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EXPLORING BIFURCATION IN A FRACTIONAL-ORDER PREDATOR-PREY SYSTEM WITH MIXED DELAYS*

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Abstract This work chiefly develops and discusses a fractional-order predator-prey model with distributed delay and discrete delay. Applying skilly an appropriate variable substitution, a novel equivalent form of the fractional-order predator-prey model with distributed delay and discrete delay is derived. By virtue of the stability theorem and bifurcation principle of fractional-order dynamical system, we establish a delay-independent stability and bifurcation criterion ensuring the stability and the onset of Hopf bifurcation for the involved predator-prey system. The role of the time delay in stabilizing system and controlling Hopf bifurcation of the considered fractional-order predator-prey model is displayed. Software simulation results are presented to support the key theoretical fruits.

Keywords Fractional-order predator-prey model, stability, Hopf bifurcation, discrete delay, distributed delay

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1. Introduction

In order to understand the variation regularity of biological population in natural world, it is very important for us to establish mathematical models to describe the relation of predator species and prey species. Through the study on the predator-prey models, we can better control the quantities of predator population and prey population and maintain the ecological balance. Generally speaking, time delay continually appears in predator-prey systems since there exists lag of the response for predator species and prey species. Many scholars argue that the delay will remain a continuous change state during the course of the response for predator species and prey species. Based on this point, the predator-prey models with distributed delays are more suitable forms that describe the interaction of predator species and prey species. During the past few decades, a great many valuable works on the study of predator-prey models with distributed delays have been published. For example, Lin and Yuan [13] discussed the existence and the stability of periodic solution to predator-prey model involving distributed delays. Yang and Ye [34] analyzed the Hopf bifurcation of a predator-prey model involving discrete time delays and distributed time delays. Liu et al. [15] considered the stationary distribution issue of **a** stochastic prey-predator system involving distributed time delays. In 2018, Liu et al. [14] focused on the dynamics of a stochastic prey-predator system involving distributed time delays. For further concrete knowledge on this aspect, one can refer to [7, 18, 19].

It is worth pointing out that all the above considered literatures only involve integer-order predator-prey systems. Fractional calculus is an important area in mathematics, but it has been kept a relatively slow state of development due to the shortage of basic knowledge on solving fractional-order differential equation and practical background [36]. Up to now, fractional calculus has received great interest from numerous researchers and great progress has been made. A lot of scholars argue that fractional calculus owns the great potential application in many disciplines such as control science, ecology, physical wave, network science, fluid mechanics and so on [22, 28, 36]. The advantage of fractional-order differential equation lies in its owned memory and hereditary peculiarity. Thus it ist a very good tool which can give a description of true natural phenomena [22, 28, 36]. Recently, many works on fractional-order dynamical models have been reported. In particular, a lot of excellent fruits on fractional-order predator-prey systems have been published (see [3, 6, 35]).

Hopf bifurcation phenomenon plays a key role in maintaining the balance of biological population. It naturally attracts great interest from many scholars. At present, the investigation on Hopf bifurcation of integer-order predator-prey systems is basically mature. However, the studies on Hopf bifurcation of fractional-order predator-prey systems are relatively rare. Some researchers have achieved some interesting results. For instance, Yuan et al. [37] studied the Hopf bifurcation for a fractional-order predator-prey system with two different time delays; Huang et al. [10] investigated the Hopf bifurcation control problem for a fractional-order predator-prey system with delays; Li et al. [12] dealt with the dynamic complexity for a fractional-order predator-prey model involving both time delays. For more detailed publications, one can refer to [1, 2, 4, 5, 8, 9, 11, 16, 17, 20, 21, 23–27, 29, 30, 33].

Here we would like to mention that all the above considered literatures only involved the predator-prey systems concerning discrete time delays. In fact, the time delay maybe extend over the whole past because of the competition among different biological populations, then there is a distribution of delays over a periodic time [23], so it is natural to introduce the distributed delays into predator-prey models. At present, the works on Hopf bifurcation of fractional-order predator-prey models involving distributed time delays are very rare. This motivates us to deal with this problem. In [29], Xu and Shao dealt with the following predator-prey model involving mixed time delays:

$$\begin{cases} \dfrac{dv_1(t)}{dt} = v_1(t)\left[\gamma_1 - \alpha_{11}\displaystyle\int_{-\infty}^{t} G(t-s)v_1(s)ds - \alpha_{12}v_2(t-\vartheta)\right], \\ \dfrac{dv_2(t)}{dt} = v_2(t)\left[-\gamma_2 + \alpha_{21}\displaystyle\int_{-\infty}^{t} G(t-s)v_1(s)ds - \alpha_{22}v_2(t-\vartheta)\right], \end{cases} \tag{1.1}$$

where $v_1(t)$ and $v_2(t)$ stand for the densities of prey population and predator population at time t, respectively, $G(s)$, which denotes the delay kernel, is a nonnegative bounded function defined on $[0,\infty)$, ϑ is the time delay that stands for the hunting time, $\vartheta, \gamma_k, \alpha_{kl} > 0 (k,l = 1,2)$ are constants. The function G satisfies the following conditions: (i) $G : [0,\infty) \to [0,\infty)$; (ii) G is piecewise continuous; (iii) $\int_0^\infty G(s)ds = 1, \int_0^\infty sG(s)ds < \infty$. The function G owns the following expression:

$$G(s) = \gamma^{k+1}\frac{s^k e^{-\gamma s}}{k!}, s \in (0,\infty), k = 0,1, \tag{1.2}$$

where $\gamma > 0$ stands for the rate of fading of past memories. If $k = 0$, then $G(s) = \gamma e^{-\gamma s}$ (weak kernel) and if $k = 1$, then $G(s) = \gamma^2 s e^{-\gamma s}$(strong kernel). In the present work, our will focus on the key issue: the stability and Hopf bifurcation of fractional-order version of predator-prey system (1.1) with the weak kernel $G(s) = \gamma e^{-\gamma s}$. On the basis of the analysis above, we set up the fractional-order predator-prey system as follow:

$$\begin{cases} \dfrac{dv_1^\varrho(t)}{dt^\varrho} = v_1(t)\left[\gamma_1 - \alpha_{11}\displaystyle\int_{-\infty}^{t} G(t-s)v_1(s)ds - \alpha_{12}v_2(t-\vartheta)\right], \\ \dfrac{dv_2^\varrho(t)}{dt^\varrho} = v_2(t)\left[-\gamma_2 + \alpha_{21}\displaystyle\int_{-\infty}^{t} G(t-s)v_1(s)ds - \alpha_{22}v_2(t-\vartheta)\right], \end{cases} \tag{1.3}$$

where $\varrho \in (0,1)$ is a real number. All the other coefficients own the same meaning as those of model (1.1).

