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UPPER SEMICONTINUITY OF UNIFORM RANDOM ATTRACTORS FOR DELAY PARABOLIC EQUATION*

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Abstract This paper concentrates on the upper semicontinuity of uniform random attractors for a class of delay parabolic equations with additive noise and nonautonomous external force terms. Firstly, through the uniform estimation of the solution, it is proved that the solution of the equation has a closed uniform pullback absorbing set with respect to the symbolic space. Then, by Arzela-Ascoli theorem, we prove uniformly pullback compactness of solutions as well as the existence and uniqueness of uniform random attractors. Finally, we prove the upper semicontinuity of the uniform random attractors when time delay approaches to zero.

Keywords Uniform random attractor, delay, upper semicontinuity.

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1. Introduction

In this paper, we investigate the asymptotic behavior of solutions to the following delay parabolic partial differential equations defined on the bounded smooth domain $\mathcal{O}$ in $\mathbb{R}^n$:

$$\begin{cases} du - \Delta u dt = (F\left(x, u\left(t, x\right)\right) + f\left(x, u\left(t-\rho, x\right)\right) + g\left(t, x\right))dt + h\left(x\right) d\omega, \\ x \in \mathcal{O},\ t \geq 0, \\ u\left(t, x\right) = 0, \quad x \in \partial\mathcal{O}, \end{cases} \tag{1.1}$$

where $\rho \in (0, 1]$ is small positive parameter, F is a superlinear source term, f is a nonlinear function that satisfies certain conditions, g is a deterministic time-dependent forcing, h is the shape of noise, and ω is a two-sided real-valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, P)$.

Attractors play an important role in the dynamical systems. As an extension of the global attractor for autonomous dynamical systems, the concept of pullback attractor for random dynamical systems was introduced in [6–8]. Since then, there is lots of literature on dynamics for stochastic partial differential equations,

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see [1–3, 10, 14]. For non-autonomous dynamical systems, the most representative attractors are pullback attractors and uniform attractors. To study the dynamical behavior of stochastic equations with deterministic non-autonomous terms, Wang [15] introduced cocycle system, i.e., two driving dynamical systems over two parameter spaces Ω_1 and Ω_2 corresponding to non-autonomous terms and random terms, respectively. He also studied the existence of cocycle attractors of stochastic differential equations with deterministic non-autonomous terms. After that, Cui and Langa [5] studied the uniform random attractors of stochastic differential equations with deterministic non-autonomous terms. The existence of pullback random attractors of non-autonomous stochastic equations without delay has been investigated by many authors, see, e.g., [11, 13, 16, 19] in the framework established in [15] and [4, 9, 20] in the framework established in [5]. In the delay case, there are only a few papers available in the literature dealing with the cocycle attractors, see [12,17,18]. However, it seems that there are very few works in the literature dealing with uniform random attractors of non-autonomous stochastic delay equations. In this work, we will address this problem.

As $\rho \to 0$, the equation (1.1) reduces to a non-delay stochastic equation and it is natural to ask the family of random dynamical systems ϕ^ρ generated by (1.1) is close to limiting random dynamical system ϕ^0 generated by the limiting equation? What is the relation between ϕ^ρ and ϕ^0? We prove the upper semicontinuity of the uniform random attractors of ϕ^ρ when time delay approaches to zero.

The outline of this paper is as follows. In Section 2, we recall the basic concept of uniform random attractors for nonautonomous random dynamical systems. In Section 3, we concentrate on studying continuous non-autonomous random dynamical system in $C\left([-\rho, 0], L^2(\mathcal{O})\right)$ generated by (1.1). In Section 4, we are devoted to the study of uniform estimates of solutions. In Section 5, the existence of uniform random attractors is obtained. Finally, the upper semicontinuity of uniform random attractors when time delay approaches to zero was established in Sections 6.

We use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norm and inner product of $L^2(\mathcal{O})$, respectively, $\|\cdot\|_p$ to denote the norm of $L^p(\mathcal{O})$, and X_ρ to denote $C\left([-\rho, 0], L^2(\mathcal{O})\right)$ with norm $\|\phi\|_\rho = \sup\limits_{s\in[-\rho,0]} \|\phi(s)\|$ for $\phi \in C\left([-\rho, 0], L^2(\mathcal{O})\right)$. The letters c and c_i $(i = 1, 2, \dots)$ represent generic positive constants.

2. Preliminaries

In this section, we will recall the basic concept of uniform random attractors for nonautonomous stochastic dynamical systems from [5].

Let (X, d) be a Polish metric space and $\mathcal{B}(X)$ be the Borel σ-algebra of X. Then we study the nonautonomous random dynamical systems ϕ on X.

Let (Σ, d_Σ) be a compact Polish metric space which is invariant in the sense that

$$\theta_t\Sigma = \Sigma, \quad \forall t \in \mathbb{R},$$

where θ is a smooth translation operator such that θ_0 is the identity on Σ, $\theta_{s+t} = \theta_s \circ \theta_t$ for all $s, t \in \mathbb{R}$ and $\theta : \mathbb{R} \times \Sigma \to \Sigma$ is continuous. We denote by $(\Omega, \mathcal{F}, P)$ a probability space endowed also with a flow $\{\vartheta_t\}_{t\in\mathbb{R}}$ satisfying that $\vartheta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(X) \times \mathcal{F}, \mathcal{F})$ -measurable, ϑ_0 is the identity on Ω, $\vartheta_{s+t} = \vartheta_s \circ \vartheta_t$ for all $s, t \in \mathbb{R}$ and $\vartheta_t P = P$ for all $t \in \mathbb{R}$.

Definition 2.1. A mapping $\phi(t, \omega, g, x): \mathbb{R}^+ \times \Omega \times \Sigma \times X \to X$ is called a continuous nonautonomous random dynamical system(NRDS) on X with base flows $\{\theta_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$, if

1) ϕ is$(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\Sigma) \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

2) $\phi(0, \omega, g, \cdot)$ is the identity on X for each $g \in \Sigma$ and $\omega \in \Omega$;

3) it holds the cocycle property that

$$\phi(t + s, \omega, g, x) = \phi(t, \vartheta_s\omega, \theta_s g) \circ \phi(s, \omega, g, x), \quad \forall t, s \in \mathbb{R}^+, g \in \Sigma, x \in X, \omega \in \Omega.$$

4) $\phi(t, \omega, g, \cdot)$ is continuous for each $t \in \mathbb{R}^+, \omega \in \Omega$ and $g \in \Sigma$.

Definition 2.2. A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $\{\vartheta_t\}_{t\in\mathbb{R}}$ if for $P-$a.e.$\omega \in \Omega$

$$\lim_{t\to\infty} e^{-\beta t} d(B(\vartheta_{-t}\omega)) = 0,$$

where $d(B) = \sup_{x\in B} \|x\|_X$ and β is a positive constant.

Let $\mathcal{D}$ be a collection of random subsets of X satisfying $\mathcal{D}$ is neighborhood-closed, i.e. for each $D \in \mathcal{D}$ there exits an $\varepsilon > 0$ such that the closed ε-neighborhood $\mathcal{N}_\varepsilon(D)$ belongs to $\mathcal{D}$, and $\mathcal{D}$ is inclusion-closed, i.e., if $D \in \mathcal{D}$ then each random set smaller than D belongs to $\mathcal{D}$.

Definition 2.3. A random set D in X is said to be uniformly $\mathcal{D}$-(pullback) absorbing under the NRDS ϕ if for each $\omega \in \Omega$ and $B \in \mathcal{D}$ there exists a time $T = T(\omega, B) > 0$ such that

$$\phi(t, \vartheta_{-t}\omega, \Sigma, B(\vartheta_{-t}\omega)) \subset D(\omega), \quad \forall t \geq T.$$

Definition 2.4. A random set D in X is said to be uniformly $\mathcal{D}$-(pullback) attracting under the NRDS ϕ if for each $B \in \mathcal{D}$,

$$\lim_{t\to+\infty} dist\left(\phi\left(t, \vartheta_{-t}\omega, \Sigma, B\left(\vartheta_{-t}\omega\right)\right), D(\omega)\right) = 0, \quad \forall \omega \in \Omega.$$

Definition 2.5. A NRDS is said to be jointly continuous in Σ and X if for each $t \in \mathbb{R}^+$ and $\omega \in \Omega$, the mapping $\phi(t, \omega, \cdot, \cdot)$ is continuous.

Theorem 2.1 ([5]). *Suppose that ϕ is a jointly continuous NRDS in both Σ and X, and Ξ is any a dense subset of Σ. If ϕ has a compact uniformly $\mathcal{D}$-attracting set K and a closed uniformly $\mathcal{D}$-absorbing set $B \in \mathcal{D}$, then it has a unique uniform random attractor $\mathcal{A} \in \mathcal{D}$ given by*

$$\mathcal{A}(\omega) = \mathcal{W}(\omega, \Sigma, B) = \mathcal{W}(\omega, \Xi, B), \quad \forall \omega \in \Omega.$$

Moreover, the uniform attractor $\mathcal{A}$ is negatively semi-invariant

$$\mathcal{A}(\vartheta_t\omega) \subseteq \phi(t, \omega, \Sigma, \mathcal{A}(\omega)) \quad \textit{for each } t \geq 0,\ \omega \in \Omega.$$

3. Existence of a continuous NRDS

In this section, we show that the stochastic delay equations (1.1) generates a jointly continuous NRDS. Suppose that the space $L^2_{loc}(\mathbb{R}, H)$, where $H = L^2(\mathcal{O})$, consists of all function g which are 2-power integrable in Bochner sense, i.e.,

$$\int_{t_1}^{t_2} \|g(s)\|^2 ds < \infty \quad \text{for any } [t_1, t_2] \subset \mathbb{R}.$$

The space $L^2_{loc}(\mathbb{R}, H)$ is endowed with the two-power mean convergence topology on any bounded segment of $\mathbb{R}$, i.e., $g_n \to g$ in $L^2_{loc}(\mathbb{R}, H)$, namely

$$\int_{t_1}^{t_2} \|g_n(s) - g(s)\|^2 ds \to 0 \quad \text{for any bounded } [t_1, t_2] \subset \mathbb{R}. \tag{3.1}$$

Let $L^{2,w}_{loc}(\mathbb{R}, H)$ denote the space $L^2_{loc}(\mathbb{R}, H)$ endowed with the local weak convergence topology, i.e., $\sigma_n \in \sigma$ in $L^{2,w}_{loc}(\mathbb{R}, H)$ namely

$$\int_{t_1}^{t_2} \langle v(s), \sigma_n(s) - \sigma(s)\rangle\, ds \to 0 \quad \text{for any bounded } [t_1, t_2] \subset \mathbb{R} \text{ and } v \in L^2_{loc}(\mathbb{R}, H^*),$$

where H^* is the dual space of H. Now we introduce two useful lemmas.

Definition 3.1 ([5]). A function $g \in L^2_{loc}(\mathbb{R}, H)$ is called translation compact in $L^{2,w}_{loc}(\mathbb{R}, H)$ if its hull $\mathcal{H}(g) = \overline{\{\theta_t g(\cdot) : t \in \mathbb{R}\}}$ is compact in $L^{2,w}_{loc}(\mathbb{R}, H)$, where

$$\theta_t g(\cdot) = g(\cdot + t), \quad \forall t \in \mathbb{R},\ g \in L^2_{loc}(\mathbb{R}, H). \tag{3.2}$$

Lemma 3.1 ([5]). *Suppose $g \in L^2_{loc}(\mathbb{R}, H)$ be translation compact in $L^{2,w}_{loc}(\mathbb{R}, H)$, then*

1) the translation operator θ_t is continuous on $\mathcal{H}(g)$ in $L^{2,w}_{loc}(\mathbb{R}, H)$;

2) the hull of g is translation invariant $\mathcal{H}(g) = \theta_t \mathcal{H}(g)$, $\forall t \in \mathbb{R}$;

3) any function $\sigma \in \mathcal{H}(g)$ is translation compact in $L^{2,w}_{loc}(\mathbb{R}, H)$ and $\mathcal{H}(\sigma) \subseteq \mathcal{H}(g)$;

4) equivalently, g is translation bounded in $L^2_{loc}(\mathbb{R}, H)$, i.e.,

$$\eta(g) := \sup_{\tau \in \mathbb{R}} \int_{\tau-1}^{\tau} \|g(s)\|_H^2\, ds < \infty; \tag{3.3}$$

5) for any $\sigma \in \mathcal{H}(g)$, $\eta(\sigma) \le \eta(g)$.

