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SOLUTIONS OF THE YANG-BAXTER-LIKE MATRIX EQUATION FOR THE MATRIX WITH NONSINGULAR JORDAN BLOCKS*

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Abstract Let A be a nonsingular matrix with only one Jordan block, we prove that the Yang-Baxter-like matrix equation $AXA = XAX$ has no nonzero singular solution. When A is a nonsingular matrix with at least two Jordan blocks, the ranks of all nonzero singular solutions are obtained. This provides a necessary condition for a matrix to be a solution of the Yang-Baxter-like matrix equation. As applications, we obtain a family of nontrivial solutions for the nonsingular Jordan block with 3×3, and further investigate the non-commuting solutions for the nonsingular matrix with $n \times n$.

Keywords Yang-Baxter-like matrix equation, group inverse, non-commuting solution, Jordan block.

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1. Introduction

Let $\mathrm{C}^{m\times n}$ denote the set of all matrices with $m \times n$ over the complex field, and

$$\mathrm{C}_r^{m\times n} = \{A \in \mathrm{C}^{m\times n} : \text{rank } A = r\}.$$

The matrix equation

$$AXA = XAX, \tag{1.1}$$

where $A \in \mathrm{C}^{n\times n}$ is a given square matrix, is called the Yang-Baxter-like matrix equation. The equation (1.1) is closely related to the Yang-Baxter equation (independently introduced by C. N. Yang and R. J. Baxter, in statiscial mechanics)

$$A(u)B(u + v)A(v) = B(v)A(u + v)B(u),$$

where A and B are the parameter function about u and v. If A and B are independent from u and v, we obtain (1.1). The Yang-Baxter equation has been extensively studied in the past decades. For more information we refer to [4, 10, 15].

The Yang-Baxter-like matrix equation (1.1) has two trivial solutions, $X = A$ and $X = 0$. But finding all solutions is a complex task, which is equivalent to solving a

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system of n^2 quadratic equations, even the following 3×3 matrix equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

The existence of nontrivial solutions to the equation (1.1) is a popular research topic, and there have been many established tools and methods to solve (1.1) with the specific matrix A (see [1–3, 5–9, 11–13, 16, 17]). For example, if A is a nonsingular quasi-stochastic matrix such that A^{-1} is a stochastic matrix, the Brouwer fixed point theorem was used to prove that (1.1) has a nontrivial solution [5]. In [7], all of the solutions which commute with A were found by the Jordan form structure of A. Using the technique of diagonalization, the authors obtained some explicit solutions for an idempotent matrix A in [1], all commuting solutions and some non-commuting solutions for the matrix A satisfying $A^2 = I$ in [2].

When the matrix A is singular, all nonzero solutions of the homogeneous equation $AX = 0$ are nontrivial singular solutions of (1.1). For any square matrix with at least two Jordan blocks, some non-trivial singular solutions can be obtained easily by a constructive way [7]. A natural problem is whether we can find some nontrivial singular solutions, when the matrix A is nonsingular with only one Jordan block. In this paper, we use a new approach based on the partitioned skill of group inverse and give a complete answer to this problem. Let us recall the notion of the group inverse.

Definition 1.1. Let $A \in \mathrm{C}^{n\times n}$. The matrix $X \in \mathrm{C}^{n\times n}$ is said to be the group inverse of A, always denoted by $A^\sharp$, if X satisfies

$$(1)\ AXA = A \quad (2)\ XAX = X \quad (3)\ AX = XA.$$

That $A^\sharp$ exists is also called that A is group invertible. It is obvious that if A is nonsingular, then A is group invertible and $A^\sharp = A^{-1}$. As we all know, a singular matrix A is group invertible if and only if rank A = rank A^2 if and only if A has the following block matrix form (see Theorem 2.2.1 and Theorem 2.2.2 in [14]).

Lemma 1.1 ([14]). *Let $A \in \mathrm{C}_r^{n\times n}$ and $1 \leq r < n$. Then A is group invertible if and only if there are two invertible matrices $P \in \mathrm{C}_n^{n\times n}$ and $C \in \mathrm{C}_r^{r\times r}$ such that*

$$A = P \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

The paper is organized as follows. In Section 2, we prove that (1.1) has no nonzero singular solution for the nonsingular matrix A with only one Jordan block. Furthermore, for any nonsingular matrix, the ranks of all nonzero singular solutions are obtained. This provides a necessary condition for a matrix to be a solution of the Yang-Baxter-like matrix equation. As applications, in Section 3, we find a family of non-commuting solutions for the nonsingular matrix A with 3×3. Finally, we obtain some sufficient conditions on the existence of non-commuting solutions for the nonsingular matrix A with $n \times n$.