This work can be planed as follows. Part two gives prerequisite basic knowledge on fractional-order dynamical system. Part three presents the key results on the stability and the onset of Hopf bifurcation for predator-prey model (1.3) with weak kernel case. Part four displays software simulations to sustain the established key conclusions. Part five draws a conclusion.

2. Indispensable theory

In this section, several definitions, lemmas and theorems on fractional differential equations are listed as follows.

Definition 2.1.([17]) The Caputo-type fractional order derivative can be defined as follows:

$$\mathcal{D}^\varrho h(\zeta) = \frac{1}{\Gamma(s-\varrho)}\int_{\zeta_0}^{\zeta} \frac{h^{(s)}(\nu)}{(\zeta-\nu)^{\varrho-s+1}}d\nu,$$

where $h(\zeta) \in ([\zeta_0, \infty), R), \Gamma(\nu) = \int_0^\infty \zeta^{\nu-1} e^{-\zeta} d\zeta$, $\zeta \geq \zeta_0$ and $s \in Z^+, s-1 \leq \varrho < s$. The Laplace transform of $\mathcal{D}^\varrho$ is given by

$$\mathcal{L}\{\mathcal{D}^\varrho h(t); s\} = s^\varrho \mathcal{H}(s) - \sum_{i=0}^{m-1} s^{\varrho-i-1} h^{(i)}(0), m-1 \leq \varrho < m \in Z^+,$$

where $\mathcal{H}(s) = \mathcal{L}\{h(t)\}$. If $h^{(i)}(0) = 0, i = 1, 2, \cdots, m$, then $\mathcal{L}\{\mathcal{D}^\varrho h(t); s\} = s^\varrho \mathcal{H}(s)$.

Definition 2.2.([2]) (v_{10}, v_{20}) is said to be an equilibrium point of system (1.3) provided that

$$\begin{cases} v_{10}\left[\gamma_1 - \alpha_{11}\int_{-\infty}^t G(t-s)v_{10}ds - \alpha_{12}v_{20}\right] = 0, \\ v_{20}\left[-\gamma_2 + \alpha_{21}\int_{-\infty}^t G(t-s)v_{10}ds - \alpha_{22}v_{20}\right] = 0. \end{cases} \tag{2.1}$$

Lemma 2.1. ([16, 20]) *Let q_* be an equilibrium point of the following system*

$$\frac{d^\varrho q(t)}{dt^\varrho} = u(t, q(t)), q(0) = q_0, \tag{2.2}$$

where $\varrho \in (0, 1]$ and $u(t, q(t)) : R^+ \times R^n \to R^n$. q_* is said to be locally asymptotically stable provided that each eigenvalue σ of $\frac{\partial u(t,q)}{\partial q}|_{q=q_*}$ satisfies $|arg(\sigma)| > \frac{\varrho\pi}{2}$.

Lemma 2.2.([4]) *Consider the following system:*

$$\begin{cases} \frac{d^{\varrho_1}\mathcal{L}_1(t)}{dt^{\varrho_1}} = d_{11}\mathcal{L}_1(t-\vartheta_{11}) + d_{12}\mathcal{L}_2(t-\vartheta_{12}) + \cdots + d_{1n}\mathcal{L}_m(t-\vartheta_{1n}), \\ \frac{d^{\varrho_2}\mathcal{L}_2(t)}{dt^{\varrho_2}} = d_{21}\mathcal{L}_1(t-\vartheta_{21}) + d_{22}\mathcal{L}_2(t-\vartheta_{22}) + \cdots + d_{2n}\mathcal{L}_m(t-\vartheta_{2n}), \\ \vdots \\ \frac{d^{\varrho_n}\mathcal{L}_n(t)}{dt^{\varrho_n}} = d_{n1}\mathcal{L}_1(t-\vartheta_{n1}) + d_{n2}\mathcal{L}_2(t-\vartheta_{n2}) + \cdots + d_{nn}\mathcal{L}_n(t-\vartheta_{nn}), \end{cases} \tag{2.3}$$

where $0 < \varrho_k < 1 (k = 1, 2, \cdots, n)$, the initial values $\mathcal{L}_k(t) = \psi_k(t) \in C[-\max_{k,l}\vartheta_{kl}, 0]$, $t \in [-\max_{k,l}\vartheta_{kl}, 0]$, $k, l = 1, 2, \cdots, n$. Denote

$$\Delta(s) = \begin{bmatrix} s^{\varrho_1} - d_{11}e^{-s\vartheta_{11}} & -d_{12}e^{-s\vartheta_{12}} & \cdots & -d_{1n}e^{-s\vartheta_{1n}} \\ -d_{21}e^{-s\vartheta_{12}} & s^{\varrho_2} - d_{22}e^{-s\vartheta_{22}} & \cdots & -d_{2n}e^{-s\vartheta_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{n1}e^{-s\vartheta_{n1}} & -d_{n2}e^{-s\vartheta_{n2}} & \cdots & s^{\varrho_n} - d_{nn}e^{-s\vartheta_{nn}} \end{bmatrix}. \tag{2.4}$$

Then the zero solution of system (2.3) is said to be Lyapunov asymptotically stable provided that each root of $\det(\Delta(s)) = 0$ has negative real parts.

Consider the following system:

$$\begin{cases} \dfrac{d^{\varrho_1}\mathcal{L}_1(t)}{dt^{\varrho_1}} = d_{11}\mathcal{L}_1(t) + d_{12}\mathcal{L}_2(t) + \cdots + d_{1n}\mathcal{L}_m(t), \\ \dfrac{d^{\varrho_2}\mathcal{L}_2(t)}{dt^{\varrho_2}} = d_{21}\mathcal{L}_1(t) + d_{22}\mathcal{L}_2(t) + \cdots + d_{2n}\mathcal{L}_m(t), \\ \quad\vdots \\ \dfrac{d^{\varrho_n}\mathcal{L}_n(t)}{dt^{\varrho_n}} = d_{n1}\mathcal{L}_1(t) + d_{n2}\mathcal{L}_2(t) + \cdots + d_{nn}\mathcal{L}_n(t), \end{cases} \tag{2.5}$$

where $\varrho_l \in (0,1](l = 1, 2, \cdots, n)$ is the rational number. The characteristic equation of system (2.5) owns the expression:

$$\det \begin{bmatrix} s^{\varrho_1} - d_{11} & -d_{12} & \cdots & -d_{1n} \\ -d_{21} & s^{\varrho_2} - d_{22} & \cdots & -d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{n1} & -d_{n2} & \cdots & s^{\varrho_n} - d_{nn} \end{bmatrix} = 0. \tag{2.6}$$

Set $\omega_l = \frac{a_l}{b_l}, a_l, b_l \in Z^+, (a_l, b_l) = 1$ and let κ be the lowest common multiple of b_l of $\omega_l, l = 1, 2, \cdots, n$.

