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VIRTUAL ELEMENT APPROXIMATIONS FOR NON-STATIONARY NAVIER-STOKES EQUATIONS ON POLYGONAL MESHES

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Abstract This article deals with the development of virtual element methods for the approximation of non-stationary Navier-Stokes equation. The proposed lowest order virtual element spaces for velocity and pressure are constructed in such a way that the inf-sup conditions holds, and easy to implement in comparison with other pair of spaces which satisfy the inf-sup condition. For time discretization, the backward Euler scheme is employed, and both semi and fully discrete schemes are discussed and analyzed. With the help of certain projection operators, error estimates are established in suitable norms for both semi and fully discretized schemes. Moreover, several numerical experiments are conducted to verify the theoretical rate of convergence and to observe the computational efficiency of the proposed schemes.

Keywords Navier-Stokes equation, lowest order, inf-sup condition, virtual element method, numerical experiments, convergence analysis.

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1. Introduction

Transient Navier-Stokes equations have remarkable applications in fluid mechanics such as weather prediction, current flow through the air, designing the aircraft, fluid flow through pipes, wastewater management, underground oil extraction and so on. In the past years, several numerical techniques such as finite element methods [4, 16, 23, 42], finite volume methods [26, 32, 37], nonconforming finite element methods [36, 48], discontinuous Galerkin methods [19] and references therein, were proposed for seeking a numerical approximation of the Navier-Stokes problems. The major difficulty lies in choosing the appropriate stable pair of discrete space based on spatial discretization, for instance, these spaces must obey the inf-sup condition [23]–a necessary condition for showing the well-posedness of the scheme as well as establishing the optimal convergence results. In order to circumvent or enforcing the inf-sup condition, we need to add a suitable stabilizer term for existence of a unique solution. In past years, several stabilized or penalty methods with various unstable finite element spaces have also been explored in [15, 25, 30, 33, 39]. We stress that the addition of an extra stabilizer term possibly would increase the computational as well as theoretical complexity of the numerical scheme. Therefore, it is

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desirable to look for suitable discrete spaces that satisfy the inf-sup condition without introducing any stabilizer. In this article, we analyze the lowest order virtual element spaces for velocity and pressure that obey the inf-sup condition (without adding any stabilization term) which is used for showing the well-posedness of the discrete formulation and in establishing the optimal error estimates for velocity and pressure.

The recently proposed virtual element method (VEM), initially introduced in [8], is proven to be very impressive and attracted the scientific community as far numerical approximation of fluid and solid mechanics on polygonal meshes is concerned. VEMs are inspired by the mimetic finite difference method, that also aim to generalize finite element methods over the very general type of polygonal meshes. In contrast with classical finite element (FE) schemes, VEM do not require explicit construction of the discrete basis functions and one needs to define suitable degrees of freedom to put the discrete formulation in the matrix form. Other fundamental properties of VEM include: making use of non-polynomial basis functions over arbitrary polygonal/polyhedral meshes [40], capability of handling the complicated geometries generally used in solid-mechanics and fluid dynamics through general meshes and usage of approximation spaces containing the higher-order polynomial with ease. In view of their computational efficiency, VEMs have been developed for various problems within a decade, and few of the basic works are on general elliptic [3, 9], parabolic [43] and semi-linear [6, 7] problems. In literature, there are few contributions that dealt with virtual approximations for Stokes [5, 10, 21], Navier-Stokes [11,12,22,34], Darcy and Brinkmann [44] and poroelasticity [18,20,41] equations. However, in these articles, a restriction on choosing the approximation order, or the degree of involved polynomials (denoted generally by k) is strictly imposed for virtual element spaces associated with velocity and pressure in order to satisfy the required inf-sup condition by the discrete spaces. In other words, it is mandatory to choose $k \geq 2$ in order to obtain stable spaces, and $k = 1$ can not be taken due to unavailability of the inf-sup condition for the discrete spaces of order $k = 1$ until a suitable stabilizer is added [10, 29]. We would like to remark that even the usage of a higher-order approximations is expected to be computationally expensive in general. Considering these points, here we aim to approach the discrete spaces that has approximation of order one, and also satisfy the required inf-sup condition [47]. Therefore, the proposed scheme is considered computationally less expensive compared to the existing higher order schemes in the context of virtual element approximations for fluid flow problems due to reduced local degrees of freedom in the case of [47].

As far the virtual element approximations of transient Navier-Stokes is concerned, VEM is not yet very well developed in the literature, and there is only one article dealing with VEM for the unsteady fluid flow problem, see [29]. However, in that article, only numerical experiments were reported, and theoretical convergence/error estimates were not analyzed. In this paper, we aim to develop the virtual element approximations for non-stationary Navier-Stokes problem with emphasize on both theoretical and computational aspects. Here, we intended to propose the semi-discrete scheme (based on spatial discretization with virtual element method) and fully discrete scheme (employing the Euler-Backward scheme for time discretization), and also discussed their well-posedness. Here we have employed an extended version of the lowest order virtual element spaces introduced first in [5] in the context of Stokes equations. We remark that these lowest order spaces satisfy the inf-sup, as well as divergence-free conditions and, are also used in [47] for unsteady Stokes problem under minimal regularity assumptions. Moreover, by following [23,47] and with the help of certain projection operator (to be introduced in Section 4), *a priori* error estimates (both semi and fully discrete) for velocity and pressure in appropriate norms are established. To the best of our knowledge, there is no article available in the literature that addresses both convergence analysis and numerical results of VEM for non-stationary Navier-Stokes equations. Hence, this article can be considered as the first contribution in this direction. We believe that the proposed analysis can be extended to more application-oriented problems consisting of time-dependent Navier-Stokes problems on polygonal meshes.

Throughout this article, we use the standard notations of Sobolev spaces and their associated norms and semi-norms. We denote the L^2 inner product and norm in domain Ω by $(\cdot,\cdot)_{0,\Omega}$ and $\|\cdot\|_{0,\Omega}$, respectively. The space $H^s(\Omega)$ denote the Sobolev space of real-valued functions with weak derivatives of order up to $s>0$ in $L^2(\Omega)$ on domain $\Omega \subset \mathbb{R}^2$, and endowed with the standard norm denoted as $\|\cdot\|_{s,\Omega}$ and semi-norm in space $H^s(\Omega)$ denoted as $|\cdot|_{s,\Omega}$. Here $H^1_0(\Omega)$ denote the space of functions that belongs to $H^1(\Omega)$ and vanishes on the boundary $\partial\Omega$. The vector valued functions will be denoted by bold letter, for instance, $\boldsymbol{v}$ stands for the velocity vector. The norm and semi-norm in the vector space $[H^s(\Omega)]^2$ are equipped with product norm and denoted same as scalar notations to keep the clarity. The square-integrable space with zero mean value is given by $L^2_0(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q\ dx = 0\}$. Moreover, the constant C denote any generic constant which vary from place to place.

The content of this paper is arranged in the following manner. We have introduced the governing equation and discuss its weak/variational formulation in Section 2. Next, we deal with virtual element formulation and well-posedness of both semi and fully discrete schemes in Section 3. With the help of Stokes and L^2 projection operators in Section 4, an optimal *a priori* error estimates for velocity and pressure in H^1 and L^2-norms are established. Lastly, we have reported our numerical experiments in Section 5 to validate the theoretical convergence rates obtained in previous Section 4.

2. Governing equations and its Variational formulation

We consider the following incompressible fluid flow problem in a domain $\Omega \subset \mathbb{R}^2$: For all $t \in (0,T]$ and $\boldsymbol{x} \in \Omega$, find the flow velocity $\boldsymbol{u}(\boldsymbol{x},t)$ and the pore pressure $p(\boldsymbol{x},t)$ such that

$$\partial_t \boldsymbol{u} - \mathbf{div}\big(\nu\ \boldsymbol{\nabla}\boldsymbol{u} - p\mathbf{I}\big) + (\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{u} = \boldsymbol{f} \qquad \text{in } \Omega \times (0,T), \tag{2.1a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega \times (0,T), \tag{2.1b}$$

$$\boldsymbol{u} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{2.1c}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \qquad \text{on } \Omega \times \{0\}, \tag{2.1d}$$

where ν is the viscosity of the fluid, $\boldsymbol{u}_0(\boldsymbol{x})$ is the initial velocity and $\boldsymbol{f}(\boldsymbol{x},t)$ is the given body force.

Let $\mathbf{V} := [H^1_0(\Omega)]^2$ and $Q := L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q\ dx = 0\}$ be the admissible spaces for velocity and pressure, respectively. We also assume that the load function $\boldsymbol{f} \in [L^2(\Omega)]^2$ and initial condition $\boldsymbol{u}_0 \in \mathbf{V}$. Multiplying the adequate test functions $\boldsymbol{v} \in \mathbf{V}$ and $q \in Q$ to the equations (2.1a) and (2.1b) respectively, with initial-boundary conditions (2.1c)-(2.1d), the weak formulation states: Find $\boldsymbol{u} : [0,T] \to \mathbf{V},\ p : [0,T] \to Q$ such that

$$\begin{aligned} &m(\partial_t \boldsymbol{u}, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) + \tilde{c}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \mathbf{F}(\boldsymbol{v}) \quad && \forall \boldsymbol{v} \in \mathbf{V}, \\ &b(\boldsymbol{u}, q) = 0 && \forall q \in Q, \end{aligned} \tag{2.2}$$

where the bilinear forms are defined as

$$m(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\ dx, \quad a(\boldsymbol{u}, \boldsymbol{v}) := \nu \int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v}\ dx,\ \mathbf{F}(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\ dx,$$

$$\tilde{c}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) := \int_\Omega (\boldsymbol{\nabla} \boldsymbol{u}\, \boldsymbol{w}) \cdot \boldsymbol{v}\ dx = \sum_{i,j=1}^{2} \left(\frac{\partial \boldsymbol{u}_i}{\partial x_j} \boldsymbol{w}_j \right) \boldsymbol{v}_i,\ b(\boldsymbol{v}, q) := -\int_\Omega \operatorname{div} \boldsymbol{v}\ q\ dx.$$

Note that the above bilinear forms are satisfying the following properties.