2. No nonzero singular solution

Theorem 2.1. *Let $A \in \mathrm{C}_n^{n\times n}$ with only one Jordan block, then (1.1) has no nonzero singular solution.*

Proof. Suppose that $X \in \mathrm{C}_r^{n\times n}(1 \leq r < n)$ is a nonzero singular solution to (1.1), then

$$AAXA = AXAX$$

and so

$$r = \operatorname{rank}(AX) = \operatorname{rank}(AAXA) = \operatorname{rank}(AXAX).$$

This implies that AX is group invertible. By Lemma 1.1, we can find two invertible matrices $P \in \mathrm{C}_n^{n\times n}$ and $C \in \mathrm{C}_r^{r\times r}$ such that

$$AX = P \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

Hence, the equation (1.1) can be written equivalently as

$$P \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} P^{-1}A = A^{-1}P \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} P^{-1}P \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} P^{-1},$$

which means

$$P^{-1}AP \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} P^{-1}AP = \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $P^{-1}AP$ be partittioned into the 2×2 block matrix

$$P^{-1}AP = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix},$$

where $Y_1 \in \mathrm{C}^{r\times r}$ and $Y_4 \in \mathrm{C}^{(n-r)\times(n-r)}$, then

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} = \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} Y_1CY_1 & Y_1CY_2 \\ Y_3CY_1 & Y_3CY_2 \end{pmatrix} = \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows from

$$r = \operatorname{rank} C^2 = \operatorname{rank}\,(Y_1CY_1) \leq \operatorname{rank} Y_1 \leq r$$

that the matrix Y_1 is invertible. So both Y_2 and Y_3 are zero matrices. This shows

$$A = P \begin{pmatrix} Y_1 & 0 \\ 0 & Y_4 \end{pmatrix} P^{-1}.$$

Let the Jordan canonical form of A be

$$J(\lambda,n)=\begin{pmatrix}\lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda\end{pmatrix},$$

by the transitivity of similar matrices, we know that

$$\begin{pmatrix}Y_1 & 0\\ 0 & Y_4\end{pmatrix} \text{ is similar to } \begin{pmatrix}\lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda\end{pmatrix}.$$

However, the elementary divisor of the partitioned matrix is composed of the ones of Y_1 and Y_4, and the elementary divisor of $J(\lambda,n)$ is only $(x-\lambda)^n$. This implies that these two matrices are not similar. This completes the proof. □

Example 2.1. Let

$$A=\begin{pmatrix}\lambda & 0\\ 1 & \lambda\end{pmatrix},\ \lambda\neq 0.$$

From Theorem 2.1, (1.1) has no nonzero singular solution. We would like to remark that this conclusion can be drawn by manual calculation. For instance, in [16], all nontrivial solutions of (1.1) are given by the one-parameter matrices

$$X(t)=\begin{pmatrix}t & -\lambda^2\\ (\frac{t}{\lambda}-1)^2 & 2\lambda-t\end{pmatrix},\quad t\in \mathrm{C},$$

and consequently, the determinant $|X(t)|=t(2\lambda-t)+\lambda^2(\frac{t}{\lambda}-1)^2=\lambda^2\neq 0$. Therefore, (1.1) has no nonzero singular solution.

Example 2.2. Let

$$A=\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix},$$

then (1.1) has nontrivial two-parameter solutions:

$$\left\{X(s,t)=\frac{1}{4}\begin{pmatrix}16+6s-4t+2s^2 & -12-4s & 8\\ 8+10s-2st+3s^2+s^3 & -4-6s-2s^2 & 4s\\ -2+3s+8t+3st+s^2-2t^2+s^2t & 2-2s-6t-2st & 4t\end{pmatrix}: s,t\in \mathrm{C}\right\}.$$

It should be noted that $X(s,t)$ is invertible with $|X(s,t)| = 1$.

