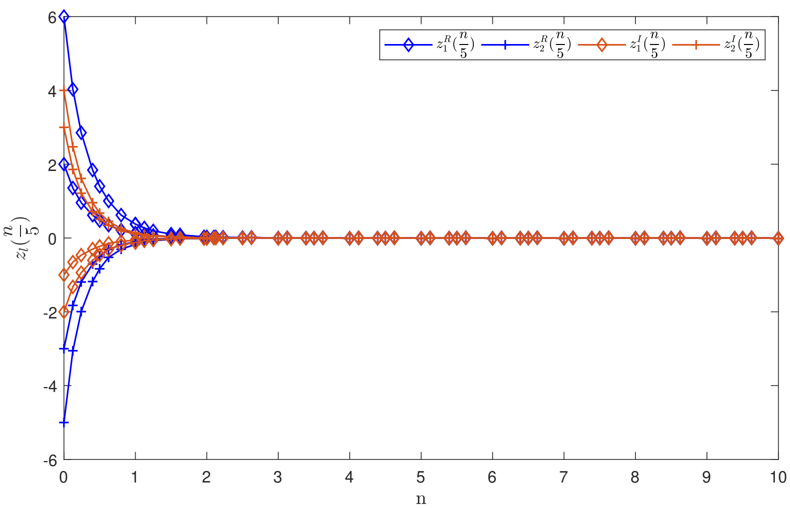

Figure 6. $\mathbb{T} = \frac{\mathbb{Z}}{5}$. Synchronization errors $z(\frac{n}{5}) = y(\frac{n}{5}) - x(\frac{n}{5})$.

to demonstrate the effectiveness of our results. We know the almost periodic synchronization for impulsive system on times scales is new. Furthermore, using the similar method, the global exponential synchronization of almost periodic solutions for the abstract impulsive ∇-dynamic equations can be applied. In the future work, some interesting results concerning impulses can be considered, such as potential impacts of delay on stability of impulsive control Systems, Exponential Stability of Delayed Systems with Average-Delay Impulses. In addition, Almost periodic synchronization in the quaternion field can be considered.

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