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FINITE-TIME STABILITY OF NON-INSTANTANEOUS IMPULSIVE SET DIFFERENTIAL EQUATIONS*

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Abstract In this paper, we investigate the finite-time stability of non-instantaneous impulsive set differential equations. By using the generalized Gronwall inequality and a revised Lyapunov method, the finite-time stability criteria for such equations are obtained. Finally, an example is given to illustrate the validity of the results.

Keywords Set differential equations, non-instantaneous impulses, finite-time stability.

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1. Introduction

For the real situation, we pay more attention to the stability of the system in a fixed finite time. Research on the issue of finite time control first started in the 1960s, the concept of finite-time stability is proposed for the first time by Weiss and Infante in [30], that is, for a given initial state of the system, its state trajectory always remains in a given range within a given time interval. Many results [5,6,18,24,33] on finite-time stability are based on the Lyapunov method under strict assumptions, that is, the derivative of Lyapunov function is negative. In 2020, Wang et al. proposed a revised Lyapunov method that weakened the constraints of the Lyapunov function to study the finite/fixed-time stability of discontinuous systems, where the derivative of Lyapunov function is indefinite [29]. Moreover, from the perspective of practical application, the finite-time stability is concerned with the transient performance of the system within a finite time interval, which has more practical significance.

Since Millman and Mishki [22, 23] first proposed instantaneous impulse differential equation in the 1960s, impulse differential equation theory has become an important research field of differential equations [25, 26, 31]. Recently, the theory of impulsive differential equation has become an important research field. Impulses can be divided into instantaneous impulses and non-instantaneous ones according to the time of action of the impulses. The non-instantaneous pulse means that the interference process depends on the state and lasts for a period of time. In real life, non-instantaneous pulse phenomenon is ubiquitous, and it has been widely used

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in pharmacokinetics, population ecological dynamics, infectious disease dynamics and so on. In 2013, Hernandez and O'Regan [12] first proposed the theory of non-instantaneous impulsive differential equations and studied the existence of weak and classical solutions. The research on the theory and application of non-instantaneous ones has just started, and has attracted the attention of scholars rapidly. The basic results can be found in the literature [1] and the papers [2–4, 11, 13, 14, 17, 19, 28].

Set differential equations are effective tools for describing uncertain systems. It has been widely used in the fields of control science, biology, computer and information processing and so on. In 1969, Pinto, Blasi and Iervolino proposed the existence and uniqueness of solutions for set differential equations [9]. For systematic work on set differential equations, see [15, 20]. For most works investigating the Lyapunov asymptotic stability of instantaneous impulsive set differential equations in an infinite time interval, see [7, 8, 10, 21, 27, 32].

It is noted that there are few results of non-instantaneous impulsive set differential equations. The main contribution of this paper is to give its finite-time stability criteria by using the generalized Gronwall inequality and an improved Lyapunov method.

2. Preliminaries

Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$. Define the Hausdorff metric

$$D[A,B] = \max\{\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\},$$

where $d(x,A) = \inf\{d(x,y) : y \in A\}$, $A, B \in K_c(\mathbb{R}^n)$. In particular,

$$D[A,\theta] = \sup_{y\in A} d(y,\theta),$$

where θ is the zero element of $K_c(\mathbb{R}^n)$.

Given any two sets $A, B \in K_c(\mathbb{R}^n)$. We called the set C is the Hukuhara difference of the sets A and B, when $A = B + C$, where $C \in K_c(\mathbb{R}^n)$, and it is denoted by $A - B$.

Definition 2.1 ([15]). The mapping $F : I = [0,T] \to K_c(\mathbb{R}^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if

$$D_H F(t_0) = \lim_{h\to 0^+} \frac{F(t_0+h) - F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0) - F(t_0-h)}{h},$$

where $h > 0$.

Definition 2.2 ([15]). The Hukuhara integral of F is given by

$$\int_I F(s)ds = \left\{ \int_I f(s)ds : f \ \text{ is measurable selector of } \ F \right\}.$$

For the properties of Hukuhara derivative and Hukuhara integral of set-valued functions, please refer to the literature [15].

Let two increasing sequences of points $\{t_k\}_{k=1}^{m-1}$ and $\{s_k\}_{k=0}^{m-1}$ be given s.t. $0 < s_0 < t_k \le s_k < t_{k+1}$, $k = 1, 2, \ldots, m-1$, $m \in \mathbb{N}$. Let $t_0 \in [0, s_0) \cup \left(\cup_{k=1}^{m-1} [t_k, s_k)\right)$ be a given arbitrary point. Without losing generality, we assume that $t_0 \in [0, s_0)$.

Consider the non-instantaneous impulsive set differential equation (NISDE) given by

$$\begin{cases} D_H X = F(t,X), t \in (t_k, s_k] \cup (t_m, +\infty), \\ X(t) = \phi_k(t, X(s_k - 0)),\ t \in (s_k, t_{k+1}], \\ X(t_0) = X_0, \end{cases} \tag{2.1}$$

where $X_0 \in K_c(\mathbb{R}^n)$, $F : \cup_{k=0}^{m-1}(t_k, s_k] \cup (t_m, +\infty) \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$, $\phi_k : (s_k, t_{k+1}] \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$, $k = 0, 1, \ldots, m-1$.

The solution $X(t, t_0, X_0)$ of the NISDE (2.1) is given by

$$X(t,t_0,X_0) = \begin{cases} X_0(t),\ t \in (t_0, s_0], \\ \phi_0(t, X_0(s_0 - 0)),\ t \in (s_0, t_1], \\ \quad \vdots \\ X_k(t),\ t \in (t_k, s_k], \\ \phi_k(t, X_k(s_k - 0)),\ t \in (s_k, t_{k+1}], \\ \quad \vdots \\ X_m(t),\ t \in (t_m, +\infty), \end{cases}$$

where $k = 1, 2, \ldots, m-1$, $X_0(t)$ is the solution of the IVP of the set differential equation

$$D_H X = F(t,X) \ \text{ for } \ t \in [\tau, s_k] \ \text{ with } \ X(\tau) = \tilde{X}_0 \in K_c(\mathbb{R}^n), \tau \geq 0, \tag{2.2}$$

for $\tau = t_0$, $s_k = s_0$, $\tilde{X}_0 = X_0$ and $X_k(t)$ is the solution of the IVP of the set differential equation (2.2) for $\tau = t_k$, $\tilde{X}_0 = \phi_{k-1}(t_k, X_{k-1}(s_{k-1} - 0))$.