- $m(\cdot,\cdot)$ is a positive definite form:

$$m(\boldsymbol{v}, \boldsymbol{v}) = \|\boldsymbol{v}\|_{0,\Omega}^2 \quad \forall \boldsymbol{v} \in \mathbf{V}.$$

- $a(\cdot,\cdot)$ is coercive:

$$a(\boldsymbol{v}, \boldsymbol{v}) = \nu |\boldsymbol{v}|_{1,\Omega}^2 \geq C\ \nu \|\boldsymbol{v}\|_{1,\Omega}^2 \quad \forall \boldsymbol{v} \in \mathbf{V}. \qquad \text{(Poincaré inequality)}$$

- $b(\cdot,\cdot)$ satisfies the inf-sup condition: there exists $\beta > 0$ such that [16, 23]

$$\sup_{\boldsymbol{v} \in \mathbf{V} \setminus \{0\}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{1,\Omega}} \geq \beta \|q\|_{0,\Omega} \qquad \forall q \in Q.$$

- $a(\cdot,\cdot)$ is continuous:

$$a(\boldsymbol{u}, \boldsymbol{v}) \leq C \|\boldsymbol{u}\|_{1,\Omega} \|\boldsymbol{v}\|_{1,\Omega} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}. \qquad \text{(Cauchy Schwarz inequality)}$$

- $\mathbf{F}(\cdot)$ is continuous:

$$\mathbf{F}(\boldsymbol{v}) \leq C \|\boldsymbol{f}\|_{0,\Omega} \|\boldsymbol{v}\|_{0,\Omega} \leq C_P \|\boldsymbol{f}\|_{0,\Omega} \|\nabla \boldsymbol{v}\|_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbf{V}.$$

- Using Cauchy Schwarz and Hölder's inequalities together with $H^1(\Omega) \subset L^4(\Omega)$), it is easy to see that $\tilde{c}(\cdot;\cdot,\cdot)$ is continuous, i.e., there exists a constant C such that

$$\tilde{c}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) \leq C \|\boldsymbol{u}\|_{1,\Omega} \|\boldsymbol{w}\|_{1,\Omega} \|\boldsymbol{v}\|_{1,\Omega} \ \ \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbf{V}.$$

- Let $\mathbf{X} := \{\boldsymbol{v} \in \mathbf{V} : b(\boldsymbol{v}, q) = 0\ \forall q \in Q\} = \{\boldsymbol{v} \in \mathbf{V} : \operatorname{div} \boldsymbol{v} = 0\}$. An application of Green's Theorem yields that $\tilde{c}(\boldsymbol{u};\cdot,\cdot)$ is skew-symmetric bilinear form on the kernel space $\mathbf{X}$, i,e., for all $\boldsymbol{u} \in \mathbf{X},\ \boldsymbol{v}, \boldsymbol{w} \in \mathbf{V}$, we have

$$\tilde{c}(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) = -\tilde{c}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})$$

Next, we introduce a new skew-symmetric trilinear form $c(\cdot;\cdot,\cdot)$ by modifying the natural trilinear form $\tilde{c}(\cdot;\cdot,\cdot)$ as follows.

$$c(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) := \frac{1}{2} \left(\tilde{c}(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) - \tilde{c}(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v}) \right) \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbf{V}.$$

It is clear from the definition that $c(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = 0$ for all $\boldsymbol{w} = \boldsymbol{v} \in \mathbf{V}$, and also trilinear form $c(\cdot;\cdot,\cdot)$ is continuous. Thus, the weak formulation (2.2) can be rewritten as: Find $\boldsymbol{u}(t) \in \mathbf{V}$ and $p(t) \in Q$ such that

$$m(\partial_t \boldsymbol{u}, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \mathbf{F}(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{V}, \tag{2.3a}$$

$$b(\boldsymbol{u}, q) = 0 \qquad \forall q \in Q. \tag{2.3b}$$

The well-posedness of the problem (2.3) follows from the coercivity and continuity of the bilinear form $a(\cdot,\cdot)$, inf-sup condition of bilinear form $b(\cdot,\cdot)$ along with the skew-symmetricity of the bilinear form $c(\boldsymbol{u};\cdot,\cdot)$ (for more details, see [23]). In addition, the solution $\boldsymbol{u} \in \mathbf{V}$ of problem (2.3) satisfies

$$\|\boldsymbol{u}(t)\|_{0,\Omega}^2 + \nu \int_0^t |\boldsymbol{u}(s)|_{1,\Omega}^2 \,\mathrm{d}s \lesssim \|\boldsymbol{u}(0)\|_{0,\Omega}^2 + \frac{1}{\nu} \int_0^t \|\boldsymbol{f}(s)\|_{0,\Omega}^2 \,\mathrm{d}s. \tag{2.4}$$

Thus, we obtain the bound (2.4) by the usage of Young's inequality.

3.Virtual element formulation and its well-posedness

In this section, by introducing the stable pair of local and global discrete spaces associated with velocity and pressure, we propose the virtual element formulation corresponding to weak formulation (2.3) of the problem (2.1). Here, we present both semi and fully discrete schemes, and address the existence of a unique virtual element solution.

3.1. Discrete spaces and their degrees of freedom

Let the domain Ω be discretized into the family of the polygonal meshes $\mathcal{T}_h$ with element K and mesh size $h := \max_{K \in \mathcal{T}_h} h_K$ where h_K denotes the diameter of element K, and edges in the polygonal mesh are denoted by e. For any natural number k, let $\mathbb{P}_k(S)$ represent the space of polynomials of degree less than or equal to k for any $S \subset \mathbb{R}^2$. Moreover, we denote $\mathcal{G}(K) \subseteq [\mathbb{P}_1(K)]^2$ as $\nabla \mathbb{P}_2(K)$, and $\mathcal{G}^\perp(K)$ as the orthogonal complement of the space $\mathcal{G}(K)$. We note that the orthogonal space $\mathcal{G}^\perp(K)$ is one dimensional and generated by vector function $\boldsymbol{g}^\perp := [\bar{y}, -\bar{x}]$, where $\bar{x}, \bar{y}$ are scaled functions in polygon K as described in [8]. We signify the classical definition for the inner product on space $H^s(K)$ as $(\cdot,\cdot)_{s,K}$ and norm as $\|\cdot\|_{s,K}$ for each $K \in \mathcal{T}_h$. The vertices of any element K are denoted as V_i with N_v^K number of vertices. The unit normal and tangent on edge e in element K are symbolized by $\boldsymbol{n}_K^e$ and $\boldsymbol{t}_K^e$, respectively.

We will suppose that the polygonal mesh satisfy the following assumptions (see [11]):

- Each K is open and simply connected set whose boundary ∂K is a non-intersecting poly-line consisting of a finite number of straight line segments;
- There exists $C_{\mathcal{T}} > 0$ such that, for every h and $K \in \mathcal{T}_h$, the ratio between the length of the shortest edge and h_K is larger than $C_{\mathcal{T}}$;
- and each $K \in \mathcal{T}_h$ is star-shaped with respect to every point within a ball of radius $C_{\mathcal{T}} h_K$.

We initiate the process of introducing the local virtual element spaces by determining two important operators: energy projection operator Π_K^{∇} and L^2 projection operator Π_K^0.

The energy operator $\Pi_K^{\nabla} : [H^1(K)]^2 \to [\mathbb{P}_1(K)]^2$ is defined as: $\forall\ \boldsymbol{p}_1 \in [\mathbb{P}_1(K)]^2$, $\boldsymbol{v} \in [H^1(K)]^2$,

$$(\boldsymbol{\nabla}(\Pi_K^{\nabla}\boldsymbol{v} - \boldsymbol{v}), \boldsymbol{\nabla}\boldsymbol{p}_1)_{0,K} = 0, \qquad P_K^0(\Pi_K^{\nabla}\boldsymbol{v} - \boldsymbol{v}) = 0,$$

where $P_K^0\boldsymbol{v} := \frac{1}{N_v^K}\sum_{i=1}^{N_v^K} \boldsymbol{v}(V_i)$. Here P_K^0 avails the projection onto constants since $\boldsymbol{p}_1$ as a constant gives no condition to help in calculation of projection Π_K^{∇}.

We determine the local L^2-projection operator $\Pi_K^0 : [L^2(K)]^2 \to [\mathbb{P}_1(K)]^2$ as,

$$(\Pi_K^0\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{p}_1)_{0,K} = 0, \quad \forall \boldsymbol{p}_1 \in [\mathbb{P}_1(K)]^2.$$

We stress that these operators will not only help us in the computation of the discrete bilinear forms, but also in deriving the optimal error estimates.

We recall the local virtual element space for the velocity introduced in [5],

$$\mathbf{W}_h(K) := \{\boldsymbol{v} \in [H^1(K)]^2 \cap \mathcal{B}(\partial K) : \begin{cases} (-\Delta\boldsymbol{v} + \nabla s)|_K = \mathbf{0}, \\ \operatorname{div}\boldsymbol{v}|_K = c_d \in \mathbb{P}_0(K) \end{cases} \text{for } s \in L^2(K)\},$$

where $c_d := \frac{1}{|K|}\sum_{e\in\partial K}\int_e \boldsymbol{v}|_e \cdot \boldsymbol{n}_K^e$, and the local boundary space $\mathcal{B}(\partial K)$ defined as

$$\mathcal{B}(\partial K) := \{\boldsymbol{v} \in [C^0(\partial K)]^2 : \boldsymbol{v}|_e \cdot \boldsymbol{n}_K^e \in \mathbb{P}_2(e), \boldsymbol{v}|_e \cdot \boldsymbol{t}_K^e \in \mathbb{P}_1(e) \quad \forall e \in \partial K\}.$$

The dimension of space $\mathbf{W}_h(K)$ is same as dimension of boundary space $\mathcal{B}(\partial K)$, that is $3N_K^v$.

For any $\boldsymbol{v} \in \mathbf{W}_h(K)$, the degrees of freedom for the space $\mathbf{W}_h(K)$ (see [5, 47]) are

(L_v1) the value of $\boldsymbol{v}$ at the vertices of element K;

(L_v2) the edge moments of $\boldsymbol{v}$ along the unit outward normal of K, that is,

$$\int_e \boldsymbol{v} \cdot \boldsymbol{n}_K^e \quad \forall e \in \partial K.$$

As seen in [47], we have non-computable term $(\boldsymbol{v}, \boldsymbol{p}_1)_{0,K}\ \forall \boldsymbol{v} \in \mathbf{W}_h(K)$, and define the extended supplementary space $\tilde{\mathbf{V}}_h$ locally as, for $\alpha \in \mathbb{R}$,

$$\tilde{\mathbf{V}}_h(K) := \{\boldsymbol{v} \in [H^1(K)]^2 \cap \mathcal{B}(\partial K) : \begin{cases} (-\Delta\boldsymbol{v} + \nabla s)|_K = \alpha\boldsymbol{g}^{\perp}, \\ \operatorname{div}\boldsymbol{v}|_K \in \mathbb{P}_0(K), \end{cases} \text{for } s \in L^2(K)\}.$$

The degrees of freedom for the local discrete space $\tilde{\mathbf{V}}_h(K)$ are: (L_v1)-(L_v2), and

(L_v3) the moment $\int_K \boldsymbol{v} \cdot \boldsymbol{g}^{\perp}\,\mathrm{d}x$ with $\boldsymbol{g}^{\perp} \in \mathcal{G}^{\perp}(K)$.

