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NEW CRITERIA FOR CLOSE-TO-CONVEXITY AND SPIRALLIKENESS*

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Abstract The aim of the present paper is to investigate several new criteria on a function to be strongly close-to-convexity, spirallikeness or starlikeness by using a new methodology.

Keywords Univalent functions, strongly close-to-convex functions, spirallike functions, starlike functions.

MSC(2010) 30C45.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and denote by $\mathcal{A}$ the class of analytic functions in $\mathcal{H}$ that are normalized by $f(0) = 0 = f'(0) - 1$. Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which is composed of functions which are univalent in $\mathbb{D}$.

A function f in $\mathcal{A}$ is said to be γ-spirallike of order α, $0 \le \alpha < 1$, if

$$\Re\left\{ e^{i\gamma} \frac{zf'(z)}{f(z)} \right\} > \alpha \cos\gamma, \quad z \in \mathbb{D},$$

for some real γ with $|\gamma| < \pi/2$. The class of the γ-spirallike functions of order α is denoted by $\mathcal{S}^\gamma(\alpha)$. The class $\mathcal{S}^\gamma(0)$, which consists of all γ-spirallike functions, was introduced by Špaček (see [10] or [2, p.52]). We recall that a set $\mathcal{G} \subset \mathbb{C}$ is called starlike with respect to the origin (or starlike) if the straight line joining any point in $\mathcal{G}$ to the origin lies in $\mathcal{G}$, i.e., $tz \in \mathcal{G}$ when $z \in \mathcal{G}$ and $t \in [0, 1]$ (*cf.* [2, p. 40]). We note that, when $\gamma = 0$ and $\alpha = 0$, the class $\mathcal{S}^\gamma(\alpha)$ reduces the class $\mathcal{S}^*$, which consists f such that $f(\mathbb{D})$ is a starlike with respect to the origin. The elements in $\mathcal{S}^*$ are called starlike functions. We also note that $\mathcal{S}^\gamma(\alpha) \subset \mathcal{S}$.

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To find conditions for γ-spirallike functions (as well as for univalent functions) is one of the main problems to study. In [5], several sufficient conditions for functions in $\mathcal{S}^{\gamma}(\alpha)$ were investigated. Some coefficients problems of these functions were raised and solved in [1] and [6]. More recent informations concerning the functions in $\mathcal{S}^{\gamma}(\alpha)$ have also found in [4] and [9].

Let $\beta \in (0,1]$. If $f \in \mathcal{A}$ satisfies

$$\left|\arg\left\{e^{i\varphi}\frac{zf'(z)}{g(z)}\right\}\right| < \frac{\pi}{2}\beta, \quad z \in \mathbb{D},$$

for some $g \in \mathcal{S}^*$ and some $\varphi \in (-\pi/2, \pi/2)$, then f is said to be strongly close-to-convex (with respect to g) in $\mathbb{D}$. Let us denote by $\mathcal{C}^{\beta}$ the class of strongly close-to-convex in $\mathbb{D}$. Especially, when $\beta = 1$, we have $\mathcal{C}^1 \equiv \mathcal{C}$ the class of close-to-convex functions was introduced by Kaplan [3]. We note that every close-to-convex function is univalent [2, p.47]. So, it holds that $\mathcal{C}^{\beta} \subset \mathcal{S}$ for $\beta \in (0,1]$. Several geometric properties of functions in a particular subclass of $\mathcal{C}^{\beta}$ were recently introduced in [8].

We say that $f \in \mathcal{A}$ is a convex function in $\mathbb{D}$ if $zf' \in \mathcal{A}$ is starlike in $\mathbb{D}$. Therefore, if $f \in \mathcal{A}$ satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D},$$

then f is convex in $\mathbb{D}$. We note that if $f \in \mathcal{A}$ satisfies

$$\Re\left\{e^{i\alpha}\frac{f'(z)}{g'(z)}\right\} > 0, \quad z \in \mathbb{D},$$

for some convex function $g \in \mathcal{A}$ and real number $\alpha \in (-\pi/2, \pi/2)$, then $f \in \mathcal{C}$.