Defining the functions $I_k(t,X) = \phi_k(t,X) - X$, $k = 0, 1, \ldots, m-1$. We know that the solution $X(t, t_0, X_0)$ of NISDE (2.1) also satisfies the following equations

$$X(t,t_0,X_0) = \begin{cases} X_0 + \int_{t_0}^{t} F(s, X(s,t_0,X_0))ds,\ t \in [t_0, s_0], \\ X_0 + I_k(t, X(s_k - 0, t_0, X_0)) + \sum_{i=0}^{k} \int_{t_i}^{s_i} F(s, X(s,t_0,X_0))ds \\ + \sum_{i=0}^{k-1} I_i(t_{i+1}, X(s_i, t_0, X_0)),\ t \in (s_k, t_{k+1}],\ k = 0, 1, \ldots, m-1, \\ X_0 + \sum_{i=0}^{k-1} \int_{t_i}^{s_i} F(s, X(s,t_0,X_0))ds + \int_{t_k}^{t} F(s, X(s,t_0,X_0))ds \\ + \sum_{i=0}^{k-1} I_i(t_{i+1}, X(s_i, t_0, X_0)),\ t \in (t_k, s_k],\ k = 1, 2, \ldots, m-1, \\ X_0 + \sum_{i=0}^{m-1} \int_{t_i}^{s_i} F(s, X(s,t_0,X_0))ds + \int_{t_m}^{t} F(s, X(s,t_0,X_0))ds \\ + \sum_{i=0}^{m-1} I_i(t_{i+1}, X(s_i, t_0, X_0)),\ t \in (t_m, +\infty). \end{cases} \tag{2.3}$$

If $F(t,\theta)=\theta$ for $t\in\cup_{k=0}^{m-1}(t_k,s_k]\cup(t_m,+\infty)$ and $\phi_k(t,\theta)\equiv\theta$ for $t\in(s_k,t_{k+1}]$, $k=0,1,\dots,m-1$, then NISDE (2.1) has a trivial solution $X(t)=\theta$.

3. Criteria for Finite-time Stability

We first discuss the finite-time stability of NISDE (2.1) by using the generalized Gronwall inequality. Secondly, we use the improved Lyapunov method to discuss its finite-time uniformly stability and fixed-time uniformly stability.

For convenience, we first give the following sets required in this paper.

$$\mathcal{K}=\{c\in C[\mathbb{R}^+,\mathbb{R}^+]: c \text{ is strictly increasing and } c(0)=0\},$$
$$\mathcal{K}_\infty=\{c\in\mathcal{K}:\lim_{s\to+\infty}c(s)=+\infty\}.$$

We now introduce the class Λ of Lyapunov-like functions.

Definition 3.1. Let $J\subset\mathbb{R}^+$ is a given interval, $\Delta\subset K_c(\mathbb{R}^n)$ be a given set, and $\theta\in\Delta$. We say that the function $V(t,X):J\times\Delta\to\mathbb{R}^+$, $V(t,\theta)\equiv 0$ belongs to the class $\Lambda(J,\Delta)$ if

(H_1) The function $V(t,X)$ is continuous on $J/\{s_k\in J\}\times\Delta$ and
$$|V(t,X)-V(t,\bar{X})|\le LD[X,\bar{X}],$$
where $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $X,\bar{X}\in\Delta$, $L>0$.

(H_2) For each $s_k\in J$ and $X\in\Delta$ there exists finite limits
$$V(s_k-0,X)=\lim_{t\to s_k^-}V(t,X),\quad V(s_k+0,X)=\lim_{t\to s_k^+}V(t,X),$$
and $V(s_k-0,X)=V(s_k+0,X)=V(s_k,X)$.

(H_3) For any $X\neq 0$, $V(t,X)>0$, $V(t,0)=0$, and $V(t,X)\to+\infty$ when $D[X,\theta]\to+\infty$.

Remark 3.1. We note that the constraint of the Lyapunov function is weakened if the derivative of the function $V\in\Lambda(\mathbb{R}^+,K_c(\mathbb{R}^n))$ is relaxed to have indefiniteness for almost every t. Such a type of Lyapunov function may be called almost indefinite Lyapunov-like function.

Definition 3.2. The trivial solution of NISDE (2.1) is said to be

(S_1) finite-time stable, if for given values $0<c_1<c_2<\infty$, $t_0\in[0,s_0)\cup\big(\cup_{k=1}^{m-1}[t_k,s_k)\big)$, $T>0$, the inequality $D[X_0,\theta]<c_1$ implies $D[X(t,t_0,X_0),\theta]<c_2$ for any $t\in[t_0,t_0+T]$;

(S_2) finite-time attractive, if there exists a $T(t_0,X_0)>0$ s.t. for any $t_0\in[0,s_0)\cup\big(\cup_{k=1}^{m-1}[t_k,s_k)\big)$, the equalities $\lim_{t\to T}X(t,t_0,X_0)=\theta$ and $X(t,t_0,X_0)=\theta$ hold for $t\ge T(t_0,X_0)$, where the settling time $T(t_0,X_0)>0$;

(S_3) Lyapunov uniformly stable, if for each $\epsilon>0$ there exists a $\delta=\delta(\epsilon)>0$ s.t. for any $t_0\in[0,s_0)\cup\big(\cup_{k=1}^{m-1}[t_k,s_k)\big)$ with $D[X_0,\theta]<\delta$, the inequality $D[X(t,t_0,X_0),\theta]<\epsilon$ holds for $t\ge t_0$;

(S_4) finite-time uniformly stable, if (S_2) and (S_3) hold.

(S_5) fixed-time uniformly stable, if (S_4) holds and $T(t_0, X_0)$ is uniformly bounded on X_0, that is, there exists $T_{\max} > 0$ s.t. $T(t_0, X_0) \leq t_0 + T_{\max}$.