Now we define the local virtual element spaces $\mathbf{V}_h(K)$ and $Q_h(K)$ associated with the velocity $\boldsymbol{u}$ and pressure p, respectively on each element K as follows (refer [47]),

$$\begin{aligned} &\mathbf{V}_h(K) := \{\boldsymbol{v}_h|_K \in \tilde{\mathbf{V}}_h(K) : (\Pi_K^{\nabla}\boldsymbol{v}_h - \boldsymbol{v}_h, \boldsymbol{g}^{\perp})_{0,K} = 0 \text{ with } \boldsymbol{g}^{\perp} \in \mathcal{G}^{\perp}(K)\}, \\ &\text{and} \qquad Q_h(K) := P_0(K). \end{aligned}$$

From definition, we have the dimension of $\mathbf{V}_h(K)$ is equal to the dimension of $\tilde{\mathbf{V}}_h(K) - 1 = 3N_K^v$. The degrees of freedom for space $\mathbf{V}_h(K)$ are same as the degrees of freedom for $\mathbf{W}_h(K)$, and for space $Q_h(K)$, the values of function q_h at any point in K.

Based on the local spaces, we define the global finite-dimensional virtual element spaces as follows.

$$\begin{aligned}\mathbf{V}_h &:= \{\boldsymbol{v}_h \in \mathbf{V} : \boldsymbol{v}_h|_K \in \mathbf{V}_h(K) \quad \forall K \in \mathcal{T}_h\},\\ Q_h &:= \{q_h \in Q : q_h|_K \in Q_h(K) \quad \forall K \in \mathcal{T}_h\}.\end{aligned}$$

In view of the definition of $\mathbf{V}_h$, it is immediate to see that following are the degrees of freedom for the global discrete space $\mathbf{V}_h$,

- Values at all the interior vertices on each polygon $K \in \mathcal{T}_h$;
- and the interior edge moments along the unit outward normal of K on each interior edge e in all polygons $K \in \mathcal{T}_h$.

The degrees of freedom for Q_h are the values of function $q_h \in Q_h$ at any point in K for each $K \in \mathcal{T}_h$.

Remark 3.1. We stress that the proposed finite-dimensional spaces for velocity and pressure are constructed in such a way that the discrete velocity is exactly divergence-free, which is desirable as far as numerical approximations of Navier-Stokes equations are concerned. On the other hand, discrete spaces used in mixed finite element settings will lead to a discrete velocity solution that is divergence-free only weakly.

Now, to define the computable discrete formulation, we define another local tensor L^2-projection $\mathbf{\Pi}_K^0 : [L^2(K)]^{2\times 2} \to [\mathbb{P}_0(K)]^{2\times 2}$ as,

$$(\mathbf{\Pi}_K^0 \boldsymbol{\nabla} \boldsymbol{v} - \boldsymbol{\nabla} \boldsymbol{v}, \boldsymbol{p})_{0,K} = 0 \qquad \forall \boldsymbol{p} \in [\mathbb{P}_0(K)]^{2\times 2},\ \boldsymbol{v} \in [H^1(\Omega)]^2.$$

Let N^V and N^Q denotes the total degrees of freedom for $\mathbf{V}_h(K)$ and $Q_h(K)$, respectively. For any $\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h \in \mathbf{V}_h(K)$ and $q_h \in Q_h(K)$, we define the local discrete forms on each element K as follows.

$$\begin{aligned}&m_h^K(\boldsymbol{u}, \boldsymbol{v}) := m^K(\Pi_K^0 \boldsymbol{u}, \Pi_K^0 \boldsymbol{v}) + S_K^0((\boldsymbol{u} - \Pi_K^0 \boldsymbol{u}), (\boldsymbol{v} - \Pi_K^0 \boldsymbol{v})),\\ &a_h^K(\boldsymbol{u}_h, \boldsymbol{v}_h) := a^K(\Pi_K^\nabla \boldsymbol{u}_h, \Pi_K^\nabla \boldsymbol{v}_h) + \nu\, S_K^\nabla((I - \Pi_K^\nabla)\boldsymbol{u}_h, (I - \Pi_K^\nabla)\boldsymbol{v}_h),\\ &\tilde{c}_h^K(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) := ((\mathbf{\Pi}_K^0 \boldsymbol{\nabla} \boldsymbol{u}_h)\ \Pi_K^0 \boldsymbol{w}_h, \Pi_K^0 \boldsymbol{v}_h)_{0,K},\\ &\mathbf{F}_h^K(\boldsymbol{v}_h) := (\Pi_K^0 \boldsymbol{f}, \boldsymbol{v}_h)_{0,K}, \quad b^K(\boldsymbol{v}, q) := -(\operatorname{div} \boldsymbol{v}, q)_{0,K}\end{aligned}$$

where the local bilinear forms are the restrictions of the continuous forms on each element K, that is

$$m^K(\boldsymbol{u}, \boldsymbol{v}) := m(\boldsymbol{u}, \boldsymbol{v})|_K, \quad a^K(\boldsymbol{u}, \boldsymbol{v}) := a(\boldsymbol{u}, \boldsymbol{v})|_K,$$

and the stabilisation terms $S_K^0(\cdot, \cdot)$ and $S_K^\nabla(\cdot, \cdot)$ are defined as, see [3]

$$S_K^0(\boldsymbol{u}_h, \boldsymbol{v}_h) := \operatorname{area}(K) \sum_{i,j=1}^{N^V} \operatorname{dof}_i(\boldsymbol{u}_h) \operatorname{dof}_j(\boldsymbol{v}_h), \qquad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in \mathbf{V}_h,$$

$$S_K^{\nabla}(\boldsymbol{u}_h,\boldsymbol{v}_h) := \sum_{i,j=1}^{N^V} \mathrm{dof}_i(\boldsymbol{u}_h)\mathrm{dof}_j(\boldsymbol{v}_h), \qquad \forall \boldsymbol{u}_h,\boldsymbol{v}_h \in \mathbf{V}_h,$$

We note that the classical stabilizer terms $S_K^0(\cdot,\cdot)$ and $S_K^{\nabla}(\cdot,\cdot)$ satisfy the following stability with respect to the continuous bilinear forms [10],

$$\begin{aligned} \zeta_* m^K(\boldsymbol{u}_h,\boldsymbol{v}_h) \le S_K^0(\boldsymbol{u}_h,\boldsymbol{v}_h) \le \zeta^* m^K(\boldsymbol{u}_h,\boldsymbol{v}_h) \quad \forall \boldsymbol{u}_h,\boldsymbol{v}_h \in \ker(\Pi_K^0),\\ \alpha_* a^K(\boldsymbol{u}_h,\boldsymbol{v}_h) \le S_K^{\nabla}(\boldsymbol{u}_h,\boldsymbol{v}_h) \le \alpha^* a^K(\boldsymbol{u}_h,\boldsymbol{v}_h) \quad \forall \boldsymbol{u}_h,\boldsymbol{v}_h \in \ker(\Pi_K^{\nabla}), \end{aligned} \tag{3.1}$$

where $\zeta_*,\zeta^*,\alpha_*,\alpha^* > 0$ are constants independent of diameter h_K of polygon K. Now considering the above defined local forms, we set the global discrete bilinear and trilinear forms for all $\boldsymbol{u}_h,\boldsymbol{v}_h \in \mathbf{V}_h$ and $q_h \in Q_h$ are simply set as sum over each polygon K as simply the sum over each polygon K,

$$\begin{aligned} m_h(\boldsymbol{u}_h,\boldsymbol{v}_h) &:= \sum_{K\in\mathcal{T}_h} m_h^K(\boldsymbol{u}_h,\boldsymbol{v}_h), \quad & b(\boldsymbol{v}_h,q_h) &:= \sum_{K\in\mathcal{T}_h} b^K(\boldsymbol{v}_h,q_h),\\ a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) &:= \sum_{K\in\mathcal{T}_h} a_h^K(\boldsymbol{u}_h,\boldsymbol{v}_h), \quad & c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h) &:= \sum_{K\in\mathcal{T}_h} c_h^K(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h) \end{aligned}$$

and the load term as

$$\mathbf{F}_h(\boldsymbol{v}_h) := \sum_{K\in\mathcal{T}_h} \mathbf{F}_h^K(\boldsymbol{v}_h).$$

Now we are in position to define our semi discrete virtual element formulation corresponding to the weak form (2.3): For each $t \in (0,T]$, find $\boldsymbol{u}_h(t) \in \mathbf{V}_h$ and $p_h(t) \in Q_h$ such that

$$m_h(\partial_t\boldsymbol{u}_h,\boldsymbol{v}_h)+a_h(\boldsymbol{u}_h,\boldsymbol{v}_h)+c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h)+b(\boldsymbol{v}_h,p_h)=\mathbf{F}_h(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h\in\mathbf{V}_h, \tag{3.2a}$$

$$b(\boldsymbol{u}_h,q_h) = 0 \quad \forall q_h \in Q_h, \tag{3.2b}$$

with given initial condition $\boldsymbol{u}_h(0)$ considered as an approximation of $\boldsymbol{u}_0$ chosen appropriately in derivation of the error analysis, and the discrete trilinear form $c_h(\cdot;\cdot,\cdot)$ is defined from $\tilde{c}_h(\cdot;\cdot,\cdot)$ analogous to the continuous trilinear form $c(\cdot,\cdot,\cdot)$. The stability properties of $S_K^0(\cdot,\cdot)$ and $S_K^{\nabla}(\cdot,\cdot)$ given in (3.1) yields

- $m_h(\cdot,\cdot)$ is positive definite form: for all $\boldsymbol{v}_h \in \mathbf{V}_h$,

$$m_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge \nu \sum_{K\in\mathcal{T}_h} \left(\|\Pi_K^0\boldsymbol{v}_h\|_{0,K}^2 + \zeta_*\|(I-\Pi_K^0)\boldsymbol{v}_h\|_{0,K}^2 \right) \ge \hat{C}_* \|\boldsymbol{v}_h\|_{0,\Omega}^2,$$

  where $\hat{C}_* := \min\{1,\zeta_*\}$.

- $a_h(\cdot,\cdot)$ is coercive: for all $\boldsymbol{v}_h \in \mathbf{V}_h$,

$$a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge \nu \sum_{K\in\mathcal{T}_h} \left(\|\Pi_K^{\nabla}\boldsymbol{v}_h\|_{1,K}^2 + \alpha_*\|(I-\Pi_K^{\nabla})\boldsymbol{v}_h\|_{1,K}^2 \right) \ge C_*\nu \|\boldsymbol{v}_h\|_{1,\Omega}^2,$$

  where $C_* := \min\{1,\alpha_*\}$.