Lemma 2.3.([4]) *The zero solution of system (2.6) is said to be locally asymptotically stable provided that each root λs of the following equation*

$$\det \begin{bmatrix} \lambda^{\kappa\omega_1} - d_{11} & -d_{12} & \cdots & -d_{1n} \\ -d_{21} & \lambda^{\kappa\omega_2} - d_{22} & \cdots & -d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{n1} & -d_{n2} & \cdots & \lambda^{\kappa\omega_n} - d_{nn} \end{bmatrix} = 0 \tag{2.7}$$

satisfies $|arg(\lambda)| > \frac{\pi}{2\kappa}$.

3. Bifurcation study for predator-prey system (1.3)

In this part, we are to analyze the stability and the appearance of Hopf bifurcation for system (1.3) with weak kernel $G(s) = \gamma e^{-\gamma s}$. Let

$$v_3(t) = \int_{-\infty}^{t} \gamma e^{-\gamma(t-s)} v_1(s) ds. \tag{3.1}$$

Then system (1.3) becomes

$$\begin{cases} \dfrac{dv_1^{\varrho}(t)}{dt^{\varrho}} = v_1(t)\left[\gamma_1 - \alpha_{11}v_3(t) - \alpha_{12}v_2(t-\vartheta)\right], \\ \dfrac{dv_2^{\varrho}(t)}{dt^{\varrho}} = v_2(t)\left[-\gamma_2 + \alpha_{21}v_3(t) - \alpha_{22}v_2(t-\vartheta)\right], \\ \dfrac{dv_3(t)}{dt} = \gamma(v_1(t) - v_3(t)). \end{cases} \tag{3.2}$$

Apparently, system (3.2) owns the following unique positive equilibrium point

$$\begin{cases} v_{10} = \dfrac{\gamma_1\alpha_{22} + \gamma_2\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \\ v_{20} = \dfrac{\gamma_1\alpha_{21} - \gamma_2\alpha_{11}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \\ v_{30} = \dfrac{\gamma_1\alpha_{22} + \gamma_2\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}} \end{cases} \tag{3.3}$$

if the following assumption

$$(\mathcal{M}1) \quad \gamma_1\alpha_{21} > \gamma_2\alpha_{11}$$

holds. The linear equation of system (3.2) around (v_{10}, v_{20}, v_{30}) can be expressed as

$$\begin{cases} \dfrac{dv_1^{\varrho}(t)}{dt^{\varrho}} = a_1 v_3(t) + a_2 v_2(t - \vartheta), \\ \dfrac{dv_2^{\varrho}(t)}{dt^{\varrho}} = b_1 v_3(t) + b_2 v_2(t - \vartheta), \\ \dfrac{dv_3(t)}{dt} = \gamma(v_1(t) - v_3(t)), \end{cases} \tag{3.4}$$

where

$$\begin{cases} a_1 = -\dfrac{\alpha_{11}(\gamma_1\alpha_{22} + \gamma_2\alpha_{12})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \\ a_2 = -\dfrac{\alpha_{12}(\gamma_1\alpha_{22} + \gamma_2\alpha_{12})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \\ b_1 = \dfrac{\alpha_{21}(\gamma_1\alpha_{21} - \gamma_2\alpha_{11})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \\ b_2 = -\dfrac{\alpha_{22}(\gamma_1\alpha_{21} - \gamma_2\alpha_{11})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}. \end{cases} \tag{3.5}$$

Hence the associated characteristic equation of system (3.4) can be expressed as:

$$\det \begin{bmatrix} s^{\varrho} & -a_2 e^{-s\vartheta} & -a_1 \\ 0 & s^{\varrho} - a_2 e^{-s\vartheta} & -b_1 \\ -\gamma & 0 & s + \gamma \end{bmatrix} = 0. \tag{3.6}$$

Let $\varrho = \frac{a}{b}$ where $a, b \in Z^+$ and $(a, b) = 1$. Set $\lambda = s^{\frac{1}{b}}$. Then Eq. (3.6) becomes

$$\det \begin{bmatrix} \lambda^a & -a_2 e^{-s\vartheta} & -a_1 \\ 0 & \lambda^a - a_2 e^{-s\vartheta} & -b_1 \\ -\gamma & 0 & \lambda^b + \gamma \end{bmatrix} = 0. \tag{3.7}$$

Lemma 3.1. *If every root λ of Eq. (3.7) satisfies $|arg(\lambda)| > \frac{\pi}{2b}$, then the equilibrium point (v_{10}, v_{20}, v_{30}) of system (3.2) is Lyapunov locally asymptotically stable.*

Proof. Clearly, Eq. (3.7) is the characteristic equation of system (3.2) with $\vartheta = 0$. Applying Lemma 2.3, one can easily concludes that Lemma 3.1 holds. □

By (3.6), one has

$$s^{2\varrho+1} + c_1 s^{2\varrho} + c_2 s^{\varrho} + (d_1 s^{\varrho+1} + d_2 s^{\varrho} + d_3)e^{-s\vartheta} = 0, \tag{3.8}$$

where

$$\begin{cases} c_1 = \gamma, c_2 = -a_1\gamma, \\ d_1 = -a_2, d_2 = -a_2\gamma, d_3 = (a_1 - b_1)a_2\gamma. \end{cases} \tag{3.9}$$

Assume that $s = i\phi = \phi\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ is the root of Eq.(3.8), then Eq. (3.8) takes the form:

$$\begin{aligned} &\phi^{2\varrho+1}\left(\cos\frac{(2\varrho+1)\pi}{2} + i\sin\frac{(2\varrho+1)\pi}{2}\right) + c_1\phi^{2\varrho}(\cos\varrho\pi + i\sin\varrho\pi) \\ &+c_2\phi^{\varrho}\left(\cos\frac{\varrho\pi}{2} + i\sin\frac{\varrho\pi}{2}\right) + \left[d_1\phi^{\varrho+1}\left(\cos\frac{(\varrho+1)\pi}{2} + i\sin\frac{(\varrho+1)\pi}{2}\right)\right. \\ &\left. + d_2\phi^{\varrho}\left(\cos\frac{\varrho\pi}{2} + i\sin\frac{\varrho\pi}{2}\right) + d_3\right](\cos\phi\vartheta - i\sin\phi\vartheta) = 0, \end{aligned} \tag{3.10}$$

which leads to

$$\begin{cases} \mathcal{A}_1\cos\phi\vartheta + \mathcal{A}_2\sin\phi\vartheta = \mathcal{B}_1, \\ \mathcal{A}_2\cos\phi\vartheta - \mathcal{A}_1\sin\phi\vartheta = \mathcal{B}_2, \end{cases} \tag{3.11}$$

