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BLOW-UP OF SOLUTIONS TO THE SEMILINEAR WAVE EQUATIONS WITH FRICTIONAL AND VISCOELASTIC DAMPING TERMS*

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Abstract Our starting point in this paper is to investigate the weakly coupled system of semilinear wave equations with two types of damping terms and combined nonlinearities $|v_t|^{p_1} + |v|^{q_1}$, $|u_t|^{p_2} + |u|^{q_2}$ on exterior domain in $n\,(n \geq 1)$ dimensions. Local existence and uniqueness of mild solutions to the problem are established. Moreover, non-existence of global solutions to the problem with power nonlinearities $|v|^p$, $|u|^q$ in the case of coupled system and the problem with power nonlinearity $|u|^p$ in the case of single equation are verified. The proofs are based on the test function technique. It is worth noticing that the frictional damping u_t is more dominant than the viscoelastic damping Δu_t when the time trend to infinity. To the best of our knowledge, the blow-up results in Theorems 1.2-1.3 are new.

Keywords Semilinear wave equations, local existence and uniqueness, exterior domain, test function technique, blow-up.

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1. Introduction

Our main aim is to study the weakly coupled system of semilinear wave equations with two types of damping terms

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = f_1(v,\, v_t), & x \in \Omega^c,\, t > 0, \\ v_{tt} - \Delta v + v_t - \Delta v_t = f_2(u,\, u_t), & x \in \Omega^c,\, t > 0, \\ (u,\, u_t,\, v,\, v_t)(x,0) = (u_0,\, u_1,\, v_0,\, v_1)(x), & x \in \Omega^c, \\ u|_{\partial\Omega^c} = 0, \quad v|_{\partial\Omega^c} = 0, & t > 0 \end{cases} \tag{1.1}$$

and single equation

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = |u|^p, & x \in \Omega^c,\, t > 0, \\ (u,\, u_t)(x,0) = (u_0,\, u_1)(x), & x \in \Omega^c, \\ u|_{\partial\Omega^c} = 0, & t > 0, \end{cases} \tag{1.2}$$

where u_t and v_t, Δu_t and Δv_t are frictional and viscoelastic damping terms. The nonlinear terms $f_1(v,\, v_t)$, $f_2(u,\, u_t)$ are presented in the forms of power type nonlinearities $f_1(v,\, v_t) = |v|^p$, $f_2(u,\, u_t) = |u|^q$, combined nonlinearities $f_1(v,\, v_t) = |v_t|^{p_1} + |v|^{q_1}$, $f_2(u,\, u_t) = |u_t|^{p_2} + |u|^{q_2}$, where the indexes satisfy $1 < p,\, p_1,\, p_2,\, q,\, q_1,\, q_2 < \infty$. Let $\Omega = B_1(0) = \{x \in \mathbb{R}^n \big| \, |x| \leq 1\}$ and $\Omega^c = \mathbb{R}^n \backslash B_1(0)$. We assume $B_R(0) = \{x \in \mathbb{R}^n \big| \, |x| \leq R\}$, where $R > 2$. The initial values $u_0,\, u_1,\, v_0,\, v_1$ are non-negative functions which satisfy $\operatorname{supp}(u_0,\, u_1,\, v_0,\, v_1) \subset \Omega^c \cap B_R(0)$.

Let us recall some contributions related to the Cauchy problem for semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u = f(u,\, u_t), \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = \varepsilon f(x), \;\; u_t(x,0) = \varepsilon g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

Problem (1.3) with $f(u,\, u_t) = |u|^p$ asserts the Strauss critical exponent $p_S(n)$, which is the positive root of quadratic equation

$$r(n,p) = -(n-1)p^2 + (n+1)p + 2 = 0.$$

The critical exponent $p_S(n)$ is the threshold between blow-up of solution even for small initial values and global existence of weak solution. We refer the interested readers to [11,26,29,42,45,46] and references therein. Problem (1.3) with $f(u,\, u_t) = |u_t|^p$ admits the Glassey critical exponent $p_G(n) = \frac{n+1}{n-1}$. The solution blows up in finite time when $p \leq p_G(n)$ and exists globally (in time) when $p > p_G(n)$ for small initial values (see [14, 44]). Problem (1.3) with combined nonlinearities $f(u,\, u_t) = |u_t|^p + |u|^q$ is discussed in [13, 15]. Upper bound lifespan estimate of solution is documented by exploiting the Kato lemma and test function approach. Ikeda et al. [17] establish blow-up results and lifespan estimates of solutions to semilinear wave equation and related weakly coupled system, where a framework of test function method is applied.