- $a_h(\cdot,\cdot)$ is continuous: for all $\boldsymbol{u}_h,\boldsymbol{v}_h \in \mathbf{V}_h$ (again by use of stability of $S_K^{\nabla}(\cdot,\cdot)$),

$$a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) \le C^* \ \nu \ \|\boldsymbol{u}_h\|_{1,\Omega}\|\boldsymbol{v}_h\|_{1,\Omega},$$

  where $C^* := \max\{1,\alpha^*\}$.

- $b(\cdot,\cdot)$ satisfies inf-sup condition on $\mathbf{V}_h \times Q_h$: There exists a $\beta_h > 0$ such that (see [47])
$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_1} \geq \beta_h \|q_h\|_0, \qquad \forall q_h \in Q_h.$$
- $\mathbf{F}_h(\cdot)$ is continuous: for all $\boldsymbol{v}_h \in \mathbf{V}_h$,
$$\mathbf{F}_h(\boldsymbol{v}_h) \leq \sum_{K \in \mathcal{T}_h} \|\Pi_K^0 \boldsymbol{f}\|_{0,K} \|\boldsymbol{v}_h\|_{0,K} \leq \|\boldsymbol{f}\|_{0,\Omega} \|\boldsymbol{v}_h\|_{0,\Omega}.$$

Now, we generate the following result to show the continuity of the trilinear form $c_h(\cdot;\cdot,\cdot)$.

Lemma 3.1. *The projection operator Π_K^0 is bounded with respect to the L^s- norm with $s \geq 2$, that is,*
$$\|\Pi_K^0 \boldsymbol{v}\|_{L^s(K)} \leq C\|\boldsymbol{v}\|_{L^s(K)} \quad \forall \boldsymbol{v} \in \mathbf{L}^s(K) \text{ and } K \in \mathcal{T}_h,$$
where C is independent of mesh size h.

Proof. The use of inverse estimates for polynomials (see [11, 17]) yields
$$\|\Pi_K^0 \boldsymbol{v}\|_{L^s(K)} \leq Ch^{2\left(\frac{1}{s}-\frac{1}{2}\right)} \|\Pi_K^0 \boldsymbol{v}\|_{0,K}.$$
In view of the definition of Π_K^0, we have $\|\Pi_K^0 \boldsymbol{v}\|_{0,K} \leq \|\boldsymbol{v}\|_{0,K}$. Now, the Hölder's inequality together with mesh regularity assumptions yields
$$\|\Pi_K^0 \boldsymbol{v}\|_{L^s(K)} \leq Ch^{2\left(\frac{1}{s}-\frac{1}{2}\right)} |K|^{\left(\frac{1}{2}-\frac{1}{s}\right)} \|\boldsymbol{v}\|_{L^s(K)} \leq C\|\boldsymbol{v}\|_{L^s(K)}.$$
$\square$

• $c_h(\cdot;\cdot,\cdot)$ is continuous: for all $\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h \in \mathbf{V}_h$ (use of Lemma 3.1 and steps from the continuity of trilinear form $c(\cdot;\cdot,\cdot)$, refer [11]),
$$\begin{aligned} c_h(\boldsymbol{u}_h; \boldsymbol{v}_h, \boldsymbol{w}_h) = &\sum_{K \in \mathcal{T}_h} \frac{1}{2} \Big(((\boldsymbol{\Pi}_K^0 \boldsymbol{\nabla} \boldsymbol{v}_h)\ \Pi_K^0 \boldsymbol{u}_h, \Pi_K^0 \boldsymbol{w}_h)_{0,K} \\ &- ((\boldsymbol{\Pi}_K^0 \boldsymbol{\nabla} \boldsymbol{w}_h)\ \Pi_K^0 \boldsymbol{u}_h, \Pi_K^0 \boldsymbol{v}_h)_{0,K} \Big) \\ \leq &C\|\boldsymbol{u}_h\|_{1,\Omega} \|\boldsymbol{w}_h\|_{1,\Omega} \|\boldsymbol{v}_h\|_{1,\Omega}. \end{aligned}$$

Now, we produce the result below on the existence of unique solution of problem (3.2) and stability of the solution.

Lemma 3.2. *The semi-discrete problem* (3.2) *has a unique solution $\boldsymbol{u}_h(t) \in \mathbf{V}_h$ all $t \in [0,T]$ and given $\boldsymbol{u}_h(0)$ and satisfies,*
$$\|\boldsymbol{u}_h(t)\|_{0,\Omega}^2 + \nu \int_0^t |\boldsymbol{u}_h(s)|_{1,\Omega}^2 \, ds \leq C\Big(\|\boldsymbol{u}_h(0)\|_{0,\Omega}^2 + \int_0^t \|\boldsymbol{f}(s)\|_{0,\Omega}^2 \, ds\Big), \tag{3.3}$$
where the constant C is independent of mesh size h.

Proof. The properties of the discrete bilinear forms $a_h(\cdot,\cdot)$, $m_h(\cdot,\cdot)$ and $b(\cdot,\cdot)$, discrete trilinear form $c_h(\cdot;\cdot,\cdot)$, and discrete linear functional $\mathbf{F}_h(\cdot)$ with the well-known wellposedness results from [42] implies that the semi-discrete problem (3.2) has a unique solution, see also [27]. Taking $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (3.2a) gives
$$\frac{d}{dt}\|\boldsymbol{u}_h\|_{0,\Omega}^2 + \nu|\boldsymbol{u}_h|_{1,\Omega}^2 \leq C\|\boldsymbol{f}\|_{0,\Omega}\|\boldsymbol{u}_h\|_{0,\Omega}.$$

Employing the Poincaré and Young's inequality, then integrating from 0 to t leads to (3.3). $\square$

3.2. Fully discrete scheme

The time interval $[0,T]$ is decomposed into subintervals $I_n := [t_{n-1}, t_n]$, where $t_n = n\Delta t$ for $n = 1,\dots,N$ and $\Delta t = \frac{T}{N}$. For the time discretization, we employ the backward Euler scheme, i.e, the approximation of the time derivative at t_n for any generic function g_h is defined as follows.

$$\delta_t g_h^n := \frac{g_h^n - g_h^{n-1}}{\Delta t}.$$

For the consistency in the notations, the solution of semi-discrete scheme and fully discrete scheme at time $t = t_n$, will be denoted by $\boldsymbol{u}_h(t_n)$ and $\boldsymbol{u}_h^n$, respectively. The fully discrete virtual element scheme corresponding to the continuous formulation (2.3) read as: Given initial conditions $\boldsymbol{u}_h^0 := \boldsymbol{u}_h(0)$, find $\boldsymbol{u}_h^n \in \mathbf{V}_h, p_h^n \in Q_h$ for each $n = 1,\dots,N$ such that

$$m_h(\delta_t \boldsymbol{u}_h^n, \boldsymbol{v}_h) + a_h(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_h^n; \boldsymbol{u}_h^n, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h^n) = \mathbf{F}_h^n(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h, \tag{3.4a}$$

$$b(\boldsymbol{u}_h^n, q_h) = 0 \quad \forall q_h \in Q_h. \tag{3.4b}$$

The following lemma provide us the well-posedness of the above fully discrete scheme.

Lemma 3.3. *There exists a unique solution $\boldsymbol{u}_h^n \in \mathbf{V}_h$, $p_h^n \in Q_h$ of the problem* (3.4) *and also satisfies the following bound,*

$$\max_{1\le j\le n} \|\boldsymbol{u}_h^j\|_{0,\Omega}^2 + \nu\Delta t \sum_{j=1}^{n} |\boldsymbol{u}_h^j|_{1,\Omega}^2 \le C\big(\|\boldsymbol{u}_h(0)\|_{0,\Omega}^2 + \Delta t \sum_{j=1}^{n} \|\boldsymbol{f}^j\|_{0,\Omega}^2\big), \tag{3.5}$$

where C is a positive constant and independent of h, Δt.

Proof. Taking $\boldsymbol{v}_h = \boldsymbol{u}_h^n, q_h = p_h^n$ in (3.4) then the coercivity of $a_h(\cdot;\cdot)$, skew-symmetry of $c_h(\boldsymbol{u}_h;\cdot,\cdot)$ and continuity of $\mathbf{F}_h^n$, and a use of Young's inequality gives

$$\begin{aligned}\frac{1}{2}(\|\boldsymbol{u}_h^n\|_{0,\Omega}^2 - \|\boldsymbol{u}_h^{n-1}\|_{0,\Omega}^2) + \nu\Delta t|\boldsymbol{u}_h^n|_{1,\Omega}^2 &\le C\Delta t\ \|\boldsymbol{f}^n\|_{0,\Omega}\|\boldsymbol{u}_h^n\|_{0,\Omega} \\ &\le C\Delta t\ \|\boldsymbol{f}^n\|_{0,\Omega}^2 + \frac{\nu\Delta t}{2}|\boldsymbol{u}_h^n|_{1,\Omega}^2.\end{aligned}$$

Summing the bound above over n leads to (3.5). Now, the existence and uniqueness can be obtained from the stability result (3.5) and the well-posedness of the discrete scheme corresponding to the steady Navier-Stokes equation, refer [10,31,34]. $\square$

4. Convergence analysis

With the help of a projection named as Stokes projection (introduced in this section by (4.8)), we establish convergence results for both semi discrete and fully discrete schemes. We derive the optimal error estimates for velocity in the H^1- norm, and for pressure in the L^2- norm under some regularity assumptions. We begin with collecting the preliminary results for the subsequent analysis.

Lemma 4.1. *The trilinear form $c_h(\cdot;\cdot,\cdot)$ satisfy the following bound:*

$$c_h(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) \leq C\|\boldsymbol{u}\|^{1/2}_{L^2(\Omega)}\|\nabla\boldsymbol{u}\|^{1/2}_{L^2(\Omega)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}, \tag{4.1}$$

where $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbf{V}$ and C is independent of h.

