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PSI, POLYGAMMA FUNCTIONS AND Q-COMPLETE MONOTONICITY ON TIME SCALES

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Abstract In this paper, we generalize psi and polygamma functions based on the Laplace transform in the field of time scales, and explore some properties of them. Next, we present the concepts of q-complete monotonicity, q-logarithmically complete monotonicity and q-absolute monotonicity with delta derivative on time scales. At last, we prove that the function

$$s \mapsto \alpha\psi_{\mathbb{R}_0,\mathbb{T}}(s) - \ln s + \frac{1}{2s} + \frac{1}{12s^2}$$

is 1-complete monotonicity on $(0,\infty)$ if $\mathbb{T} = \mathbb{N}$ and $\alpha \in [\frac{3-2\sqrt{3}}{6}, \frac{3+2\sqrt{3}}{6}]$, and it is decreasing on $(0,\infty)$ if $\mathbb{T} = h\mathbb{N} \cup \{1\} (h \geq 1)$ and $\alpha = 1$, where $\mathbb{R}_0 = [0,\infty)$ and $\psi_{\mathbb{R}_0,\mathbb{T}}$ is a psi function on time scales.

Keywords Psi function, polygamma function, completely monotonic, gamma function, time scale.

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1. Introduction

Gamma function, the most important special function, has been explored for hundreds of years. Traditional gamma function was defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t, \quad x > 0.$$

The psi or digamma function was defined by the logarithmic derivative of gamma function

$$\psi(x) = \big(\ln\Gamma(x)\big)' = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0. \tag{1.1}$$

Taking derivatives of the psi function several times yields so-called polygamma function $\psi^{(i)}(x)$, $i = 1, 2, \cdots$. A great number of interesting works involving gamma, psi and polygamma functions have been provided [20, 22, 26, 29–31, 35, 36, 38].

A well-know relationship between gamma function $\Gamma(x)$ and Laplace transform $\mathcal{L}\{h\}(y) := \int_0^\infty h(t)e^{-yt}\mathrm{d}t$ is

$$\Gamma(x) = \mathcal{L}\{I^{x-1}\}(1), \quad x > 0, \tag{1.2}$$

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where I is the identity function.

The time scale theory was established by Hilger [13] and it has received a rapid development in recent years. Enormous interesting results have been presented [4, 5, 7, 8, 11, 12, 15, 17–19, 27, 28, 33].

For convenience, we list some symbols that appeared in this article: $\mathbb{T}$ and $\mathbb{T}_x$ represent two arbitrary time scales of $\mathbb{R}_0 := [0, \infty)$, namely, they are nonempty closed subsets of $\mathbb{R}_0$, and satisfy that $\sup \mathbb{T} = \sup \mathbb{T}_x = \infty$; $\mathbb{T}_h := h\mathbb{N} \cup \{1\}(h \geq 1)$ is a time scale; $(0, \infty)_{\mathbb{T}} := (0, \infty) \cap \mathbb{T}$ and $(0, \infty)_{\mathbb{T}_x} := (0, \infty) \cap \mathbb{T}_x$; $\sigma(s) = \inf\{t \in \mathbb{T} : t > s\}$ ($\sigma_x(s) = \inf\{t \in \mathbb{T}_x : t > s\}$) is the forward jump operator on $\mathbb{T}$ ($\mathbb{T}_x$); $\mu(s) = \sigma(s) - s$ ($\mu_x(s) = \sigma_x(s) - s$) is the graininess function on $\mathbb{T}$ ($\mathbb{T}_x$); Δ (Δ_x), $\nabla(\nabla_x)$ and $\diamond_\alpha(\diamond_{\alpha x})$ are the delta derivative, nabla derivative and diamond-alpha derivative on $\mathbb{T}(\mathbb{T}_x)$, respectively; $\mathbb{T}^* = \begin{cases} \mathbb{T} & \text{if } \sigma(\inf \mathbb{T}) = \inf \mathbb{T} \\ \mathbb{T}\backslash \inf \mathbb{T} & \text{if } \sigma(\inf \mathbb{T}) > \inf \mathbb{T} \end{cases}$ and $\mathbb{T}_x^* = \begin{cases} \mathbb{T}_x & \text{if } \sigma(\inf \mathbb{T}_x) = \inf \mathbb{T}_x \\ \mathbb{T}_x\backslash \inf \mathbb{T}_x & \text{if } \sigma(\inf \mathbb{T}_x) > \inf \mathbb{T}_x \end{cases}$; $\ominus_\mu f = \frac{-f}{1+\mu f}$ is the circle minus, where f is a function defined on $\mathbb{T}$, μ is the graininess function; $e_f(t, s) = \exp\big(\int_s^t \frac{1}{\mu(\eta)} \ln\big(1+ f(\eta)\mu(\eta)\big)\Delta\eta\big)$ is the exponential function, where f is a function defined on $\mathbb{T}$. We also denote that $\mathbb{N} = \{0, 1, 2, \cdots\}$.

In 2002, by using delta derivative, Bohner and Peterson [9] generalized Laplace transform on time scales as follows

$$\mathcal{L}_{\mathbb{T}}\{h\}(y) = \int_0^\infty h(t) e^\sigma_{\ominus_\mu y}(t, 0)\Delta t, \tag{1.3}$$

where h is regulated function. On the basis of [9] and the relationship $\Gamma_{\mathbb{T}}(x) = \mathcal{L}_{\mathbb{T}}\{e_{\frac{x-1}{\sigma}}(\cdot, s)\}(1)$ which similar to (1.2), in 2013, Bohner and Karpuz [6] further generalized gamma function on time scales as follows

$$\Gamma_{\mathbb{T}}(x) = \int_0^\infty e_{(x-1)/\sigma}(t, s) e^\sigma_{\ominus_\mu 1}(t, 0)\Delta t, \quad x > 0, \tag{1.4}$$

where $s \in \mathbb{T}$ is a given constant. For convenience, we call (1.3) and (1.4) delta Laplace transform and delta gamma function, respectively.

Inspired by these above, in this paper, we presented the psi and polygamma functions on time scales and then explored some properties of their.

Complete monotonicity, which generalizes monotonicity and convexity, can better show the properties of a function. Meanwhile, logarithmically complete monotonicity and absolute monotonicity are also of great significance. Scholars are committed to presenting some results that are completely monotonic or logarithmically completely monotonic or absolutely completely monotonic involving gamma, psi, polygamma and other special functions [10, 14, 20, 24, 25, 30, 32, 37, 39, 40].

Recall that complete monotonicity, logarithmically complete monotonicity, and absolute monotonicity are defined as follows, respectively.

Definition 1.1 ([3, 34])**.** Suppose that the function ϕ has derivatives of all orders and satisfies

$$(-1)^n \phi^{(n)}(x) \geq 0$$

for all $x \in (0, \infty)$ and $n \geq 0$, then ϕ is called completely monotonic on $(0, \infty)$.

Definition 1.2 ([2, 23]). Suppose that the function ϕ satisfies

$$(-1)^n(\ln \phi(x))^{(n)} \geq 0$$

for all $x \in (0,\infty)$ and $n \geq 0$, then ϕ is called logarithmically completely monotonic on $(0,\infty)$.