Lemma 3.2 ([5]). *Let $g \in L^2_{loc}(\mathbb{R}, H)$ be translation compact in $L^{2,w}_{loc}(\mathbb{R}, H)$. Then*

$$\sup_{\sigma \in \mathcal{H}(g)} \int_{-\infty}^{0} e^{\lambda s} \|\sigma(s)\|_H^2\, ds \le \frac{\eta(g)}{1 - e^{-\lambda}}, \quad \forall \lambda > 0, \tag{3.4}$$

where $\eta(g)$ is the constant given by (3.3).

Let $\Sigma = \mathcal{H}(g_0)$, the hull of a given translation bounded function $g_0 \in L^2_{loc}(\mathbb{R}, H)$, endowed with the local weak convergence topology and a group of translation operator $\{\theta_t\}_{t\in\mathbb{R}}$ defined by (3.2) acting on Σ, which is Polish. The group $\{\theta_t\}_{t\in\mathbb{R}}$ is a base flow on Σ.

Consider the following equation:

$$du - \Delta u dt = F(x, u(t,x))dt + f(x, u(t-\rho, x))dt + g(t,x)dt + h(x)d\omega, \quad x \in \mathcal{O},\ t \geq 0, \tag{3.5}$$

endowed with the boundary condition

$$u(t,x) = 0, \quad x \in \partial\mathcal{O},\ t \geq 0, \tag{3.6}$$

and the initial condition

$$u_0(s,x) = u(s,x), \quad x \in \mathcal{O},\ s \in [-\rho, 0], \tag{3.7}$$

where $\rho \in (0,1]$, $g \in \Sigma$, $h \in L^{2p-2}(\mathcal{O}) \cap H_0^1(\mathcal{O}) \cap W^{2,p}(\mathcal{O})$ and ω is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, P)$. The nonlinear term F and f satisfy the follwing standard conditions:

(H_1) $F : \mathcal{O} \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous function, and for all $x \in \mathcal{O},\ s \in \mathbb{R}$,

$$F(x,s)s \leq -\alpha_1 |s|^p + \beta_1(x), \tag{3.8}$$

$$|F(x,s)| \leq \alpha_2 |s|^{p-1} + \beta_2(x), \tag{3.9}$$

$$\frac{\partial}{\partial s} F(x,s) \leq \alpha_3, \tag{3.10}$$

$$\left| \frac{\partial}{\partial x} F(x,s) \right| \leq \beta_3(x), \tag{3.11}$$

where $p \geq 2, \alpha_i (i = 1,2,3)$ is positive constant, $\beta_i (i = 1,2,3)$ is nonnegative function on $\mathcal{O}$ satisfying $\beta_1 \in L^{\frac{2p-2}{p}}(\mathcal{O})$ and $\beta_2, \beta_3 \in L^2(\mathcal{O})$.

(H_2) $f : \mathcal{O} \times \mathbb{R} \mapsto \mathbb{R}$ is continuous, and for all $x \in \mathcal{O},\ s_1, s_2 \in \mathbb{R}$,

$$|f(x, s_1) - f(x, s_2)| \leq C_f |s_1 - s_2|, \tag{3.12}$$

$$|f(x, s_1)|^2 \leq L_f^2 |s_1|^2 + |\eta_1(x)|^2, \tag{3.13}$$

where C_f and L_f are positive constants, and $\eta_1 \in L^2(\mathcal{O})$.

By using Poincare's inequality: there exists a positive constant λ_1 such that

$$\|\nabla u\|^2 \geq \lambda_1 \|u\|^2, \quad \forall u \in H_0^1(\mathcal{O}).$$

Suppose that $\lambda_1 > 4L_f$, choosing a positive constant m such that

$$m - \frac{\lambda_1}{4} + \frac{4L_f^2}{\lambda_1} e^{m\rho} < 0. \tag{3.14}$$

Let $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ be the Borel σ-algebra included by the compact-open topololgy of Ω, and $\{\vartheta_t\}_{t \in \mathbb{R}}$ be the measure-preserving transformations on $(\Omega, \mathcal{F}, P)$ as defined by

$$\vartheta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \forall \omega \in \Omega,\ t \in \mathbb{R}.$$

Considering the one-dimensional Ornstein-Uhlenbeck equation

$$dz(\vartheta_t \omega) + z(\vartheta_t \omega)dt = d\omega. \tag{3.15}$$

It is easy to verify that a sulution to (3.15) is given by

$$z(\omega) = -\int_{-\infty}^{0} e^{\tau}\omega(\tau)d\tau, \quad \forall\omega \in \Omega.$$

Meanwhile, there exists a $\{\vartheta_t\}_{t\in\mathbb{R}}$-invariant subset $\tilde{\Omega} \subseteq \Omega$ of full measure such that $z(\vartheta_t\omega)$ is continuous in t for every $\omega \in \Omega$ and the random variable $|z(\cdot)|$ is tempered. For convenience, $\tilde{\Omega}$ and Ω will not be distinguished.

Let $v(t) = u(t) - hz(\vartheta_t\omega)$, where u is a solution of system (3.5)-(3.7), then v satisfies

$$\frac{dv}{dt} - \Delta v = F(x, u(t,x)) + f(x, u(t-\rho, x)) + g(t,x) + z(\vartheta_t\omega)(\Delta h + h),\ x \in \mathcal{O}, t \geq 0, \tag{3.16}$$

with boundary condition

$$v(t,x) = 0, \quad x \in \partial\mathcal{O},\ t \geq 0, \tag{3.17}$$

and initial condition

$$v_0(s,x) = u(s,x) - hz(\vartheta_s\omega), \quad x \in \mathcal{O},\ s \in [-\rho, 0]. \tag{3.18}$$

By the Galerkin method, for $\omega \in \Omega$ and for all $v_0 \in X_\rho$, (3.16)-(3.18) has a unique solution v :

$$v(\cdot, \omega, g, v_0) \in C\left([-\rho, \infty], L^2(\mathcal{O})\right) \cap L^2_{loc}\left((0,\infty), H^1_0(\mathcal{O})\right) \cap L^p_{loc}\left((0,\infty), L^p(\mathcal{O})\right).$$

Moreover, v is continuous in v_0 and g, and $v_t(\cdot,\cdot,g,v_0)$ is $(\mathcal{F}, \mathcal{B}(X_\rho))$-measurable in ω.

For each $t \geq 0,\ \omega \in \Omega,\ g \in \Sigma,\ u_0 \in X_\rho$, let

$$\phi(t, \omega, g, u_0) = u_t(\cdot, \omega, g, u_0) = v_t(\cdot, \omega, g, v_0) + h(x)z(\vartheta_{t+\cdot}\omega),$$

where $u_t(s, \omega, g, u_0) = u(t+s, \omega, g, u_0),\ s \in [-\rho, 0]$. Then $\phi(t, \omega, g, u_0)$ is the solution of (3.5)-(3.7) at time t with initial data u_0. It is easy to check that ϕ satisfies conditions in Definition 2.1, and hence ϕ is a jointly continuous NRDS in X_ρ and Σ.

For studying the tempered uniform attractors, take the universe of tempered random sets in X_ρ as the attraction universe $\mathcal{D}$, i.e.,

$$\mathcal{D} = \{D : D \text{ is a bounded random set in } X_\rho \text{ satisfing } \lim_{t\to\infty} e^{-\frac{1}{2}mt}\|D(\vartheta_{-t}\omega)\|^2_\rho = 0\}.$$

The universe $\mathcal{D}$ is both inclusion-closed and neighborhood-closed.

4. Uniform estimates of solutions

In this section, we prove uniform estimates of solutions of (3.5)-(3.7).

Lemma 4.1. *Assume that assumptions* $(H_1), (H_2)$ *and* (3.14) *hold, then for every* $D \in \mathcal{D}, \omega \in \Omega$, *there exists* $T = T(D, \omega) > 0$ *such that for all* $t \geq T$, $g \in \Sigma$ *and* $v_0 \in D(\vartheta_{-t}\omega)$, *the solution* u *of* (3.5)-(3.7) *satisfies*

$$\begin{aligned} &\|u_t(\cdot, \vartheta_{-t}\omega, \theta_{-t}g, u_0)\|^2_\rho + \int_0^t e^{m(r-t)}\|\nabla u(r, \vartheta_{-t}\omega, \theta_{-t}g, u_0)\|^2 dr \\ &+ \int_0^t e^{m(r-t)}\|u(r, \vartheta_{-t}\omega, \theta_{-t}g, u_0)\|^p_{L^p} dr \leq R_0(\omega, \rho), \end{aligned} \tag{4.1}$$

where $R_0(\omega,\rho)$ is determined by

$$R_0(\omega,\rho) = M_0\frac{\eta(g_0)}{1-e^{-m}} + M_0\int_{-\infty}^{0} e^{mr}|z(\vartheta_r\omega)|^p dr + M_0\int_{-\infty}^{0} e^{mr}|z(\vartheta_{r-\rho}\omega)|^2 dr + M_0\sup_{-\rho\le s\le 0}|z(\vartheta_s\omega)|^2 + M_0, \tag{4.2}$$

where M_0 is a positive constant independent of ω, ρ, Σ and D.

Proof. Taking the inner product of (3.16) with v in $L^2(\mathcal{O})$, we find that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \|\nabla v\|^2 = \int_{\mathcal{O}} F(x,u(t,x))v dx + \int_{\mathcal{O}} f(x,u(t-\rho,x))v dx + \int_{\mathcal{O}} g(t,x)v dx + \int_{\mathcal{O}} z(\vartheta_t\omega)(\Delta h(x)+h(x))v dx. \tag{4.3}$$

By assumption (3.8)(3.9) and Young's inequality, we obtain

$$\begin{aligned}\int_{\mathcal{O}} F(x,u(t))v(t)dx &= \int_{\mathcal{O}} F(x,u(t))u(t)dx - z(\vartheta_t\omega)\int_{\mathcal{O}} F(x,u(t))h(x)dx \\ &\le -\frac{\alpha_1}{2}\|u(t)\|_{L^p}^p + \|\beta_1\|_{L^1} + \|\beta_2\|_{L^{\frac{p}{p-1}}}^{\frac{p}{p-1}} + c_1|z(\vartheta_t\omega)|^p.\end{aligned} \tag{4.4}$$

By (3.13) and Young's inequality, we have

$$\begin{aligned}\int_{\mathcal{O}} f(x,u(t-\rho,x))v dx &\le \frac{\lambda_1}{4}\|v(t)\|^2 + \frac{1}{\lambda_1}\int_{\mathcal{O}}\left(L_f^2|u(t-\rho)|^2 + |\eta_1(x)|^2\right)dx \\ &\le \frac{2L_f^2}{\lambda_1}\|v(t-\rho)\|^2 + \frac{\lambda_1}{4}\|v(t)\|^2 + c_2|z(\vartheta_{t-\rho}\omega)|^2 + c_2\|\eta_1\|^2.\end{aligned} \tag{4.5}$$

For the last two terms on the right-hand side of (4.3), we get

$$\begin{aligned}&\int_{\mathcal{O}} g(t,x)v dx + \int_{\mathcal{O}} z(\vartheta_t\omega)(\Delta h(x)+h(x))v dx \\ \le&\frac{\lambda_1}{16}\|v(t)\|^2 + \frac{8}{\lambda_1}\|g(t)\|^2 + \frac{1}{16}\|\nabla v\|^2 + 4\|\nabla h\|^2|z(\vartheta_t\omega)|^2 + \frac{8}{\lambda_1}\|h\|^2|z(\vartheta_t\omega)|^2.\end{aligned} \tag{4.6}$$