Let $0 \le \beta < 1$. A function $p \in \mathcal{H}$ is called a Carathéodory function of order β if $p(0) = 1$ and it satisfies the condition

$$\Re\{p(z)\} > \beta, \quad z \in \mathbb{D}.$$

The class of the Carathéodory functions of order β will be denoted by $\mathcal{P}(\beta)$. Especially, we put $\mathcal{P}(0) \equiv \mathcal{P}$, which is the class of all Carathéodory functions.

In the present paper, we investigate several new criteria for strongly close-to-convexity, spirallikeness and starlikeness of functions in $\mathcal{A}$ using various new methods.

2. Main Results

Applying the same idea in [12], we can obtain the following result on Carathéodory functions.

Theorem 2.1. *Let $\beta \in (0,1]$. Let p be analytic in $\mathbb{D}$, $p(0) = 1$ and suppose that*

$$-\frac{\alpha\beta}{2\alpha - \beta} < \Re\left\{\frac{zp'(z)}{p(z)}\right\} < \alpha, \quad z \in \mathbb{D}, \tag{2.1}$$

where α is real and $\alpha > \beta/2$. Then $\left|\arg\left\{e^{i\varphi}p(z)\right\}\right| < \pi\beta/2$ for some φ with $|\varphi| < \pi\beta/2$.

Proof. Let $0 \le r < 1$ be given. From the hypothesis (2.1), we have $p(z) \neq 0$ in $\mathbb{D}$ and

$$\int_{|z|=r} \Re\left\{\frac{zp'(z)}{p(z)}\right\} \mathrm{d}\theta = \int_0^{2\pi} \frac{\mathrm{d}\arg p(z)}{\mathrm{d}\theta} \mathrm{d}\theta = 0. \tag{2.2}$$

Let $\partial\mathbb{D}_r := \{z \in \mathbb{C} : |z| = r\}$ and let us put C_1 the part of $\partial\mathbb{D}_r$ on which

$$\Re\left\{\frac{zp'(z)}{p(z)}\right\} > 0.$$

Put

$$l = \int_{C_1} \mathrm{d}\arg z \quad \text{and} \quad y_1 = \int_{C_1} \Re\left\{\frac{zp'(z)}{p(z)}\right\} \mathrm{d}\theta.$$

On the other hand, let us put C_2 the part of $\partial\mathbb{D}_r$ on which

$$\Re\left\{\frac{zp'(z)}{p(z)}\right\} \le 0,$$

and put

$$-y_2 = \int_{C_2} \Re\left\{\frac{zp'(z)}{p(z)}\right\} \mathrm{d}\theta.$$

Then, we have

$$\int_{C_2} \mathrm{d}\arg z = 2\pi - l.$$

From the hypothesis (2.1), we have

$$y_1 < \alpha l \tag{2.3}$$

and

$$y_2 < (2\pi - l)\frac{\alpha\beta}{2\alpha - \beta}.$$

Also, from (2.2), we have

$$y_1 - y_2 = 0. \tag{2.4}$$

Now, we shall show that $y_1 < \pi\beta$. Suppose that $y_1 \ge \pi\beta$. Then, from (2.3) and (2.4), we have

$$y_1 \ge \pi\beta,\ y_2 \ge \pi\beta \quad \text{and} \quad \pi\beta < \alpha l. \tag{2.5}$$

It follows from (2.3) that

$$y_2 < (2\pi - l)\frac{\alpha\beta}{2\alpha - \beta} < \pi\beta.$$

This contradicts (2.5). Therefore, we have $y_1 < \pi\beta$ and $y_2 < \pi\beta$. Hence, we have

$$\begin{aligned} y_1 + y_2 &= \int_{|z|=r} \left|\Re\left\{\frac{zp'(z)}{p(z)}\right\}\right| \mathrm{d}\theta \\ &= \int_{|z|=r} \left|\frac{\mathrm{d}\arg p(z)}{\mathrm{d}\theta}\right| \mathrm{d}\theta < 2\pi\beta. \end{aligned}$$

This shows that

$$\left|\arg\left\{e^{i\varphi}p(z)\right\}\right| < \pi\beta/2, \quad z \in \mathbb{D},$$

for some φ with $|\varphi| < \pi\beta/2$. $\square$

Let $0 \le \gamma < 1$. Putting $\beta = 1/2$ and replacing the function p by $(p-\gamma)/(1-\gamma)$ in Theorem 2.1, we can easily obtain the following result.