In order to get the main results of this paper, the following necessary conditions are given:

(A_1) The function $F \in C\big(\cup_{k=0}^{m-1}(t_k, s_k] \cup (t_m, +\infty)), K_c(\mathbb{R}^n)\big)$ is such that for any $(\tau, X_0) : \tau \in [t_k, s_k)$, $X_0 = \tilde{X}_0 \in K_c(\mathbb{R}^n)$, $k = 0, 1, \ldots, m-1$, the system of SDE (2.2) has a solution $X(t, \tau, X_0) \in C^1([\tau, s_k], K_c(\mathbb{R}^n))$.

(A_2) The function $\phi_k \in C((s_k, t_{k+1}] \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n))$ is such that for any $t \in (s_k, t_{k+1}]$ there exists at the least one function $Z : (s_k, t_{k+1}] \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ such that $Z(t, X) = \phi_k(t, X)$ and $Z(t, \theta) \equiv \theta$.

Lemma 3.1 ([16]). *Assume that the following conditions are satisfied:*

(B_1) *The function $u(t)$ is nonnegative, piecewise continuous and left continuous at the discontinuities of the first type for each $t \geq t_0$.*

(B_2) *The inequality $u(t) \leq u(t_0) + \int_{t_0}^{t} \lambda(s)u(s)ds + \sum_{t_0<t_i\leq t} \beta_i u(t_i)$, $t \geq t_0$ holds, where the continuous function $\lambda(t)$ is nonnegative and the constant $\beta_i \geq 0$.*

Then, $u(t) \leq u(t_0) \prod_{t_0<t_i\leq t}(1+\beta_i)\exp(\int_{t_0}^{t}\lambda(s)ds)$, $t \geq t_0$.

Theorem 3.1. *Assume that the conditions of (A_1) and (A_2) are satisfied, and*

(A_3) *There exists a continuous function $\lambda(t) > 0$ such that $D[F(t, X(t)), \theta] \leq \lambda(t)D[X(t), \theta]$, $t \notin (s_k, t_{k+1}], X \in K_c(\mathbb{R}^n)$ holds, where $k = 0, 1, \cdots, m-1$.*

(A_4) *The function $I_k(t, X)$ is non-increasing in t, $k = 0, 1, \ldots, m-1$.*

(A_5) *There exist constants $\beta_i \geq 0$ such that*

$$D[\sum_{t_0\leq t_i<t} I_i(t_{i+1}, X(s_i - 0)), \theta] \leq \sum_{t_0\leq t_i<t} \beta_i D[X(t_i), \theta],$$

$t \in (s_k, t_{k+1}], X \in K_c(\mathbb{R}^n)$ holds, where $k = 0, 1, \cdots, m-1$;

(A_6) *The inequality $\prod_{t_0\leq t_i<t}(1+\beta_i)\exp(\int_{t_0}^{t}\lambda(s)ds) < \frac{c_2}{c_1}$, $t \in [t_0, t_0+T]$ holds, where $0 < c_1 < c_2 < \infty$.*

Then the solution $X(t, t_0, X_0)$ of NISDE (2.1) is finite-time stable.

Proof. Let $t \in [t_0, s_0]$. From (2.3), we get $X(t) = X_0 + \int_{t_0}^{t} F(s, X(s, t_0, X_0))ds$. Then from condition ($A_3$), we obtain

$$D[X(t), \theta] \leq D[X_0, \theta] + \int_{t_0}^{t} \lambda(s)D[X(s), \theta]ds.$$

Due to Lemma 3.1, one has

$$D[X(t), \theta] \leq D[X_0, \theta]\exp(\int_{t_0}^{t}\lambda(s)ds).$$

Let $t \in (s_0, t_1]$. From (2.3) and condition (A_4), we obtain

$$X(t) \le X_0 + \int_{t_0}^{s_0} F(s, X(s, t_0, X_0))ds + I_0(t_1, X(s_0 - 0, t_0, X_0)).$$

From condition (A_3), (A_5), we get

$$D[X(t), \theta] \le D[X_0, \theta] + \int_{t_0}^{t} \lambda(s) D[X(s), \theta]ds + \beta_0 D[X(t_0), \theta].$$

Due to Lemma 3.1, one derives

$$D[X(t), \theta] \le D[X_0, \theta](1 + \beta_0) \exp(\int_{t_0}^{t} \lambda(s)ds).$$

Let $t \in (t_1, s_1]$. Similarly, from (2.3), (A_3) and (A_5), one has

$$\begin{aligned} X(t) =& X_0 + \int_{t_0}^{s_0} F(s, X(s, t_0, X_0))ds + \int_{t_1}^{t} F(s, X(s, t_0, X_0))ds \\ &+ I_0(t_1, X(s_0 - 0, t_0, X_0)), \\ D[X(t), \theta] \le& D[X_0, \theta] + \int_{t_0}^{s_0} \lambda(s) D[X(s), \theta]ds + \int_{t_1}^{t} \lambda(s) D[X(s), \theta]ds + \beta_0 D[X(t_0), \theta] \\ \le& D[X_0, \theta] + \int_{t_0}^{t} \lambda(s) D[X(s), \theta]ds + \beta_0 D[X(t_0), \theta]. \end{aligned}$$

Thus, by Lemma 3.1, one derives

$$D[X(t), \theta] \le D[X_0, \theta](1 + \beta_0) \exp(\int_{t_0}^{t} \lambda(s)ds).$$