By (4.3)-(4.6) and Poincare's inequality, we have

$$\begin{aligned}\frac{d}{dt}\|v(t)\|^2 + \|\nabla v\|^2 + \alpha_1\|u(t)\|_{L^p}^p \le& -\frac{\lambda_1}{4}\|v(t)\|^2 + \frac{4L_f^2}{\lambda_1}\|v(t-\rho)\|^2 + c_3\|g(t)\|^2 \\ &+ c_3\left(|z(\vartheta_t\omega)|^p + |z(\vartheta_{t-\rho}\omega)|^2 + 1\right).\end{aligned} \tag{4.7}$$

Multiply (4.7) by e^{mt}, where m satisfies (3.14),

$$\begin{aligned}&\frac{d}{dt}(e^{mt}\|v(t)\|^2) + e^{mt}\|\nabla v\|^2 + \alpha_1 e^{mt}\|u(t)\|_{L^p}^p - e^{mt}\left(m-\frac{\lambda_1}{4}\right)\|v(t)\|^2 \\ \le& e^{mt}\left(\frac{4L_f^2}{\lambda_1}\|v(t-\rho)\|^2 + c_3\|g(t)\|^2 + c_3\left(|z(\vartheta_t\omega)|^p + |z(\vartheta_{t-\rho}\omega)|^2 + 1\right)\right).\end{aligned} \tag{4.8}$$

Replacing ω and g with $\vartheta_{-t}\omega$ and $\theta_{-t}g$, and then integrating over $(0, t+s)$ for any fixed $s \in [-\rho, 0]$ with $t > \rho$, we obtain

$$\begin{aligned}
&e^{m(t+s)}\|v(t+s,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 + \int_0^{t+s} e^{mr}\|\nabla v(r,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 dr \\
&+ \alpha_1 \int_0^{t+s} e^{mr}\|u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_{L^p}^p dr \\
\le&\|v_0\|^2 + \left(m - \frac{\lambda_1}{4}\right)\int_0^{t+s} e^{mr}\|v(r,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 dr \\
&+ \frac{4L_f^2}{\lambda_1}\int_0^{t+s} e^{mr}\|v(r-\rho,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 dr + c_3\int_0^{t+s} e^{mr}\|g(r-t)\|^2 dr \\
&+ c_3\int_0^{t+s} e^{mr}\left(|z(\vartheta_{r-t}\omega)|^p + |z(\vartheta_{r-\rho-t}\omega)|^2 + 1\right)dr.
\end{aligned} \tag{4.9}$$

We now estimate the third term on the right-hand side of (4.9)

$$\begin{aligned}
&\frac{4L_f^2}{\lambda_1}\int_0^{t+s} e^{mr}\|v(r-\rho,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 dr \\
\le&\frac{4L_f^2}{\lambda_1}\int_0^{t+s} e^{m(r+\rho)}\|v(r,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 dr + \frac{4L_f^2}{m\lambda_1}e^{m\rho}\|v_0\|_\rho^2.
\end{aligned} \tag{4.10}$$

Then, it follows from (3.14) and (4.9)-(4.10) that

$$\begin{aligned}
&\|v(t+s,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 + \int_0^{t+s} e^{m(r-t-s)}\|\nabla v(r,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 dr \\
&+ \alpha_1\int_0^{t+s} e^{m(r-t-s)}\|u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_{L^p}^p dr \\
\le&c_4\frac{4L_f^2 e^{m(\rho-t-s)}}{m\lambda_1}\|v_0\|_\rho^2 + c_4\int_0^{t+s} e^{m(r-t-s)}\|g(r-t)\|^2 dr \\
&+ c_4\int_0^{t+s} e^{m(r-t-s)}|z(\vartheta_{r-t}\omega)|^p dr + c_4\int_0^{t+s} e^{m(r-t-s)}\left(|z(\vartheta_{r-\rho-t}\omega)|^2 + 1\right)dr.
\end{aligned} \tag{4.11}$$

Since $s \in [-\rho, 0]$, we have for $t > \rho$,

$$\begin{aligned}
&\|v(t+s,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 + \int_0^{t+s} e^{m(r-t)}\|\nabla v(r,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 dr \\
&+ \alpha_1\int_0^{t+s} e^{m(r-t)}\|u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_{L^p}^p dr \\
\le&c_4\frac{4L_f^2 e^{m(2\rho-t)}}{m\lambda_1}\|v\|_\rho^2 + c_4 e^{m\rho}\int_0^{t} e^{m(r-t)}\|g(r-t)\|^2 dr \\
&+ c_4 e^{m\rho}\int_0^{t} e^{m(r-t)}|z(\vartheta_{r-t}\omega)|^p dr + c_4 e^{m\rho}\int_0^{t} e^{m(r-t)}\left(|z(\vartheta_{r-\rho-t}\omega)|^2 + 1\right)dr.
\end{aligned} \tag{4.12}$$

Note that

$$v(t+s,\vartheta_{-t}\omega,\theta_{-t}g,v_0) = u(t+s,\vartheta_{-t}\omega,\theta_{-t}g,u_0) - z(\vartheta_s\omega)h(x). \tag{4.13}$$

So, by (4.12)-(4.13), we obtain

$$\begin{aligned}
&\|u(t+s,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2+\int_0^{t+s}e^{m(r-t)}\|\nabla u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2dr\\
&+\int_0^{t+s}e^{m(r-t)}\|u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_{L^p}^p\,dr\\
\le&c_5e^{-mt}\left(\|u_0\|_\rho^2+\sup_{-\rho\le s\le 0}|z(\vartheta_s\omega)|^2\right)+c_5\int_0^te^{m(r-t)}\|g(r-t)\|^2dr\\
&+c_5\int_0^te^{m(r-t)}(|z(\vartheta_{r-t}\omega)|^p+|z(\vartheta_{r-\rho-t}\omega)|^2)dr+c_5|z(\vartheta_s\omega)|^2+c_5.
\end{aligned}\tag{4.14}$$

Since $u_0\in D(\vartheta_{-t}\omega),D\in\mathcal{D}$, the tempered condition of $\mathcal{D}$ and continuity of $z(\vartheta_t\omega)$, there exists a $T=T(\omega,D)>1$ such that

$$\begin{aligned}
&\|u(t+s,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2+\int_0^{t+s}e^{m(r-t)}\|\nabla u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2dr\\
&+\int_0^{t+s}e^{m(r-t)}\|u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_{L^p}^p\,dr\\
\le&1+c_5\int_{-t}^0e^{mr}\|g(r)\|^2dr+c_5\int_{-t}^0e^{mr}|z(\vartheta_r\omega)|^pdr\\
&+c_5\int_{-t}^0e^{mr}|z(\vartheta_{r-\rho}\omega)|^2dr+c_5\sup_{-\rho\le s\le 0}|z(\vartheta_s\omega)|^2+c_5,\quad t\ge T.
\end{aligned}\tag{4.15}$$

By (3.4), we get that

$$\begin{aligned}
&\|u(t+s,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2+\int_0^{t+s}e^{m(r-t)}\|\nabla u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2dr\\
&+\int_0^{t+s}e^{m(r-t)}\|u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_{L^p}^p\,dr\\
\le&c_6\frac{\eta(g_0)}{1-e^{-m}}+c_6\int_{-\infty}^0e^{mr}|z(\vartheta_r\omega)|^pdr+c_6\int_{-\infty}^0e^{mr}|z(\vartheta_{r-\rho}\omega)|^2dr\\
&+c_6\sup_{-\rho\le s\le 0}|z(\vartheta_s\omega)|^2+c_6,\quad t\ge T.
\end{aligned}\tag{4.16}$$

The proof of Lemma 4.1 is cmpleted. □

By Lemma 4.1, we next get uniform estimates for v in $H_0^1(\mathcal{O})$.

Lemma 4.2. *Assume that $(H_1),(H_2)$ and (3.14) hold. Then for every $D\in\mathcal{D},\omega\in\Omega$, there exists $T=T(D,\omega)>\rho+1$ such that for all $t\ge T$, $g\in\Sigma$ and $v_0\in D(\vartheta_{-t}\omega)$, the solution v of (3.16)-(3.18) satisfies*

$$\|\nabla v(t+s,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2+\int_{t-\rho}^t\|\Delta v(r,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2dr\le R_1(\omega,\rho),\tag{4.17}$$

where $R_1(\omega,\rho)$ given by

$$R_1(\omega,\rho)=M_1\frac{\eta(g_0)}{1-e^{-m}}+M_1\int_{-\infty}^0e^{mr}|z(\vartheta_r\omega)|^pdr+M_1\int_{-\infty}^0e^{mr}|z(\vartheta_{r-\rho}\omega)|^2dr$$

$$+ M_1 \sup_{-\rho\le s\le 0} |z(\vartheta_s\omega)|^2 + M_1, \tag{4.18}$$

where M_1 *is a positive constant independent of* ω, ρ, Σ *and* D.

Proof. Taking the inner product of (3.16) with $-\Delta v$ in $L^2(\mathcal{O})$, we derive that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2+\|\Delta v\|^2 = &-\int_{\mathcal{O}} F(x,u(t))\,\Delta v dx - \int_{\mathcal{O}} f(x,u(t-\rho))\,\Delta v dx\\ &- z(\vartheta_t\omega)\int_{\mathcal{O}}\Delta v(\Delta h(x)+h(x))dx - \int_{\mathcal{O}} g(t,x)\,\Delta v dx.\end{aligned} \tag{4.19}$$

Each term of (4.19) is now estimated. By (3.9)-(3.11) and Young's inequality, we have

$$\begin{aligned}&-\int_{\mathcal{O}} F(x,u(t))\,\Delta v dx\\ =&-\int_{\mathcal{O}} F(x,u(t))\,\Delta u dx + z(\vartheta_t\omega)\int_{\mathcal{O}} F(x,u(t))\,\Delta h(x)\,dx\\ \le&\ \|\beta_3(x)\|\,\|\nabla u\| + \alpha_3\|\nabla u\|^2 + |z(\vartheta_t\omega)|\int_{\mathcal{O}}\Big(\alpha_2|u|^{p-1}+\beta_2(x)\,|\Delta h|\Big)\,dx\\ \le&\ c_1\Big(\|\nabla u(t)\|^2+\|u(t)\|_{L^p}^p+|z(\vartheta_t\omega)|^p+1\Big).\end{aligned} \tag{4.20}$$

By (3.13) and Young's inequality, we obtain

$$\begin{aligned}-\int_{\mathcal{O}} f(x,u(t-\rho))\,\Delta v dx &\le \int_{\mathcal{O}}\left(|f(x,u(t-\rho))|^2+\frac{1}{4}|\Delta v|^2\right)dx\\ &\le L_f^2\|u(t-\rho)\|^2+\|\eta_1\|^2+\frac{1}{4}\|\Delta v(t)\|^2.\end{aligned} \tag{4.21}$$

By Young's inequality, we have

$$\begin{aligned}&- z(\vartheta_t\omega)\int_{\mathcal{O}}\Delta v(\Delta h(x)+h(x))dx - \int_{\mathcal{O}} g(t,x)\,\Delta v dx\\ \le&\int_{\mathcal{O}}(\frac{1}{8}|\Delta u|^2+2|z(\vartheta_t\omega)|^2|\Delta h(x)+h(x)|^2)dx+\int_{\mathcal{O}}\left(\frac{1}{8}|\Delta v|^2+2|g(t,x)|^2\right)dx\\ \le&\frac{1}{4}\|\Delta v(t)\|^2+c_2\left(\|g(t,x)\|^2+|z(\vartheta_t\omega)|^2\right).\end{aligned} \tag{4.22}$$

By (4.19)-(4.22) and Poincare's inequality, we get

$$\begin{aligned}&\frac{d}{dt}\|\nabla v\|^2+\|\Delta v\|^2\\ \le& c_3\left(\|\nabla u(t)\|^2+\|u(t-\rho)\|^2+\|u(t)\|_{L^p}^p+\|g(t)\|^2+|z(\vartheta_t\omega)|^p+1\right).\end{aligned} \tag{4.23}$$

Now integrating (4.23) from σ to $t+s$ with $s\in[-\rho,0]$, $\sigma\in(t+s-1,t+s)$ and