Corollary 2.1. *Let $A \in C_n^{n\times n}$ with only one Jordan block, then the commuting solutions of (1.1) are trivial.*

Proof. Let X be a nonzero commuting solution to (1.1), then by Theorem 2.1, X is invertible and so it follows from $AXA = XAX$ and $AX = XA$ that $X = A$. This completes the proof. □

From Corollary 2.1, we can see that the matrix $X(s,t)$ in Example 2.2 is a non-commuting solution. The next corollary is an immediate consequence of Theorem 2.1, we omit the proof.

Corollary 2.2. *Let $A \in \mathrm{C}_n^{n\times n}$, then (1.1) has nonzero singular solutions if and only if A has at least two Jordan blocks.*

The following theorem points out the ranks of all nonzero singular solutions, which provides a necessary condition for a matrix to be a solution of the Yang-Baxter-like matrix equation.

Theorem 2.2. *Let $A \in \mathrm{C}_n^{n\times n}$, if (1.1) has nonzero singular solutions, i.e., the Jordan canonical form of A is*

$$J = \begin{pmatrix} J(\lambda_1,k_1) & & & \\ & J(\lambda_2,k_2) & & \\ & & \ddots & \\ & & & J(\lambda_s,k_s) \end{pmatrix}, \ s \geq 2.$$

Then the set to ranks of all nonzero singular solutions is

$$\left\{ \sum_{i\in I} k_i : \ \forall I \subset \{1,2,...,s\}, \ \emptyset \neq I \neq \{1,2,...,s\} \right\}.$$

Proof. On one hand, let $X \in \mathrm{C}_r^{n\times n}$ be a nonzero singular solution to (1.1). From the proof of Theorem 2.1, the matrix A is similar to

$$\begin{pmatrix} Y_1 & 0 \\ 0 & Y_4 \end{pmatrix},$$

where $Y_1 \in C^{r\times r}$ and $Y_4 \in C^{(n-r)\times(n-r)}$. Thus, all elementary divisors of Y_1 come from ones of A and suppose that elementary divisors of Y_1 are

$$(x-\lambda_{i_1})^{k_{i_1}}, (x-\lambda_{i_2})^{k_{i_2}}, \cdots, (x-\lambda_{i_t})^{k_{i_t}},$$

where $\{i_1, i_2, ..., i_t\} \subset \{1,2,...,s\}$. Considering $Y_1 \in C^{r\times r}$, we can get $r = \sum_{j=1}^{t} k_{i_j}$. On the other hand, we can claim that for any nonempty proper subset $I \subset \{1,2,...,s\}$, $\sum_{i\in I} k_i$ is the rank of some nontrivial singular solution. In fact, we can take the same Jordan block $J(\lambda_i,k_i)$ for any $i \in I$ and the Jordan block as 0 for all $i \notin I$. Hence we obtain a nontrivial singular solution with the rank $\sum_{i \in I} k_i$. For example, suppose $I = \{1, 3\}$, then

$$P \begin{pmatrix} J(\lambda_1, k_1) & & & & & \\ & 0 & & & & \\ & & J(\lambda_3, k_3) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} P^{-1},$$

is a nontrivial singular solution to (1.1), where $P \in \mathrm{C}_n^{n \times n}$ satisfying $A = PJP^{-1}$. This completes the proof. $\square$

Example 2.3. Let

$$A = \begin{pmatrix} 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 3 & -1 \\ 5 & -2 & 2 & -4 & -2 \\ 1 & -1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 5 & 2 \end{pmatrix}$$

and its Jordan canonical form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Thus, the set of the ranks of all nonzero singular solutions to (1.1) is $\{2, 3\}$, and so, any matrix with rank 1 or rank 4 can not be the solution of (1.1).

Example 2.4. Suppose that the elementary divisors of the nonsingular matrix A with 20×20 are $(x-1)^2, (x-1)^8, (x-2)^5, (x-3)^5$. From Theorem 2.2, the set of ranks of all nonzero singular solutions is

$$\{2,\ 5,\ 8,\ 2+5,\ 5+5,\ 5+8,\ 2+5+5,\ 2+5+8,\ 5+5+8\}.$$

Thus, the set of ranks of all nonzero singular solutions to (1.1) is $\{2, 5, 7, 8, 10, 12, 13, 15, 18\}$, and (1.1) has no solution with the rank in $\{1, 3, 4, 6, 9, 11, 14, 16, 17, 19\}$.