Proof. Let $p_1 = 2,\ q_1 = 3,\ r_1 = 6$ then repeated application of generalized version of Hölder's inequality with $1/p_1 + 1/q_1 + 1/r_1 = 1$ along with Lemma 3.1 implies that

$$\begin{aligned}
\tilde{c}_h(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) &\leq \sum_{i,j=1}^{2}\sum_{K\in\mathcal{T}_h}\left\|\Pi^0_K\frac{\partial \boldsymbol{v}_i}{\partial x_j}\right\|_{L^2(K)}\left\|\Pi^0_K\boldsymbol{u}_j\right\|_{L^3(K)}\left\|\Pi^0_K\boldsymbol{w}_i\right\|_{L^6(K)}\\
&\leq \sum_{i,j=1}^{2}\left(\sum_{K\in\mathcal{T}_h}\left\|\Pi^0_K\frac{\partial \boldsymbol{v}_i}{\partial x_j}\right\|^2_{L^2(K)}\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}\left\|\Pi^0_K\boldsymbol{u}_j\right\|^3_{L^3(K)}\right)^{\frac{1}{3}}\\
&\quad\times\left(\sum_{K\in\mathcal{T}_h}\left\|\Pi^0_K\boldsymbol{w}_i\right\|^6_{L^6(K)}\right)^{\frac{1}{6}}\\
&\leq C\sum_{i,j=1}^{2}\left\|\frac{\partial \boldsymbol{v}_i}{\partial x_j}\right\|_{L^2(\Omega)}\left\|\boldsymbol{u}_j\right\|_{L^3(\Omega)}\left\|\boldsymbol{w}_i\right\|_{L^6(\Omega)}.
\end{aligned}$$

Employing the Sobolev embedding $W^{m,p}(\Omega) \subset L^q(\Omega)$, for $1 \leq q \leq \frac{2p}{2-mp}$, $mp < 2$ (see [2]) with $p = 2,\ q = 3,\ m = 1/2$, and for $mp = 2$, we have $W^{m,p}(\Omega) \subset L^q(\Omega)$, $q \in [1,\infty)$ with $p = 2$, $m = 1$, we arrive at

$$\tilde{c}_h(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) \leq C\sum_{i,j=1}^{2}\left\|\frac{\partial \boldsymbol{v}_i}{\partial x_j}\right\|_{L^2(\Omega)}\left\|\boldsymbol{u}_j\right\|_{W^{\frac{1}{2},2}(\Omega)}\left\|\boldsymbol{w}_i\right\|_{H^1(\Omega)}. \tag{4.2}$$

The interpolation estimates (see [2, Theorem 4.17] on page 79), for all $\boldsymbol{v} \in W^{m,p}(\Omega)$, $1 \leq j \leq m$, gives

$$\|\boldsymbol{v}\|_{W^{j,p}(\Omega)} \leq C\|\boldsymbol{v}\|^{j/m}_{W^{m,p}(\Omega)}\|\boldsymbol{v}\|^{(m-j)/m}_{L^p(\Omega)}. \tag{4.3}$$

The choice of $j = 1/2$, $m = 1$, $p = 2$ in (4.3) and using Poincaré inequality, we get

$$\|\boldsymbol{v}\|_{W^{1/2,2}(\Omega)} \leq C\|\boldsymbol{v}\|^{1/2}_{W^{1,2}(\Omega)}\|\boldsymbol{v}\|^{1/2}_{L^2(\Omega)} \leq C_P\|\nabla\boldsymbol{v}\|^{1/2}_{0,\Omega}\|\boldsymbol{v}\|^{1/2}_{0,\Omega}.$$

Thus, the use of above bound in (4.2) leads to

$$\tilde{c}_h(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) \leq C\left\|\nabla\boldsymbol{v}\right\|_{L^2(\Omega)}\|\nabla\boldsymbol{u}\|^{1/2}_{L^2(\Omega)}\|\boldsymbol{u}\|^{1/2}_{L^2(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}.$$

Proceeding in the similar fashion, we can derive the same bounds for the term $\tilde{c}_h(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})$ and then we conclude the bound (4.1) using the definition of $c_h(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})$. □

Lemma 4.2. *Let $\boldsymbol{u}_\pi \in [\mathbb{P}_1(K)]^2$ be the polynomial approximation of $\boldsymbol{u}$ on each $K \in \mathcal{T}_h$. Under the regularity assumption on the polygonal mesh $\mathcal{T}_h$ (mentioned in Section 3), there exists a positive constant C independent of h such that (see [8, 17])*

$$\sum_{K\in\mathcal{T}_h}\left(\|\boldsymbol{u}-\boldsymbol{u}_\pi\|_{0,K} + h\ |\boldsymbol{u}-\boldsymbol{u}_\pi|_{1,K}\right) \leq Ch^2|\boldsymbol{u}|_{2,\Omega}. \tag{4.4}$$

Lemma 4.3. *For each $\boldsymbol{u} \in \mathbf{V} \cap \mathbf{H}^{r+1}(\Omega)$ with $0 \leq r \leq 1$ and under the regularity assumption on the polygonal mesh (mentioned in Section 3), there exist an interpolant $\boldsymbol{u}_I \in \mathbf{V}_h$ satisfying (see [47])*

$$\|\boldsymbol{u} - \boldsymbol{u}_I\|_{0,\Omega} + h_K \ |\boldsymbol{u} - \boldsymbol{u}_I|_{1,\Omega} \leq Ch^{r+1}|\boldsymbol{u}|_{r+1,\Omega}. \tag{4.5}$$

Lemma 4.4. *The bilinear form $b(\cdot, \cdot)$ satisfies the discrete inf-sup condition on $\mathbf{V}_h \times Q_h$, that is, there exists a $\beta_h > 0$ such that (see [47])*

$$\sup_{\boldsymbol{v}_h (\neq 0) \in \mathbf{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geq \beta_h \|q_h\|_{0,\Omega} \qquad \forall q_h \in Q_h. \tag{4.6}$$

For the proof of Lemma 4.3 and 4.4, we refer to [10, 11, 47].

Defining the discrete kernel space $\mathbf{X}_h$ with use of the fact that $\operatorname{div} \mathbf{V}_h \subset Q_h$, as

$$\mathbf{X}_h := \{\boldsymbol{v}_h \in \mathbf{V}_h : b(\boldsymbol{v}_h, q_h) = 0 \ \ \forall q_h \in Q_h\} = \{\boldsymbol{v}_h \in \mathbf{V}_h : \operatorname{div} \boldsymbol{v}_h = 0\}.$$

For a given $\boldsymbol{v} \in \mathbf{X}$, we have the following approximation property for the discrete space $\mathbf{X}_h$ as a consequence of the discrete inf-sup condition from Lemma 4.4 (see in [16] and also [11]):

$$\inf_{\boldsymbol{z}_h \in \mathbf{X}_h, \boldsymbol{z}_h \neq 0} \|\boldsymbol{v} - \boldsymbol{z}_h\|_1 \leq C \inf_{\boldsymbol{v}_h \in \mathbf{V}_h, \boldsymbol{v}_h \neq 0} \|\boldsymbol{v} - \boldsymbol{v}_h\|_1. \tag{4.7}$$

Next, we define the classical Stokes projection $S_h(\boldsymbol{u}, p) := (S_h^u \boldsymbol{u}, S_h^p p) \in \mathbf{V}_h \times Q_h$ as a solution of the following equation (see also [16] and [23]).

$$a_h(S_h^u \boldsymbol{u}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, S_h^p p) = a(\boldsymbol{u}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p) \qquad \forall \boldsymbol{v}_h \in \mathbf{V}_h, \tag{4.8a}$$

$$b(\boldsymbol{u} - S_h^u \boldsymbol{u}, q_h) = 0 \qquad \forall q_h \in Q_h. \tag{4.8b}$$

Use of the inf-sup condition (4.6), and continuity of the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ in equation (4.8) for $\boldsymbol{v}_h \neq 0$ gives

$$\frac{a_h(S_h^u \boldsymbol{u}, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_1} + \|S_h^p p\|_{0,\Omega} \leq C(\|\nabla \boldsymbol{u}\|_{0,\Omega} + \|p\|_{0,\Omega}).$$

Choosing $\boldsymbol{v}_h = S_h^u \boldsymbol{u}$ in (4.8) then the following bounds are easily available by employing coercivity of the discrete bilinear form $a_h(\cdot, \cdot)$.

$$\|\nabla S_h^u \boldsymbol{u}\|_{0,\Omega} + \|S_h^p p\|_{0,\Omega} \leq C(\|\nabla \boldsymbol{u}\|_{0,\Omega} + \|p\|_{0,\Omega}). \tag{4.9}$$

By definition of S_h^u in (4.8b) and use of (2.3b) implies $b(S_h^u \boldsymbol{u}, q_h) = 0$ for all $q_h \in Q_h$ and thus $S_h^u \boldsymbol{u} \in \mathbf{X}_h$. Then the following error estimates of the operator S_h^u can be easily derived by using the properties of the bilinear forms $a_h(\cdot, \cdot)$, $b(\cdot, \cdot)$, Lemma 4.2 and Lemma 4.3, and appealing to the duality arguments (refer [47]).

Lemma 4.5. *Let $(\boldsymbol{u}, p) \in \mathbf{V} \times Q$ be the solution of the continuous problem* (2.3) *and $(S_h^u \boldsymbol{u}, S_h^p p) \in \mathbf{V}_h \times Q_h$ satisfies the equation* (4.8) *then there exists a positive constant C, independent of h, such that*

$$\|\boldsymbol{u} - S_h^u \boldsymbol{u}\|_{0,\Omega} + h(|\boldsymbol{u} - S_h^u \boldsymbol{u}|_{1,\Omega} + \|p - S_h^p p\|_{0,\Omega}) \leq Ch^2(|\boldsymbol{u}|_{2,\Omega} + |p|_{1,\Omega}). \tag{4.10}$$

In the following lemma, we estimate the error between the trilinear forms $c(\cdot; \cdot, \cdot)$ and $c_h(\cdot; \cdot, \cdot)$. The main ideas in following lemma are borrowed from [11].

Lemma 4.6. *For all $\boldsymbol{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$ and $\boldsymbol{v}_h \in \mathbf{V}_h$, the following holds.*

$$|c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h) - c_h(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)| \le Ch|\boldsymbol{u}|_{2,\Omega}\|\nabla\boldsymbol{u}\|_{0,\Omega}\|\nabla\boldsymbol{v}_h\|_{0,\Omega}, \tag{4.11}$$

where C is independent of h.