Let $t \in (s_1, t_2]$. From (2.3) and conditions $(A_3) - (A_5)$, one has

$$\begin{aligned} X(t) &= X_0 + \sum_{i=0}^{1} \int_{t_i}^{s_i} F(s, X(s, t_0, X_0))ds + I_0(t_1, X(s_0 - 0, t_0, X_0)) \\ &\quad + I_1(t, X(s_1 - 0, t_0, X_0)) \\ &\le X_0 + \sum_{i=0}^{1} \int_{t_i}^{s_i} F(s, X(s, t_0, X_0))ds + \sum_{i=0}^{1} I_i(t_{i+1}, X(s_i, t_0, X_0)), \\ D[X(t), \theta] &\le D[X_0, \theta] + \sum_{i=0}^{1} \int_{t_i}^{s_i} \lambda(s) D[X(s), \theta]ds + \sum_{i=0}^{1} \beta_i D[X(t_i), \theta] \\ &\le D[X_0, \theta] + \int_{t_0}^{t} \lambda(s) D[X(s), \theta]ds + \sum_{i=0}^{1} \beta_i D[X(t_i), \theta]. \end{aligned}$$

Furthermore, one derives

$$D[X(t), \theta] \le D[X_0, \theta] \prod_{i=0}^{1} (1 + \beta_i) \exp(\int_{t_0}^{t} \lambda(s)ds).$$

Continue this process and from induction argument, one obtains

$$D[X(t),\theta] \leq D[X_0,\theta] \prod_{t_0 \leq t_i < t} (1+\beta_i) \exp(\int_{t_0}^{t} \lambda(s)ds), \quad t \geq 0. \tag{3.1}$$

Whenever $D[X_0, \theta] < c_1$, from (3.1) and condition (A_6), one derives

$$D[X(t),\theta] < c_2, \ t \in [t_0, t_0+T].$$

Therefore, Theorem 3.1 is proved. □

Theorem 3.2. *Assume that the conditions of* (A_1), (A_2) *are satisfied, the functions* $r(t)$, $p(t)$ *are respectively indefinite and negative, and*

(A_7) *For the function* $V \in \Lambda(\mathbb{R}^+, K_c(\mathbb{R}^n))$,
(i) the inequality $\dot{V}(t,X(t)) \leq r(t)V(t,X(t)) + p(t)V^a(t,X(t)), t \notin (s_k, t_{k+1}]$, $X \in K_c(\mathbb{R}^n)$ *holds, where* $0 < a < 1$, $k = 0, 1, \cdots, m-1$;
(ii) the inequality $V(t, \phi_k(t, X(t))) \leq \eta^{\frac{1}{1-a}} V(s_k^-, X(s_k^-)), t \in (s_k, t_{k+1}], X \in K_c(\mathbb{R}^n)$ *holds, where* $\eta > 0$, $k = 0, 1, \cdots, m-1$;
(iii) the inequality $\varphi_1(D[X(t),\theta]) \leq V(t,X(t)) \leq \varphi_2(D[X(t),\theta])$, $t \in \mathbb{R}^+$, $X \in K_c(\mathbb{R}^n)$ *holds, where* $\varphi_1, \varphi_2 \in \mathcal{K}_\infty$.

(A_8) *There exists* $G \geq 0$, $\gamma > 0$ *and* $H \geq 0$ *such that*

$$\int_0^{+\infty} |r(s)|ds < G, \quad \int_{t_0}^{t} |p(s)|ds < -\gamma(t-t_0) + H.$$

Then the solution $X(t, t_0, X_0)$ *of NISDE (2.1) is finite-time uniformly stable. Moreover, the settling time* $T(t_0, X_0)$ *is given by*

$$T(t_0, X_0) = t_0 + \frac{V^{1-a}(t_0, X_0)e^{G(1-a)} + \eta^m(1-a)H}{\eta^m(1-a)\gamma}, \quad 0 < \eta < 1$$

and

$$T(t_0, X_0) = t_0 + \frac{\eta^m V^{1-a}(t_0, X_0)e^{G(1-a)} + (1-a)H}{(1-a)\gamma}, \quad \eta \geq 1.$$

Proof. Let $t_0 \in [0, s_0)$. From the conditions $(A_7)(i)$, (ii), one gets

$$\dot{V}(t,X(t)) \cdot V^{-a}(t,X(t)) \leq r(t)V^{1-\alpha}(t,X(t)) + p(t), \quad t \in (t_k, s_k], \tag{3.2}$$

$$V^{1-a}(t, \phi_k(t,X(t))) \leq \eta V^{1-a}(s_k^-, X(s_k^-)), \quad t \in (s_k, t_{k+1}]. \tag{3.3}$$

Defining $Y(t) = V^{1-a}(t, X(t))$. From (3.2), (3.3) and $0 < a < 1$, one gets

$$\dot{Y}(t) \leq (1-a)|r(t)|Y(t) + (1-a)p(t), \quad t \in (t_k, s_k], \tag{3.4}$$

$$Y(t) \leq \eta Y(s_k^-), \quad t \in (s_k, t_{k+1}]. \tag{3.5}$$

Let $t \in [t_0, s_0]$. According to inequality (3.4), one gets

$$Y(t) \leq Y(0)e^{(1-a)\int_{t_0}^{t}|r(\varrho)|d\varrho} + (1-a)\int_{t_0}^{t} p(s)e^{(1-a)\int_{s}^{t}|r(\varrho)|d\varrho}ds. \tag{3.6}$$

From the continuity of $Y(t)$ and inequality (3.6), one gets

$$Y(s_0^-) \le Y(0)e^{(1-a)\int_{t_0}^{s_0}|r(\varrho)|d\varrho} + (1-a)\int_{t_0}^{s_0} p(s)e^{(1-a)\int_s^{s_0}|\mu(\varrho)|d\varrho}ds. \tag{3.7}$$

Let $t \in (s_0, t_1]$. From inequalities (3.5) and (3.7), we obtain

$$Y(t_1) \le \eta Y(s_0^-) \le \eta Y(0)e^{(1-a)\int_{t_0}^{s_0}|r(\varrho)|d\varrho} + (1-a)\eta\int_{t_0}^{s_0} p(s)e^{(1-a)\int_s^{s_0}|r(\varrho)|d\varrho}ds. \tag{3.8}$$