3. Non-commuting solutions

Theorem 3.1. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \textit{and} \quad J = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}.$$

If the matrix

$$\begin{pmatrix} a & b & c \\ d & 0 & f \\ 0 & h & i \end{pmatrix}$$

is a solution of (1.1), then

$$\begin{pmatrix} a\lambda & b\lambda^2 & c\lambda^3 \\ d & 0 & f\lambda^2 \\ 0 & h & i\lambda \end{pmatrix}$$

is a solution of the equation $XJX = JXJ$.

Proof. A direct computation yields

$$\begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} \begin{pmatrix} a\lambda & b\lambda^2 & c\lambda^3 \\ d & 0 & f\lambda^2 \\ 0 & h & i\lambda \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}$$
$$= \begin{pmatrix} (a+b)\lambda^3 & (b+c)\lambda^4 & c\lambda^5 \\ (a+b+d)\lambda^2 & (b+c+f)\lambda^3 & (c+f)\lambda^4 \\ (d+h)\lambda & (f+h+i)\lambda^2 & (f+i)\lambda^3 \end{pmatrix}$$

and

$$\begin{pmatrix} a\lambda & b\lambda^2 & c\lambda^3 \\ d & 0 & f\lambda^2 \\ 0 & h & i\lambda \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} \begin{pmatrix} a\lambda & b\lambda^2 & c\lambda^3 \\ d & 0 & f\lambda^2 \\ 0 & h & i\lambda \end{pmatrix}$$
$$= \begin{pmatrix} (a^2+ab+bd+cd)\lambda^3 & (ab+b^2+ch)\lambda^4 & (ac+bc+bf+cf+ci)\lambda^5 \\ (ad+df)\lambda^2 & (bd+fh)\lambda^3 & (dc+f^2+fi)\lambda^4 \\ (ah+hd+id)\lambda & (bh+hi)\lambda^2 & (ch+hf+if+i^2)\lambda^3 \end{pmatrix}.$$

Considering that

$$\begin{pmatrix} a & b & c \\ d & 0 & f \\ 0 & h & i \end{pmatrix}$$

is a solution of (1.1), we can complete the proof. □

Corollary 3.1. Let

$$J = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix},$$

then

$$\begin{pmatrix} 2\lambda & -2\lambda^2 & 2\lambda^3 \\ \frac{1}{2} & 0 & -\lambda^2 \\ 0 & 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & -2\lambda^2 & 2\lambda^3 \\ 1 & 0 & -\lambda^2 \\ 0 & -1 & 2\lambda \end{pmatrix}$$

and

$$\begin{pmatrix} 2\lambda & -\lambda^2 & 2\lambda^3 \\ -1 & 0 & -2\lambda^2 \\ 0 & 1 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & -\lambda^2 & 2\lambda^3 \\ 0 & 0 & -2\lambda^2 \\ 0 & \frac{1}{2} & 2\lambda \end{pmatrix}$$

are non-commuting solutions of $XJX = JXJ$.

Proof. In Example 2.2, we take

$$(s,t) = (-1,1),\ (-1,2),\ (-2,1) \text{ and } (-2,2)$$

and then

$$-4 - 6s - 2s^2 = 0, \quad -2 + 3s + 8t + s^2 + 3st - 2t^2 + s^2 t = 0.$$

Hence

$$\begin{pmatrix} 2 & -2 & 2 \\ \frac{1}{2} & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 & 2 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 2 \\ -1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & -2 \\ 0 & \frac{1}{2} & 2 \end{pmatrix}$$

are four solutions of (1.1). Therefore, by Theorem 3.1 and Corollary 2.1, we can get what we desired. □

In order to find more solutions to $AXA = XAX$, we need the following proposition.

Proposition 3.1. *Let $A \in \mathbb{C}_n^{n\times n}$ and X be a nonzero solution of (1.1), then*

(i) If X is nonsingular, X is similar to A;

(ii) If A has only one Jordan block, X is similar to A;

(iii) For any integer k, A^kXA^{-k} is a solution of (1.1).

Proof. We only need to prove that (i) and (iii). If X is nonsingular, then the equation (1.1) can be written in the form

$$A = (XA)X(XA)^{-1},$$

hence X is similar to A. For any integer k,

$$\begin{aligned}(A^kXA^{-k})A(A^kXA^{-k}) &= A^kXAXA^{-k}\\ &= A^kAXAA^{-k}\\ &= A(A^kXA^{-k})A.\end{aligned}$$

This completes the proof. □

Summarizing Proposition 3.1, if a nonzero solution of (1.1) is found, we can obtain a family of infinite many similar solutions.