$t \geq \rho + 1$, we get

$$\begin{aligned}
&\|\nabla v(t+s,\omega,g,v_0)\|^2 \\
\leq&\|\nabla v(\sigma,\omega,g,v_0)\|^2 + c_3\int_{t-\rho-1}^{t}\|\nabla u(r,\omega,g,u_0)\|^2 dr \\
&+ c_3\int_{t-\rho-1}^{t}\|u(r-\rho,\omega,g,u_0)\|^2 dr + c_3\int_{t-\rho-1}^{t}\|u(r,\omega,g,u_0)\|_{L^p}^p dr \\
&+ c_3\int_{t-\rho-1}^{t}\|g(r)\|^2 dr + c_3\int_{t-\rho-1}^{t}|z(\vartheta_r\omega)|^p dr + c_3.
\end{aligned} \tag{4.24}$$

Replacing ω and g with $\vartheta_{-t}\omega$ and $\theta_{-t}g$, and then integrate over $(t+s-1, t+s)$ for any fixed $s \in [-\rho, 0]$ with $t > \rho + 1$, and by (4.13), we obtain

$$\begin{aligned}
&\|\nabla v(t+s,\vartheta_{-t}\omega,\theta_{-t}g,v_0)\|^2 \\
\leq& c_4\int_{t-\rho-1}^{t}\|\nabla u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2 dr + c_4\int_{t-\rho-1}^{t}\|u(r-\rho,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2 dr \\
&+ c_4\int_{t-\rho-1}^{t}\|g(r-t)\|^2 dr + c_4\int_{t-\rho-1}^{t}\|u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_{L^p}^p dr \\
&+ c_4\int_{t-\rho-1}^{t}|z(\vartheta_{r-t}\omega)|^p dr + c_4.
\end{aligned} \tag{4.25}$$

By (4.1), there exists $T = T(\omega, D) > \rho + 1$ such that for all $t \geq T$

$$\begin{aligned}
&e^{-m(\rho+1)}\int_{t-\rho-1}^{t}\left(\|\nabla u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2 + \|u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_{L^p}^p\right) dr \\
\leq& \int_{t-\rho-1}^{t} e^{m(r-t)}(\|\nabla u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2 + \|u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_{L^p}^p) dr \\
\leq& \int_{0}^{t} e^{m(r-t)}(\|\nabla u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2 + \|u(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_{L^p}^p) dr \leq R_0(\omega,\rho).
\end{aligned} \tag{4.26}$$

By (4.15), we find that

$$\begin{aligned}
&\int_{t-\rho-1}^{t}\|u(r-\rho,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|^2 dr \leq \int_{t-\rho-1}^{t}\|u_r(\cdot,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_\rho^2 dr \\
&\leq (\rho+1)\sup_{t-\rho-1\leq r\leq t}\|u_r(\cdot,\vartheta_{-t}\omega,\theta_{-t}g,u_0)\|_\rho^2 \\
&\leq c_5 e^{m(\rho+1)}\int_{-\infty}^{0} e^{mr}\|g(r)\|^2 dr + c_5 e^{m(\rho+1)}\int_{-\infty}^{0} e^{mr}|z(\vartheta_r\omega)|^p dr \\
&\quad + c_5 e^{m(\rho+1)}\int_{-\infty}^{0} e^{mr}|z(\vartheta_{r-\rho}\omega)|^2 dr + c_5\sup_{-\rho\leq s\leq 0}|z(\vartheta_s\omega)|^2 + c_5.
\end{aligned} \tag{4.27}$$

Moreover,

$$\begin{aligned}
&\int_{t-\rho-1}^{t}\|g(r-t)\|^2 dr + \int_{t-\rho-1}^{t}|z(\vartheta_{r-t}\omega)|^p dr \\
\leq& c_6 e^{m(\rho+1)}\int_{-\infty}^{0} e^{mr}\|g(r)\|^2 dr + c_6 e^{m(\rho+1)}\int_{-\infty}^{0} e^{mr}|z(\vartheta_r\omega)|^p dr.
\end{aligned} \tag{4.28}$$

Finally, by Lemma 4.1 and (4.25)-(4.28), we get that for all $s \in [-\rho, 0]$ and $t > T$,

$$\begin{aligned} &\left\|\nabla v\left(t+s, \vartheta_{-t}\omega, \theta_{-t}g, v_0\right)\right\|^2 \\ \leq & c_7 e^{m(\rho+1)} \frac{\eta\left(g_0\right)}{1-e^{-m}} + c_7 e^{m(\rho+1)} \int_{-\infty}^{0} e^{mr} \left|z\left(\vartheta_r\omega\right)\right|^p dr \\ &+ c_7 e^{m(\rho+1)} \int_{-\infty}^{0} e^{mr} \left|z\left(\vartheta_{r-\rho}\omega\right)\right|^2 dr + c_7 \sup_{-\rho \leq s \leq 0} \left|z\left(\vartheta_s\omega\right)\right|^2 + c_7. \end{aligned} \tag{4.29}$$

Now integrating (4.23) from $t-\rho$ to t, there exists $T = T(\omega, D) \geq \rho + 1$ such that for all $t \geq T$,

$$\begin{aligned} &\int_{t-\rho}^{t} \left\|\Delta v\left(r, \vartheta_{-t}\omega, \theta_{-t}g, v_0\right)\right\|^2 dr \\ \leq & \left\|\nabla v\left(t-\rho, \vartheta_{-t}\omega, \theta_{-t}g, v_0\right)\right\|^2 \\ &+ c_3 \int_{t-\rho}^{t} \left(\left\|\nabla u\left(r, \vartheta_{-t}\omega, \theta_{-t}g, u_0\right)\right\|^2 + \left\|u\left(r, \vartheta_{-t}\omega, \theta_{-t}g, u_0\right)\right\|_{L^p}^p\right) dr \\ &+ c_3 \int_{t-\rho}^{t} \left\|u\left(r-\rho, \vartheta_{-t}\omega, \theta_{-t}g, u_0\right)\right\|^2 dr + c_3 \int_{t-\rho}^{t} \left\|g\left(r-t\right)\right\|^2 dr \\ &+ c_3 \int_{t-\rho}^{t} \left|z\left(\vartheta_{r-t}\omega\right)\right|^p dr + c_3\rho. \end{aligned} \tag{4.30}$$

By Lemma 4.1 and (4.26)-(4.30), we obtain

$$\begin{aligned} &\left\|\nabla v\left(t+s, \vartheta_{-t}\omega, \theta_{-t}g, v_0\right)\right\|^2 + \int_{t-\rho}^{t} \left\|\Delta v\left(r, \vartheta_{-t}\omega, \theta_{-t}g, v_0\right)\right\|^2 dr \\ \leq & c_8 \frac{\eta\left(g_0\right)}{1-e^{-m}} + c_8 \int_{-\infty}^{0} e^{mr} \left|z\left(\vartheta_r\omega\right)\right|^p dr + c_8 \int_{-\infty}^{0} e^{mr} \left|z\left(\vartheta_{r-\rho}\omega\right)\right|^2 dr \\ &+ c_8 \sup_{-\rho \leq s \leq 0} \left|z\left(\vartheta_s\omega\right)\right|^2 + c_8, \end{aligned} \tag{4.31}$$

which together with (4.29) completes the proof. □

By using (4.13) and Lemma 4.2, we can derive uniform estimates for the solution u of (3.5)-(3.7) in $H_0^1(\mathcal{O})$.

Lemma 4.3. *Assume that $(H_1), (H_2)$ and (3.14) hold. Then for every $D \in \mathcal{D}, \omega \in \Omega$, there exists $T = T(D, \omega) > \rho + 1$ such that for all $t \geq T$, $g \in \Sigma$ and $u_0 \in D(\vartheta_{-t}\omega)$, the solution u of (3.5)-(3.7) satisfies*

$$\left\|\nabla u\left(t+s, \vartheta_{-t}\omega, \theta_{-t}g, u_0\right)\right\|^2 \leq R_2(\omega, \rho), \tag{4.32}$$

where

$$\begin{aligned} R_2(\omega, \rho) = & M_2 \frac{\eta\left(g_0\right)}{1-e^{-m}} + M_2 \int_{-\infty}^{0} e^{mr} \left|z\left(\vartheta_r\omega\right)\right|^p dr + M_2 \int_{-\infty}^{0} e^{mr} \left|z\left(\vartheta_{r-\rho}\omega\right)\right|^2 dr \\ &+ M_2 \sup_{-\rho \leq s \leq 0} \left|z\left(\vartheta_s\omega\right)\right|^2 + M_2, \end{aligned} \tag{4.33}$$

where M_2 is a positive constant independent of ω, ρ, Σ and D.

Lemma 4.4. *Assume that* (H_1), (H_2) *and* (3.14) *hold. Then for every* $D \in \mathcal{D}$, $\omega \in \Omega$, *there exists* $T = T(D,\omega) > \rho + 1$ *such that for all* $t \geq T$, $g \in \Sigma$ *and* $v_0 \in D(\vartheta_{-t}\omega)$, *the solution* v *of* (3.16)-(3.18) *satisfies*

$$\|\nabla v(t+s, \vartheta_{-t}\omega, \theta_{-t}g, v_0)\|_{L^p}^p + \int_{t-\rho}^{t} \|v(r, \vartheta_{-t}\omega, \theta_{-t}g, v_0)\|_{L^{2p-2}}^{2p-2} dr \leq R_3(\omega,\rho), \quad (4.34)$$

where $R_3(\omega,\rho)$ *is given by:*

$$\begin{aligned} R_3(\omega,\rho) = {} & M_3 \frac{\eta(g_0)}{1-e^{-m}} + M_3 \int_{-\infty}^{0} e^{mr} |z(\vartheta_r\omega)|^{2p-2} dr + M_3 \int_{-\infty}^{0} e^{mr} |z(\vartheta_{r-\rho}\omega)|^2 dr \\ & + M_3 \int_{-\infty}^{0} e^{mr} |z(\vartheta_r\omega)|^p dr + M_3 \sup_{-\rho \leq s \leq 0} |z(\vartheta_s\omega)|^2 + M_3, \end{aligned} \quad (4.35)$$

where M_3 *is a positive constant independent of* ω, ρ, Σ *and* D.

Proof. Taking the inner product of (3.16) with $|v|^{p-2}v$ in $L^2(\mathcal{O})$, we get that

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_{\mathcal{O}} |v|^p dx = {} & \int_{\mathcal{O}} |v|^{p-2} v \Delta v dx + \int_{\mathcal{O}} |v|^{p-2} v F(x, u(t,x)) dx \\ & + \int_{\mathcal{O}} |v|^{p-2} v f(x, u(t-\rho, x)) dx + \int_{\mathcal{O}} |v|^{p-2} v g(t,x) dx \\ & + z(\vartheta_t\omega) \int_{\mathcal{O}} |v|^{p-2} v (\Delta h(x) + h(x)) dx. \end{aligned} \quad (4.36)$$

Integration by parts of the first term on the right-hand side of (4.36), we get that

$$\int_{\mathcal{O}} |v|^{p-2} v \Delta v dx = -(p-1) \int_{\mathcal{O}} |\nabla v|^2 |v|^{p-2} dx \leq 0. \quad (4.37)$$

By using Young's inequality, for $p \geq 2$ we obtain

$$-|u(t)|^p \leq -2^{1-p} \left(|v(t)|^p + |z(\vartheta_t\omega) h(x)|^p\right). \quad (4.38)$$

It follows from (3.8), (3.9) and (4.38) that,

$$\begin{aligned} & \int_{\mathcal{O}} |v|^{p-2} v F(x, u(t,x)) dx \\ = {} & \int_{\mathcal{O}} |v|^{p-2} u F(x, u(t,x)) dx - \int_{\mathcal{O}} |v|^{p-2} z(\vartheta_t\omega) h(x) F(x, u(t,x)) dx \\ \leq {} & -\frac{2k}{p} \|v(t)\|_{L^{2p-2}}^{2p-2} + c_1 \left(|z(\vartheta_t\omega)|^{2p-2} + 1 \right). \end{aligned} \quad (4.39)$$

where $k = \frac{\alpha_1 p}{2^{p+2}}$. By (3.12) and Young's inequality, the third term on the right-hand side of (4.36) are bounded by

$$\begin{aligned} \int_{\mathcal{O}} |v|^{p-2} v f(x, u(t-\rho, x)) dx & \leq \frac{k}{2p} \int_{\mathcal{O}} |v|^{2p-2} dx + \frac{p}{2k} \int_{\mathcal{O}} |f(x, u(t-\rho, x))|^2 dx \\ & \leq \frac{k}{2p} \|v(t)\|_{L^{2p-2}}^{2p-2} + c_2 \|u(t-\rho, x)\|^2 + c_2. \end{aligned} \quad (4.40)$$