where

$$\begin{cases} \mathcal{A}_1 = d_1\phi^{\varrho+1}\cos\frac{(\varrho+1)\pi}{2} + d_2\phi^{\varrho}\cos\frac{\varrho\pi}{2} + d_3, \\ \mathcal{A}_2 = d_1\phi^{\varrho+1}\sin\frac{(\varrho+1)\pi}{2} + d_2\phi^{\varrho}\sin\frac{\varrho\pi}{2}, \\ \mathcal{B}_1 = -\phi^{2\varrho+1}\cos\frac{(2\varrho+1)\pi}{2} - c_1\phi^{2\varrho}\cos\varrho\pi - c_2\phi^{\varrho}\cos\frac{\varrho\pi}{2}, \\ \mathcal{B}_2 = -\phi^{2\varrho+1}\sin\frac{(2\varrho+1)\pi}{2} - c_1\phi^{2\varrho}\sin\varrho\pi - c_2\phi^{\varrho}\sin\frac{\varrho\pi}{2}. \end{cases} \tag{3.12}$$

By (3.11), we derive

$$\mathcal{A}_1^2 + \mathcal{A}_2^2 = \mathcal{B}_1^2 + \mathcal{B}_2^2 \tag{3.13}$$

Let

$$\begin{cases} f_1 = d_1\cos\frac{(\varrho+1)\pi}{2}, \quad f_2 = d_2\cos\frac{\varrho\pi}{2}, \quad f_3 = d_3, \\ f_4 = d_1\sin\frac{(\varrho+1)\pi}{2}, \quad f_5 = d_2\sin\frac{\varrho\pi}{2}, \quad f_6 = -\cos\frac{(2\varrho+1)\pi}{2}, \\ f_7 = -c_1\cos\varrho\pi, \quad f_8 = -c_2\cos\frac{\varrho\pi}{2}, \quad f_9 = -\sin\frac{(2\varrho+1)\pi}{2}, \\ f_{10} = -c_1\sin\varrho\pi, \quad f_{11} = -c_2\sin\frac{\varrho\pi}{2}, \end{cases} \tag{3.14}$$

then (3.12) takes the following form:

$$\begin{cases} \mathcal{A}_1 = f_1\phi^{\varrho+1} + f_2\phi^{\varrho} + f_3, \\ \mathcal{A}_2 = f_4\phi^{\varrho+1} + f_5\phi^{\varrho}, \\ \mathcal{B}_1 = f_6\phi^{2\varrho+1} + f_7\phi^{2\varrho} + f_8\phi^{\varrho}, \\ \mathcal{B}_2 = f_9\phi^{2\varrho+1} + f_{10}\phi^{2\varrho} + f_{11}\phi^{\varrho}. \end{cases} \tag{3.15}$$

Applying (3.13) and (3.15), one gets

$$\begin{aligned} &\sigma_1\phi^{4\varrho+2} + \sigma_2\phi^{4\varrho+1} + \sigma_3\phi^{4\varrho} + \sigma_4\phi^{3\varrho+1} + \sigma_5\phi^{3\varrho} + \sigma_6\phi^{2\varrho+2} \\ &+\sigma_7\phi^{2\varrho+1} + \sigma_8\phi^{2\varrho} + \sigma_9\phi^{\varrho+1} + \sigma_{10}\phi^{\varrho} + \sigma_{11} = 0, \end{aligned} \tag{3.16}$$

where

$$\begin{cases} \sigma_1 = f_6^2 + f_9^2, \\ \sigma_2 = 2(f_6f_7 + f_9f_{10}), \\ \sigma_3 = f_7^2 + f_{10}^2, \\ \sigma_4 = 2(f_6f_8 + f_9f_{11}), \\ \sigma_5 = 2(f_7f_8 + f_{10}f_{11}), \\ \sigma_6 = -f_1^2 - f_4^2, \\ \sigma_7 = -2(f_1f_2 + f_4f_5), \\ \sigma_8 = f_8^2 + f_{11}^2 - f_2^2 - f_5^2, \\ \sigma_9 = -f_1f_3, \\ \sigma_{10} = -2f_2f_3, \\ \sigma_{11} = -f_3^2. \end{cases} \tag{3.17}$$

Let

$$\begin{aligned} \mathcal{C}(v) = {}& \sigma_1v^{4\varrho+2} + \sigma_2v^{4\varrho+1} + \sigma_3v^{4\varrho} + \sigma_4v^{3\varrho+1} + \sigma_5v^{3\varrho} + \sigma_6v^{2\varrho+2} \\ &+ \sigma_7v^{2\varrho+1} + \sigma_8v^{2\varrho} + \sigma_9v^{\varrho+1} + \sigma_{10}v^{\varrho} + \sigma_{11}. \end{aligned} \tag{3.18}$$

Lemma 3.2. *Eq. (3.8) owns at least a pair of purely imaginary roots.*

Proof. Apparently, $\mathcal{C}(0) = \sigma_{11} = -f_3^2 < 0$ and $\lim_{v\to\infty}\mathcal{C}(v) = +\infty$. Then we know that Eq. (3.16) owns at least one positive root. Then Eq. (3.8) owns at least a pair of purely imaginary roots. □

We find that it is not easy to obtain the roots of Eq. (3.16) because the powers of Eq. (3.16) are not integer number, now we may change Eq.(3.16) as an equivalent form involving integer power by means of variable substitution. Let $y = \phi^{\frac{1}{b}}$, then $\phi = y^b$. Thus Eq.(3.16) takes the form:

$$\begin{aligned} &\sigma_1y^{4a+2b} + \sigma_2y^{4a+b} + \sigma_3y^{4a} + \sigma_4y^{3a+b} + \sigma_5y^{3a} + \sigma_6y^{2a+2b} \\ &+\sigma_7y^{2a+b} + \sigma_8y^{2a} + \sigma_9y^{a+b} + \sigma_{10}y^{a} + \sigma_{11} = 0, \end{aligned} \tag{3.19}$$

By means of computer, one can derive the roots of Eq. (3.19). Now we suppose that Eq. (3.19) owns the positive root which is denoted by y_k, then Eq. (3.16) possess the positive root $\phi_k = y_k^b$. Suppose that Eq. (3.19) owns j positive roots $y_k, k = 1, 2, \cdots, j$. Based on (3.11), one gets

$$\vartheta_k^l = \frac{1}{\phi_l}\left[\arccos\frac{\mathcal{A}_1\mathcal{B}_1 + \mathcal{A}_2\mathcal{B}_2}{\mathcal{A}_1^2 + \mathcal{A}_2^2} + 2l\pi\right], k = 1, 2, \cdots, j; l = 0, 1, 2, \cdots. \tag{3.20}$$

Denote

$$\vartheta_0 = \vartheta_{k0}^{(0}\min_{k=1,2,\cdots,j}\{\vartheta_k^0\}, \phi_0 = \phi|_{\vartheta=\vartheta_0}. \tag{3.21}$$

In the sequel, the following assumption is needed.