Corollary 3.2. Let

$$A = \begin{pmatrix} \lambda & 0 & 0\\ 1 & \lambda & 0\\ 0 & 1 & \lambda \end{pmatrix},\ \lambda \neq 0,$$

then for any integer k,

$$\begin{pmatrix} (2+3k+k^2)\lambda & (-2-2k)\lambda^2 & 2\lambda^3\\ \frac{1+3k+5k^2+2k^3}{2} & (-k-2k^2)\lambda & (-1+2k)\lambda^2\\ \frac{-k^2+k^3+k^4}{2\lambda} & k^2-k^3 & (1-2k+k^2)\lambda \end{pmatrix}$$

is a solution of the equation $XAX = AXA$.

Proof. Since

$$X_1 = \begin{pmatrix} 2\lambda & -2\lambda^2 & 2\lambda^3\\ \frac{1}{2} & 0 & -\lambda^2\\ 0 & 0 & \lambda \end{pmatrix}$$

is a solution of the equation $XAX = AXA$ and

$$A^k = \begin{pmatrix} \lambda^k & 0 & 0\\ k\lambda^{k-1} & \lambda^k & 0\\ \frac{k^2-k}{2}\lambda^{k-2} & k\lambda^{k-1} & \lambda^k \end{pmatrix},\ A^{-k} = \begin{pmatrix} \lambda^{-k} & 0 & 0\\ -k\lambda^{-k-1} & \lambda^{-k} & 0\\ \frac{k^2+k}{2}\lambda^{-k-2} & -k\lambda^{-k-1} & \lambda^{-k} \end{pmatrix},$$

we know that

$$A^kX_1A^{-k} = \begin{pmatrix} (2+3k+k^2)\lambda & (-2-2k)\lambda^2 & 2\lambda^3\\ \frac{1+3k+5k^2+2k^3}{2} & (-k-2k^2)\lambda & (-1+2k)\lambda^2\\ \frac{-k^2+k^3+k^4}{2\lambda} & k^2-k^3 & (1-2k+k^2)\lambda \end{pmatrix}$$

is a solution of $XAX = AXA$. This completes the proof. $\square$

Obviously, if $A = \lambda I$ ($\lambda \neq 0$), then (1.1) has no non-commuting solutions, since it has only the commuting solution X which satisfies $X^2 = \lambda X$. Now we are interested in looking for some non-commuting solutions for the matrix $A \neq \lambda I$.

Theorem 3.2. *Let A be a nonsingular matrix with 3×3, then (1.1) has non-commuting solutions if and only if $A \neq \lambda I$.*

Proof. We only consider the sufficiency. Let λ, μ and ν be three distinct nonzero complex numbers, then the Jordan canonical forms J of A must be one of the followings:

$$(1)\ J = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix};\ (2)\ J = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix};\ (3)\ J = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix};$$

$$(4)\ J = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \text{ and } (5)\ J = \begin{pmatrix} \nu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

In case (1), if

$$A = P \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} P^{-1},$$

then

$$X = P \begin{pmatrix} (2+3k+k^2)\lambda & (-2-2k)\lambda^2 & 2\lambda^3 \\ \frac{1+3k+5k^2+2k^3}{2} & (-k-2k^2)\lambda & (-1+2k)\lambda^2 \\ \frac{-k^2+k^3+k^4}{2\lambda} & k^2-k^3 & (1-2k+k^2)\lambda \end{pmatrix} P^{-1},\ k \in \mathrm{Z}$$

is a non-commuting invertible solution.

The cases (2) and (3) are similar, we only prove the case (3), if

$$A = P \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} P^{-1},$$

then

$$X = P \begin{pmatrix} t & -\lambda^2 & 0 \\ (\frac{t}{\lambda}-1)^2 & 2\lambda - t & 0 \\ 0 & 0 & a \end{pmatrix} P^{-1},\ t \in \mathrm{C}$$

is a non-commuting singular solution when $a = 0$ and a non-commuting nonsingular solution when $a = \mu$.