Proof. We begin with splitting the skew-symmetric terms into simpler trilinear forms $\tilde{c}(\cdot;\cdot,\cdot)$ and $\tilde{c}_h(\cdot;\cdot,\cdot)$ in the following manner.

$$\begin{aligned} c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h) - c_h(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h) &= \frac{1}{2}\Big(\big(\tilde{c}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h) - \tilde{c}_h(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)\big) \\ &\qquad + \big(\tilde{c}(\boldsymbol{u};\boldsymbol{v}_h,\boldsymbol{u}) - \tilde{c}_h(\boldsymbol{u};\boldsymbol{v}_h,\boldsymbol{u})\big)\Big) \\ &:= \frac{1}{2}\sum_{i=1}^{2}\mathcal{C}_i(\boldsymbol{v}_h). \end{aligned}$$

We proceed to estimate $\mathcal{C}_i, i = 1,2$. An application of generalised Hölder's inequality, Lemma 3.1, Sobolev embedding $W^{r,4}(\Omega) \subset H^{r+1}(\Omega)$, $r \ge 0$ and estimates of projections Π_K^0, $\boldsymbol{\Pi}_K^{0,0}$ gives

$$\begin{aligned} \mathcal{C}_1(\boldsymbol{v}_h) &= \sum_K \sum_{i,j=1}^{2} \int_K \Big(\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}\boldsymbol{u}_j\ (\boldsymbol{v}_{h,i} - \Pi_K^0\boldsymbol{v}_{h,i}) + \frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}(\boldsymbol{u}_j - \Pi_K^0\boldsymbol{u}_j)\ \Pi_K^0\boldsymbol{v}_{h,i} \\ &\qquad - \Big((I - \boldsymbol{\Pi}_K^{0,0})\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}\Big)\Pi_K^0\boldsymbol{u}_j\ \Pi_K^0\boldsymbol{v}_{h,i}\Big) \\ &\le \sum_K \sum_{i,j=1}^{2} \Bigg(\left\|\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}\right\|_{L^4(K)} \|\boldsymbol{u}_j\|_{L^4(K)}\|\boldsymbol{v}_{h,i} - \Pi_K^0\boldsymbol{v}_{h,i}\|_{L^2(K)} \\ &\qquad + \left\|\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}\right\|_{L^4(K)} \|(I - \Pi_K^0)\boldsymbol{u}_j\|_{L^2(K)}\|\Pi_K^0\boldsymbol{v}_{h,i}\|_{L^4(K)} \\ &\qquad + \|\boldsymbol{u}_j\|_{L^4(K)}\left\|(I - \boldsymbol{\Pi}_K^{0,0})\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}\right\|_{L^4(K)}\|\Pi_K^0\boldsymbol{v}_{h,i}\|_{L^2(K)}\Bigg) \\ &\le Ch\,|\boldsymbol{u}|_{2,\Omega}\|\nabla\boldsymbol{u}\|_{0,\Omega}\|\nabla\boldsymbol{v}_h\|_{0,\Omega}. \end{aligned}$$

Proceeding in the similar fashion, we can easily obtain the following bounds for $\mathcal{C}_2(\boldsymbol{v}_h)$.

$$\mathcal{C}_2(\boldsymbol{v}_h) \le Ch|\boldsymbol{u}|_{2,\Omega}\|\nabla\boldsymbol{u}\|_{0,\Omega}\|\nabla\boldsymbol{v}_h\|_{0,\Omega}.$$

Collecting all the bounds of $\mathcal{C}_i(\boldsymbol{v}_h), i = 1,2$, we finally obtain the bound (4.11). □

4.1. Estimates for semi-discrete scheme

We collect all the derived/recalled results to state the estimates below.

Theorem 4.1. *Let $(\boldsymbol{u},p) \in \mathbf{V} \times Q$ and $(\boldsymbol{u}_h,p_h) \in \mathbf{V}_h \times Q_h$ be the solutions of the continuous problem (2.3) and discrete problem (3.2) respectively. Assuming the additional regularity $\boldsymbol{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$ and $p \in H^1(\Omega) \cap Q$, then there exists a positive constant C independent of h such that*

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^\infty(0,t;[L^2(\Omega)]^2)} + \nu\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(0,t;[H^1(\Omega)]^2)} \\ &+ \|p - p_h\|^2_{L^2(0,t;L^2(\Omega))} \le C\,h^2. \end{aligned} \tag{4.12}$$

Proof. Split the error as $(\boldsymbol{u}-\boldsymbol{u}_h)(t) := \boldsymbol{e}_I(t)+\boldsymbol{e}_A(t)$, where $\boldsymbol{e}_I(t) := (\boldsymbol{u}-S_h^u\boldsymbol{u})(t)$ and $\boldsymbol{e}_A(t) := (S_h^u\boldsymbol{u}-\boldsymbol{u}_h)(t)$. Now since the estimates for $\boldsymbol{e}_I(t)$ are known from Lemma 4.5, we proceed to establish the estimates for term $\boldsymbol{e}_A(t)$.

The error equation with the help of Stokes projection (4.8), weak form (2.3b) and semi-discrete form (3.2b) in terms of $\boldsymbol{e}_A$ is given as

$$\begin{aligned} m_h(\partial_t\boldsymbol{e}_A,\boldsymbol{v}_h)+a_h(\boldsymbol{e}_A,\boldsymbol{v}_h) &= (\mathbf{F}-\mathbf{F}_h)(\boldsymbol{v}_h)+b(\boldsymbol{v}_h,p_h-S_h^p p)\\ &\quad -\big(m(\partial_t\boldsymbol{u},\boldsymbol{v}_h)-m_h(\partial_t S_h^u\boldsymbol{u},\boldsymbol{v}_h)\big)\\ &\quad -\big(c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)-c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h)\big). \end{aligned} \tag{4.13}$$

From equations (4.8), (2.3b) and (3.2b), we have for all $q_h\in Q_h$,

$$b(\boldsymbol{e}_A,q_h)=b(\boldsymbol{u}-\boldsymbol{u}_h,q_h)=0. \tag{4.14}$$

Using (4.8) and taking $\boldsymbol{v}_h=\boldsymbol{e}_A$ in (4.13) together with (4.14) implies

$$\begin{aligned} m_h(\partial_t\boldsymbol{e}_A,\boldsymbol{e}_A)+a_h(\boldsymbol{e}_A,\boldsymbol{e}_A) &= \underbrace{(\mathbf{F}-\mathbf{F}_h)(\boldsymbol{e}_A)}_{:=T_1}-\underbrace{\big(m(\partial_t\boldsymbol{u},\boldsymbol{e}_A)-m_h(\partial_t S_h^u\boldsymbol{u},\boldsymbol{e}_A)\big)}_{:=T_2}\\ &\quad -\underbrace{\big(c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{e}_A)-c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{e}_A)\big)}_{:=T_3}. \end{aligned} \tag{4.15}$$

The Cauchy Schwarz inequality, estimates of the projection Π_K^0 and Poincaré inequality infer that

$$|T_1|\le \|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,\Omega}\|\boldsymbol{e}_A\|_{0,\Omega}\le Ch|\boldsymbol{f}|_{1,\Omega}\|\nabla\boldsymbol{e}_A\|_{0,\Omega}.$$

The consistency of $m_h(\cdot,\cdot)$, Cauchy Schwarz inequality, triangle's inequality, repeated application of estimates of Π_K^0 and (4.10) together with Poincaré inequality enable us

$$\begin{aligned} T_2 &= \sum_{K\in\mathcal{T}_h} m^K(\partial_t(I-\Pi_K^0)\boldsymbol{u},\boldsymbol{e}_A)-m_h^K(\partial_t(S_h^u\boldsymbol{u}-\Pi_K^0\boldsymbol{u}),\boldsymbol{e}_A)\\ &\le \Big(\sum_{K\in\mathcal{T}_h}(\|\partial_t(I-\Pi_K^0)\boldsymbol{u}\|_{0,K}+\|\partial_t(S_h^u\boldsymbol{u}-\Pi_K^0\boldsymbol{u})\|_{0,K})\Big)\|\boldsymbol{e}_A\|_{0,\Omega}\\ &\le C\,h|\partial_t\boldsymbol{u}|_{1,\Omega}\|\nabla\boldsymbol{e}_A\|_{0,\Omega}. \end{aligned}$$

The estimates for the term T_3 is quite involved and we proceed by separating the terms as

$$T_3=(c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{e}_A)-c_h(\boldsymbol{u};\boldsymbol{u},\boldsymbol{e}_A))+(c_h(\boldsymbol{u};\boldsymbol{u},\boldsymbol{e}_A)-c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{e}_A)):=\sum_{i=1}^{2}T_{3,i}.$$

The consequence of Lemma 4.6 gives

$$T_{3,1}\le C\,h|\boldsymbol{u}|_{2,\Omega}\|\nabla\boldsymbol{u}\|_{0,\Omega}\|\nabla\boldsymbol{e}_A\|_{0,\Omega}.$$

For $T_{3,2}$, we employ Lemma 4.1, estimate of Stokes projection (4.10), stability bound of Stokes projection (4.9), continuity of trilinear form $c_h(\cdot,\cdot,\cdot)$, Poincaré inequality and bound (2.4) to obtain

$$T_{3,2}=c_h(\boldsymbol{u};\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{e}_A)+c_h(\boldsymbol{u}-\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{e}_A)$$

$$
\begin{aligned}
&= c_h(\boldsymbol{u};\boldsymbol{e}_I,\boldsymbol{e}_A)+c_h(u;\boldsymbol{e}_A,\boldsymbol{e}_A)+c_h(\boldsymbol{e}_I;\boldsymbol{u}_h,\boldsymbol{e}_A)+c_h(\boldsymbol{e}_A;\boldsymbol{u}_h,\boldsymbol{e}_A)\\
&= c_h(\boldsymbol{u};\boldsymbol{e}_I,\boldsymbol{e}_A)+c_h(\boldsymbol{e}_I;\boldsymbol{u},\boldsymbol{e}_A)-c_h(\boldsymbol{e}_I;\boldsymbol{e}_I,\boldsymbol{e}_A)+c_h(\boldsymbol{e}_A;\boldsymbol{u},\boldsymbol{e}_A)\\
&\quad + c_h(\boldsymbol{e}_A;\boldsymbol{e}_A,\boldsymbol{e}_A)-c_h(\boldsymbol{e}_A;\boldsymbol{e}_I,\boldsymbol{e}_A)\\
&\le Ch\|\nabla\boldsymbol{u}\|_{0,\Omega}(|\boldsymbol{u}|_{2,\Omega}+|p|_{1,\Omega})\|\nabla\boldsymbol{e}_A\|_{0,\Omega}+h^2(|\boldsymbol{u}|_{2,\Omega}+|p|_{1,\Omega})^2\|\nabla\boldsymbol{e}_A\|_{0,\Omega}\\
&\quad + \Big(\|\nabla\boldsymbol{u}\|_{0,\Omega}+\|p\|_{0,\Omega}\Big)\|\boldsymbol{e}_A\|_{0,\Omega}^{\frac12}\|\nabla\boldsymbol{e}_A\|_{0,\Omega}^{\frac32}.
\end{aligned}
$$

Collecting the bounds of $T_{3,i},\ i=1,2$ and use of Young's inequality, we finally obtain the following bound for T_3.

$$
\begin{aligned}
T_3 &\le Ch\Big((|\boldsymbol{u}|_{2,\Omega}+|p|_{1,\Omega})(\|\nabla\boldsymbol{u}\|_{0,\Omega}+h|\boldsymbol{u}|_{2,\Omega}+h|p|_{1,\Omega})\Big)\|\nabla\boldsymbol{e}_A\|_{0,\Omega}\\
&\quad + \Big(\|\nabla\boldsymbol{u}\|_{0,\Omega}+\|p\|_{0,\Omega}\Big)\|\boldsymbol{e}_A\|_{0,\Omega}^{\frac12}\|\nabla\boldsymbol{e}_A\|_{0,\Omega}^{\frac32}\\
&\le C\Big(\|\nabla\boldsymbol{u}\|_{0,\Omega}+\|p\|_{0,\Omega}\Big)^2\|\nabla\boldsymbol{e}_A\|_{0,\Omega}\|\boldsymbol{e}_A\|_{0,\Omega}+\frac{\nu}{4}\|\nabla\boldsymbol{e}_A\|_{0,\Omega}^2\\
&\le C\Big(\|\nabla\boldsymbol{u}\|_{0,\Omega}+\|p\|_{0,\Omega}\Big)^4\|\boldsymbol{e}_A\|_{0,\Omega}^2+\frac{\nu}{2}\|\nabla\boldsymbol{e}_A\|_{0,\Omega}^2.
\end{aligned}
\tag{4.16}
$$