Definition 1.3 ([34]). Suppose that the function ϕ has derivatives of all orders and satisfies

$$\phi^{(n)}(x) \geq 0$$

for all $x \in (0,\infty)$ and $n \geq 0$, then ϕ is called absolutely monotonic on $(0,\infty)$.

Clearly, a completely monotonic function is non-negative, decreasing and convex; an absolutely monotonic function is non-negative, increasing and convex.

In this paper, we will present the concepts of q-complete monotonicity, q-logarithmically complete monotonicity, and q-absolute monotonicity on time scales in section 3. At last, we will provide an application of the psi function and complete monotonicity on time scales, which generalizes a result in real analysis.

2. Psi and polygamma functions on time scales

In order to present psi and polygamma functions on time scales, we first generalize the delta gamma function to

$$\Gamma_{\mathbb{T}_x,\mathbb{T}}(x) = \int_0^\infty g(x,t)\Delta t, \quad x \in (0,\infty)_{\mathbb{T}_x}, \tag{2.1}$$

where $g(x,t) := e_{(x-1)/\sigma}(t,s)e^{\sigma}_{\ominus_\mu 1}(t,0)$ is defined on $(0,\infty)_{\mathbb{T}_x} \times \mathbb{T}$.

Clearly, $\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)$ converges for any $x \in (0,\infty)_{\mathbb{T}_x}$ based on [6, Theorem 3]. Then we give examples of $\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)$ under some time scales.

Example 2.1 ([6, Table 4]). Let $s \in \mathbb{T}$. If $\mathbb{T} = \mathbb{R}_0$, $h\mathbb{N}$, $q^{\mathbb{Z}}$ respectively, then the delta gamma function $\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)$ reduces to

$$\Gamma_{\mathbb{T}_x,\mathbb{R}_0}(x) = \int_0^\infty \Big(\frac{\eta}{s}\Big)^{x-1} e^{-\eta}\mathrm{d}\eta,$$

$$\Gamma_{\mathbb{T}_x,h\mathbb{N}}(x) = h\sum_{\eta=0}^{\infty}\Big(\prod_{k=s/h}^{\eta-1}\frac{k+x}{k+1}\Big)\frac{1}{(h+1)^{\eta+1}}, \quad h>0,$$

$$\Gamma_{\mathbb{T}_x,q^{\mathbb{Z}}}(x) = \frac{(q-1)s}{(1+(q-1)x)^{\log_q(s)}}\sum_{\eta=-\infty}^{\infty}\frac{(1+(q-1)x)^\eta}{\prod_{k=-\infty}^{\eta}(1+(q-1)q^k)}, \quad q>1.$$

Example 2.2. Let $s = 1 \in \mathbb{T}$. If $\mathbb{T} = \mathbb{T}_h$, then the delta gamma function $\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)$ reduces to

$$\Gamma_{\mathbb{T}_x,\mathbb{T}_h}(x) = \frac{h-1}{2xh} + \frac{(h-1)x+1}{2xh}\Big(1+\frac{1}{h}\Big)^x, \quad x \in (0,\infty)_{\mathbb{T}_x}.$$

Proof. Noting that

$$e_{(x-1)/\sigma}(t,1) = \exp\Big(\int_1^t \frac{\ln(1+\frac{x-1}{\sigma(\tau)}\mu(\tau))}{\mu(\tau)}\Delta\tau\Big),$$

and

$$e^{\sigma}_{\ominus_\mu 1}(t,0)=\exp\Big(\int_0^{\sigma(t)}\frac{\ln(1-\frac{1}{1+\mu(\tau)}\mu(\tau))}{\mu(\tau)}\Delta\tau\Big)=\exp\Big(-\int_0^{\sigma(t)}\frac{\ln(1+\mu(\tau))}{\mu(\tau)}\Delta\tau\Big),$$

using the the definition of $\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)$, we have

$$\begin{aligned}
&\Gamma_{\mathbb{T}_x,\mathbb{T}_h}(x)\\
&=\int_0^{\infty} e_{\frac{x-1}{\sigma}}(t,1)e^{\sigma}_{\ominus_\mu 1}(t,0)\Delta t\\
&=e_{\frac{x-1}{\sigma}}(0,1)e^{\sigma}_{\ominus_\mu 1}(0,0)+e_{\frac{x-1}{\sigma}}(1,1)e^{\sigma}_{\ominus_\mu 1}(1,0)\mu(1)+\sum_{t=1}^{\infty}e_{(x-1)/\sigma}(th,1)e^{\sigma}_{\ominus_\mu 1}(th,0)\mu(th)\\
&=\frac{1}{2x}+\frac{h-1}{2h}+h\sum_{t=1}^{\infty}\exp\Big(\int_1^{th}\frac{\ln(1+\frac{x-1}{\sigma(\tau)}\mu(\tau))}{\mu(\tau)}\Delta\tau\Big)\exp\Big(-\int_0^{\sigma(th)}\frac{\ln(1+\mu(\tau))}{\mu(\tau)}\Delta\tau\Big)\\
&=\frac{1}{2x}+\frac{h-1}{2h}+h\Big(1+\frac{(h-1)(x-1)}{h}\Big)\frac{1}{2h(h+1)}\\
&\quad+h\sum_{t=2}^{\infty}\Big(\Big(1+\frac{(h-1)(x-1)}{h}\Big)\prod_{k=1}^{t-1}\frac{k+x}{k+1}\frac{1}{2h(h+1)^t}\Big)\\
&=\frac{1}{2x}+\frac{h^2+hx-x}{2h^2+2h}+\frac{(h-1)x+1}{2h}\sum_{t=2}^{\infty}\Big(\prod_{k=1}^{t-1}\frac{k+x}{k+1}\frac{1}{(h+1)^t}\Big)\\
&=\frac{1}{2x}+\frac{h^2+hx-x}{2h^2+2h}+\frac{(h-1)x+1}{2h}\sum_{t=2}^{\infty}\Big(\frac{\Gamma(t+x)}{\Gamma(t+1)\Gamma(x+1)}\frac{1}{(h+1)^t}\Big)\\
&=\frac{1}{2x}+\frac{h^2+hx-x}{2h^2+2h}+\frac{(h-1)x+1}{2xh}\Big(\Big(1+\frac{1}{h}\Big)^x-1-\frac{x}{h+1}\Big)\\
&=\frac{h-1}{2hx}+\frac{(h-1)x+1}{2xh}\Big(1+\frac{1}{h}\Big)^x.
\end{aligned}$$

□

According to the definitions of derivatives on time scales and the convergence of $\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)$, we present the following definitions.

Definition 2.1. Let $s\in\mathbb{T}$ be given. Then delta-psi function is defined by

$$\psi_{\Delta_x,\mathbb{T}_x,\mathbb{T}}(x)=\frac{\Gamma^{\Delta_x}_{\mathbb{T}_x,\mathbb{T}}(x)}{\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)},\quad x\in(0,\infty)_{\mathbb{T}_x},\tag{2.2}$$

where $\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)$ is given by (2.1).