Let $t \in (t_1, s_1]$. Similarly, due to inequalities (3.4) and (3.8), one has

$$\begin{aligned} Y(t) &\le Y(t_1)e^{(1-a)\int_{t_1}^{t}|r(\varrho)|d\varrho} + (1-a)\int_{t_1}^{t} p(s)e^{(1-a)\int_s^{t}|r(\varrho)|d\varrho}ds \\ &\le \eta Y(0)e^{(1-a)\int_{t_0}^{t}|r(\varrho)|d\varrho} + (1-a)\eta\int_{t_0}^{s_0} p(s)e^{(1-a)\int_s^{t}|r(\varrho)|d\varrho}ds \\ &\quad + (1-a)\int_{t_1}^{t} p(s)e^{(1-a)\int_s^{t}|r(\varrho)|d\varrho}ds, \end{aligned} \tag{3.9}$$

$$\begin{aligned} Y(s_1^-) &\le \eta Y(0)e^{(1-a)\int_{t_0}^{s_1}|r(\varrho)|d\varrho} + (1-a)\eta\int_{t_0}^{s_0} p(s)e^{(1-a)\int_s^{s_1}|r(\varrho)|d\varrho}ds \\ &\quad + (1-a)\int_{t_1}^{s_1} p(s)e^{(1-a)\int_s^{s_1}|r(\varrho)|d\varrho}ds. \end{aligned} \tag{3.10}$$

Let $t \in (s_1, t_2]$. From formulas (3.5) and (3.10), we obtain

$$\begin{aligned} Y(t_2) \le \eta Y(s_1^-) &\le \eta^2 Y(0)e^{(1-a)\int_{t_0}^{s_1}|r(\varrho)|d\varrho} + (1-a)\eta^2\int_{t_0}^{s_0} p(s)e^{(1-a)\int_s^{s_1}|r(\varrho)|d\varrho}ds \\ &\quad + (1-a)\eta\int_{t_1}^{s_1} p(s)e^{(1-a)\int_s^{s_1}|r(\varrho)|d\varrho}ds. \end{aligned}$$

Continue this process. By the method of induction, we get

$$\begin{aligned} Y(t) &\le \eta^k Y(0)e^{(1-a)\int_{t_0}^{t}|r(\varrho)|d\varrho} + (1-a)\eta^k\int_{t_0}^{s_0} p(s)e^{(1-a)\int_s^{t}|r(\varrho)|d\varrho}ds \\ &\quad + (1-a)\sum_{j=1}^{k-1}\eta^{k-j}\int_{t_j}^{s_j} p(s)e^{(1-a)\int_s^{t}|r(\varrho)|d\varrho}ds \\ &\quad + (1-a)\int_{t_k}^{t} p(s)e^{(1-a)\int_s^{t}|r(\varrho)|d\varrho}ds \end{aligned}$$

holds for $t \in [t_k, s_k]$, $k = 2, 3, \cdots, m-1$, and

$$\begin{aligned} Y(t) &\le \eta^{k+1} Y(0)e^{(1-a)\int_{t_0}^{s_k}|r(\varrho)|d\varrho} + (1-a)\eta^{k+1}\int_{t_0}^{s_0} p(s)e^{(1-a)\int_s^{s_k}|r(\varrho)|d\varrho}ds \\ &\quad + (1-a)\sum_{j=1}^{k-1}\eta^{k+1-j}\int_{t_j}^{s_j} p(s)e^{(1-a)\int_s^{s_k}|r(\varrho)|d\varrho}ds \\ &\quad + (1-a)\eta\int_{t_k}^{s_k} p(s)e^{(1-a)\int_s^{s_k}|r(\varrho)|d\varrho}ds \end{aligned}$$

holds for $t \in (s_k, t_{k+1}]$, $k = 2, 3, \cdots, m-1$.

Therefore, we have

$$\begin{aligned} Y(t) \le\ & \eta^m Y(0) e^{(1-a)\int_{t_0}^{t} |r(\varrho)| d\varrho} + (1-a)\eta^m e^{(1-a)\int_{t_0}^{t} |r(s)| ds} \int_{t_0}^{s_0} p(s) ds \\ & + (1-a) \sum_{j=1}^{m-1} \eta^{m-j} e^{(1-a)\int_{t_j}^{t} |r(s)| ds} \int_{t_j}^{s_j} p(s) ds \\ & + (1-a) e^{(1-a)\int_{t_m}^{t} |r(s)| ds} \int_{t_m}^{t} p(s) ds \end{aligned} \tag{3.11}$$

holds for $t \in [t_m, +\infty]$.

The following discussion is divided into two cases according to the value of η.

Case I. $0 < \eta < 1$. From (A_8), (3.11) and $0 < a < 1$, one gets

$$Y(t) \le Y(0) e^{(1-a)\int_0^{+\infty} |r(\varrho)| d\varrho} \le e^{G(1-a)} Y(0).$$

Furthermore, one gets

$$V(t, X(t)) \le e^G V(t_0, X_0). \tag{3.12}$$

Combined with condition $(A_7)(iii)$, we get

$$V(t_0, X_0) \le \varphi_2(D[X_0, \theta]), \tag{3.13}$$

$$D[X(t), \theta] \le \varphi_1^{-1}(V(t, X(t))). \tag{3.14}$$

From inequalities (3.12) and (3.13), one gets

$$V(t, X(t)) \le e^G \varphi_2(D[X_0, \theta]). \tag{3.15}$$

Combined with formulas (3.14) and (3.15), we have

$$D[X(t), \theta] \le \varphi_1^{-1}(e^G \varphi_2(D[X_0, \theta])). \tag{3.16}$$

For $\epsilon > 0$, we choose $\delta = \varphi_2^{-1}(\frac{\varphi_1(\epsilon)}{e^G}) > 0$, then, for $D[X_0, \theta] < \delta$, one has

$$D[X(t), \theta] \le \varphi_1^{-1}(e^G \varphi_2(\varphi_2^{-1}(\frac{\varphi_1(\epsilon)}{e^G}))) = \epsilon.$$

That is, the solution of NISDE (2.1) is Lyapunov uniformly stable.