On substituting the bounds of T_1, T_2 and T_3 in (4.15) and applying the Young's inequality, we arrive at

$$
\frac12\frac{d}{dt}\|\boldsymbol{e}_A\|_{0,\Omega}^2+\frac{\nu}{2}\|\nabla\boldsymbol{e}_A\|_{0,\Omega}^2\le C\Big(h^2+\Big(\|\nabla\boldsymbol{u}\|_{0,\Omega}+\|p\|_{0,\Omega}\Big)^4\|\boldsymbol{e}_A\|_{0,\Omega}^2\Big).
$$

Now integrating over time from 0 to t then taking $\boldsymbol{u}_h(0):=\boldsymbol{u}_I(0)$, we get

$$
\begin{aligned}
&\|\boldsymbol{e}_A(t)\|_{0,\Omega}^2+\nu\int_0^t\|\nabla\boldsymbol{e}_A(s)\|_{0,\Omega}^2\,\mathrm{d}s\\
\le&\|\boldsymbol{e}_A(0)\|_{0,\Omega}^2+C\Big(h^2+\int_0^t\Big(\|\nabla\boldsymbol{u}(s)\|_{0,\Omega}+\|p(s)\|_{0,\Omega}\Big)^4\|\boldsymbol{e}_A(s)\|_{0,\Omega}^2\,\mathrm{d}s\Big).
\end{aligned}
$$

An application of Gronwall's lemma together with the additional regularities of $\boldsymbol{u}$ and p yields

$$
\|\boldsymbol{e}_A(t)\|_{0,\Omega}^2+\nu\int_0^t\|\nabla\boldsymbol{e}_A(s)\|_{0,\Omega}^2\,\mathrm{d}s\le Ch^2.
\tag{4.17}
$$

For pressure estimates, we split again the error as: $(p-p_h)(t)=(p-S_h^pp)(t)+(S_h^pp-p_h)(t):=e_S(t)+e_Q(t)$, and then proceed to derive estimate for $e_Q(t)$.

Now, an application of discrete inf-sup condition from Lemma 4.4 implies

$$
\beta_h\|e_Q\|_{0,\Omega}\le\sup_{(0\ne)\boldsymbol{v}_h\in\mathbf{V}_h}\frac{b(\boldsymbol{v}_h,e_Q)}{\|\boldsymbol{v}_h\|_{1,\Omega}}.
\tag{4.18}
$$

From equations (2.3), (3.2) and (4.8), we get

$$
\begin{aligned}
b(\boldsymbol{v}_h,e_Q)&=a_h(\boldsymbol{e}_A,\boldsymbol{v}_h)+(\boldsymbol{f}-\boldsymbol{f}_h,\boldsymbol{v}_h)+\big(m_h(\partial_t\boldsymbol{u}_h,\boldsymbol{v}_h)-m(\partial_t\boldsymbol{u},\boldsymbol{v}_h)\big)\\
&\quad+\big(c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h)-c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)\big).
\end{aligned}
$$

The inequality (4.18) and integration from 0 to t implies

$$
\int_0^t\|e_Q(s)\|_{0,\Omega}^2\,\mathrm{d}s\le C\int_0^t\Big(\|\boldsymbol{e}_A(s)\|_{1,\Omega}^2+\|(\boldsymbol{f}-\boldsymbol{f}_h)(s)\|_{0,\Omega}^2
$$

$$+ \|\partial_t(\boldsymbol{u}-\boldsymbol{u}_h)(s)\|_{0,\Omega}^2 + \|\partial_t(\boldsymbol{u}-\Pi_K^0\boldsymbol{u})(s)\|_{0,\Omega}^2$$
$$+ \frac{1}{\|\boldsymbol{v}_h\|_{1,\Omega}^2}\big(c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h) - c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)\big)^2\Big)\,\mathrm{d}s. \qquad (4.19)$$

The following bound for $\partial_t\boldsymbol{e}_A$ is achieved by differentiating the error equation (4.13) with respect to time, choosing $\boldsymbol{v}_h = \partial_t\boldsymbol{e}_A$ and then imitating the proof of (4.17) analogously to obtain,

$$\|\partial_t\boldsymbol{e}_A\|_{0,\Omega}^2 + \nu\int_0^t \|\nabla(\partial_t\boldsymbol{e}_A)(s)\|_{0,\Omega}^2\,ds \le Ch^2. \qquad (4.20)$$

In view of triangle's inequality, bound of T_3 (4.16), estimates (4.20) and bound (4.19) with estimates from Lemma 4.5, the desired result follows. □

4.2. Estimates for fully-discrete scheme

Following analogously to the semi-discrete scheme in this section, we provide a sketch of the proof estimating the total error occurred through time discretization (by employing the backward Euler scheme) and space discretization. We introduce the following discrete l^2-norm for any bounded function $v(t) \in H^m(\Omega)$ on interval $[0,T]$ as

$$\|v\|_{l^2(0,T;H^m(\Omega))}^2 := \sum_{i=1}^N (\Delta t)\|v(t_i)\|_{H^m(\Omega)}^2, \quad t_i = i\ \Delta t.$$

Theorem 4.2. *Let $(\boldsymbol{u}^n, p^n) \in \mathbf{V}\times Q$ and $(\boldsymbol{u}_h^n, p_h^n) \in \mathbf{V}_h \times Q_h$ be the solutions of the continuous problem* (2.3) *and fully discrete problem* (3.4)*, respectively for each $n = 1,\ldots,N$. Assuming the additional regularity that $\boldsymbol{u} \in \mathbf{H}^2(\Omega)\cap\mathbf{V}$ and $p \in H^1(\Omega)\cap Q$ then,*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{l^\infty(0,t_n;[L^2(\Omega)]^2)}^2 + \nu\|\boldsymbol{u}-\boldsymbol{u}_h\|_{l^2(0,t_n;[H^1(\Omega)]^2)}^2$$
$$+ \|p-p_h\|_{l^2(0,t_n;L^2(\Omega))}^2 \le C(h^2+\Delta t^2), \qquad (4.21)$$

for constant C independent of h.

Proof. Decompose the error as: $\boldsymbol{u}^n - \boldsymbol{u}_h^n = E_I^n + E_A^n$, where

$$E_I^n := \boldsymbol{u}^n - S_h^u\boldsymbol{u}^n \quad \text{and} \quad E_A^n := S_h^u\boldsymbol{u}^n - \boldsymbol{u}_h^n.$$

Using the estimates of Stokes projection in Lemma 4.5 at each time t_n, one obtains

$$\|E_I^n\|_{0,\Omega} + h\|\nabla E_I^n\|_{1,\Omega} \le Ch^2(|\boldsymbol{u}^n|_{2,\Omega} + |p^n|_{1,\Omega}).$$

We proceed to obtain the estimates E_A^n. The following error equation in terms of E_A^n can be easily written with the help of Stokes projection (4.8), weak form (2.3) and fully discrete form (3.4).

$$\begin{aligned} m_h(\delta_t E_A^n, \boldsymbol{v}_h) + a_h(E_A^n, \boldsymbol{v}_h) &= (\mathbf{F}^n - \mathbf{F}_h^n)(\boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h^n - S_h^p p^n) \\ &\quad + \big(a_h(S_h^u\boldsymbol{u}^n, \boldsymbol{v}_h) - a(\boldsymbol{u}^n, \boldsymbol{v}_h)\big) \\ &\quad + (m_h(\delta_t(S_h^u\boldsymbol{u}^n), \boldsymbol{v}_h) - m(\partial_t\boldsymbol{u}^n, \boldsymbol{v}_h)) \\ &\quad + (c(\boldsymbol{u}^n;\boldsymbol{u}^n,\boldsymbol{v}_h) - c_h(\boldsymbol{u}_h^n;\boldsymbol{u}_h^n,\boldsymbol{v}_h)). \qquad (4.22) \end{aligned}$$

Choosing $\boldsymbol{v}_h = E_A^n$ and using coercivity of $m_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$, we infer that

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\|E_A^n\|_{0,\Omega}^2 - \|E_A^{n-1}\|_{0,\Omega}^2\right) + \nu\|\nabla E_A^n\|_{0,\Omega}^2 \\
&\lesssim (m_h(\delta_t(S_h^u \boldsymbol{u}^n), E_A^n) - m(\partial_t \boldsymbol{u}^n, E_A^n)) + (\mathbf{F}^n - \mathbf{F}_h^n)(E_A^n) \\
&\quad + \left(a_h(S_h^u \boldsymbol{u}^n, E_A^n) - a(\boldsymbol{u}^n, E_A^n)\right) \\
&\quad + (c(\boldsymbol{u}^n;\boldsymbol{u}^n, E_A^n) - c_h(\boldsymbol{u}_h^n;\boldsymbol{u}_h^n, E_A^n)).
\end{aligned}$$

Multiplying the above inequality with Δt and then summing over n gives

$$\begin{aligned}
&\frac{1}{2}\left(\|E_A^n\|_{0,\Omega}^2 - \|E_A^0\|_{0,\Omega}^2\right) + \nu\Delta t\sum_{j=1}^n \|\nabla E_A^j\|_{0,\Omega}^2 \\
&\lesssim \sum_{j=1}^n \left(m_h\left(S_h^u\boldsymbol{u}^j - S_h^u\boldsymbol{u}^{j-1}, E_A^j\right) - (\Delta t)m(\partial_t\boldsymbol{u}^j, E_A^j)\right) \\
&\quad + (\Delta t)\sum_{j=1}^n(\mathbf{F}^j - \mathbf{F}_h^j)(E_A^j) + (\Delta t)\sum_{j=1}^n\left(a_h(S_h^u\boldsymbol{u}^j, E_A^j) - a(\boldsymbol{u}^j, E_A^j)\right) \\
&\quad + (\Delta t)\sum_{j=1}^n(c(\boldsymbol{u}^j;\boldsymbol{u}^j, E_A^j) - c_h(\boldsymbol{u}_h^j;\boldsymbol{u}_h^j, E_A^j)) := \sum_{i=1}^4 G_i.
\end{aligned} \tag{4.23}$$