Corollary 2.1. *Let p be analytic in $\mathbb{D}$ and suppose that*

$$-\frac{\alpha}{4\alpha-1} < \Re\left\{\frac{zp'(z)}{p(z)-\gamma}\right\} < \alpha, \quad z \in \mathbb{D},$$

where $1/4 \le \alpha$ and $0 \le \gamma < 1$. Then $p \in \mathcal{P}(\gamma)$.

Letting $\alpha \to 1/4^-$ or $\alpha \to \infty$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $\beta \in (0,1]$. Let p be analytic in $\mathbb{D}$ and $p(0)=1$. If p satisfies one of the following conditions*

$$\Re\left\{\frac{zp'(z)}{p(z)}\right\} < \frac{\beta}{2}, \quad z \in \mathbb{D},$$

or

$$-\frac{\beta}{2} < \Re\left\{\frac{zp'(z)}{p(z)}\right\}, \quad z \in \mathbb{D}.$$

Then $\left|\arg\left\{e^{i\varphi}p(z)\right\}\right| < \pi\beta/2$ for some φ with $|\varphi| < \pi\beta/2$.

If we put $p(z) = zf'(z)/g(z)$, where f and $g \in \mathcal{A}$, in Theorem 2.1, then we can obtain the following corollary.

Corollary 2.3. *If $f \in \mathcal{A}$ and $g \in \mathcal{S}^*$ satisfy*

$$-\frac{\alpha\beta}{2\alpha-\beta} < \Re\left\{1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right\} < \alpha, \quad z \in \mathbb{D},$$

where $0 < \beta \le 1$ and $\alpha > \beta/2$, then f is a strongly close-to-convex function of order β.

Now, we find another sufficient conditions for functions in $\mathcal{A}$ to be close-to-convex.

Theorem 2.2. *Let $\beta \in (0,1]$ and $f \in \mathcal{A}$. Suppose that there exists a convex function $g \in \mathcal{A}$ such that*

$$\left|\frac{zf''(z)}{f'(z)}-\frac{zg''(z)}{g'(z)}\right| < \sqrt{2}\beta, \quad z \in \mathbb{D}. \tag{2.6}$$

Then f is a strongly close-to-convex function of order β.

Proof. Let $0 \le r < 1$. From the hypothesis (2.6), we have

$$\int_{|z|=r}\left|\frac{zf''(z)}{f'(z)}-\frac{zg''(z)}{g'(z)}\right|^2 \mathrm{d}\theta < 4\pi\beta^2.$$

Since

$$\int_{|z|=r}\left(\frac{zf''(z)}{f'(z)}-\frac{zg''(z)}{g'(z)}\right)^2 \mathrm{d}\theta = 0,$$

we have

$$\begin{aligned}
&\int_{|z|=r} \left\{\Re\left\{\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right\}\right\}^2 \mathrm{d}\theta \\
=&\frac{1}{4}\int_{|z|=r}\left[\left(\frac{zf''(z)}{f'(z)}-\frac{zg''(z)}{g'(z)}\right)^2+2\left|\frac{zf''(z)}{f'(z)}-\frac{zg''(z)}{g'(z)}\right|^2+\left(\overline{\frac{zf''(z)}{f'(z)}-\frac{zg''(z)}{g'(z)}}\right)^2\right]\mathrm{d}\theta \\
<&2\pi\beta^2.
\end{aligned} \tag{2.7}$$

It follows from Cauchy-Schwarz inequality and (2.7) that

$$\begin{aligned}
&\int_{|z|=r} \left|\mathrm{d}\arg\left(\frac{f'(z)}{g'(z)}\right)\right| \mathrm{d}\theta \\
=&\int_{|z|=r} \left|\Re\left\{\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right\}\right| \mathrm{d}\theta \\
\le&\sqrt{2\pi\int_{|z|=r} \left\{\Re\left\{\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right\}\right\}^2 \mathrm{d}\theta} \\
<&2\pi\beta.
\end{aligned}$$