$(\mathcal{M}2)$ $\quad \mathcal{W}_{11}\mathcal{W}_{21} + \mathcal{W}_{12}\mathcal{W}_{22} > 0$, where

$$\begin{cases} \mathcal{W}_{11} = (2\varrho+1)\phi_0^{2\varrho}\cos\varrho\pi + 2c_1\varrho\phi_0^{2\varrho-1}\cos\dfrac{(2\varrho-1)\pi}{2} + c_2\varrho\phi_0^{\varrho-1}\cos\dfrac{(\varrho-1)\pi}{2} \\ \qquad + \left[(\varrho+1)d_1\phi_0^{\varrho}\cos\dfrac{\varrho\pi}{2} + \varrho d_2\phi_0^{\varrho-1}\cos\dfrac{(\varrho-1)\pi}{2}\right]\cos\phi_0\vartheta_0 \\ \qquad + \left[(\varrho+1)d_1\phi_0^{\varrho}\sin\dfrac{\varrho\pi}{2} + \varrho d_2\phi_0^{\varrho-1}\sin\dfrac{(\varrho-1)\pi}{2}\right]\sin\phi_0\vartheta_0, \\ \mathcal{W}_{12} = (2\varrho+1)\phi_0^{2\varrho}\sin\varrho\pi + 2c_1\varrho\phi_0^{2\varrho-1}\sin\dfrac{(2\varrho-1)\pi}{2} + c_2\varrho\phi_0^{\varrho-1}\sin\dfrac{(\varrho-1)\pi}{2} \\ \qquad + \left[(\varrho+1)d_1\phi_0^{\varrho}\cos\dfrac{\varrho\pi}{2} + \varrho d_2\phi_0^{\varrho-1}\cos\dfrac{(\varrho-1)\pi}{2}\right]\sin\phi_0\vartheta_0 \\ \qquad - \left[(\varrho+1)d_1\phi_0^{\varrho}\sin\dfrac{\varrho\pi}{2} + \varrho d_2\phi_0^{\varrho-1}\sin\dfrac{(\varrho-1)\pi}{2}\right]\cos\phi_0\vartheta_0, \\ \mathcal{W}_{21} = \left[d_1\phi_0^{\varrho+1}\cos\dfrac{(\varrho+1)\pi}{2} + d_2\phi_0^{\varrho}\cos\dfrac{\varrho\pi}{2} + d_3\right]\phi_0\sin\phi_0\vartheta_0 \\ \qquad - \left[d_1\phi_0^{\varrho+1}\sin\dfrac{(\varrho+1)\pi}{2} + d_2\phi_0^{\varrho}\sin\dfrac{\varrho\pi}{2}\right]\phi_0\cos\phi_0\vartheta_0, \\ \mathcal{W}_{22} = \left[d_1\phi_0^{\varrho+1}\cos\dfrac{(\varrho+1)\pi}{2} + d_2\phi_0^{\varrho}\cos\dfrac{\varrho\pi}{2} + d_3\right]\phi_0\cos\phi_0\vartheta_0 \\ \qquad + \left[d_1\phi_0^{\varrho+1}\sin\dfrac{(\varrho+1)\pi}{2} + d_2\phi_0^{\varrho}\sin\dfrac{\varrho\pi}{2}\right]\phi_0\sin\phi_0\vartheta_0. \end{cases} \tag{3.22}$$

Lemma 3.3. *Suppose that $s(\vartheta) = \eta_1(\vartheta) + i\eta_2(\vartheta)$ is the root of Eq. (3.8) at $\vartheta = \vartheta_0$ which satisfies $\eta_1(\vartheta_0) = 0, \eta_2(\vartheta_0) = \phi_0$, then we derive Re $\left[\frac{ds}{d\vartheta}\right]\Big|_{\vartheta=\vartheta_0,\phi=\phi_0} > 0$.*

Proof. By means of Eq. (3.8), we get

$$\begin{aligned} &\left[(2\varrho+1)s^{2\varrho} + 2c_1\varrho s^{2\varrho-1} + \varrho c_2 s^{\varrho-1}\right]\frac{ds}{d\vartheta} + \left[(\varrho+1)d_1 s^{\varrho} + \varrho d_2 s^{\varrho-1}\right]e^{-s\vartheta}\frac{ds}{d\vartheta} \\ &-e^{-s\vartheta}\left(\frac{ds}{d\vartheta}\vartheta + s\right)\left(d_1 s^{\varrho+1} + d_2 s^{\varrho} + d_3\right) = 0, \end{aligned} \tag{3.23}$$

which leads to

$$\left[(2\varrho+1)s^{2\varrho} + 2c_1\varrho s^{2\varrho-1} + \varrho c_2 s^{\varrho-1} + \left((\varrho+1)d_1 s^{\varrho} + \varrho d_2 s^{\varrho-1}\right)e^{-s\vartheta}\right.$$

$$-\left(d_1 s^{\varrho+1}+d_2 s^{\varrho}+d_3\right) e^{-s\vartheta}\vartheta\Big]\frac{ds}{d\vartheta}=e^{-s\vartheta}s\left(d_1 s^{\varrho+1}+d_2 s^{\varrho}+d_3\right). \tag{3.24}$$

Then

$$\left[\frac{ds}{d\vartheta}\right]^{-1}=\frac{(2\varrho+1)s^{2\varrho}+2c_1\varrho s^{2\varrho-1}+\varrho c_2 s^{\varrho-1}+\left((\varrho+1)d_1 s^{\varrho}+\varrho d_2 s^{\varrho-1}\right)e^{-s\vartheta}}{e^{-s\vartheta}s\left(d_1 s^{\varrho+1}+d_2 s^{\varrho}+d_3\right)}-\frac{\vartheta}{s}. \tag{3.25}$$

So

$$\begin{aligned}
&\operatorname{Re}\left[\frac{ds}{d\vartheta}\right]^{-1}_{\vartheta=\vartheta_0,\phi=\phi_0}\\
=&\operatorname{Re}\left[\frac{(2\varrho+1)s^{2\varrho}+2c_1\varrho s^{2\varrho-1}+\varrho c_2 s^{\varrho-1}+\left((\varrho+1)d_1 s^{\varrho}+\varrho d_2 s^{\varrho-1}\right)e^{-s\vartheta}}{e^{-s\vartheta}s\left(d_1 s^{\varrho+1}+d_2 s^{\varrho}+d_3\right)}\right]_{\vartheta=\vartheta_0,\phi=\phi_0}\\
=&\frac{\mathcal{W}_{11}\mathcal{W}_{21}+\mathcal{W}_{12}\mathcal{W}_{22}}{\mathcal{W}_{21}^2+\mathcal{W}_{22}^2}.
\end{aligned} \tag{3.26}$$

By $(\mathcal{M}2)$, one derives

$$\operatorname{Re}\left[\frac{ds}{d\vartheta}\right]^{-1}_{\vartheta=\vartheta_0,\phi=\phi_0}>0,$$

which ends the proof.