Definition 2.2. Let $s\in\mathbb{T}$ be given. Then nabla-psi function is defined by

$$\psi_{\nabla_x,\mathbb{T}_x,\mathbb{T}}(x)=\frac{\Gamma^{\nabla_x}_{\mathbb{T}_x,\mathbb{T}}(x)}{\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)},\quad x\in\mathbb{T}_x^{*^2}\backslash\{0\},\tag{2.3}$$

where $\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)$ is given by (2.1).

Definition 2.3. Let $s\in\mathbb{T}$ be given. Then diamond-alpha-psi function is defined by

$$\psi_{\diamond_{\alpha x},\mathbb{T}_x,\mathbb{T}}(x)=\frac{\Gamma^{\diamond_{\alpha x}}_{\mathbb{T}_x,\mathbb{T}}(x)}{\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)},\quad x\in\mathbb{T}_x^{*^2}\backslash\{0\},\tag{2.4}$$

where $\Gamma_{\mathbb{T}_x,\mathbb{T}}(x)$ is given by (2.1).

Remark 2.1. Here, we call delta-psi function, nabla-psi function and diamond alpha-psi function all psi function on time scales. One important psi function on time scales is obtained by taking $\mathbb{T}_x = \mathbb{R}_0$:

$$\psi_{\mathbb{R}_0,\mathbb{T}}(x) = \frac{\Gamma'_{\mathbb{R}_0,\mathbb{T}}(x)}{\Gamma_{\mathbb{R}_0,\mathbb{T}}(x)}, \quad x > 0. \tag{2.5}$$

Next, we calculate some expressions of psi functions under some specific time scales.

Example 2.3. Let $\mathbb{T}_x = \mathbb{R}_0$, $h \geq 1, q > 1$ and $s = 1$. Then $\psi_{\mathbb{R}_0,\mathbb{R}_0}(x)$ reduces to (1.1),

$$\psi_{\mathbb{R}_0,\mathbb{T}_h}(x) = -\frac{\left(\frac{1}{h}+1\right)^x - x((h-1)x+1)\left(\frac{1}{h}+1\right)^x \ln\left(\frac{1}{h}+1\right) + h - 1}{x\left(hx\left(\frac{1}{h}+1\right)^x - x\left(\frac{1}{h}+1\right)^x + \left(\frac{1}{h}+1\right)^x + h - 1\right)},$$

and

$$\psi_{\mathbb{R}_0,q^{\mathbb{Z}}}(x) = \frac{\sum_{t=-\infty}^{\infty} \frac{(q-1)t((q-1)x+1)^{t-1}}{\left((1-q)q^t;\frac{1}{q}\right)_\infty}}{\sum_{t=-\infty}^{\infty} \frac{((q-1)x+1)^{t}}{\left((1-q)q^t;\frac{1}{q}\right)_\infty}},$$

where $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$ is the q-pochhammer symbol.

Example 2.4. Let $\mathbb{T}_x = h_1\mathbb{N}$, $h_1 > 0, h_2 \geq 1, q > 1$ and $s = 1$. Then we have

$$\begin{aligned}
\psi_{\Delta_x,h_1\mathbb{N},\mathbb{R}_0}(x) &= \frac{\Gamma(h_1+x) - \Gamma(x)}{h_1\Gamma(x)},\\
\psi_{\nabla_x,h_1\mathbb{N},\mathbb{R}_0}(x) &= \frac{\Gamma(x) - \Gamma(x-h_1)}{h_1\Gamma(x)},\\
\psi_{\diamond_{\alpha x},h_1\mathbb{N},\mathbb{R}_0}(x) &= \frac{\alpha\Gamma(x+h_1) + (1-2\alpha)\Gamma(x) + (\alpha-1)\Gamma(x-h_1)}{h_1\Gamma(x)},
\end{aligned}$$

and

$$\begin{aligned}
&\psi_{\Delta_x,h_1\mathbb{N},\mathbb{T}_{h_2}}(x)\\
&= \frac{\left(\begin{aligned} &x\Big(\big(\frac{1}{h_2}+1\big)^{h_1}-1\Big)\big((h_2-1)x+1\big)\big(\frac{1}{h_2}+1\big)^x\\ &\quad + h_1x\Big(1-\big(\frac{1}{h_2}+1\big)^{h_1}\big(\frac{1}{h_2}+1\big)^x\Big)\\ &\quad + h_1h_2\bigg(\Big(x\Big(\big(\frac{1}{h_2}+1\big)^{h_1}-1\Big)\big(\frac{1}{h_2}+1\big)^x-1\Big)-\big(\frac{1}{h_2}+1\big)^x+1\bigg)\end{aligned}\right)}{h_1(h_1+x)\left(h_2x\left(\frac{1}{h_2}+1\right)^x - x\left(\frac{1}{h_2}+1\right)^x + \left(\frac{1}{h_2}+1\right)^x + h_2 - 1\right)}.
\end{aligned}$$

According to the definition of delta, nabla and diamond-alpha derivative as well as [6, Theorem 7], it is easy to obtain that the psi functions on time scales have the following properties.

Property 2.1. *Let $h_1 > 0$, $q > 1$, $\psi_{\Delta_x,\mathbb{T}_x,\mathbb{T}}(x)$, $\psi_{\nabla_x,\mathbb{T}_x,\mathbb{T}}(x)$ and $\psi_{\diamond_{\alpha x},\mathbb{T}_x,\mathbb{T}}(x)$ be defined by (2.2), (2.3) and (2.4), respectively. Then we have*

$$\psi_{\diamond_{\alpha x},\mathbb{T}_x,\mathbb{T}}(x) = \alpha\psi_{\Delta_x,\mathbb{T}_x,\mathbb{T}}(x) + (1-\alpha)\psi_{\nabla_x,\mathbb{T}_x,\mathbb{T}}(x), \quad x \in \mathbb{T}_x^{*^2}\backslash\{0\},$$

$$\psi_{\Delta_x,h_1\mathbb{N},\mathbb{T}}(x)=\psi_{\nabla_x,h_1\mathbb{N},\mathbb{T}}(x+h_1),\quad x\in h_1\mathbb{N}\backslash\{0\},$$

and

$$\psi_{\Delta_x,q^{\mathbb{Z}},\mathbb{T}}(x)=\psi_{\nabla_x,q^{\mathbb{Z}},\mathbb{T}}(qx),\quad x\in q^{\mathbb{Z}}.$$

Property 2.2. *Let $s\in\mathbb{T}$. If $\mathbb{T}_x=\{x_0\}+\mathbb{N}$, then*

$$\psi_{\Delta_x,\mathbb{T}_x,\mathbb{R}_0}(x+1)=\frac{x+1}{x}\psi_{\Delta_x,\mathbb{T}_x,\mathbb{R}_0}(x)+\frac{1}{x},\quad x\in\{x_0\}+\mathbb{N},$$

where $x_0>0$ is a given constant. If $\mathbb{T}_x=\mathbb{T}^{x_0}_{(q,1)}$, then

$$\psi_{\Delta_x,\mathbb{T}_x,q^{\mathbb{Z}}}(qx+1)=\frac{qx+1}{x}\psi_{\Delta_x,\mathbb{T}_x,\mathbb{R}_0}(x)+\frac{1}{x},\quad x\in\mathbb{T}^{x_0}_{(q,1)},$$

where $x_0>0$ is a given constant, $\mathbb{T}^{x_0}_{(q,h)}:=\{x_0q^k+[k]_qh,k\in\mathbb{N}\}(q>1,h\geq 0)$ and $[k]_q:=\frac{q^k-1}{q-1}$.