This shows that

$$\left|\arg\left\{e^{i\gamma}\frac{f'(z)}{g'(z)}\right\}\right| < \frac{\pi}{2}\beta, \quad z \in \mathbb{D},$$

for some $\gamma \in \mathbb{R}$ with $|\gamma| < \pi\beta/2$. Thus, f is a strongly close-to-convex function of order β. □

Theorem 2.3. *Let $f \in \mathcal{A}$ and there exists a function $g \in \mathcal{S}^*$ such that*

$$\int_0^{2\pi} \left|\frac{zf'(z)}{g(z)}\right|^2 \mathrm{d}\theta \le 4\pi, \quad z \in \mathbb{D}. \tag{2.8}$$

Then $f \in \mathcal{C}$.

Proof. Let $0 < r < 1$. From (2.8), we have

$$\int_0^{2\pi} \left|\frac{zf'(z)}{g(z)}\right|^2 \mathrm{d}\theta \le 4\pi,$$

for $|z| = r$. And so, we have

$$\begin{aligned}
&\int_0^{2\pi} \left(\Re\left\{\frac{zf'(z)}{g(z)}\right\}\right)^2 \mathrm{d}\theta \\
=&\frac{1}{4}\int_0^{2\pi} \left[\left(\frac{zf'(z)}{g(z)}\right)^2 + 2\left|\frac{zf'(z)}{g(z)}\right|^2 + \left(\overline{\frac{zf'(z)}{g(z)}}\right)^2\right] \mathrm{d}\theta \\
\le&2\pi,
\end{aligned}$$

since

$$\int_0^{2\pi} \left(\frac{zf'(z)}{g(z)}\right)^2 \mathrm{d}\theta = \int_0^{2\pi} \left(\overline{\frac{zf'(z)}{g(z)}}\right)^2 \mathrm{d}\theta = 0,$$

for $|z| = r$. Therefore, applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_0^{2\pi}\left|\Re\left\{\frac{zf'(z)}{g(z)}\right\}\right|\mathrm{d}\theta\\ \le&\left(\int_0^{2\pi}\mathrm{d}\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\left(\Re\left\{\frac{zf'(z)}{g(z)}\right\}\right)^2\mathrm{d}\theta\right)^{\frac{1}{2}}\\ \le&2\pi.\end{aligned}\tag{2.9}$$

On the other hand, from the hypothesis, we have

$$\int_0^{2\pi}\frac{zf'(z)}{g(z)}\mathrm{d}\theta=\frac{1}{\mathrm{i}}\int_{|z|=r}\frac{f'(z)}{g(z)}\mathrm{d}z=2\pi.$$

Therefore, from (2.9), we have

$$2\pi=\int_0^{2\pi}\Re\left\{\frac{zf'(z)}{g(z)}\right\}\mathrm{d}\theta\le\int_0^{2\pi}\left|\Re\left\{\frac{zf'(z)}{g(z)}\right\}\right|\mathrm{d}\theta\le 2\pi.$$

This shows that for arbitrary r with $0 < r < 1$,

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\}=\left|\Re\left\{\frac{zf'(z)}{g(z)}\right\}\right|\ge 0,\tag{2.10}$$

on $|z| = r$, or

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\}\ge 0,\quad z\in\mathbb{D}.$$

Now, if there exists a point $z_0 = r_0e^{\mathrm{i}\theta_0}$, $0 < r_0 < 1$, for which

$$\Re\left\{\frac{z_0f'(z_0)}{g(z_0)}\right\}=0,$$

then let us take a sufficiently small neighborhood $N_\delta(z_0)$ which is a disc of center $z = z_0$ and radius $\delta < 1 - |z_0|$, then there exists a point $z_1 \in N_\delta(z_0)$ for which

$$\Re\left\{\frac{z_1f'(z_1)}{g(z_1)}\right\}<0.$$

This contradicts (2.10) and therefore, we have

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\}>0,\quad z\in\mathbb{D}.$$

It completes the proof of Theorem 2.3. □

Now, we find sufficient conditions for functions in $\mathcal{A}$ to be γ-spirallike functions or starlike functions.