According to the analysis above, one can easily obtain the following assertion:

Theorem 3.1. *Under the conditions of Lemma 3.1. If* $(\mathcal{M}1)$ *and* $(\mathcal{M}2)$ *are fulfilled, then the positive equilibrium point* (v_{10}, v_{20}, v_{30}) *of system (3.2) is locally asymptotically stable provided that* ϑ *falls into the range of* $[0, \vartheta_0)$ *and a Hopf bifurcation will take place around the positive equilibrium point* (v_{10}, v_{20}, v_{30}) *when* $\vartheta = \vartheta_0$.

Remark 3.1. In this paper, we establish a new fractional-order predator-prey model with distributed delays. In order to explore the stability and Hopf bifurcation issue conveniently, we introduce a variable substitution and then obtain an equivalent form including two fractional-order equations and one integer-order equation.

Remark 3.2. In this paper, we deal with the bifurcation anti-control of a fractional-order stable finance model by virtue of a suitable washout filter controller involving time delay. We can also deal with the bifurcation anti-control of a fractional-order stable finance model by virtue of time delay feedback controller. We will focus on this aspect in near future.

Remark 3.3. although the assumption $(\mathcal{M}2)$ is very complex, we can easily check it by computer.

Remark 3.4. We obtain the characteristic equation of integer-order differential equation via matrix theory and integer-order differential equation theory. However we obtain the characteristic equation of fractional-order differential equation via matrix theory, Laplace transform and fractional-order differential equation theory.

4. Computer simulations

Consider the following predator-prey system:

$$\begin{cases} \dfrac{dv_1^{\varrho}(t)}{dt^{\varrho}} = v_1(t)\left[1 - 1v_3(t) - 0.6v_2(t-\vartheta)\right], \\ \dfrac{dv_2^{\varrho}(t)}{dt^{\varrho}} = v_2(t)\left[-1 + 2v_3(t) - 0.5v_2(t-\vartheta)\right], \\ \dfrac{dv_3(t)}{dt} = 2(v_1(t) - v_3(t)). \end{cases} \tag{4.1}$$

One can easily obtain that system (4.1) owns the positive equilibrium point $(v_{10}, v_{20}, v_{30}) = (0.65, 0.59, 0.65)$. Select $\varrho = 0.92$. By means of Matlab software, one derives that $\varphi_0 = 0.9174$ and $\vartheta_0 = 0.805$, the conditions of Lemma 3.1 hold and the conditions $(\mathcal{M}1)$ and $(\mathcal{M}2)$ of Theorem 3.1 are fulfilled. So one can know that $(v_{10}, v_{20}, v_{30}) = (0.65, 0.59, 0.65)$ of system (4.1) keeps locally asymptotically stable state when ϑ falls into the range of $[0, 0.805)$. Choosing $\vartheta = 0.78 < \vartheta_0 = 0.805$, one derives the simulation results that are given in Figure 1. Apparently, Figure 1 confirms that when the time delay ϑ falls into the range of $[0, 0.805)$, then the three state variables will tend to $0.65, 0.59, 0.65$, respectively. From the viewpoint of biology, when the hunting time falls into the range of $[0, 0.805)$, the densities of prey population and predator population will be close to $0.65, 0.59$, respectively. When the time delay ϑ passes through $\vartheta_0 = 0.805$, the three state variables will lose their stability and a Hopf bifurcation appears at once. Choosing $\vartheta = 0.92 > \vartheta_0 = 0.805$, one gets the simulation diagram Figure 2 which shows that the three state variables will keep periodic oscillation near $(v_{10}, v_{20}, v_{30}) = (0.65, 0.59, 0.65)$. From the viewpoint of biology, when the hunting time exceeds the value 0.805, the densities of prey population and predator population keep periodic oscillation around 0.65 and 0.59, respectively. In addition, the relation between ϱ and ϑ_0 is given in Table 1 and the bifurcation figures are presented in Figures 3-5.

Table 1. The relation between ϱ and ϑ_0 of predator-prey system (4.1).

ϱ	ϑ_0
0.26	0.342
0.39	0.453
0.48	0.543
0.57	0.611
0.69	0.681
0.74	0.704
0.86	0.792
0.92	0.805