In addition, from (A_8), (3.11) and $0 < a < 1$, we can also get

$$\begin{aligned} Y(t) & \le Y(0) e^{(1-a)\int_0^{+\infty} |r(\varrho)| d\varrho} + (1-a)\eta^m \int_{t_0}^{t} p(s) ds \\ & \le Y(0) e^{(1-a)G} + (1-a)\eta^m(-\gamma(t-t_0) + H). \end{aligned}$$

Obviously, we get that $V(t, X(t)) \equiv 0$ when $t \ge t_0 + \frac{V^{1-a}(t_0, X_0) e^{(1-a)G} + \eta^m (1-a) H}{\eta^m (1-a)\gamma}$. Therefore the solution of NISDE (2.1) is finite-time attractive.

Case II. $\eta \ge 1$. From (A_8), (3.11) and $0 < a < 1$, one gets

$$Y(t) \le \eta^m Y(0) e^{(1-a)\int_0^{+\infty} |r(\varrho)| d\varrho} \le \eta^m e^{(1-a)G} Y(0).$$

Furthermore, one gets

$$V(t,X(t)) \le e^G \eta^{\frac{m}{1-a}} V(t_0,X_0). \tag{3.17}$$

From inequalities (3.13), (3.17), one gets

$$V(t,X(t)) \le e^G \eta^{\frac{m}{1-a}} \varphi_2(D[X_0,\theta]). \tag{3.18}$$

Combined with inequalities (3.14) and (3.18), we have

$$D[X(t),\theta] \le \varphi_1^{-1}(\eta^{\frac{m}{1-a}} e^G \varphi_2(D[X_0,\theta])). \tag{3.19}$$

Thus, for $\epsilon > 0$, we can choose $\delta = \varphi_2^{-1}(\frac{\varphi_1(\epsilon)}{\eta^{\frac{m}{1-a}} e^G}) > 0$, then, for $D[X_0,\theta] < \delta$, one gets

$$D[X(t),\theta] \le \varphi_1^{-1}(\eta^{\frac{m}{1-a}} e^G \varphi_2(\varphi_2^{-1}(\frac{\varphi_1(\epsilon)}{\eta^{\frac{m}{1-a}} e^G}))) = \epsilon.$$

That is, the solution of NISDE (2.1) is Lyapunov uniformly stable.

From condition (A_8), formula (3.11) and $0 < a < 1$, one has

$$\begin{aligned} Y(t) &\le Y(0)\eta^m e^{(1-a)\int_0^{+\infty}|r(\varrho)|d\varrho} + (1-a)\int_{t_0}^t p(s)ds \\ &\le Y(0)\eta^m e^{(1-a)G} + (1-a)(-\gamma(t-t_0)+H). \end{aligned}$$

Similarly, one gets that $V(t,X(t)) \equiv 0$ when $t \ge t_0 + \frac{\eta^m V^{1-a}(t_0,X_0)e^{(1-a)G}+(1-a)H}{(1-a)\gamma}$. Therefore the solution of NISDE (2.1) is finite-time attractive.

Taken together, the solution of NISDE (2.1) is finite-time uniformly stable. □

Theorem 3.3. *Assume that the conditions of (A_1), (A_2) are satisfied, the function $r(t)$ is indefinite, $p(t)$ and $q(t)$ are negative functions, and*

(A_9) *For the function $V \in \Lambda(\mathbb{R}^+, K_c(\mathbb{R}^n))$,*
(i) the inequality $\dot V(t,X(t)) \le r(t)V(t,X(t)) + p(t)V^a(t,X(t)) + q(t)V^b(t,X(t))$, $t \notin (s_k,t_{k+1}]$, $X \in K_c(\mathbb{R}^n)$ holds, where $0 < a < 1$, $b > 1$, $k = 0,1,\cdots,m-1$;
(ii) the inequality $V(t,\phi_k(t,X(t))) \le \eta^{\frac{1}{(1-a)(1-b)}} V(s_k^-, X(s_k^-)), t \in (s_k,t_{k+1}]$, $X \in K_c(\mathbb{R}^n)$ holds, where $\eta > 0$, $k = 0,1,\cdots,m-1$;
(iii) the inequality $\varphi_1(D[X(t),\theta]) \le V(t,X(t)) \le \varphi_2(D[X(t),\theta])$, $t \in \mathbb{R}^+$, $X \in K_c(\mathbb{R}^n)$ holds, where $\varphi_1,\varphi_2 \in \mathcal{K}_\infty$.

(A_{10}) *There exists positive numbers G,γ,H,ξ and K such that*

$$\int_0^{+\infty}|r(s)|ds < G,\ \int_{t_0}^t |p(s)|ds < -\gamma(t-t_0)+H,\ \int_{t_0}^t |q(s)|ds < -\xi(t-t_0)+K.$$

Then the solution $X(t,t_0,X_0)$ of NISDE (2.1) is fixed-time uniformly stable, and the settling time $T(t_0,X_0)$ is given by

$$T(t_0,X_0) = t_0 + \frac{e^{(1-a)G}\eta^{\frac{m}{1-b}} + H(1-a)}{\gamma(1-a)} + \frac{1+(b-1)Ke^{(1-b)G}\eta^{\frac{m}{1-a}}}{(b-1)\xi e^{(1-b)G}\eta^{\frac{m}{1-a}}},\quad 0<\eta<1$$

and

$$T(t_0,X_0) = t_0 + \frac{e^{(1-a)G} + (1-a)H\eta^{\frac{m}{1-b}}}{(1-a)\gamma\eta^{\frac{m}{1-b}}} + \frac{1+(b-1)Ke^{(1-b)G}}{(b-1)\xi e^{(1-b)G}},\quad \eta \ge 1.$$

Proof. First, we proved that the solution of NISDE (2.1) is fixed-time attractive, that is, there exists positive numbers T_1 and $\tilde{t}$ with $t_0 < \tilde{t} < T_1$ s.t. $V(\tilde{t}, \tilde{X}) \leq 1$. Instead, we assume $V(t, X) > 1$ for $t \in [t_0, T_1]$.