Theorem 2.4. *Let $f \in \mathcal{A}$ and suppose that*

$$-\frac{(1-\alpha)\beta}{5\alpha+4\beta-5}<\Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}<\beta,\quad z\in\mathbb{D},\tag{2.11}$$

where $0 \le \alpha < 1$, $\beta > 0$ *and* $5\alpha + 4\beta - 5 > 0$. *Then, for arbitrary* r *with* $0 \le r < 1$, *we have*

$$\int_{|z|=r} \left| \Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \right| \mathrm{d}\theta < 3(1-\alpha)\pi.$$

Furthermore, for the case $1/3 < \alpha < 1$, *we have* $f \in \mathcal{S}^{\gamma}(\alpha)$ *for some* $\gamma \in \mathbb{R}$ *with* $|\gamma| < \pi(1-\alpha)/2$.

Proof. Let $0 \le r < 1$. Since $f \in \mathcal{A}$, we have

$$\begin{aligned}\int_{|z|=r} \left(\frac{zf'(z)}{f(z)} - \alpha \right) \mathrm{d}\theta &= 2(1-\alpha)\pi \\ &= \int_{|z|=r} \left\{ \Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \right\} \mathrm{d}\theta.\end{aligned}$$

Now, let $\partial\mathbb{D}_r := \{z \in \mathbb{C} : |z| = r\}$ and let us put C_1 by the part of $\partial\mathbb{D}_r$ on which

$$\Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > 0$$

and

$$\int_{C_1} \mathrm{d}\arg z = l.$$

And put C_2 by the part of $\partial\mathbb{D}_r$ on which

$$\Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \le 0$$

and

$$\int_{C_2} \mathrm{d}\arg z = 2\pi - l.$$

Furthermore, let us put

$$y_1 = \int_{C_1} \left\{ \Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \right\} \mathrm{d}\theta$$

and

$$-y_2 = \int_{C_2} \left\{ \Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \right\} \mathrm{d}\theta,$$

then it follows that

$$y_1 - y_2 = \int_{|z|=r} \left\{ \Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \right\} \mathrm{d}\theta = 2(1-\alpha)\pi \tag{2.12}$$

and

$$\int_{|z|=r} \left| \Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \right| \mathrm{d}\theta = y_1 + y_2.$$

From hypothesis (2.11), we have

$$y_1 = \int_{C_1} \left\{ \Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \right\} \mathrm{d}\theta < \beta l$$

and

$$y_2 = \int_{C_2} \left\{ -\Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \right\} \mathrm{d}\theta < (2\pi - l)\frac{(1-\alpha)\beta}{5\alpha+4\beta-5} = \frac{(2\pi\beta - \beta l)(1-\alpha)}{5\alpha+4\beta-5}. \tag{2.13}$$

We shall show that $y_1 < 5(1-\alpha)\pi/2$. On the Contrary, we suppose that $y_1 \geq 5(1-\alpha)\pi/2$. Then, we have

$$\beta l > \frac{5}{2}(1-\alpha)\pi$$

and this inequality and (2.13) give us that

$$\begin{aligned} y_2 &< \frac{(2\pi\beta - \beta l)(1-\alpha)}{5\alpha+4\beta-5} \\ &< \frac{(2\pi\beta - \frac{5}{2}(1-\alpha)\pi)(1-\alpha)}{5\alpha+4\beta-5} \\ &= \frac{\pi}{2}(1-\alpha). \end{aligned}$$

Therefore, we have

$$y_1 - y_2 > \frac{5}{2}(1-\alpha)\pi - \frac{1}{2}(1-\alpha)\pi = 2(1-\alpha)\pi,$$

which is a contradiction to (2.12). Therefore, we have

$$y_1 < \frac{5}{2}(1-\alpha)\pi,$$

and so, we have

$$y_1 + y_2 = 2y_1 - 2(1-\alpha)\pi < 3(1-\alpha)\pi.$$

This shows that for arbitrary r with $0 \leq r < 1$,

$$\int_{|z|=r} \left| \Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \right| \mathrm{d}\theta < 3(1-\alpha)\pi, \quad z \in \mathbb{D}.$$