For the last two terms on the right-hand side of (4.36), we obtain

$$\begin{aligned}&\int_{\mathcal{O}}|v|^{p-2}vg\left(t,x\right)dx+z\left(\vartheta_t\omega\right)\int_{\mathcal{O}}|v|^{p-2}v(\Delta h\left(x\right)+h(x))dx\\ \le&\frac{k}{2p}\left\|v\left(t\right)\right\|_{L^{2p-2}}^{2p-2}+c_3\|g\left(t\right)\|^2+c_3|z\left(\vartheta_t\omega\right)|^2.\end{aligned}\tag{4.41}$$

By (4.37) and (4.39)-(4.41), we obtain that

$$\frac{d}{dt}\left\|v\left(t\right)\right\|_{L^p}^p+k\left\|v\left(t\right)\right\|_{L^{2p-2}}^{2p-2}\le c_4\left(\|u\left(t-\rho\right)\|^2+\|g\left(t\right)\|^2+|z\left(\vartheta_t\omega\right)|^{2p-2}+1\right).\tag{4.42}$$

Integrating (4.42) over $(\sigma,t+s)\,,s\in[-\rho,0]$, where $\sigma\in(t+s-1,t+s)\,,t\ge\rho+1$, then we have

$$\begin{aligned}&\left\|v\left(t+s,\omega,g,v_0\right)\right\|_{L^p}^p-\left\|v\left(\sigma,\omega,g,v_0\right)\right\|_{L^p}^p\\ \le&c_4\int_{t-\rho-1}^t\|u\left(r-\rho,\omega,g,u_0\right)\|^2dr+c_4\int_{t-\rho-1}^t\|g\left(r\right)\|^2dr\\ &+c_4\int_{t-\rho-1}^t|z\left(\vartheta_r\omega\right)|^{2p-2}dr+c_4\left(\rho+1\right).\end{aligned}\tag{4.43}$$

Replacing ω and g with $\vartheta_{-t}\omega$ and $\theta_{-t}g$, and then integrating with respect to σ over $(t+s-1,t+s)$ for any fixed $s\in[-\rho,0]$,and by (4.13), we obtain

$$\begin{aligned}&\left\|v\left(t+s,\vartheta_{-t}\omega,\theta_{-t}g,v_0\right)\right\|_{L^p}^p\\ \le&\int_{t-\rho-1}^t\left\|u\left(r,\vartheta_{-t}\omega,\theta_{-t}g,u_0\right)\right\|_{L^p}^p dr+c_5\int_{t-\rho-1}^t\left\|u_r\left(\cdot,\vartheta_{-t}\omega,\theta_{-t}g,u_0\right)\right\|_{\rho}^2dr\\ &+c_5\int_{t-\rho-1}^t\|g\left(r-t\right)\|^2dr+c_5\int_{t-\rho-1}^t|z\left(\vartheta_{r-t}\omega\right)|^{2p-2}dr\\ &+c_5\int_{t-\rho-1}^t|z\left(\vartheta_{r-t}\omega\right)|^pdr+c_5\left(\rho+1\right).\end{aligned}\tag{4.44}$$

By (4.26)-(4.28) and (4.44), there exists $T=T\left(\omega,D\right)>\rho+1$, we have for all $t\ge T$,

$$\begin{aligned}&\left\|v\left(t+s,\vartheta_{-t}\omega,\theta_{-t}g,v_0\right)\right\|_{L^p}^p\\ \le&c_6\frac{\eta\left(g_0\right)}{1-e^{-\lambda}}+c_6\int_{-\infty}^0e^{mr}|z\left(\vartheta_r\omega\right)|^{2p-2}dr+c_6\int_{-\infty}^0e^{mr}|z\left(\vartheta_{r-\rho}\omega\right)|^2dr\\ &+c_6\int_{-\infty}^0e^{mr}|z(\vartheta_r\omega)|^pdr+c_6\sup_{-\rho\le s\le0}|z\left(\vartheta_s\omega\right)|^2+c_6.\end{aligned}\tag{4.45}$$

Finally, integrating (4.42) over $(t-\rho,t)$ and replacing ω and g with $\vartheta_{-t}\omega$ and $\theta_{-t}g$, there exists $T=T\left(\omega,D\right)>\rho+1$, we find for all $t\ge T$,

$$\begin{aligned}&k\int_{t-\rho}^t\left\|v\left(r,\vartheta_{-t}\omega,\theta_{-t}g,v_0\right)\right\|_{L^{2p-2}}^{2p-2}\\ \le&\left\|v\left(t-\rho,\vartheta_{-t}\omega,\theta_{-t}g,v_0\right)\right\|_{L^p}^p+c_4\int_{t-\rho}^t\|u\left(r-\rho,\vartheta_{-t}\omega,\theta_{-t}g,u_0\right)\|^2dr\\ &+c_4\int_{t-\rho}^t\|g\left(r-t\right)\|^2dr+c_4\int_{t-\rho}^t|z\left(\vartheta_{r-t}\omega\right)|^{2p-2}d+c_4\rho.\end{aligned}\tag{4.46}$$

By (4.26)-(4.28) and (4.45)-(4.46), there exists $T = T(\omega, D) > \rho + 1$, we get for all $t \geq T$,

$$\begin{aligned} & k \int_{t-\rho}^{t} \|v(r, \vartheta_{-t}\omega, \theta_{-t}g, v_0)\|_{L^{2p-2}}^{2p-2} \\ \leq & c_7 \frac{\eta(g_0)}{1-e^{-m}} + c_7 \int_{-\infty}^{0} e^{mr} |z(\vartheta_r\omega)|^{2p-2} dr + c_7 \int_{-\infty}^{0} e^{mr} |z(\vartheta_{r-\rho}\omega)|^2 dr \\ & + c_7 \int_{-\infty}^{0} e^{mr} |z(\vartheta_r\omega)|^p dr + c_7 \sup_{-\rho \leq s \leq 0} |z(\vartheta_s\omega)|^2 + c_7. \end{aligned} \tag{4.47}$$

(4.45) and (4.47) lead to conclusion (4.34), which completes the proof. □

5. Uniform random attractors

In this section, we prove the existence of tempered uniform random attractors for (3.5)-(3.7) in X_ρ.

Lemma 5.1. *Assume that* $(H_1), (H_2)$ *and* (3.14) *hold. Then the nonautonomous random dynamical system* ϕ *has a closed measurable uniformly* $\mathcal{D}$*-absorbing set* $K = \{K(\omega) : \omega \in \Omega\} \in \mathcal{D}$*, that is, for any* $\omega \in \Omega$*,* $D = \{D(\omega) : \omega \in \Omega\} \in \mathcal{D}$*, there is* $T = T(D, \omega) > 0$ *such that for all* $t \geq T$*,* $g \in \Sigma$ *and* $u_0 \in D(\vartheta_{-t}\omega)$*,*

$$\phi(t, \vartheta_{-t}\omega, \theta_{-t}g, u_0) \subseteq K(\omega),$$

where $K(\omega)$ *is given by:*

$$K(\omega) = \left\{ u \in \mathcal{D} : \|u_t\|_\rho^2 \leq R_0(\omega, \rho) \right\}. \tag{5.1}$$

Note that

$$\begin{aligned} R_0(\omega, \rho) = & M_0 \frac{\eta(g_0)}{1-e^{-m}} + M_0 \int_{-\infty}^{0} e^{mr} |z(\vartheta_r\omega)|^p dr + M_0 \int_{-\infty}^{0} e^{mr} |z(\vartheta_{r-\rho}\omega)|^2 dr \\ & + M_0 \sup_{-\rho \leq s \leq 0} |z(\vartheta_s\omega)|^2 + M_0. \end{aligned} \tag{5.2}$$

Proof. Given $D \in \mathcal{D}$, $\omega \in \Omega$, by Lemma 4.1, for all $t \geq T$, $g \in \Sigma$ and $u_0 \in D(\vartheta_{-t}\omega)$, we find that

$$\phi(t, \vartheta_{-t}\omega, \theta_{-t}g, D(\vartheta_{-t}\omega)) \subseteq K(\omega).$$

Therefore K uniform attract all elements in $\mathcal{D}$, and then we check that K is tempered. For all $\gamma > 0$, we have

$$\begin{aligned} & e^{-\gamma t} \|K(\vartheta_{-t}\omega)\|_\rho^2 \leq e^{-\gamma t} R_0(\vartheta_{-t}\omega, \rho) \\ = & M_0 e^{-\gamma t} \frac{\eta(g_0)}{1-e^{-m}} + M_0 e^{-\gamma t} \int_{-\infty}^{0} e^{mr} |z(\vartheta_r\omega)|^p dr + M_0 e^{-\gamma t} \int_{-\infty}^{0} e^{mr} |z(\vartheta_{r-\rho}\omega)|^2 dr \\ & + M_0 e^{-\gamma t} \sup_{-\rho \leq s \leq 0} |z(\vartheta_s\omega)|^2 + M_0 e^{-\gamma t}. \end{aligned} \tag{5.3}$$

By (3.3)-(3.4) and the tempered of $z(\vartheta_t\omega)$

$$\lim_{t \to \infty} e^{-\gamma t} \|K(\vartheta_{-t}\omega)\|_\rho^2 = 0. \tag{5.4}$$

Then $K \in \mathcal{D}$, which completes the proof. $\square$

The the Arzela-Ascoli theorem is then used to prove the uniformly compactness of the NRDS ϕ.

Lemma 5.2. *Assume that $(H_1),(H_2)$ and (3.14) hold. Then the nonautonomous random dynamical system ϕ has a uniformly $\mathcal{D}$-pullback compactness absorbing set in X_ρ.*

Proof. Set for $\omega \in \Omega$, for all $\eta > 0$, there exists $\delta = \delta(\eta,\omega) > 0$, such that

$$Y_1(\omega) = \left\{ y \in H_0^1(\mathcal{O}) : \|y\|_{H_0^1(\mathcal{O})} \leq R(\omega) \right\}, \tag{5.5}$$

$$Y_2(\omega) = \left\{ y \in X_\rho : \sup_{-\rho \leq s_1 < s_2 \leq 0, s_2 - s_1 < \delta} \|y(s_2) - y(s_1)\| \leq \eta \right\}, \tag{5.6}$$

and

$$Y(\omega) = Y_1(\omega) \cap Y_2(\omega), \tag{5.7}$$

where for each $\omega \in \Omega$, $R(\omega)$ is a sufficiently large constant and $\delta(\omega)$ is a sufficiently small constant. By Arzela-Ascoli theorem and the compactness of embedding $H_0^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$, Y is compact in X_ρ.