Let $t_0 \in [0, s_0)$. Since $p(t) < 0$, $b > 1$, from conditions $(A_9)(i)$, (ii), one gets

$$\dot{V}(t, X(t)) \cdot V^{-b}(t, X(t)) \leq q(t) + |r(t)|V^{1-b}(t, X(t)), \quad t \in (t_k, s_k], \tag{3.20}$$

$$V^{1-b}(t, \phi_k(t, X(t))) \geq \eta^{\frac{1}{1-a}} V^{1-b}(s_k^-, X(s_k^-)), \quad t \in (s_k, t_{k+1}]. \tag{3.21}$$

Defining $Z(t) = V^{1-b}(t, X(t))$. From formulas (3.20), (3.21) and $b > 1$, we have

$$\dot{Z}(t) \geq (1-b)q(t) + (1-b)|r(t)|Z(t), \quad t \in (t_k, s_k], \tag{3.22}$$

$$Z(t) \geq \eta^{\frac{1}{1-a}} Z(s_k^-), \quad t \in (s_k, t_{k+1}]. \tag{3.23}$$

Let $t \in [t_0, s_0]$. According to formula (3.22), one derives

$$Z(t) \geq Z(0)e^{(1-b)\int_{t_0}^{t} |r(\varrho)|d\varrho} + (1-b)\int_{t_0}^{t} q(s)e^{(1-b)\int_s^t |r(\varrho)|d\varrho} ds. \tag{3.24}$$

From the continuity of $Z(t)$ and formula (3.24), we obtain

$$Z(s_0^-) \geq Z(0)e^{(1-b)\int_{t_0}^{s_0} |r(\varrho)|d\varrho} + (1-b)\int_{t_0}^{s_0} q(s)e^{(1-b)\int_s^{s_0} |r(\varrho)|d\varrho} ds. \tag{3.25}$$

Let $t \in (s_0, t_1]$. From inequalities (3.23) and (3.25), we obtain

$$\begin{aligned} Z(t_1) &\geq \eta^{\frac{1}{1-a}} Z(s_0^-) \geq \eta^{\frac{1}{1-a}} Z(0)e^{(1-b)\int_{t_0}^{s_0} |r(\varrho)|d\varrho} \\ &\quad + (1-b)\eta^{\frac{1}{1-a}} \int_{t_0}^{s_0} q(s)e^{(1-b)\int_s^{s_0} |r(\varrho)|d\varrho} ds. \end{aligned} \tag{3.26}$$

Let $t \in (t_1, s_1]$. Due to formulas (3.22) and (3.26), one has

$$\begin{aligned} Z(t) &\geq Z(t_1)e^{(1-b)\int_{t_1}^{t} |r(\varrho)|d\varrho} + (1-b)\int_{t_1}^{t} q(s)e^{(1-b)\int_s^t |r(\varrho)|d\varrho} ds \\ &\geq Z(0)\eta^{\frac{1}{1-a}} e^{(1-b)\int_{t_0}^{t} |r(\varrho)|d\varrho} + (1-b)\eta^{\frac{1}{1-a}} \int_{t_0}^{s_0} q(s)e^{(1-b)\int_s^t |r(\varrho)|d\varrho} ds \\ &\quad + (1-b)\int_{t_1}^{t} q(s)e^{(1-b)\int_s^t |r(\varrho)|d\varrho} ds. \end{aligned} \tag{3.27}$$

and combined with the continuity of $Z(t)$, we have

$$\begin{aligned} Z(s_1^-) &\geq \eta^{\frac{1}{1-a}} Z(0)e^{(1-b)\int_{t_0}^{s_1} |r(\varrho)|d\varrho} + (1-b)\eta^{\frac{1}{1-a}} \int_{t_0}^{s_0} q(s)e^{(1-b)\int_s^{s_1} |r(\varrho)|d\varrho} ds \\ &\quad + (1-b)\int_{t_1}^{s_1} q(s)e^{(1-b)\int_s^{s_1} |r(\varrho)|d\varrho} ds. \end{aligned} \tag{3.28}$$

Let $t \in (s_1, t_2]$. From inequalities (3.23) and (3.28), we obtain

$$\begin{aligned} Z(t_2) &\geq \eta^{\frac{1}{1-a}} Z(s_1^-) \geq Z(0)\eta^{\frac{2}{1-a}} e^{(1-b)\int_{t_0}^{s_1} |r(\varrho)|d\varrho} \\ &\quad + (1-b)\eta^{\frac{2}{1-a}} \int_{t_0}^{s_0} q(s)e^{(1-b)\int_s^{s_1} |r(\varrho)|d\varrho} ds \\ &\quad + (1-b)\eta^{\frac{1}{1-a}} \int_{t_1}^{s_1} q(s)e^{(1-b)\int_s^{s_1} |r(\varrho)|d\varrho} ds. \end{aligned}$$

Repeating the above process, we get

$$\begin{aligned}Z(t) \geq &\eta^{\frac{k}{1-a}} Z(0) e^{(1-b)\int_{t_0}^{t} |r(\varrho)| d\varrho} + (1-b)\eta^{\frac{k}{1-a}} \int_{t_0}^{s_0} q(s) e^{(1-b)\int_{s}^{t} |r(\varrho)| d\varrho} ds \\ &+ (1-b)\sum_{j=1}^{k-1} \eta^{\frac{k-j}{1-a}} \int_{t_j}^{s_j} q(s) e^{(1-b)\int_{s}^{t} |r(\varrho)| d\varrho} ds + (1-b)\int_{t_k}^{t} q(s) e^{(1-b)\int_{s}^{t} |r(\varrho)| d\varrho} ds\end{aligned}$$

holds for $t \in [t_k, s_k]$, $k = 2, 3, \cdots, m-1$, and

$$\begin{aligned}Z(t) \geq &\eta^{\frac{k+1}{1-a}} Z(0) e^{(1-b)\int_{t_0}^{s_k} |r(\varrho)| d\varrho} + (1-b)\eta^{\frac{k+1}{1-a}} \int_{t_0}^{s_0} q(s) e^{(1-b)\int_{s}^{s_k} |r(\varrho)| d\varrho} ds \\ &+ (1-b)\sum_{j=1}^{k-1} \eta^{\frac{k+1-j}{1-a}} \int_{t_j}^{s_j} q(s) e^{(1-b)\int_{s}^{s_k} |r(\varrho)| d\varrho} ds \\ &+ (1-b)\eta^{\frac{1}{1-a}} \int_{t_k}^{s_k} q(s) e^{(1-b)\int_{s}^{s_k} |r(\varrho)| d\varrho} ds\end{aligned}$$

holds for $t \in (s_k, t_{k+1}]$, $k = 2, 3, \cdots, m-1$.