Furthermore, for the case $1/3 < \alpha < 1$, we have

$$\Re\left\{ \mathrm{e}^{\mathrm{i}\gamma} \left(\frac{zf'(z)}{f(z)} - \alpha \right) \right\} > 0, \quad z \in \mathbb{D},$$

for some $\gamma \in \mathbb{R}$ with $|\gamma| < (1-\alpha)\pi/2$. That is, $f \in \mathcal{S}^\gamma(\alpha)$. □

Theorem 2.5. *Let $f \in \mathcal{A}$ and suppose that*

$$\int_0^{2\pi} \left(\Re\left\{ \frac{zf'(z)}{f(z)} \right\} \right)^2 \mathrm{d}\theta \leq 2\pi, \quad z \in \mathbb{D}. \tag{2.14}$$

Then $f \in \mathcal{S}^*$.

Proof. Applying (2.14) and Cauchy-Schwarz inequality, we have

$$\int_0^{2\pi} \left| \Re\left\{ \frac{zf'(z)}{f(z)} \right\} \right| \mathrm{d}\theta \leq \left(\int_0^{2\pi} \mathrm{d}\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} \left(\Re\left\{ \frac{zf'(z)}{f(z)} \right\} \right)^2 \mathrm{d}\theta \right)^{\frac{1}{2}} \leq 2\pi, \ z \in \mathbb{D}. \tag{2.15}$$

On the other hand, from the hypothesis (2.14), we have that $f(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$, because if there exists a point z_0, $0 < |z_0| < 1$, for which $f(z_0) = 0$, then it contradicts (2.14). Therefore, we have $f(z) \neq 0$ in $0 < |z| < 1$. We also note that

$$\int_0^{2\pi} \frac{zf'(z)}{f(z)} \mathrm{d}\theta = \int_{|z|=r} \frac{zf'(z)}{f(z)} \frac{\mathrm{d}z}{\mathrm{i}z} = 2\pi.$$

Applying (2.15), we have

$$2\pi = \int_0^{2\pi} \left(\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \right) \mathrm{d}\theta \leq \int_0^{2\pi} \left| \Re \left\{ \frac{zf'(z)}{f(z)} \right\} \right| \mathrm{d}\theta \leq 2\pi. \tag{2.16}$$

From (2.16), we have

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} = \left| \Re \left\{ \frac{zf'(z)}{f(z)} \right\} \right| \geq 0, \quad z \in \mathbb{D}. \tag{2.17}$$

On the other hand, if there exists a point z_1, $|z_1| < 1$, for which

$$\Re \left\{ \frac{z_1 f'(z_1)}{f(z_1)} \right\} = 0,$$

then, let us take a very small neighborhood of the point z_1, $N_\delta(z_1)$ with the center z_1 and radius δ, $0 < \delta < 1 - |z_1|$, then there exists a point $z_2 \in N_\delta(z_1)$ for which

$$\Re \left\{ \frac{z_2 f'(z_2)}{f(z_2)} \right\} < 0.$$

This contradicts (2.17), therefore, we have

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in \mathbb{D}.$$

It completes the proof of Theorem 2.5. □

Applying the same idea in [12], we get the following result.

Theorem 2.6. *Let p be analytic in $\mathbb{D}$, $p(0) = 1$ and suppose that*

$$-|z| \Re \left(\frac{1+z}{1-z} \right) \leq \Re \left(p(z) + \frac{zp'(z)}{p(z)} \right) \leq (1 - |z|) \Re \left(\frac{1+z}{1-z} \right), \quad z \in \mathbb{D}. \tag{2.18}$$