Firstly, take the inner product of (3.16) with v in $L^2(\mathcal{O})$, and by (3.8), (3.13) and Lemmas 4.1–4.4, for every $D \in \mathcal{D}, \omega \in \Omega$, there exists $T_1 = T_1(D,\omega) > 0$ such that for all $t \geq T_1$, $g \in \Sigma$ and $u_0 \in D(\vartheta_{-t}\omega)$,

$$\int_{t-\rho}^{t} \left\| \frac{d}{dr} v(r, \vartheta_{-t}\omega, \theta_{-t}g, v_0) \right\|^2 dr \leq c, \tag{5.8}$$

where $c = c(\omega)$ is positive number. Then for any $t \geq T_1$ and $s_1, s_2 \in [-\rho, 0]$, we have

$$\begin{aligned}
&\|v_t(s_2, \vartheta_{-t}\omega, \theta_{-t}g, v_0) - v_t(s_1, \vartheta_{-t}\omega, \theta_{-t}g, v_0)\| \\
=&\int_{t+s_1}^{t+s_2} \frac{d}{dr} v(r, \vartheta_{-t}\omega, \theta_{-t}g, v_0)\, dr \\
\leq&|s_2 - s_1|^{\frac{1}{2}} \left(\int_{t+s_1}^{t+s_2} \left\| \frac{d}{dr} v(r, \vartheta_{-t}\omega, \theta_{-t}g, v_0) \right\|^2 dr \right)^{\frac{1}{2}} \\
\leq&|s_2 - s_1|^{\frac{1}{2}} \left(\int_{t-\rho}^{t} \left\| \frac{d}{dr} v(r, \vartheta_{-t}\omega, \theta_{-t}g, v_0) \right\|^2 dr \right)^{\frac{1}{2}} \leq c_1 |s_2 - s_1|^{\frac{1}{2}}.
\end{aligned} \tag{5.9}$$

By using (4.13) and (5.9), we have that for all $t \geq T_1$ and $s_1, s_2 \in [-\rho, 0]$,

$$\|u_t(s_2, \vartheta_{-t}\omega, \theta_{-t_n}g, u_0) - u_t(s_1, \vartheta_{-t}\omega, \theta_{-t}g, u_0)\| \leq c_1 |s_2 - s_1|^{\frac{1}{2}} + c_2 \left| z(\vartheta_{s_2 - s_1}\omega) \right|. \tag{5.10}$$

Therefore, for all $t \geq T_1$, $u_t(\cdot, \vartheta_{-t}\omega, \Sigma, D) \in Y_2$. By Lemma 4.3, for every $D \in \mathcal{D}$, $\omega \in \Omega$, there exists $T_2 = T_2(D,\omega) > T_1$ such that for all $t \geq T_2$, we have that $u(t, \vartheta_{-t}\omega, \Sigma, D) \in Y_1$. We get that for all $t \geq T_2$, $u_t(\cdot, \vartheta_{-t}\omega, \Sigma, D) \in Y$. This completes the proof. $\square$

Now we prove the existence of $\mathcal{D}$-uniform random attractors.

Theorem 5.1. *Assume that* (H_1), (H_2) *and* (3.14) *hold. Then the nonautonomous random dynamical system* ϕ *has a unique* $\mathcal{D}$*-uniform random attractors* $\mathcal{A} \in \mathcal{D}$ *in* X_ρ.

Proof. By Lemmas 5.1, 5.2 and Theorem 2.1, we are able to prove that the NRDS ϕ has a unique $\mathcal{D}$-uniform random attractors $\mathcal{A} = \{\mathcal{A}(\omega) : \omega \in \Omega\} \in \mathcal{D}$. □

6. Upper semicontinuity of attractors as delay approaches zero

In order to study the upper semicontinuity of attractors system (3.5)-(3.7) as delay approaches to zero, we assume that the g_0 is an almost periodic function in $t \in \mathbb{R}$ with values in H. Since an almost periodic function is bounded and uniformly continuous on $\mathbb{R}$, it follows that $g_0 \in C_b(\mathbb{R}, H)$, where $C_b(\mathbb{R}, H)$ is the space of bounded continuous functions on $\mathbb{R}$ with values in H. Given $g \in C_b(\mathbb{R}, H)$, denote the norm of g by $\|\mathrm{g}\|_{C_b(\mathbb{R},H)} = \sup\limits_{t\in\mathbb{R}} \|g(t)\|$. Obviously, $g \in L^2_{loc}(\mathbb{R}, H)$ and is translation compact in $L^{2,w}_{loc}(\mathbb{R}, H)$. Note that all results in the previous sections are valid for the case that g_0 is an almost periodic function.

The following is a basis for judging the upper semicontinuity of attractors when delay of stochastic delay equation approaches to zero.

Theorem 6.1. *Let* X *be a Banach space. Suppose that for every* $\rho \geq 0$, *let* $\mathcal{D}_0$ *and* $\mathcal{D}_\rho$ *be collections of families of some subsets of* X *and* $C([-\rho, 0], X)$, ϕ_0 *and* ϕ_ρ *are continuous nonautonomous random dynamical systems on* X *and* $C([-\rho, 0], X)$. *If*

(i) for every $t \in \mathbb{R}^+$, $\omega \in \Omega$,

$$\lim_{n\to\infty} \sup_{-\rho_n \leq s \leq 0} \|\phi_{\rho_n}(t, \omega, g_n, u_n)(s) - \phi_0(t, \omega, g, x)\|_X = 0, \tag{6.1}$$

for any $\rho_n \to 0$, $g_n, g \in \Sigma$ *with* $g_n \to g$ *in* $(\Sigma, \|\cdot\|_{C_b(\mathbb{R},X)})$, $u_n \in C([-\rho_n, 0], X)$ *and* $x \in X$ *with* $\sup\limits_{-\rho_n \leq s \leq 0} \|u_n(s) - x\| \to 0$;

(ii) ϕ_ρ *has a uniformly* $\mathcal{D}_\rho$*-absorbing set* B_ρ *and a* $\mathcal{D}_\rho$*-uniform random attractors* $\mathcal{A}_\rho \subseteq B_\rho$, ϕ_0 *has a uniformly* $\mathcal{D}_0$*-absorbing set* B_0 *and a* $\mathcal{D}_0$*-uniform random attractors* $\mathcal{A}_0 \subseteq B_0$, *where*

$$B_\rho = \{B_\rho(\omega) = \{u \in C([-\rho, 0], X) : \|u\|_{C([-\rho,0],X)} \leq R_\rho(\omega)\}, \omega \in \Omega\} \in \mathcal{D}_\rho,$$
$$B_0 = \{B_0(\omega) = \{x \in X : \|x\|_X \leq R_0(\omega)\}, \omega \in \Omega\} \in \mathcal{D}_0,$$

and $R_\rho(\omega) : \Omega \to \mathbb{R}^+ (\rho \geq 0)$ *such that for all* $\omega \in \Omega$,

$$\limsup_{\rho\to 0} \sup_{u \in B_\rho(\omega)} \|u\|_{C([-\rho,0],X)} \leq R_0(\omega); \tag{6.2}$$

(iii) for every $\omega \in \Omega$, *if* $\rho_n \to 0$ *and* $u_n \in \mathcal{A}_{\rho_n}(\omega)$, *there exist* $x \in X$ *and a subsequence* $\{u_{n_m}\}_{m=1}^\infty$ *such that*

$$\lim_{m\to\infty} \sup_{-\rho_{n_m} \leq s \leq 0} \|u_{n_m}(s) - x\|_X = 0; \tag{6.3}$$

Then for every $\omega \in \Omega$,

$$d_H(\mathcal{A}_\rho(\omega), \mathcal{A}_0(\omega)) \to 0 \quad as\ \rho \to 0, \tag{6.4}$$

where the distance d_H is defined for any subset E of $C([-\rho,0],X)$ and S of X by

$$d_H(E,S) = \sup_{u\in E}\inf_{x\in S}\sup_{-\rho\le s\le 0}\|u(s)-x\|_X. \tag{6.5}$$

Proof. If (6.4) is not true, then there exist $\eta>0$ and $\rho_n\to 0$ such that for all $n\in\mathbb{N}$,

$$d_H(\mathcal{A}_{\rho_n}(\omega),\mathcal{A}_0(\omega))\ge 2\eta,$$

which implies that there exist $u_n\in\mathcal{A}_{\rho_n}(\omega)$ such that for every fixed n

$$d_H(u_n,\mathcal{A}_0(\omega))\ge\eta. \tag{6.6}$$

By (6.3), there exist $x_0\in X$ and a subsequence (which is not relabeled) such that

$$\lim_{m\to\infty}\sup_{-\rho_n\le s\le 0}\|u_n(s)-x_0\|_X=0. \tag{6.7}$$

Take a sequence $\{t_m\}_{m=1}^{\infty}$ with $t_m\to\infty$. By the negatively semi-invariance of $\mathcal{A}_{\rho_n}$, we derive that for each $n\in\mathbb{N}$, there exists $u_{1,n}\in\mathcal{A}_{\rho_n}(\vartheta_{-t_1}\omega)$ and $g_{1,n}\in\Sigma$ such that

$$u_n=\phi_{\rho_n}(t_1,\vartheta_{-t_1}\omega,\theta_{-t_1}g_{1,n},u_{1,n}). \tag{6.8}$$

Since $u_{1,n}\in\mathcal{A}_{\rho_n}(\vartheta_{-t_1}\omega)$, by (6.3) we get that there exists $x_1\in X$ such that

$$\lim_{n\to\infty}\sup_{\rho_n\le s\le 0}\|u_{1,n}(s)-x_1\|_X=0. \tag{6.9}$$

Meanwhile, since $\{g_{1,n}\}\subset\Sigma$ and Σ is compact, there exists $g_1\in\Sigma$ and a subsequence of $\{g_{1,n}\}$ (which is not relabeled) such that

$$g_{1,n}\to g_1 \quad\text{as}\quad n\to\infty. \tag{6.10}$$

By (6.1) and (6.9)-(6.10) we derive that

$$\lim_{n\to\infty} d_H(\phi_{\rho_n}(t_1,\vartheta_{-t_1}\omega,\theta_{-t_1}g_{1,n},u_{1,n}),\phi_0(t_1,\vartheta_{-t_1}\omega,\theta_{-t_1}g_1,x_1))=0. \tag{6.11}$$

By (6.7)-(6.8) and (6.11) we obtain

$$x_0=\phi_0(t_1,\vartheta_{-t_1}\omega,\theta_{-t_1}g_1,x_1). \tag{6.12}$$

Since $\mathcal{A}_{\rho_n}(\vartheta_{-t_1}\omega)\subseteq B_{\rho_n}(\vartheta_{-t_1}\omega)$ and $u_{1,n}\in\mathcal{A}_{\rho_n}(\vartheta_{-t_1}\omega)$, by (6.2) we derive that

$$\limsup_{n\to\infty}\|u_{1,n}\|_{C([-\rho_n,0],X)}\le R_0(\vartheta_{-t_1}\omega). \tag{6.13}$$

By (6.9) and (6.13) we get

$$\|x_1\|_X\le R_0(\vartheta_{-t_1}\omega).$$

Repeating this process for every $m\ge 1$, we infer that there exist $x_m\in X$ and $g_m\in\Sigma$ such that for all $m\ge 1$,

$$x_0=\phi_0(t_m,\vartheta_{-t_m}\omega,\theta_{-t_m}g_m,x_m), \tag{6.14}$$

and

$$\|x_m\|_X\le R_0(\vartheta_{-t_m}\omega). \tag{6.15}$$

Since $\mathcal{A}_0$ is a $\mathcal{D}_0$-uniform random attractors of ϕ_0, by (6.14)-(6.15) we get

$$d_X(x_0,\mathcal{A}_0(\omega)) \leq d_X(\phi_0(t_m,\vartheta_{-t_m}\omega,\theta_{-t_m}g_m,B_0(\vartheta_{-t_m}\omega)),\mathcal{A}_0(\omega)) \to 0 \quad \text{as } m \to 0. \tag{6.16}$$

Therefore we have $x_0 \in \mathcal{A}_0(\omega)$. By (6.7) we derive that

$$d_H(u_n,\mathcal{A}_0(\omega)) \leq d_H(u_n,x_0) \to 0 \quad \text{as } n \to \infty.$$

So it contradicts (6.6). The proof of Theorem 6.1 is completed. □

For $\rho \in (0,1]$, we write the solution and NRDS of system of (3.5)-(3.7) as u^ρ and ϕ^ρ, and let $\mathcal{A}^\rho = \{\mathcal{A}^\rho(\omega) : \omega \in \Omega\}$ be the uniform random attractors of ϕ^ρ in X_ρ. By Lemma 5.1, we find that the uniformly $\mathcal{D}$-absorbing set B^ρ of ϕ^ρ satisfies that for all $\omega \in \Omega$,

$$B^\rho(\omega) = \left\{ u \in \mathcal{D} : \|u_t\|_\rho^2 \leq R_0(\omega,\rho) \right\}, \tag{6.17}$$

where $R_0(\omega,\rho)$ is given by

$$\begin{aligned} R_0(\omega,\rho) =& M_0\frac{\eta(g_0)}{1-e^{-m}} + M_0\int_{-\infty}^0 e^{mr}|z(\vartheta_r\omega)|^p dr + M_0\int_{-\infty}^0 e^{mr}|z(\vartheta_{r-\rho}\omega)|^2 dr \\ &+ M_0\sup_{-\rho\leq s\leq 0}|z(\vartheta_s\omega)|^2 + M_0. \end{aligned}$$