The cases (4) and (5) are similar, and we consider the case (5). If

$$A = P \begin{pmatrix} \nu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} P^{-1},$$

then

$$X = \frac{1}{(\lambda-\mu)^2 s} P \begin{pmatrix} b & 0 & 0 \\ 0 & \mu^2(\mu-\lambda)s & (\lambda-\mu)^2 s^2 \\ 0 & -\lambda\mu(\lambda^2-\lambda\mu+\mu^2) & \lambda^2(\lambda-\mu)s \end{pmatrix} P^{-1}, \ 0 \neq s \in \mathrm{C}$$

is a non-commuting singular solution when $b = 0$ and a non-commuting invertible solution when $b = (\lambda-\mu)^2 s\nu \neq 0$. This completes the proof. □

Theorem 3.3. *Let $A \in \mathrm{C}_n^{n\times n}$ and suppose that the Jordan canonical form of A is*

$$J = \begin{pmatrix} J(\lambda_1, k_1) & & & \\ & J(\lambda_2, k_2) & & \\ & & \ddots & \\ & & & J(\lambda_s, k_s) \end{pmatrix}, \ s \geq 2,$$

then the following statements are true:

(i) if there are two distinct eigenvalues λ_i and λ_j such that $k_i = k_j = 1$, then (1.1) has non-commuting solutions;

(ii) if there is a $J(\lambda_i, k_i)$ such that $k_i = 2$ or 3, then (1.1) has non-commuting solutions.

Proof. In case (i), we can choose the suitable nonsingular matrix P such that

$$A = P \begin{pmatrix} \lambda_i & & & & \\ & \lambda_j & & & \\ & & * & & \\ & & & \ddots & \\ & & & & * \end{pmatrix} P^{-1}$$

and verify that

$$X_1 = \frac{1}{(\lambda_i-\lambda_j)^2} \begin{pmatrix} \lambda_j^2(\lambda_j-\lambda_i) & (\lambda_i-\lambda_j)^2 \\ -\lambda_i\lambda_j(\lambda_i^2-\lambda_i\lambda_j+\lambda_j^2) & \lambda_i^2(\lambda_i-\lambda_j) \end{pmatrix}$$

is a non-commuting solution of the equation

$$\mathrm{diag}(\lambda_{\mathrm{i}}, \lambda_{\mathrm{j}}) X \mathrm{diag}(\lambda_{\mathrm{i}}, \lambda_{\mathrm{j}}) = X \mathrm{diag}(\lambda_{\mathrm{i}}, \lambda_{\mathrm{j}}) X.$$

Then

$$X = P \begin{pmatrix} X_1 & & & \\ & * & & \\ & & \ddots & \\ & & & * \end{pmatrix} P^{-1}$$

is a non-commuting solution of (1.1).

In case (ii), for the sake of simplicity, we suppose $k_1 = 3$ and

$$A = P \begin{pmatrix} J(\lambda_1, k_1) & & & \\ & J(\lambda_2, k_2) & & \\ & & \ddots & \\ & & & J(\lambda_s, k_s) \end{pmatrix} P^{-1}.$$

From Corollary 3.1, we can find a non-commuting solution X_1 of the equation $J(\lambda_1, k_1) X J(\lambda_1, k_1) = X J(\lambda_1, k_1) X$. It follows that

$$X = P \begin{pmatrix} X_1 & & & \\ & J(\lambda_2, k_2) & & \\ & & \ddots & \\ & & & J(\lambda_s, k_s) \end{pmatrix} P^{-1},$$

is a non-commuting solution of (1.1). This completes the proof. □

Example 3.1. Suppose that the Jordan canonical form of A_1 and A_2 is

$$J_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \lambda \end{pmatrix}, \ \lambda \neq 0,$$

respectively, then both $A_1 X A_1 = X A_1 X$ and $A_2 X A_2 = X A_2 X$ has non-commuting solutions.

4. Conclutions

We have proved that the Yang-Baxter-like matrix equation $AXA = XAX$ has no nonzero singular solution, when the nonsingular matrix A has only one Jordan block. If the nonsingular matrix A has at least two Jordan block, the ranks of all nonzero singular solutions are given. We find some non-commuting solutions for any nonsingular 3×3 matrix $A \neq \lambda I$ and some nonsingular $n \times n$ matrices. For more matrices and further applications, we shall discuss them in our consequent papers.

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