Then we have

$$\Re \left(p(z) + \frac{zp'(z)}{p(z)} \right) > 0, \quad z \in \mathbb{D}.$$

Proof. Let $\partial\mathbb{D}_r := \{z \in \mathbb{C} : |z| = r\}$ and let us put C_1 the part of $\partial\mathbb{D}_r$ on which

$$\Re \left(p(z) + \frac{zp'(z)}{p(z)} \right) > 0.$$

Then from (2.18), we have

$$\begin{aligned} y_1 &= \int_{C_1} \Re \left(p(z) + \frac{zp'(z)}{p(z)} \right) \mathrm{d}\theta \\ &\leq (1 - |z|) \int_{C_1} \Re \left(\frac{1+z}{1-z} \right) \mathrm{d}\theta \\ &= (1 - |z|) \int_{C_1} \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \mathrm{d}\theta \\ &= 2\pi(1 - |z|), \end{aligned} \tag{2.19}$$

where $z = re^{i\theta}$ and $0 \le \theta \le 2\pi$. On the other hand, let us put C_2 the part of $\partial\mathbb{D}_r$ on which

$$\Re\left(p(z) + \frac{zp'(z)}{p(z)}\right) \le 0,$$

then we have

$$\begin{aligned}
-y_2 &= \int_{C_2} \Re\left(p(z) + \frac{zp'(z)}{p(z)}\right) \mathrm{d}\theta \\
&\ge -|z| \int_{C_2} \Re\left(\frac{1+z}{1-z}\right) \mathrm{d}\theta \\
&= -2\pi|z|.
\end{aligned} \tag{2.20}$$

Now then, from the hypothesis (2.18), we have $p(z) \ne 0$ in $\mathbb{D}$, because if $p(z)$ has a zero $z = a_0$, $0 < |a_0| < 1$, it contradicts (2.18). Therefore, we have

$$\begin{aligned}
y_1 - y_2 &= \int_{|z|=r} \Re\left(p(z) + \frac{zp'(z)}{p(z)}\right) \mathrm{d}\theta \\
&= \int_{|z|=r} \left(p(z) + \frac{zp'(z)}{p(z)}\right) \frac{\mathrm{d}z}{\mathrm{i}z} \\
&= -\mathrm{i} \int_{|z|=r} \left(\frac{p(z)}{z} + \frac{p'(z)}{p(z)}\right) \mathrm{d}z \\
&= 2\pi.
\end{aligned}$$

Then, from (2.19) and (2.20), we have

$$\begin{aligned}
y_1 + y_2 &= \int_{|z|=r} \left|\Re\left(p(z) + \frac{zp'(z)}{p(z)}\right)\right| \mathrm{d}\theta \\
&\le 2\pi(1-|z|) + 2\pi|z| \\
&= 2\pi.
\end{aligned}$$

This shows that

$$\int_{|z|=r} \Re\left(p(z) + \frac{zp'(z)}{p(z)}\right) \mathrm{d}\theta = \int_{|z|=r} \left|\Re\left(p(z) + \frac{zp'(z)}{p(z)}\right)\right| \mathrm{d}\theta$$

and therefore, we have

$$\Re\left(p(z) + \frac{zp'(z)}{p(z)}\right) \ge 0, \tag{2.21}$$

for all r, $0 < r < 1$. Now then, if there exists a point z_0, $0 \le |z_0| < 1$, for which

$$\Re\left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right) = 0,$$

then $p(z_0) + (z_0 p'(z_0))/(p(z_0))$ is a pure imaginary number and the function $p(z) + (zp'(z))/(p(z))$ is continuous at the point $z = z_0$, and so $\Re(p(z) + (zp'(z))/(p(z)))$ takes a negative real number at the very small neighborhood of the point $z = z_0$. It contradicts (2.21) and so, it completes the proof. □

It is well-known that a convex univalent function in $\mathcal{A}$ is starlike of order $1/2$ (see [7,11]). Using this fact and Theorem 2.6, we can obtain the following corollary.

Corollary 2.4. *Let $f \in \mathcal{A}$ and suppose that*

$$-|z|\Re\left(\frac{1+z}{1-z}\right) \le \Re\left(1+\frac{zf''(z)}{f'(z)}\right) \le (1-|z|)\Re\left(\frac{1+z}{1-z}\right), \quad z \in \mathbb{D}.$$

Then we have

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D},$$

and so, $f \in \mathcal{S}^*(1/2)$.

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