Then for every $\omega \in \Omega$, we get that

$$\mathcal{A}^\rho(\omega) \subseteq B^\rho(\omega). \tag{6.18}$$

For $\rho = 0$, from (3.5)-(3.7) we obtain

$$du - \Delta u dt = F(x,u(t,x))\,dt + f(x,u(t,x))\,dt + g(t,x)\,dt + h(x)\,d\omega, \quad x \in \mathcal{O},\ t \geq 0, \tag{6.19}$$

with boundary condition

$$u(t,x) = 0, \quad x \in \partial\mathcal{O},\ t \geq 0, \tag{6.20}$$

and initial data

$$u_0(x) = u(0,x), \quad x \in \mathcal{O}. \tag{6.21}$$

The NRDS generated by (6.19)-(6.21) is denoted by ϕ^0, and the collection of all tempered families of nonempty subsets of $L^2(\mathcal{O})$ is denoted by $\mathcal{D}^0$:

$$\mathcal{D}^0 = \{\{D(\omega) \subseteq L^2(\mathcal{O}) : \omega \in \Omega\} : \lim_{t\to-\infty} e^{ct}\,\|D(t,\theta_t\omega)\| = 0, \quad \forall c > 0\}.$$

By section 4, we obtain that ϕ^0 has a $\mathcal{D}^0$-random uniform attractors $\mathcal{A}^0 = \{\mathcal{A}^0(\omega) : \omega \in \Omega\}$ in $L^2(\mathcal{O})$ and a uniformly $\mathcal{D}^0$-absorbing set $B^0 = \{B^0(\omega) : \omega \in \Omega\}$ given by

$$B^0(\omega) = \{u \in L^2(\mathcal{O}) : \|u\|^2 \leq R^0(\omega)\}, \tag{6.22}$$

where $R^0(\omega)$ is given by

$$R^0(\omega) = M_0\frac{\eta(g_0)}{1-e^{-m}} + M_0\int_{-\infty}^0 e^{mr}|z(\vartheta_r\omega)|^p dr + M_0\int_{-\infty}^0 e^{mr}|z(\vartheta_r\omega)|^2 dr + M_0.$$

By (6.17) and (6.22), we get

$$\limsup_{\rho\to 0} \|B^\rho(\omega)\|_\rho = \left\|B^0(\omega)\right\|. \tag{6.23}$$

Now we study the convergence of solution of (6.19)-(6.21) as $\rho \to 0$, for which we need the following assumption:

(H_3) There exist $\alpha_4 > 0$ and $\beta_4 \in L^{p^*}(\mathcal{O})$ such that for all $x \in \mathcal{O}$ and $s \in \mathbb{R}$,

$$\left|\frac{\partial F}{\partial s}(x,s)\right| \le \alpha_4 |s|^{p-2} + \beta_4(x), \tag{6.24}$$

where $p^* = \infty$ for $p = 2$ and $p^* = \frac{p}{p-2}$ for $p > 2$.

Lemma 6.1. *Assume that* (H_1)-(H_3) *hold. Let* u^ρ *and* u *be the solutions of* (3.5)-(3.7) *and* (6.19)-(6.21), *respectively, then for every* $\omega \in \Omega, T > 0, \eta \in (0,1]$ *and* $g_n, g \in \Sigma$, $n \in N$, *there exists* $\rho_0 = \rho_0(\omega, T, \eta) \in (0,1]$ *such that for all* $\rho \le \rho_0$ *and* $t \in [0,T]$,

$$\begin{aligned}&\sup_{-\rho\le s\le 0} \|u^\rho(t+s,\omega,g_n,u_0^\rho) - u(t,\omega,g,u_0)\|^2\\ \le& c(\sup_{-\rho\le s\le 0}\|u_0^\rho - u_0\|^2 + \|g_n - g\|_{C_b(\mathbb{R},H)}) + c\eta(1 + \|u_0^\rho\|_\rho^2 + \|u_0\|^2).\end{aligned} \tag{6.25}$$

Proof. Let $v^\rho(t,\omega,g,v_0^\rho) = u^\rho(t,\omega,g_n,u_0^\rho) - hz(\vartheta_t\omega)$ and $v(t,\omega,g,v_0) = u(t,\omega,g_n, u_0) - z(\vartheta_t\omega)$ where $u_0^\rho(s) = v_0^\rho(s) + h(x)z(\vartheta_s\omega)$, $u_0(x) = v_0(x)$, $s \in [-\rho, 0]$. Fix $s \in [-\rho, 0]$, and let $\widetilde{v}(t) = v^\rho(t+s) - v(t)$. Then $\widetilde{v}$ satisfies that for $t > -s$, $s \in [-\rho, 0]$,

$$\begin{aligned}\frac{d\widetilde{v}}{dt} - \Delta\widetilde{v} =& F(x, u^\rho(t+s)) - F(x, u(t)) + f(x, u^\rho(t+s-\rho)) - f(x, u(t))\\ &+ g_n(t+s,x) - g(t,x) + (z(\vartheta_{t+s}\omega) - z(\vartheta_t\omega))(\Delta h(x) + h(x)).\end{aligned} \tag{6.26}$$

Taking the inner product of (6.26) with $\widetilde{v}$ in $L^2(\mathcal{O})$, we obtain for $t > -s$, $s \in [-\rho, 0]$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\widetilde{v}(t)\|^2 + \|\nabla\widetilde{v}(t)\|^2\\ =& \int_{\mathcal{O}}(F(x,u^\rho(t+s)) - F(x,u(t)))\widetilde{v}dx + \int_{\mathcal{O}}(f(x,u^\rho(t+s-\rho)) - f(x,u(t)))\widetilde{v}dx\\ &+ \int_{\mathcal{O}}(g_n(t+s,x) - g(t,x))\widetilde{v}dx + (z(\vartheta_{t+s}\omega) - z(\vartheta_t\omega))\int_{\mathcal{O}}(\Delta h(x) + h(x))\widetilde{v}dx.\end{aligned} \tag{6.27}$$

For the first term on the right-hand side of (6.27), from (3.10) and (6.24) we get

$$\begin{aligned}&\int_{\mathcal{O}}(F(x,u^\rho(t+s)) - F(x,u(t)))\widetilde{v}dx = \int_{\mathcal{O}}\frac{\partial F}{\partial s}(x,s)(u^\rho(t+s) - u(t))\widetilde{v}dx\\ =& \int_{\mathcal{O}}\frac{\partial F}{\partial s}(x,s)\widetilde{v}^2dx + (z(\vartheta_{t+s}\omega) - z(\vartheta_t\omega))\int_{\mathcal{O}}\frac{\partial F}{\partial s}(x,s)h(x)\widetilde{v}dx\\ \le& \alpha_3\|\widetilde{v}\|^2 + |z(\vartheta_{t+s}\omega) - z(\vartheta_t\omega)|\int_{\mathcal{O}}\Big(\alpha_4(|u^\rho(t+s)| + |u(t)|)^{p-2} + \beta_4(x)\Big)h(x)\widetilde{v}dx\\ \le& \alpha_3\|\widetilde{v}\|^2 + c_1|z(\vartheta_{t+s}\omega) - z(\vartheta_t\omega)|(\|u^\rho(t+s)\|_{L^p}^p + \|u(t)\|_{L^p}^p + 1)\\ &+ c_1|z(\vartheta_{t+s}\omega) - z(\vartheta_t\omega)|^2.\end{aligned} \tag{6.28}$$

For the second term on the right-hand side of (6.27), from (3.12) we have

$$\begin{aligned}\int_{\mathcal{O}}(f(x,u^{\rho}(t+s-\rho))-f(x,u(t)))\widetilde{v}dx\leq&\int_{\mathcal{O}}C_f\left|u^{\rho}(t+s-\rho)-u(t)\right|\left|\widetilde{v}\right|dx\\ \leq& c_2\|\widetilde{v}\|^2+c_2\|u^{\rho}(t+s-\rho)-u(t)\|^2.\end{aligned}\tag{6.29}$$

For the last two term on the right-hand side of (6.27), we have

$$\int_{\mathcal{O}}(g_n(t+s,x)-g(t,x))\widetilde{v}dx\leq\frac{1}{2}\|\widetilde{v}\|^2+\frac{1}{2}\|g_n(t+s,x)-g(t,x)\|^2.\tag{6.30}$$

and

$$(z(\vartheta_{t+s}\omega)-z(\vartheta_t\omega))\int_{\mathcal{O}}(\Delta h(x)+h(x))\widetilde{v}dx\leq\|\widetilde{v}\|^2+c_3|z(\vartheta_{t+s}\omega)-z(\vartheta_t\omega)|^2.\tag{6.31}$$

In conclusion, from (6.27)-(6.31) we get for $t>-s,\ s\in[-\rho,0]$,

$$\begin{aligned}\frac{d}{dt}\|\widetilde{v}(t)\|^2\leq& c_4\|\widetilde{v}(t)\|^2+c_4|z(\vartheta_{t+s}\omega)-z(\vartheta_t\omega)|^2\\&+c_4\left|z(\vartheta_{t+s}\omega)-z(\vartheta_t\omega)\right|(\|u^{\rho}(t+s)\|_{L^p}^p+\|u(t)\|_{L^p}^p+1)\\&+c_4\|u^{\rho}(t+s-\rho)-u(t)\|^2+\|g_n(t+s,x)-g(t,x)\|^2.\end{aligned}\tag{6.32}$$

Let $t\in[0,T],\ t\geq-s$. Integration (6.32) over $(-s,t)$, we obtain

$$\begin{aligned}\|\widetilde{v}(t)\|^2\leq&\|\widetilde{v}(-s)\|^2+c_4\int_{-s}^t\|\widetilde{v}(r)\|^2dr+c_4\int_{-s}^t|z(\vartheta_{r+s}\omega)-z(\vartheta_r\omega)|^2dr\\&+c_4\int_{-s}^t|z(\vartheta_{r+s}\omega)-z(\vartheta_r\omega)|(\|u^{\rho}(r+s)\|_{L^p}^p+\|u(r)\|_{L^p}^p+1)\,dr\\&+c_4\int_{-s}^t\|u^{\rho}(r+s-\rho)-u(r)\|^2dr+\int_{-s}^t\|g_n(r+s,x)-g(r,x)\|^2dr.\end{aligned}\tag{6.33}$$

For the fifth term on the right-hand side of (6.33), we have

$$\begin{aligned}&\int_{-s}^t\|u^{\rho}(r+s-\rho)-u(r)\|^2dr\\ \leq&\int_{-s}^{\rho-s}\|u^{\rho}(r+s-\rho)-u(r)\|^2dr+\int_{\rho-s}^t\|u^{\rho}(r+s-\rho)-u(r)\|^2dr\\ \leq&2\int_{-s}^{\rho-s}\|u^{\rho}(r+s-\rho)-u_0\|^2dr+2\int_{-s}^{\rho-s}\|u(r)-u_0\|^2dr\\&+\int_{-s}^{t-\rho}\|u^{\rho}(r+s)-u(r+\rho)\|^2dr\\ \leq&2\int_{-\rho}^0\|u^{\rho}(r)-u_0\|^2dr+2\int_{-s}^{\rho-s}\|u(r)-u_0\|^2dr+2\int_{-s}^t\|u^{\rho}(r+s)-u(r)\|^2dr\\&+2\int_{-s}^t\|u(r+\rho)-u(r)\|^2dr\\ \leq&2\rho\sup_{-\rho\leq s\leq0}\|u_0^{\rho}(s)-u_0\|^2+2\int_{-s}^{\rho-s}\|u(r)-u_0\|^2dr+4\int_{-s}^t\|\widetilde{v}\|^2dr\end{aligned}$$

$$+\,4\|h\|^2\int_{-s}^{t}\left|z\left(\vartheta_{r+s}\omega\right)-z\left(\vartheta_r\omega\right)\right|^2dr+2\int_{-s}^{t}\left\|u\left(r+\rho\right)-u\left(r\right)\right\|^2dr. \tag{6.34}$$