Use of the polynomial approximation $\Pi_K^0\boldsymbol{u}$, Cauchy-Schwarz inequality and Taylor's expansion for any continuous function $f(t)$ is

$$f^j - f^{j-1} = (\Delta t)\partial_t f^j + \int_{t_{j-1}}^{t_j}(s - t_{j-1})\partial_{tt}f(s)\,\mathrm{d}s,$$

and thus implies

$$\begin{aligned}
G_1 &= \sum_{j=1}^n\Big(\sum_{K\in\mathcal{T}_h} m_h^K((S_h^u\boldsymbol{u}^j - S_h^u\boldsymbol{u}^{j-1}) - \Pi_K^0(\boldsymbol{u}^j - \boldsymbol{u}^{j-1}), E_A^j) \\
&\quad + m^K(\Pi_K^0(\boldsymbol{u}^j - \boldsymbol{u}^{j-1}) - (\Delta t)\partial_t\boldsymbol{u}(t_j), E_A^j)\Big) \\
&\le C\sum_{j=1}^n\left(h|\boldsymbol{u}^j - \boldsymbol{u}^{j-1}|_{1,\Omega} + \left\|(\boldsymbol{u}^j - \boldsymbol{u}^{j-1}) - (\Delta t)\partial_t\boldsymbol{u}^j\right\|_{0,\Omega}\right)\|E_A^j\|_{0,\Omega} \\
&\le C\sum_{j=1}^n\left(h\left((\Delta t)\int_{t_{j-1}}^{t_j}|\partial_t\boldsymbol{u}(s)|_{1,\Omega}^2\,\mathrm{d}s\right)^{1/2}\right. \\
&\quad \left. + \Delta t\left((\Delta t)\int_{t_{j-1}}^{t_j}\|\partial_{tt}\boldsymbol{u}(s)\|_{0,\Omega}^2\,\mathrm{d}s\right)^{1/2}\right)\left((\Delta t)\ \|E_A^j\|_{0,\Omega}^2\right)^{1/2} \\
&\le C(\Delta t)^{1/2}\left(h\|\partial_t\boldsymbol{u}\|_{L^2(0,t_n;H^1(\Omega))} + (\Delta t)\|\partial_{tt}\boldsymbol{u}\|_{L^2(0,t_n;L^2(\Omega))}\right)\|E_A\|_{l^2(0,t_n;L^2(\Omega))}.
\end{aligned}$$

The bounds for the other terms, i.e., $G_i, i = 2,3,4$ can be easily obtain as we have estimated the terms $T_i, 1 \le 3$ in the proof of Theorem 4.1. Now collecting all the bounds of G_i in (4.23), we conclude that

$$\sum_{i=2}^4 G_i \lesssim (h + \Delta t)\|E_A\|_{l^2(0,t_n;H^1(\Omega))}$$

$$+\left((\Delta t)\sum_{j=1}^{n}\Big(h(|\boldsymbol{u}^j|_{2,\Omega}+h|p^j|_{1,\Omega})+\|\nabla \boldsymbol{u}(t_j)\|_{0,\Omega}\Big)\right.$$
$$\left.\times\,\|E_A^j\|_{0,\Omega}^{\frac{1}{2}}\|\nabla E_A^j\|_{0,\Omega}^{\frac{1}{2}}\right)\|E_A\|_{l^2(0,t_n;\mathbf{H}^1(\Omega))}.$$

Choosing $\boldsymbol{u}_h^0 := \boldsymbol{u}_I^0$ and employing the Young's inequality, we finally arrive at

$$\frac{1}{2}\|E_A^n\|_{0,\Omega}^2+\frac{\nu}{2}\;\|E_A\|_{l^2(0,t_n;H^1(\Omega))}^2$$
$$\lesssim h^2+\Delta t^2+\Delta t\sum_{j=1}^{n}\Big(h(|\boldsymbol{u}^j|_{2,\Omega}+h|p^j|_{1,\Omega})+\|\nabla \boldsymbol{u}^j\|_{0,\Omega}\Big)^4\|E_A^j\|_{0,\Omega}^2.$$

Now an application of the triangle's inequality and discrete Gronwall's lemma [28] concludes (4.21).

Proceeding analogously to the semi-discrete case, the estimates for pressure can be easily obtain by writing the error equations in terms of $E_Q^n := S_h^p p^n - p_h^n$ and employing the inf-sup condition together with the properties of discrete forms $a_h(\cdot,\cdot)$, $b_h(\cdot,\cdot)$ and $c_h(\cdot;\cdot,\cdot)$ (also refer to [1, 24, 31]). □

5. Numerical tests

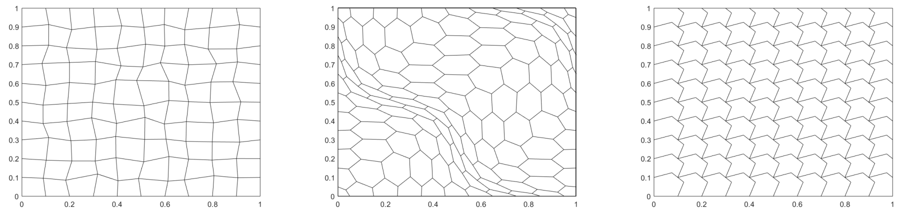
In this section, we illustrate the numerical verification of the theoretical rate of convergence of the proposed method. In order to see the computational efficiency of the virtual element methods used for space discretizations, we have considered here three different meshes: distorted square, distorted hexagonal and non-convex mesh (see Fig. 1). After employing the backward Euler method (for time discretization) and the proposed virtual element methods, the resultant non-linear system of equations is solved by Newton's method. We compute the error for velocity and pressure in the following norms.

$$E_1(\boldsymbol{u}) := \Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}(\boldsymbol{u}-\Pi_K^{\nabla}\boldsymbol{u}_h)\|_{0,K}^2\Big)^{\frac{1}{2}},\; E_0(\boldsymbol{u}) := \Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{u}-\Pi_K^{0}\boldsymbol{u}_h\|_{0,K}^2\Big)^{\frac{1}{2}}$$
$$\text{and}\qquad E_0(p) := \Big(\sum_{K\in\mathcal{T}_h}\|p-p_h\|_{0,K}^2\Big)^{\frac{1}{2}}.$$

For assessing the experimental convergence of the proposed scheme applied to (2.1) defined over square domain $\Omega=[0,1]\times[0,1]$, we consider the exact velocity of the fluid flow and pressure as follows.

$$p = t\Big(x^3y^3-\frac{1}{16}\Big),$$
$$\boldsymbol{u} = t^2\begin{bmatrix} x^2(1-x)^4y^2(1-y)(3-5y)\\ -2x(1-x)^3(1-3x)y^3(1-y)^2\end{bmatrix}.$$

Then the load function $\boldsymbol{f}$ is enforced from the equation (2.1). Moreover, we have taken viscosity $\nu=1$, time step $\Delta t=0.01$ and final time $T=1$. The Table 1



(a) (b) (c)

Figure 1. Samples of (a) Distorted square, (b) Distorted hexagonal, and (c) Non-convex meshes employed for the numerical tests in this section.

Table 1. Errors and convergence rates r for fluid velocity and pressure.

Ndof	h^{-1}	$E_1(\boldsymbol{u})$	r	$E_0(\boldsymbol{u})$	r	$E_0(p)$	r
222	4	0.0108080	-	0.0024850	-	0.0295061	-
842	8	0.0060613	0.83	0.0010103	1.30	0.0174711	0.76
3282	16	0.0031157	0.96	0.0003372	1.58	0.0092992	0.91
12962	32	0.0015168	1.04	0.0000972	1.79	0.0047183	0.98
51522	64	0.0007375	1.04	0.0000260	1.90	0.0023658	1.00

displays the computed order of convergence (r) for velocity and pressure in the estimated errors $E_1(\boldsymbol{u})$, $E_0(\boldsymbol{u})$ and $E_0(p)$.

The computed order of convergence for all three meshes are reported in Fig. 2. From Table 1 and Fig. 2, we observe that the computed rate of convergence and theoretical rate of convergence are in good agreement irrespective of the mesh type.

6. Conclusions

In this article, we have extended the analysis of [47] that deals with lowest order virtual element approximations for Stokes problems. Establishment of optimal *a priori* error estimates and the well-posedness for both semi and fully discrete schemes can be considered as novelty and major contributions of this work. We have also verified the theoretical rate of convergence with the help of numerical tests. The proposed analysis can also be extended to more realistic problems with discontinuous viscosity of fluid [13, 35, 38] such as interface Stokes and Navier-Stokes equations, and we will study these problems in near future. Even more broad sections, including the coupled fluid flow problems such as coupled Stokes-Darcy problem, and coupled poroelastic-Advection-Diffusion-Reaction problems [45, 46], can be studied using the lowest virtual element approximation, which is proposed here. Future directions also includes the development of stabilized virtual element methods by following [14, 29, 30, 39, 49] for the non-stationary Navier-Stokes problem for general order k.

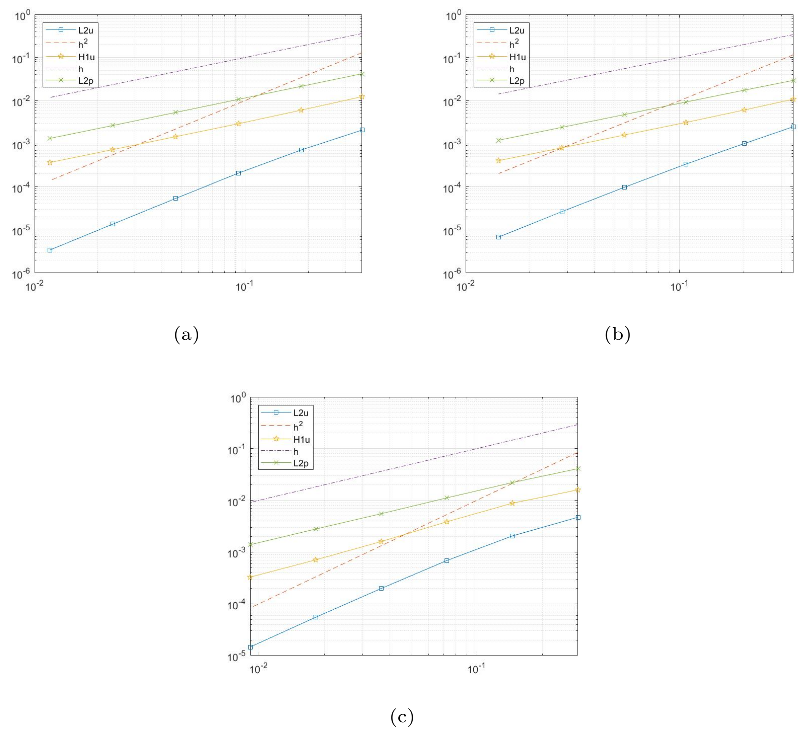


Figure 2. Convergence in space for three different meshes: (a) Distorted square, (b) Distorted Hexagonal, and (c) Non-convex mesh.

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