Let $s\in\mathbb{T}$, we define the higher derivative of $\psi_{\Delta_x,\mathbb{T}_x,\mathbb{T}}(x)$ as delta-polygamma function, namely, we call $\frac{\mathrm{d}^n}{\Delta_x x^n}\psi_{\Delta_x,\mathbb{T}_x,\mathbb{T}}(x)$ delta-polygamma function, where $x\in(0,\infty)_{\mathbb{T}_x}$. Likewise, we call $\frac{\mathrm{d}^n}{\nabla_x x^n}\psi_{\nabla_x,\mathbb{T}_x,\mathbb{T}}(x)$ nabla-polygamma function, where $x\in\mathbb{T}^{*^{n+1}}_x\backslash\{0\}$. Both delta-polygamma function and nabla-polygamma function are called polygamma function on time scales. The following example lists some polygamma functions on time scales.

Example 2.5. Let $s=1\in\mathbb{T}$. Then

$$\psi^{(n)}_{\mathbb{R}_0,\mathbb{N}}(x)=(-1)^{n+1}\Gamma(n+1)x^{-1-n},\quad x>0,$$
$$\psi'_{\mathbb{R}_0,\mathbb{T}_2}(x)=\frac{6^xx^3\ln^2\frac{3}{2}+6^xx^2\left(\ln^2\frac{3}{2}+\ln\frac{9}{4}\right)+2x3^x\left(2^x+3^x\right)+\left(2^x+3^x\right)^2}{x^2\left(3^xx+2^x+3^x\right)^2},\ x>0,$$

and

$$\begin{aligned}&\psi'_{\mathbb{R}_0,\mathbb{T}_h}(x)\\=&\frac{(h-1)x^2((h-1)x+1)\left(\frac{1}{h}+1\right)^x\ln^2\left(\frac{1}{h}+1\right)+2(h-1)^2x^2\left(\frac{1}{h}+1\right)^x\ln\left(\frac{1}{h}+1\right)}{x^2\left(hx\left(\frac{1}{h}+1\right)^x-x\left(\frac{1}{h}+1\right)^x+\left(\frac{1}{h}+1\right)^x+h-1\right)^2}\\&+\frac{\left(\left(\frac{1}{h}+1\right)^x+h-1\right)\left(2hx\left(\frac{1}{h}+1\right)^x-2x\left(\frac{1}{h}+1\right)^x+\left(\frac{1}{h}+1\right)^x+h-1\right)}{x^2\left(hx\left(\frac{1}{h}+1\right)^x-x\left(\frac{1}{h}+1\right)^x+\left(\frac{1}{h}+1\right)^x+h-1\right)^2},\quad x>0.\end{aligned}$$

3. Q-complete monotonicity, q-logarithmically complete monotonicity and q-absolute monotonicity on time scales

Now we will present q-complete monotonicity, q-logarithmically complete monotonicity and q-absolute monotonicity with delta derivative on time scales.

Definition 3.1. Let $n,q\in\mathbb{N}$ and q be given. Suppose that the function ϕ has delta-derivatives of all orders and satisfies

$$(-1)^n\phi^{\Delta^n}(x)\geq 0$$

for all $x \in (0,\infty)_{\mathbb{T}}$ and $n \geq q$, then ϕ is called q-completely monotonic on time scales.

Definition 3.2. Let $n, q \in \mathbb{N}$ and q be given. Suppose that the function ϕ satisfies

$$(-1)^n (\ln \phi(x))^{\Delta^n} \geq 0,$$

for all $x \in (0,\infty)_{\mathbb{T}}$ and $n \geq q$, then ϕ is called q-logarithmically completely monotonic on time scales.

Definition 3.3. Let $n, q \in \mathbb{N}$ and q be given. Suppose that the function ϕ has delta-derivatives of all orders and satisfies

$$\phi^{\Delta^n}(x) \geq 0,$$

for all $x \in (0,\infty)_{\mathbb{T}}$ and $n \geq q$, then ϕ is called q-absolutely monotonic on time scales.

Remark 3.1. Let $q = 0$ and $\mathbb{T} = \mathbb{R}_0$. Then q-completely monotonic functions, q-logarithmically completely monotonic functions, and q-absolutely monotonic functions on time scales reduce to completely monotonic functions, logarithmically completely monotonic functions, and absolutely monotonic functions, respectively. Let $q = 0$ and $\mathbb{T} = \mathbb{N}$. Then q-completely monotonic functions, q-logarithmically completely monotonic functions, and q-absolutely monotonic functions on time scales reduce to completely monotonic sequences, logarithmically completely monotonic sequences, and absolutely monotonic sequences, respectively. Acquiescently, we denote 0-completely monotonic on time scales as completely monotonic on time scales, and so it is the same for the other two.

Remark 3.2. If $\mathbb{T} = \mathbb{R}$, then a function ϕ is q-completely monotonic on time scales means $(-1)^n \phi^{(n)}(x) \geq 0$ for all $x > 0$ and $q \leq n \in \mathbb{N}$, where q is a given integer. The same for q-logarithmically completely monotonic function on time scales and q-absolutely monotonic function on time scales.

The polynomial on time scales [1], for all $k = 0, 1, 2, \cdots$, was defined by

$$h_0(t,s) = 1, \qquad h_{k+1}(t,s) = \int_s^t h_k(\eta, s)\Delta\eta, \qquad s, t \in \mathbb{T}.$$

Proposition 3.1. *For given $m \in \mathbb{N}$, the function $h_m(t,0)$ is absolutely monotonic on time scales with respect to t.*

Proof. We claim that $h_m(t,0) > 0$ for all $m \in \mathbb{N}$. In fact, $h_0(t,0) = 1$ and $h_1(t,0) = t \geq 0$. Under the assumption that $h_m(t,0) > 0$, we have

$$h_{m+1}(t,0) = \int_s^t h_m(\eta, 0)\Delta\eta > 0.$$

Thus, we get the conclusion by using induction.

Clearly, we have

$$h_m^{\Delta^k}(t,0) = h_{m-k}(t,0) \geq 0 \qquad \text{if} \quad k \leq m,$$

and

$$h_m^{\Delta^k}(t,0) = h_0^{\Delta^{k-m}}(t,0) = 0 \geq 0 \qquad \text{if} \quad k \geq m+1.$$

□

4. Application

In 2005, Qi, Cui and Chen, et al. [21] posed the following theorem.

Theorem 4.1. *The function*

$$s \mapsto \psi(s) - \ln s + \frac{1}{2s} + \frac{1}{12s^2}$$

is strictly completely monotonic in $\mathbb{R}^+$, where $\psi(s)$ is shown in (1.1).

Naturally, we would ask whether this conclusion holds on arbitrary time scales. We denote that

$$Q(\mathbb{T};\alpha;s) := \alpha\psi_{\mathbb{R}_0,\mathbb{T}}(s) - \ln s + \frac{1}{2s} + \frac{1}{12s^2}, \qquad \alpha \in \mathbb{R}. \tag{4.1}$$

First we consider the case of $\mathbb{T} = \mathbb{N}$.