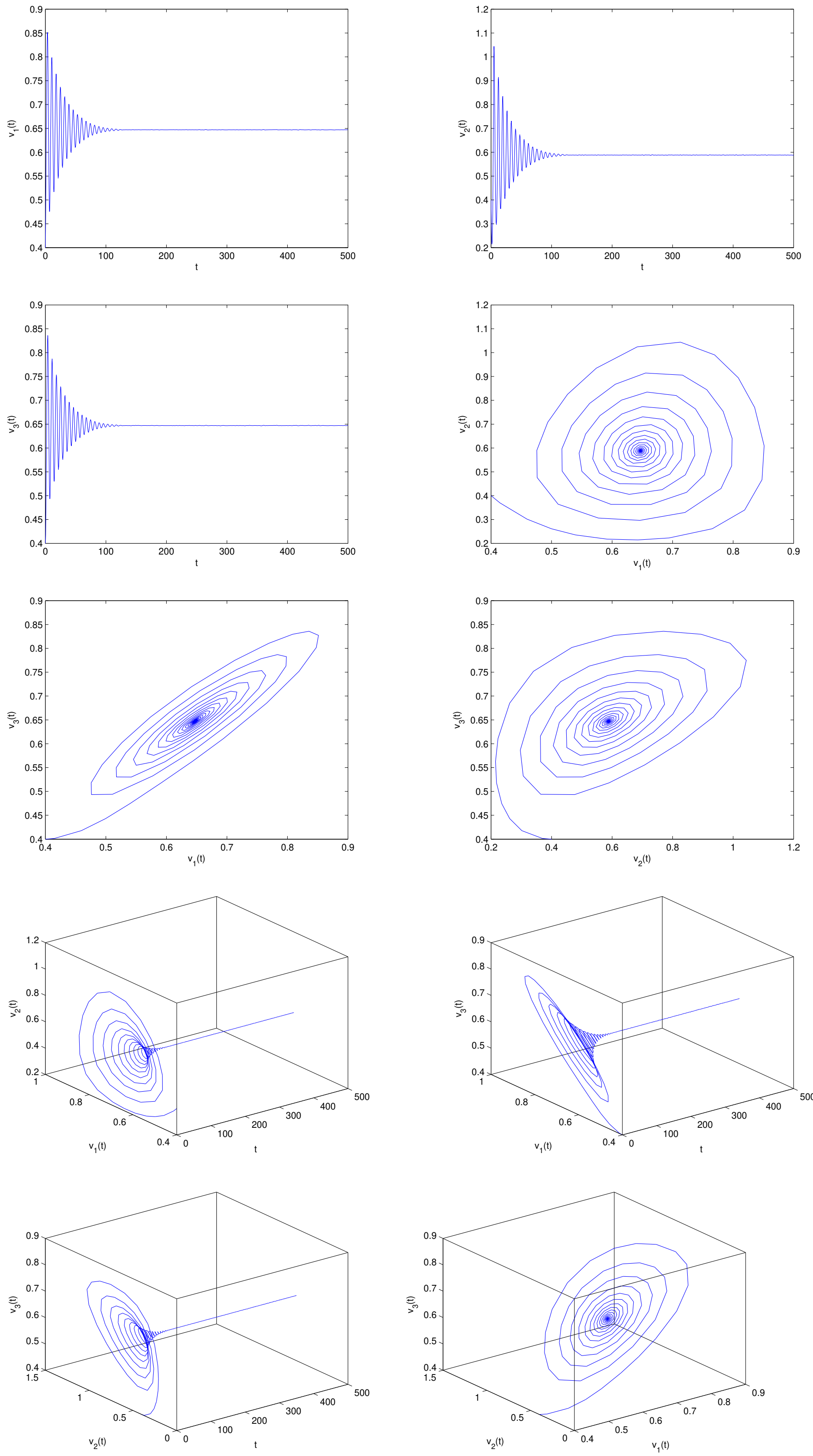

Figure 1. Simulation results of predator-prey system (4.1) when $\vartheta = 0.78 < \vartheta_0 = 0.805$.

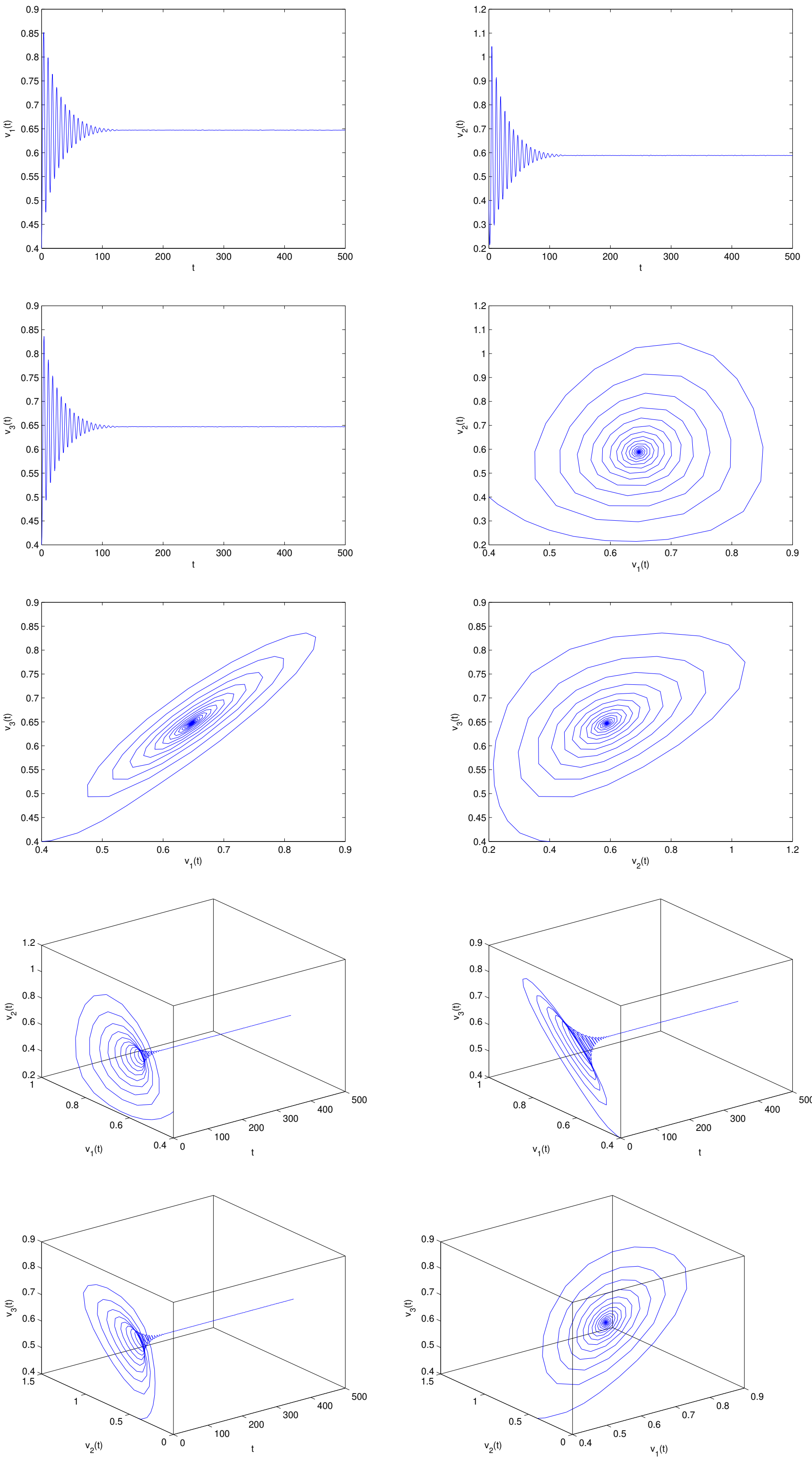

Figure 2. Simulation results of predator-prey system (4.1) when $\vartheta = 0.78 < \vartheta_0 = 0.805$.