Therefore, one gets

$$\begin{aligned}Z(t) \geq &\eta^{\frac{m}{1-a}} Z(0) e^{(1-b)\int_{t_0}^{t} |r(\varrho)| d\varrho} + (1-b)\eta^{\frac{m}{1-a}} e^{(1-b)\int_{t_0}^{t} |r(s)| ds} \int_{t_0}^{s_0} q(s) ds \\ &+ (1-b)\sum_{j=1}^{m-1} \eta^{\frac{m-j}{1-a}} e^{(1-b)\int_{t_j}^{t} |r(s)| ds} \int_{t_j}^{s_j} q(s) ds \\ &+ (1-b) e^{(1-b)\int_{t_m}^{t} |r(s)| ds} \int_{t_m}^{t} q(s) ds\end{aligned} \tag{3.29}$$

holds for $t \in [t_m, +\infty]$.

The following discussion is divided into two cases according to the value of η.

Case I. $0 < \eta < 1$. From condition (A_{10}), formula (3.29) and $b > 1$, we have

$$\begin{aligned}Z(t) &\geq \eta^{\frac{m}{1-a}} (1-b) e^{(1-b)\int_{0}^{+\infty} |r(\varrho)| d\varrho} \int_{t_0}^{t} q(s) ds \\ &\geq \eta^{\frac{m}{1-a}} (1-b) e^{(1-b)G} (-\xi(t-t_0) + K).\end{aligned} \tag{3.30}$$

Set $T_1 = t_0 + \frac{1+(b-1)\eta^{\frac{m}{1-a}} e^{(1-b)G} K}{(b-1)\xi\eta^{\frac{m}{1-a}} e^{(1-b)G}}$, inequality (3.30) implies that $Z(T_1) \geq 1$. From $Z(T_1) = V^{1-b}(T_1, X(T_1))$ and $b > 1$, we get $V(T_1, X(T_1)) \leq 1$. This contradicts $V(t, X(t)) > 1$ for all $t \in [t_0, T_1]$.

Case II. $\eta \geq 1$. From condition (A_{10}), formula (3.29) and $b > 1$, we have

$$\begin{aligned}Z(t) &\geq (1-b) e^{(1-b)\int_{0}^{+\infty} |r(\varrho)| d\varrho} \int_{t_0}^{t} q(s) ds \\ &\geq (1-b) e^{(1-b)G} (-\xi(t-t_0) + K).\end{aligned} \tag{3.31}$$

Set $T_1 = t_0 + \frac{1+(b-1)e^{(1-b)G} K}{(b-1)\xi e^{(1-b)G}}$, formula (3.31) implies that $Z(T_1) \geq 1$, that is $V(T_1, X(T_1)) \leq 1$. This contradicts $V(t, X(t)) > 1$ for all $t \in [t_0, T_1]$.

Similar to the proof of Theorem 3.2, we derive the solution of NISDE (2.1) is finite-time uniformly stable. Therefore, according to the result of settling time in Theorem 3.2, we derive that since $V^{1-a}(\tilde{t}, \tilde{X}) < 1$, $t_0 < \tilde{t} \le T_1$, the inequality $V(t, X(t)) \equiv 0$ holds, for

$$t \geq t_0 + \frac{e^{(1-a)G}\eta^{\frac{m}{1-b}} + H(1-a)}{\gamma(1-a)} + \frac{1 + (b-1)Ke^{(1-b)G}\eta^{\frac{m}{1-a}}}{(b-1)\xi e^{(1-b)G}\eta^{\frac{m}{1-a}}}, \quad 0 < \eta < 1$$

and

$$t \geq t_0 + \frac{e^{(1-a)G} + (1-a)H\eta^{\frac{m}{1-b}}}{(1-a)\gamma\eta^{\frac{m}{1-b}}} + \frac{1 + (b-1)Ke^{(1-b)G}}{(b-1)\xi e^{(1-b)G}}, \quad \eta \geq 1.$$

Therefore, the solution of NISDE (2.1) is fixed-time uniformly stable. □

To verify the validity of the result in this paper, we give the following example.

Example 3.1. Consider the following system:

$$\begin{cases} D_H X = r(t)X(t) + p(t), \ t \in (t_k, s_k] \cup (t_m, +\infty), \\ X(t) = \eta X(s_k - 0), \ t \in (s_k, t_{k+1}], \\ X(t_0) = X_0, \end{cases} \tag{3.32}$$

where $r(t) = \frac{1}{1+t^2}$, $X(t) = [x_1(t), x_2(t)]$ and $p(t) = [-t|\sin t|, -t|\cos t|]$ are interval functions, $k = 0, 1, \ldots, m-1$.

Choosing the Lyapunov function as $V(t) = (D[X(t), \theta])^2 = x_1^2(t) + x_2^2(t)$, then one has $\dot{V}(t) \leq 2r(t)V(t) + 2\sqrt{2}q(t)V^{\frac{1}{2}}(t)$ for $t \in (t_k, s_k] \cup (t_m, +\infty)$ and $V(t) \leq \eta^2 V(s_k - 0)$ for $t \in (s_k, t_{k+1}]$, where $q(t) = \max\{-t|\sin t|, -t|\cos t|\}$, $k = 0, 1, \ldots, m-1$.

Since $\int_0^{+\infty} 2r(\varrho)d\varrho = \pi$ and $\int_{t_0}^{t} 2\sqrt{2}q(\varrho)d\varrho \leq -\frac{8\sqrt{2}}{3\pi}(t-t_0)+16\sqrt{2}$, we can find that the conditions of Theorem 3.2 are satisfied, where $a = 0.5$, $G = \pi$, $\gamma = \frac{8\sqrt{2}}{3\pi}$, $H = 16\sqrt{2}$. Therefore, the trivial solution of system (3.32) is finite-time uniformly stable.

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