Theorem 4.2. *Let $\alpha \in [\frac{3-2\sqrt{3}}{6}, \frac{3+2\sqrt{3}}{6}]$. Then the function $Q(\mathbb{N};\alpha;s)$ is 1-completely monotonic on time scales.*

Proof. Since

$$Q(\mathbb{N};\alpha;s) = -\frac{\alpha}{s} + \alpha\ln 2 - \ln s + \frac{1}{2s} + \frac{1}{12s^2},$$

we have

$$Q'(\mathbb{N};\alpha;s) = -\frac{1}{6s^3} + \frac{\alpha-\frac{1}{2}}{s^2} - \frac{1}{s} = -\frac{1}{s^3}\Big(\big(s-(\frac{\alpha}{2}-\frac{1}{4})\big)^2 + \frac{1}{6} - (\frac{\alpha}{2}-\frac{1}{4})^2\Big) < 0,$$
$$Q''(\mathbb{N};\alpha;s) = \frac{1}{2s^4} - \frac{2\alpha-1}{s^3} + \frac{1}{s^2} = \frac{1}{s^4}\Big(s^2 - (2\alpha-1)s + \frac{1}{2}\Big) > 0,$$

and

$$Q'''(\mathbb{N};\alpha;s) = -\frac{2}{s^5} + \frac{3(2\alpha-1)}{s^4} - \frac{2}{s^3} = -\frac{1}{s^5}\Big(2s^2 - 3(2\alpha-1)s + 2\Big) < 0.$$

Noting that

$$\begin{aligned}
&Q^{(n)}(\mathbb{N};\alpha;s)\\
&= -\frac{1}{6}\prod_{k=0}^{n-2}(-3-k)s^{-2-n} + \big(\alpha-\frac{1}{2}\big)\prod_{k=0}^{n-2}(-2-k)s^{-1-n} - \prod_{k=0}^{n-2}(-1-k)s^{-n}\\
&= \prod_{k=0}^{n-4}(-3-k)s^{-2-n}\Big(-\frac{1}{6}(-n)(-1-n) + n(2\alpha-1)s - 2s^2\Big),
\end{aligned}$$

and the discriminant

$$n^2(2\alpha-1)^2 - 4\times\frac{2}{6}n(n+1) \le 0, \quad \alpha \in \left[\frac{3-2\sqrt{3}}{6}, \frac{3+2\sqrt{3}}{6}\right],$$

we can deduce that $(-1)^n Q^{(n)}(\mathbb{N};\alpha;s) \ge 0$ for all $n \ge 4$ and $s \in (0,\infty)$. Hence the conclusion is obtained. □

Taking $\alpha = 1,\ \frac{1}{2}$, we have the following corollary.

Corollary 4.1. *The functions* $Q(\mathbb{N};1;s)$ *and* $Q(\mathbb{N};\frac{1}{2};s)$ *are* 1*-completely monotonic on time scales.*

Now we consider the monotonicity of $Q(\mathbb{T}_h;1;s)$. The following lemmas are needed.

Lemma 4.1. *The inequality*

$$\ln^2\Big(\frac{1}{h}+1\Big)\geq(h-1)^3\Big(-\frac{3}{8}-\frac{7}{12}\ln 2\Big)+(h-1)^2\Big(\frac{1}{4}+\frac{3\ln 2}{4}\Big)-(h-1)\ln 2+\ln^2 2$$

holds for all $h\geq 1$.

Proof. Denote that

$$T_1(q)=\ln^2\big(\frac{1}{q+1}+1\big)-\Big(-\frac{3}{8}-\frac{7}{12}\ln 2\Big)q^3-\Big(\frac{1}{4}+\frac{3\ln 2}{4}\Big)q^2+q\ln 2-\ln^2 2,\ \ q>0.$$

It is easy to check that $T_1(0)=0$ and

$$T_1'(q)=-\frac{2}{(q+1)(q+2)}\ln\big(\frac{q+2}{q+1}\big)+3\Big(\frac{3}{8}+\frac{7}{12}\ln 2\Big)q^2-2\Big(\frac{1}{4}+\frac{3\ln 2}{4}\Big)q+\ln 2.$$

Clearly, we have $T_1'(0)=-\ln 2+\ln 2=0$ and

$$T_1''(q)=\frac{2(2q+3)\ln\big(\frac{q+2}{q+1}\big)+2}{(q+1)^2(q+2)^2}+3\Big(\frac{3}{4}+\frac{7}{6}\ln 2\Big)q-\Big(\frac{1}{2}+\frac{3\ln 2}{2}\Big).$$

Likewise, $T_1''(0)=0$, taking derivative again yields

$$T_1'''(q)=\frac{4(3q^2+9q+7)\ln\big(\frac{q+1}{q+2}\big)-6(2q-3)}{(q+1)^3(q+2)^3}+3\Big(\frac{3}{4}+\frac{7}{6}\ln 2\Big),$$

and $T_1'''(0)=0$ as well as

$$T_1^{(4)}(q)=6\left(\frac{1}{(q+1)^2}-\frac{1}{(q+2)^2}\right)^2+16\left(\frac{1}{q+2}-\frac{1}{q+1}\right)\left(\frac{1}{(q+2)^3}-\frac{1}{(q+1)^3}\right)$$
$$+12\left(\frac{1}{(q+1)^4}-\frac{1}{(q+2)^4}\right)\ln\big(\frac{q+2}{q+1}\big)>0,\quad q>0.$$

$T_1'''(0)=0$ and $T_1^{(4)}(q)>0$ deduce that $T_1'''(q)$ is increasing and positive. Then we know that $T_1''(q)$ and $T_1'(q)$ are increasing and positive. Finally, we get $T_1(q)>0$ which is the desired result. □

Lemma 4.2. *The inequality*

$$\ln\big(\frac{1}{h}+1\big)\geq-\frac{7}{24}(h-1)^3+\frac{3}{8}(h-1)^2-\frac{1}{2}(h-1)+\ln 2$$

holds for all $h\geq 1$.

Proof. Denote that

$$T_2(q)=\ln\big(\frac{1}{q+1}+1\big)+\frac{7}{24}q^3-\frac{3}{8}q^2+\frac{1}{2}q-\ln 2,\quad q\geq 0.$$

Differentiation leads to

$$T_2'(q) = -\frac{1}{q^2+3q+2} + \frac{7}{8}q^2 - \frac{3}{4}q + \frac{1}{2},$$
$$T_2''(q) = \frac{2q+3}{(q+1)^2(q+2)^2} + \frac{7}{4}q - \frac{3}{4},$$
$$T_2'''(q) = -\frac{2(3q^2+9q+7)}{(q+1)^3(q+2)^3} + \frac{7}{4},$$

and

$$T_2^{(4)}(q) = \frac{6\left(4q^3+18q^2+28q+15\right)}{(q+1)^4(q+2)^4}.$$

Noting that $T_2'(0) = T_2''(0) = T_2'''(0) = 0$ and $T_2^{(4)}(q) > 0$ for all $q > 0$, the desired conclusion follows. □

Lemma 4.3. *The inequality*

$$\left(\frac{1}{h}+1\right)^5 \leq 1865(h-1)^8 - 1375(h-1)^7 + 985(h-1)^6 - 681(h-1)^5 + 450(h-1)^4 - 280(h-1)^3 + 160(h-1)^2 - 80(h-1) + 32$$

holds for all $h \geq 1$.