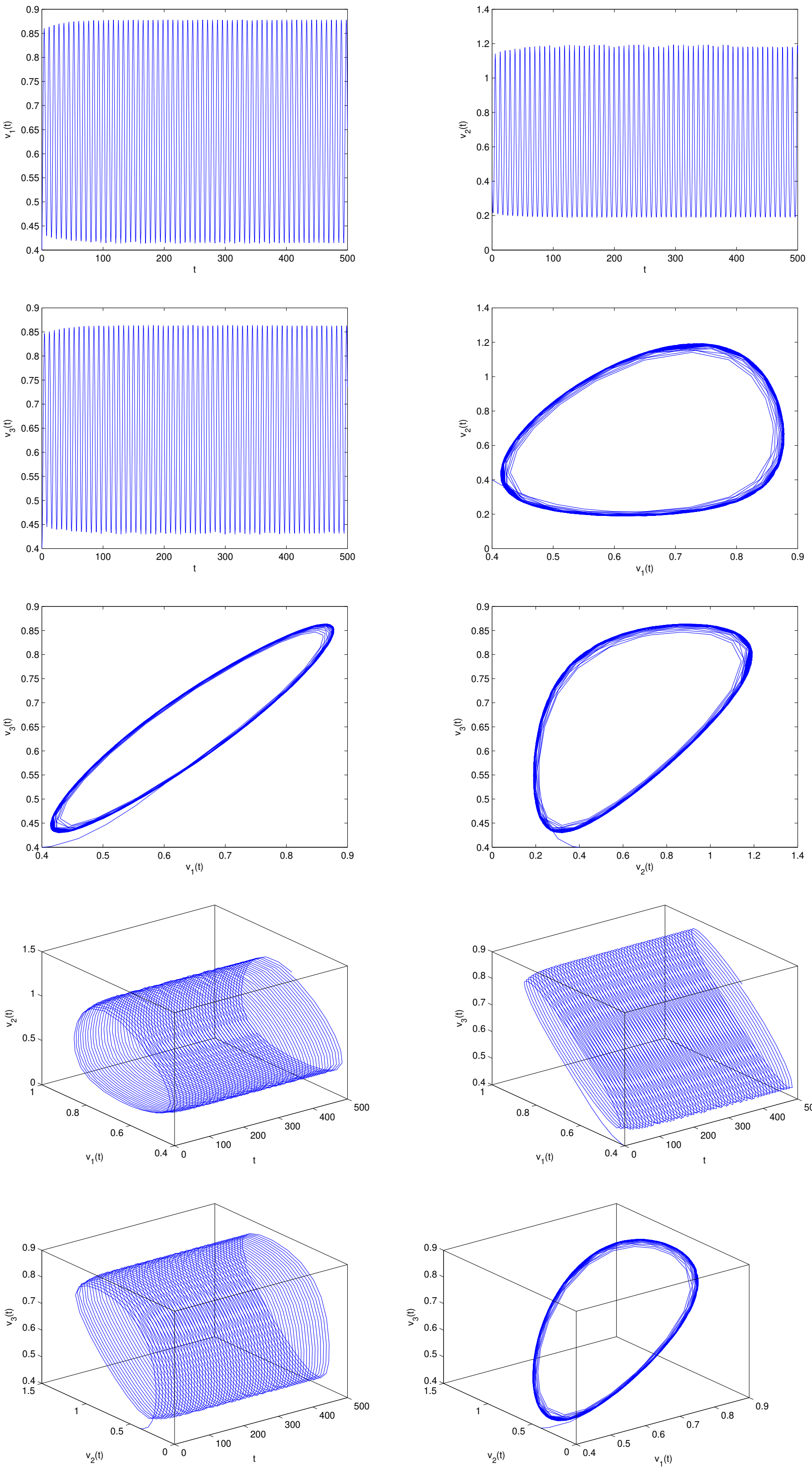

Figure 3. Simulation results of predator-prey system (4.1) when $\vartheta = 0.92 > \vartheta_0 = 0.805$.

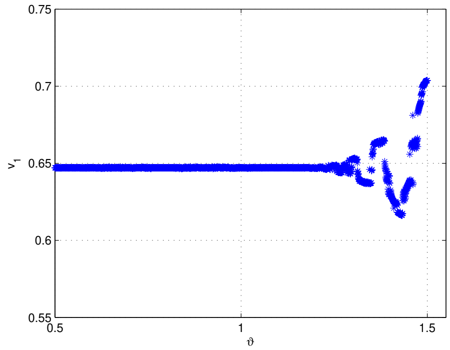

Figure 4. The bifurcation plot of system (4.1): t-v_1.

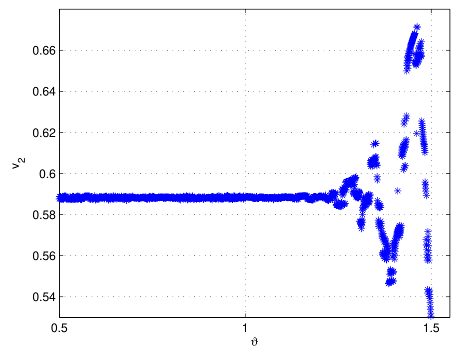

Figure 5. The bifurcation plot of system (4.1): t-v_2.

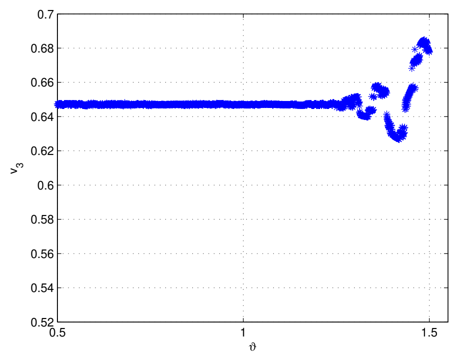

Figure 6. The bifurcation plot of system (4.1): t-v_3.

5. Conclusions

Nowadays the study on fractional-order predator-prey models has attracted great attention from many scholars in mathematics and biology due to the potential value in maintaining the ecological balance. In this present research, on the basis of the previous work of [21], we set up a new fractional-order predator-prey model with distributed delay and discrete delay. By using a suitable substitution of variable, we obtain an equivalent system which includes two fractional-order equations and one integer-order equations. Applying the time delay as bifurcation parameter, we derive a delay-independent stability and bifurcation criterion to guarantee the stability and the emergence of Hopf bifurcation of the involved predator-prey model. The study shows that time delay is a significant role that affects the Hopf bifurcation of the considered fractional-order predator-prey model. In 2012, Xu and Shao [21] discussed the bifurcation problem of integer-order predator-prey model by selecting the time delay as bifurcation parameter. They did not analyze the fractional-order form of this predator-prey model. In the present article, we establish a new fractional-order predator-prey system (1.3) and focus on the stability behavior and bifurcation phenomenon of predator-prey system (1.3) including weak kernel. From this viewpoint, we think that this work supplements the study of Xu and Shao [21]. Although the works of [5, 10, 20, 26, 37] have studied the Hopf bifurcation of fractional-order dynamical models, every equation of all systems is fractional-order cases. However, by a suitable variable substitution, we obtain the equivalent predator-prey system (3.2) which includes two fractional-order differential equations and a integer-order differential equation. So, the investigation on the corresponding characteristic equation of predator-prey system (3.2) becomes more difficult. So far, only a few works involve this aspect. Based on this point, we think that the present investigation improves the earlier publications(e.g., [5,10,20,26,37]). Also, the derived fruits enrich the bifurcation theory of fractional-order dynamical system to some degree. In addition, in many biological systems, there are different delays. We will try to deal with the Hopf bifurcation of fractional-order predator-prey model with different delays in near future.

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