By (6.33)-(6.34) we obtain that for $t\in[0,T]$, $t\geq -s$,

$$\begin{aligned}\|\widetilde{v}(t)\|^2\leq&\|\widetilde{v}(-s)\|^2+c_5\int_{-s}^{t}\|\widetilde{v}(r)\|^2dr+c_5\int_{-s}^{t}\left|z\left(\vartheta_{r+s}\omega\right)-z\left(\vartheta_r\omega\right)\right|^2dr\\&+c_5\int_{-s}^{t}\left|z\left(\vartheta_{r+s}\omega\right)-z\left(\vartheta_r\omega\right)\right|\left(\|u^\rho\left(r+s\right)\|_{L^p}^p+\|u\left(r\right)\|_{L^p}^p+1\right)dr\\&+c_5\rho\sup_{-\rho\leq s\leq 0}\left\|u_0^\rho\left(s\right)-u_0\right\|^2+c_5\int_0^{2\rho}\left\|u\left(r\right)-u_0\right\|^2dr\\&+c_5\int_{-s}^{t}\left\|u\left(r+\rho\right)-u\left(r\right)\right\|^2dr+c_5\int_{-s}^{t}\left\|g_n\left(r+s,x\right)-g\left(r,x\right)\right\|^2dr.\end{aligned} \tag{6.35}$$

Since $z\left(\vartheta_t\omega\right)$ is uniformly continuous on $[-1,T]$, given $\eta>0$, there exists $\rho_1\in(0,1]$, such that for all $\rho<\rho_1,\ s\in[-\rho,0]$ and $r\in[0,T]$,

$$\left|z\left(\vartheta_{r+s}\omega\right)-z\left(\vartheta_r\omega\right)\right|\leq\eta. \tag{6.36}$$

Since $\lim\limits_{\rho\to\infty}\int_0^{2\rho}\|u(r)-u_0\|^2dr=0$, we get that there exists $\rho_2\leq\rho_1$ such that for all $\rho<\rho_2$,

$$\int_0^{2\rho}\|u(r)-u_0\|^2dr\leq\eta. \tag{6.37}$$

Since u is uniformly continuous on $[0,T+1]$,we obtain that there exists $\rho_3\leq\rho_2$ such that for all $\rho<\rho_3$ and $r\in[0,T]$,

$$\|u(r+\rho)-u(r)\|\leq\eta. \tag{6.38}$$

Since $g\in L^2_{loc}\left(\mathbb{R},L^2\left(\mathcal{O}\right)\right)$, we get that there exists $\rho_4\leq\rho_3$ such that for all $\rho<\rho_4$ and $s\in[-\rho,0]$,

$$\int_0^{T}\left\|g\left(r+s\right)-g\left(r\right)\right\|^2dr\leq\eta,$$

which implies that for $s\in[-\rho,0]$,

$$\begin{aligned}&\int_0^{T}\left\|g_n\left(r+s\right)-g\left(r\right)\right\|^2dr\\\leq&2\int_0^{T}\left\|g_n\left(r+s\right)-g\left(r+s\right)\right\|^2dr+2\int_0^{T}\left\|g\left(r+s\right)-g\left(r\right)\right\|^2dr\\\leq&2T\left\|g_n-g\right\|_{C_b(\mathbb{R},H)}+2\eta.\end{aligned} \tag{6.39}$$

By (4.8), we obtain for $\rho\in(0,1]$,

$$\begin{aligned}&\int_{-s}^{t}\|u^\rho\left(r+s\right)\|_{L^p}^pdr=\int_0^{t+s}\|u^\rho\left(t\right)\|_{L^p}^pdr\\\leq&c_6\left(\|u_0^\rho\|_\rho^2+\int_0^{T}\|g\left(r\right)\|^2dr+\int_0^{T}\left(\left|z\left(\vartheta_t\omega\right)\right|^p+\left|z\left(\vartheta_{t-\rho}\omega\right)\right|^p\right)dr+1\right).\end{aligned} \tag{6.40}$$

By (6.19), we have that

$$\int_0^T \|u(r)\|_{L^p}^p dr \le c_7 \left(\|u_0\|^2 + \int_0^T \|g(r)\|^2 dr + \int_0^T |z(\vartheta_t\omega)|^p dr + 1 \right). \quad (6.41)$$

By (6.35)-(6.41), we get that for all $\rho \le \rho_4,\ t \in [0,T],\ t \ge -s$ and $s \in [-\rho, 0]$,

$$\begin{aligned} \|\widetilde{v}(t)\|^2 \le& \|\widetilde{v}(-s)\|^2 + c_5 \int_{-s}^t \|\widetilde{v}(r)\|^2 dr + c_8\eta(1 + \|u_0^\rho\|_\rho^2 + \|u_0\|^2) \\ &+ c_5\rho \sup_{-\rho\le s\le 0} \|u_0^\rho(s) - u_0\|^2 + c_8 \|g_n - g\|_{C_b(\mathbb{R},H)}. \end{aligned} \quad (6.42)$$

By Gronwall's lemma, we obtain that for all $\rho \le \rho_4,\ t \in [0,T],\ t \ge -s$ and $s \in [-\rho, 0]$,

$$\begin{aligned} \|\widetilde{v}(t)\|^2 \le& c_9\|\widetilde{v}(-s)\|^2 + c_9\eta(1 + \|u_0^\rho\|_\rho^2 + \|u_0\|^2) \\ &+ c_9(\sup_{-\rho\le s\le 0} \|u_0^\rho(s) - u_0\|^2 + \|g_n - g\|_{C_b(\mathbb{R},H)}). \end{aligned} \quad (6.43)$$

Since

$$\begin{aligned} \|\widetilde{v}(-s)\|^2 =& \|v^\rho(0) - v(-s)\|^2 \\ \le& 2\|u^\rho(0) - u(-s)\| + 2\|h\|^2 |z(\vartheta_{-s}\omega) - z(\omega)|^2 \\ \le& 4\|u^\rho(0) - u_0\|^2 + 4\|u(-s) - u_0\|^2 + 2\|h\|^2 |z(\vartheta_{-s}\omega) - z(\omega)|^2. \end{aligned} \quad (6.44)$$

By the continuity of u and $z(\vartheta_t\omega)$ at $t = 0$, we obtain that there exists $\rho_5 \le \rho_4$ such that for all $\rho \le \rho_5$,

$$\|\widetilde{v}(-s)\|^2 \le \eta + 4 \sup_{-\rho\le s\le 0} \|u_0^\rho(s) - u_0\|^2. \quad (6.45)$$

By (6.43)-(6.45), we have that for all $\rho \le \rho_5,\ t \in [0,T],\ t \ge -s$ and $s \in [-\rho, 0]$,

$$\|\widetilde{v}(t)\|^2 \le c_{10}\eta(1+\|u_0^\rho\|_\rho^2+\|u_0\|^2)+c_{10}(\sup_{-\rho\le s\le 0} \|u_0^\rho(s)-u_0\|^2+\|g_n-g\|_{C_b(\mathbb{R},H)}). \quad (6.46)$$

By (6.36)-(6.46), we have that for all $\rho \le \rho_5,\ t \in [0,T],\ t \ge -s$ and $s \in [-\rho, 0]$,

$$\begin{aligned} &\|u^\rho(t+s) - u(t)\|^2 \\ \le& c_{11}\eta(1 + \|u_0^\rho\|_\rho^2 + \|u_0\|^2) + c_{11}(\sup_{-\rho\le s\le 0} \|u_0^\rho(s) - u_0\|^2 + \|g_n - g\|_{C_b(\mathbb{R},H)}). \end{aligned} \quad (6.47)$$

By the continuity of u at $t = 0$, we obtain that there exists $\rho_6 \le \rho_5$ such that for all $\rho \le \rho_6, 0 \le t \le -s$,

$$\begin{aligned} \|u^\rho(t+s) - u(t)\|^2 \le& 2\|u^\rho(t+s) - u_0\|^2 + 2\|u(t) - u_0\|^2 \\ \le& 2 \sup_{-\rho\le s\le 0} \|u_0^\rho(s) - u_0\|^2 + \eta. \end{aligned} \quad (6.48)$$

By (6.47)-(6.48), we get that for all $\rho \le \rho_6,\ t \in [0,T]$ and $s \in [-\rho, 0]$,

$$\begin{aligned} &\|u^\rho(t+s) - u(t)\|^2 \\ \le& c_{12}\eta(1 + \|u_0^\rho\|_\rho^2 + \|u_0\|^2) + c_{12}(\sup_{-\rho\le s\le 0} \|u_0^\rho(s) - u_0\|^2 + \|g_n - g\|_{C_b(\mathbb{R},H)}). \end{aligned} \quad (6.49)$$

The proof of Lemma 6.1 is completed. □

We now study that the uniform compactness of attractors with respect to ρ.

Lemma 6.2. *Assume that* $(H_1), (H_2)$ *and* (3.14) *hold. Let* u^ρ *and* u *be the solutions of* (3.5)-(3.7) *and* (6.19)-(6.21), *respectively. If* $\omega \in \Omega, \rho_n \in (0, \rho_0], \rho_n \to 0, u^{\rho_n} \in \mathcal{A}^{\rho_n}(\omega)$, *then there exists* $u \in L^2(\mathcal{O})$ *and a subsequence* $\{u^{\rho_{n_m}}\}_{m=1}^{\infty}$ *of* $\{u^{\rho_n}\}$ *such that*

$$\lim_{m\to\infty} \sup_{\rho_{n_m} \le s \le 0} \|u^{\rho_{n_m}}(s) - u\| = 0. \tag{6.50}$$

Proof. Take a sequence of positive real numbers $\{t_n\}_{n=1}^{\infty}, t_n \to \infty$. By the negatively semi-invariance of $\mathcal{A}^{\rho_n}$, there exists $\hat{u}_n = \hat{u}_n(\vartheta_{-t_n}\omega) \in \mathcal{A}^{\rho_n}(\vartheta_{-t_n}\omega)$ and $g_n \in \Sigma$ such that

$$\begin{aligned} u^{\rho_n}(\omega) =& u^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n) \\ =& \phi^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n(\vartheta_{-t_n}\omega)). \end{aligned} \tag{6.51}$$

By (6.18), we get $\hat{u}_n \in B^{\rho_n}(\vartheta_{-t_n}\omega)$. Due to all estimates in Section 4 are uniform with respect to ρ and Lemma 5.2, we obtain

1) $\phi^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n)(0)$ is precompact in $L^2(\mathcal{O})$.

2) Given any $\eta > 0$, there exists $N_1 \ge 1$ such that for all $n \ge N_1, s \in [-\rho_n, 0]$,

$$\|\phi^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n)(s) - \phi^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n)(0)\| \le \eta.$$

By 1), there exists a subsequence of $\phi^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n)(0)$ (not relabeled) is convergent to $u \in L^2(\mathcal{O})$. We obtain that there exists $N_2 \ge N_1$ such that for all $n \ge N_2$,

$$\|\phi^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n)(0) - u\| \le \eta. \tag{6.52}$$

By 2) and (6.52), we get that for all $n \ge N_2$, $s \in [-\rho_n, 0]$,

$$\begin{aligned} &\|\phi^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n)(s) - u\| \\ \le &\|\phi^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n)(s) - \phi^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n)(0)\| \\ &+ \|\phi^{\rho_n}(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}g_n, \hat{u}_n)(0) - u\| \le 2\eta. \end{aligned}$$

By (6.51), we get that for all $n \ge N_2$, $s \in [-\rho_n, 0]$,

$$\|u^{\rho_n}(s) - u\| \le 2\eta.$$

The proof of Lemma 6.2 is completed. □

By Lemma 4.3, Lemma 6.1, Lemma 6.2 and Theorem 6.1, we get that the upper semicontinuity of attractors as delay approaches to zero.

Theorem 6.2. *Assume that* $(H_1), (H_2), (H_3)$ *and* (3.14) *hold. Then for every* $\omega \in \Omega$,

$$\lim_{\rho\to 0} d_H\left(\mathcal{A}^\rho(\omega), \mathcal{A}^0(\omega)\right) = 0,$$

where d_H *is the distance as defined by* (6.5) *with* $X = L^2(\mathcal{O})$.

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