Proof. Set

$$T_3(h) = \left(\frac{1}{h}+1\right)^5 - 1865(h-1)^8 + 1375(h-1)^7 - 985(h-1)^6 + 681(h-1)^5 - 450(h-1)^4 + 280(h-1)^3 - 160(h-1)^2 + 80(h-1) - 32.$$

Differentiation leads to

$$T_3'(h) = -\frac{5\left(\frac{1}{h}+1\right)^4}{h^2} - 14920(h-1)^7 + 9625(h-1)^6 - 5910(h-1)^5 + 3405(h-1)^4 - 1800(h-1)^3 + 840(h-1)^2 - 320(h-1) + 80,$$
$$T_3''(h) = \frac{10(h+1)^3(h+3)}{h^7} - 104440(h-1)^6 + 57750(h-1)^5 - 29550(h-1)^4 + 13620(h-1)^3 - 5400(h-1)^2 + 1680(h-1) - 320,$$
$$T_3'''(h) = -\frac{30(h+1)^2\left(h^2+6h+7\right)}{h^8} - 626640(h-1)^5 + 288750(h-1)^4 - 118200(h-1)^3 + 40860(h-1)^2 - 10800(h-1) + 1680,$$
$$T_3^{(4)}(h) = \frac{120(h+1)\left(h^3+9h^2+21h+14\right)}{h^9} - 3133200(h-1)^4 + 1155000(h-1)^3 - 354600(h-1)^2 + 81720(h-1) - 10800,$$
$$T_3^{(5)}(h) = -\frac{120\left(5h^4+60h^3+210h^2+280h+126\right)}{h^{10}} - 12532800(h-1)^3 + 3465000(h-1)^2 - 709200(h-1) + 81720,$$
$$T_3^{(6)}(h) = \frac{3600\left(h^4+14h^3+56h^2+84h+42\right)}{h^{11}} - 37598400(h-1)^2 + 6930000(h-1)$$

$$- 709200,$$

$$T_3^{(7)}(h) = -\frac{25200\left(h^4+16h^3+72h^2+120h+66\right)}{h^{12}} - 75196800(h-1) + 6930000,$$

$$T_3^{(8)}(h) = \frac{201600\left(h^4+18h^3+90h^2+165h+99\right)}{h^{13}} - 75196800,$$

and

$$T_3^{(9)}(h) = -\frac{1814400\left(h^4+20h^3+110h^2+220h+143\right)}{h^{14}}.$$

Noting that $T_3'(1) = T_3''(1) = T_3'''(1) = T_3^{(4)}(1) = T_3^{(5)}(1) = T_3^{(6)}(1) = T_3^{(7)}(1) = T_3^{(8)}(1) = 0$ and $T_3^{(9)}(h) < 0$ for all $h > 1$, we obtain that $T_3(h) < 0$ for all $h > 1$. □

Lemma 4.4. *The inequality*

$$\begin{aligned} &-(h-1) + \ln^2\big(\frac{1}{h}+1\big) + 2(h-1)\ln\big(\frac{1}{h}+1\big) \\ \leq &\left(\frac{15\ln 2}{32} - \frac{29}{192}\right)(h-1)^4 + \left(\frac{3}{8} - \frac{7\ln 2}{12}\right)(h-1)^3 \\ &+\frac{3}{4}(\ln 2 - 1)(h-1)^2 + (\ln 2 - 1)(h-1) + \ln^2 2 \end{aligned}$$

holds for all $h \in [1, 1.35]$.

Proof. Set

$$\begin{aligned} T_4(q) = &-q + \ln^2\big(\frac{1}{q+1}+1\big) + 2\ln\big(\frac{1}{q+1}+1\big)q - \left(\frac{15\ln 2}{32} - \frac{29}{192}\right)q^4 \\ &- \left(\frac{3}{8} - \frac{7\ln 2}{12}\right)q^3 - \frac{3}{4}(\ln 2 - 1)q^2 - (\ln 2 - 1)q - \ln^2 2. \end{aligned}$$

Taking derivatives lead to

$$\begin{aligned} T_4'(q) = &-\frac{q^2 - 2\left(q^2+3q+1\right)\ln\left(\frac{q+2}{q+1}\right) + 5q + 2}{(q+1)(q+2)} \\ &- \left(\frac{15\ln 2}{8} - \frac{29}{48}\right)q^3 - 3\left(\frac{3}{8} - \frac{7\ln 2}{12}\right)q^2 - \frac{3}{2}(\ln 2 - 1)q - (\ln 2 - 1), \\ T_4''(q) = &\frac{2(2q+3)\ln\left(\frac{q+2}{q+1}\right) - 6(q+1)}{(q+1)^2(q+2)^2} - 3\left(\frac{15\ln 2}{8} - \frac{29}{48}\right)q^2 \\ &- 3\left(\frac{3}{4} - \frac{7\ln 2}{6}\right)q - \frac{3}{2}(\ln 2 - 1), \\ T_4'''(q) = &\frac{18q^2 + 38q + 18 - 4\left(3q^2+9q+7\right)\ln\left(\frac{q+2}{q+1}\right)}{(q+1)^3(q+2)^3} - 3\left(\frac{15\ln 2}{4} - \frac{29}{24}\right)q \\ &- 3\left(\frac{3}{4} - \frac{7\ln 2}{6}\right), \\ T_4^{(4)}(q) = &\frac{12\left(4q^3+18q^2+28q+15\right)\ln\left(\frac{q+2}{q+1}\right) - 2\left(36q^3+116q^2+114q+29\right)}{(q+1)^4(q+2)^4} \\ &- 3\left(\frac{15\ln 2}{4} - \frac{29}{24}\right)q, \end{aligned}$$

and

$$T_4^{(5)}(q)=4\left(\frac{90q^4+390q^3+585q^2+313q+15-12\left(5q^4+30q^3+70q^2+75q+31\right)\ln\left(\frac{q+2}{q+1}\right)}{(q+1)^5(q+2)^5}\right).$$

Clearly, $T_4(0)=T_4'(0)=T_4''(q)=T_4'''(q)=T_4^{(4)}(0)=0$. We claim that $T_4^{(5)}(q)<0$ for all $q\in(0,0.35)$. In fact, we have

$$\begin{aligned}
&90q^4+390q^3+585q^2+313q+15-12\left(5q^4+30q^3+70q^2+75q+31\right)\ln\big(\frac{q+2}{q+1}\big)\\
\le\ &90q^4+390q^3+585q^2+313q+15-12\left(5q^4+30q^3+70q^2+75q+31\right)\ln\big(\frac{2.35}{1.35}\big)\\
\le\ &90q^4+390q^3+585q^2+313q+15-12\left(5q^4+30q^3+70q^2+75q+31\right)\frac{1}{2}\\
=\ &60q^4+210q^3+165q^2-137q-171<0.
\end{aligned}$$

Hence the required conclusion holds. □

Theorem 4.3. *The function $Q(\mathbb{T}_h;1;s)$ is decreasing on $(0,\infty)$.*

Proof. Noting that

$$Q(\mathbb{T}_h;1;s)=\frac{3}{40s}+\frac{\left(\frac{1}{h}+1\right)^s\left(((h-1)s+1)\ln\left(\frac{1}{h}+1\right)+h-1\right)}{(hs-s+1)(\frac{1}{h}+1)^s+h-1}-\ln s-\frac{23}{40s}+\frac{1}{12s^2},$$

and the function $s\mapsto-\ln s-\frac{23}{40s}+\frac{1}{12s^2}$ is decreasing, it is enough to prove

$$\varphi'(s)<0,$$

where

$$\varphi(s):=\frac{3}{40s}+\frac{\left(\frac{1}{h}+1\right)^s\left(((h-1)s+1)\ln\left(\frac{1}{h}+1\right)+h-1\right)}{(hs-s+1)(\frac{1}{h}+1)^s+h-1}.$$

We denote that

$$\varphi_1(s):=\big(\frac{1}{h}+1\big)^s\Big(((h-1)s+1)\ln\big(\frac{1}{h}+1\big)+h-1\Big),$$

and

$$\varphi_2(s):=(hs-s+1)\big(\frac{1}{h}+1\big)^s+h-1.$$

A direct calculation yields

$$\begin{aligned}
\varphi'(s)&=\Big(\frac{3}{40s}+\frac{\varphi_1(s)}{\varphi_2(s)}\Big)'=-\frac{3}{40s^2}+\frac{\varphi_1'(s)\varphi_2(s)-\varphi_1(s)\varphi_2'(s)}{\varphi_2^2(s)}\\
&=\frac{40s^2\Big(\varphi_1'(s)\varphi_2(s)-\varphi_1(s)\varphi_2'(s)\Big)-3\varphi_2^2(s)}{40s^2\varphi_2^2(s)},
\end{aligned}$$

and

$$\begin{aligned}
&40s^2\Big(\varphi_1'(s)\varphi_2(s)-\varphi_1(s)\varphi_2'(s)\Big)-3\varphi_2^2(s)\\
=\ &40s^2(h-1)\big(\frac{1}{h}+1\big)^s\Big(-(h-1)\big(\frac{1}{h}+1\big)^s+(hs-s+1)\ln^2\big(\frac{1}{h}+1\big)
\end{aligned}$$

$$+2(h-1)\ln\big(\frac{1}{h}+1\big)\Big)-3\Big((hs-s+1)\big(\frac{1}{h}+1\big)^s+h-1\Big)^2. \tag{4.2}$$

Let

$$W_h(s):=-(h-1)\big(\frac{1}{h}+1\big)^s+(hs-s+1)\ln^2\big(\frac{1}{h}+1\big)+2(h-1)\ln\big(\frac{1}{h}+1\big).$$

Noting that

$$\begin{aligned}W_h'(s)&=-(h-1)\big(\frac{1}{h}+1\big)^s\ln\big(\frac{1}{h}+1\big)+(h-1)\ln^2\big(\frac{1}{h}+1\big)\\&\le-(h-1)\ln\big(\frac{1}{h}+1\big)+(h-1)\ln^2\big(\frac{1}{h}+1\big)<0,\end{aligned}$$

we obtain

$$W_h(s)\le W_h(0)=-(h-1)+\ln^2\big(\frac{1}{h}+1\big)+2(h-1)\ln\big(\frac{1}{h}+1\big).$$

So we have

$$\begin{aligned}&40s^2\Big(\varphi_1'(s)\varphi_2(s)-\varphi_1(s)\varphi_2'(s)\Big)-3\varphi_2^2(s)\\&=40s^2(h-1)\big(\frac{1}{h}+1\big)^sW_h(s)-3\Big((hs-s+1)\big(\frac{1}{h}+1\big)^s+h-1\Big)^2\\&\le 40s^2(h-1)\big(\frac{1}{h}+1\big)^s\left(-(h-1)+\ln^2\big(\frac{1}{h}+1\big)+2(h-1)\ln\big(\frac{1}{h}+1\big)\right)\\&\quad-3\Big((hs-s+1)\big(\frac{1}{h}+1\big)^s+h-1\Big)^2.\end{aligned}$$

We denote that

$$L(h):=(h-1)\big(\frac{1}{h}+1\big)^5\left(-(h-1)+\ln^2\big(\frac{1}{h}+1\big)+2(h-1)\ln\big(\frac{1}{h}+1\big)\right).$$

We claim that $L(h)\le 2$. In fact, taking derivative of it leads to

$$L'(h)=\frac{(h+1)^4}{h^6}L_1(h),$$

where

$$\begin{aligned}L_1(h)=&-2h^3+3h^2-4h+3+\big(h^2-4h+5\big)\ln^2\big(\frac{1}{h}+1\big)\\&+2\big(2h^3-5h^2+7h-4\big)\ln\big(\frac{1}{h}+1\big).\end{aligned} \tag{4.3}$$

According to the inequalities

$$h^2-4h+5=(h-2)^2+1\ge 0,\quad h\in\mathbb{R},$$

and

$$2h^3-5h^2+7h-4=2(h-1)^3+(h-1)^2+3(h-1)\ge 0,\quad h\ge 1,$$

as well as Lemmas 4.1 and 4.2

$$\ln^2\big(\frac{1}{h}+1\big)\geq (h-1)^3\Big(-\frac{3}{8}-\frac{7}{12}\ln 2\Big)+(h-1)^2\Big(\frac{1}{4}+\frac{3\ln 2}{4}\Big)$$
$$-(h-1)\ln 2+\ln^2 2,\quad h\geq 1,$$
$$\ln\big(\frac{1}{h}+1\big)\geq -\frac{7}{24}(h-1)^3+\frac{3}{8}(h-1)^2-\frac{h-1}{2}+\ln 2,\quad h\geq 1,$$

we have

$$\begin{aligned}
L_1(h) &= -2(h-1)^3-3(h-1)^2-4(h-1)+\big(2-2(h-1)+(h-1)^2\big)\ln^2\big(\frac{1}{h}+1\big)\\
&\quad +2\left(2(h-1)^3+(h-1)^2+3(h-1)\right)\ln\big(\frac{1}{h}+1\big)\\
&\geq -2(h-1)^3-3(h-1)^2-4(h-1)+\big(2-2(h-1)+(h-1)^2\big)\\
&\quad \times\Big((h-1)^3\Big(-\frac{3}{8}-\frac{7}{12}\ln 2\Big)+(h-1)^2\Big(\frac{1}{4}+\frac{3\ln 2}{4}\Big)-(h-1)\ln 2+\ln^2 2\Big)\\
&\quad +2\left(2(h-1)^3+(h-1)^2+3(h-1)\right)\\
&\quad \times\Big(-\frac{7}{24}(h-1)^3+\frac{3}{8}(h-1)^2-\frac{h-1}{2}+\ln 2\Big)\\
&= -\frac{7}{6}q^6+\big(\frac{13}{24}-\frac{7\ln 2}{12}\big)q^5+\big(\frac{23\ln 2}{12}-2\big)q^4+\big(\frac{\ln 2}{3}-2\big)q^3\\
&\quad +\big(\ln^2 2+\frac{11\ln 2}{2}-\frac{11}{2}\big)q^2+(4\ln 2-2\ln^2 2-4)q+2\ln^2 2\\
&= \frac{1}{24}\Big(-28q^6+(13-14\ln 2)q^5+(46\ln 2-48)q^4+8(\ln 2-6)q^3\\
&\quad +12\left(-11+2\ln^2 2+\ln 2048\right)q^2-48\left(2+\ln^2 2-\ln 4\right)q+48\ln^2 2\Big),
\end{aligned}$$

where $q=h-1$. Expanding the numerator at $q=0.35$, we have

$$\begin{aligned}
&-28q^6+(13-14\ln 2)q^5+(46\ln 2-48)q^4+8(\ln 2-6)q^3\\
&+12\left(-11+2\ln^2 2+\ln 2048\right)q^2-48\left(2+\ln^2 2-\ln 4\right)q+48\ln^2 2\\
\approx &-0.965639-91.8022(q-0.35)-90.2854(q-0.35)^2-84.9886(q-0.35)^3\\
&-61.7973(q-0.35)^4-55.5041(q-0.35)^5-28(q-0.35)^6<0,\quad q>0.35.
\end{aligned}$$

Combining the conclusions above, we deduce that $L(h)$ is decreasing on $h>1.35$.

In order to get an upper bound of $L(h)$, we need to focus on $h\in[1,1.35]$. Using Lemmas 4.3 and 4.4, we have

$$\begin{aligned}
L(h)\leq &\Big(1865(h-1)^9-1375(h-1)^8+985(h-1)^7-681(h-1)^6+450(h-1)^5\\
&-280(h-1)^4+160(h-1)^3-80(h-1)^2+32(h-1)\Big)\\
&\times\left(\left(\frac{15\ln 2}{32}-\frac{29}{192}\right)(h-1)^4+\left(\frac{3}{8}-\frac{7\ln 2}{12}\right)(h-1)^3\right.\\
&\left.+\frac{3}{4}(\ln 2-1)(h-1)^2+(\ln 2-1)(h-1)+\ln^2 2\right)=R(q),
\end{aligned}$$

where $q = h - 1$ and

$$\begin{aligned}
R(q) = &\left(\frac{27975\ln 2}{32} - \frac{54085}{192}\right) q^{13} + \left(\frac{174155}{192} - \frac{166315\ln 2}{96}\right) q^{12} \\
&+ \left(\frac{255605\ln 2}{96} - \frac{396125}{192}\right) q^{11} + \left(-\frac{23137}{64} - \frac{5765}{96}\ln 2\right) q^{10} \\
&+ \left(\frac{10013}{32} + 1865\ln^2 2 - \frac{449\ln 2}{16}\right) q^9 + \left(-\frac{6317}{24} - 1375\ln^2 2 + \frac{161\ln 2}{2}\right) q^8 \\
&+ \left(\frac{643}{3} + 985\ln^2 2 - \frac{631\ln 2}{6}\right) q^7 + \left(-\frac{2015}{12} - 681\ln^2 2 + \frac{655\ln 2}{6}\right) q^6 \\
&+ \left(\frac{751}{6} + 450\ln^2 2 - \frac{295\ln 2}{3}\right) q^5 + \left(-88 - 280\ln^2 2 + \frac{244\ln 2}{3}\right) q^4 \\
&+ \left(56 + 160\ln^2 2 - 56\ln 2\right) q^3 + \left(-32 - 80\ln^2 2 + 32\ln 2\right) q^2 + 32q\ln^2 2.
\end{aligned}$$

A direct calculation yields

$$\begin{aligned}
R'(q) = &\,13\left(\frac{27975\ln 2}{32} - \frac{54085}{192}\right) q^{12} + 12\left(\frac{174155}{192} - \frac{166315\ln 2}{96}\right) q^{11} \\
&+11\left(\frac{255605\ln 2}{96} - \frac{396125}{192}\right) q^{10} + 10\left(-\frac{23137}{64} - \frac{5765}{96}\ln 2\right) q^9 \\
&+9\left(\frac{10013}{32} + 1865\ln^2 2 - \frac{449\ln 2}{16}\right) q^8 + 8\left(-\frac{6317}{24} - 1375\ln^2 2 + \frac{161\ln 2}{2}\right) q^7 \\
&+7\left(\frac{643}{3} + 985\ln^2 2 - \frac{631\ln 2}{6}\right) q^6 + 6\left(-\frac{2015}{12} - 681\ln^2 2 + \frac{655\ln 2}{6}\right) q^5 \\
&+5\left(\frac{751}{6} + 450\ln^2 2 - \frac{295\ln 2}{3}\right) q^4 + 4\left(-88 - 280\ln^2 2 + \frac{244\ln 2}{3}\right) q^3 \\
&+3\left(56 + 160\ln^2 2 - 56\ln 2\right) q^2 + 2\left(-32 - 80\ln^2 2 + 32\ln 2\right) q + 32\ln^2 2 \geq 0.
\end{aligned}$$

Hence, $L(h) \leq R(q) \leq R(0.35) \approx 1.95298 < 2$.

Thus, (4.2) leads to

$$\begin{aligned}
&40s^2\Big(\varphi_1'(s)\varphi_2(s) - \varphi_1(s)\varphi_2'(s)\Big) - 3\varphi_2^2(s) \\
\leq\, &40s^2(h-1)\big(\frac{1}{h}+1\big)^s \left(-(h-1) + \ln^2\big(\frac{1}{h}+1\big) + 2(h-1)\ln\big(\frac{1}{h}+1\big)\right) \\
&-3\Big((hs-s+1)\big(\frac{1}{h}+1\big)^s + h - 1\Big)^2 \\
\leq\, &40s^2\big(\frac{1}{h}+1\big)^{s-5} \times 2 - 3\Big((hs-s+1)\big(\frac{1}{h}+1\big)^s + h - 1\Big)^2 \\
=\, &s^2\big(\frac{1}{h}+1\big)^s\Big(\frac{80}{\big(\frac{1}{h}+1\big)^5} - 3\frac{(hs-s+1)^2}{s^2}\big(\frac{1}{h}+1\big)^s\Big) \\
\leq\, &s^2\big(\frac{1}{h}+1\big)^s\Big(\frac{80}{2^5} - 3\frac{1}{s^2}2^s\Big) < 0.
\end{aligned}$$

So we get the conclusion that $\varphi'(s) < 0$. □

Remark 4.1. Theorem 4.1 shows that $Q(\mathbb{R}; 1; s)$ is completely monotonic in $\mathbb{R}^+$.

Based on the results in this section, we provide a conjecture and an open problem.

Conjecture 4.1. *The function* $Q(\mathbb{T}_h; 1; s)$ *is* 1*-completely monotonic on time scales.*

Problem 4.1. *Under what conditions will the function* $Q(\mathbb{T}; \alpha; s)$ *be completely monotonic on time scales.*

Remark 4.2. This paper is a revised version of the preprint [16].

References

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