The Cauchy problem of semilinear wave equation with structural damping

$$\begin{cases} u_{tt} - \Delta u + c(x,t)(-\Delta)^\theta u_t = f(u,\, u_t), & x \in \mathbb{R}^n,\, t > 0, \\ (u,\, u_t)(x,0) = (u_0,\, u_1)(x), & x \in \mathbb{R}^n \end{cases} \tag{1.4}$$

has been widely studied by many mathematicians (see detailed illustrations in previous works [2,6,8–10,18,23,27,28,31–34,36–38,43]), where $c(x,t)(-\Delta)^{\theta}u_t$ ($\theta \in [0,1]$) is the structural damping term. Fino [10] investigates blow-up of solution to problem (1.4) with $c(x,t)=1$, $\theta=1$ and $f(u,\, u_t)=|u|^p$ in the sub-critical and critical cases on exterior domain by utilizing the test function method. Lifespan estimate of solution to the Cauchy problem of semilinear wave equation with scale invariant damping ($c(x,t)=\frac{2}{1+t}$, $\theta=0$) in two dimensions is established in [23]. Non-existence of global solution to problem (1.4) with $c(x,t)=1$ and $f(u,\, u_t)=|u|^p,\ |u_t|^p$ are deduced in [8,9], where the test function method is performed. Problem (1.4) with $c(x,t)=1$, $\theta=1$ and $f(u,\, u_t)=|u_t|^p,\ |u_t|^p+|u|^q$ on exterior domain is discussed (see [2]). Local existence of mild solution is investigated by exploiting the Banach fixed point theorem. Blow-up result of solution is obtained by applying the test function method. Taking advantage of the rescaled test function approach and iteration method, Ming et al. [34] illustrate lifespan estimate of solution to variable coefficient semilinear wave equation with scattering damping term ($c(x,t)=\frac{\mu}{(1+t)^{\beta}}$, $\theta=0$) and divergence form nonlinearities in the sub-critical and critical cases.

Let us turn to the Cauchy problem of semilinear wave equation with double damping terms

$$\begin{cases} u_{tt}-\Delta u+u_t+(-\Delta)^{\theta}u_t=f(u), & x\in\mathbb{R}^n,\ t>0,\\ (u,\, u_t)(x,0)=(u_0,\, u_1)(x), & x\in\mathbb{R}^n. \end{cases} \tag{1.5}$$

Ikehata et al. [22] prove non-existence of global solution to problem (1.5) with $\theta=1$ and $f(u)=|u|^p$ in the sub-critical and critical cases, where the initial values satisfy $\int_{\mathbb{R}^n}u_i dx>0$ for $i=0,\,1$. Blow-up of solution to problem (1.5) with $\theta=1$ and $f(u)=|u|^p$ on exterior domain in $n\,(n\geq 2)$ dimensions is documented by employing the test function technique (see [5]). Ikehata et al. [21] consider problem (1.5) with $\theta=1$, $f(u)=0$. The asymptotic behavior of solution satisfies $u(x,t)\sim (P_0+P_1)G(x,t)$ as $t\to\infty$ in the $L^2(\mathbb{R}^n)$ sense, where $P_j=\int_{\mathbb{R}^n}u_j(x)dx\,(j=0,\,1)$ and $G(x,t)=\frac{1}{(\sqrt{4\pi t})^n}e^{-\frac{|x|^2}{4t}}$ is the Gauss kernel. Optimal decay rates of the total energy for problem (1.5) with $\theta>1$ and $f(u)=0$ is discussed in [20]. D'Abbicco et al. [4] investigate linear estimates of global solutions to problem (1.5) with $\theta=1$ and three types of nonlinearities $|u|^p$, $|u_t|^p$ and $|\nabla u|^p$, where small initial values in the energy space possess $L^1(\mathbb{R}^n)$ regularity. Large time behavior of solution to problem (1.5) with $\theta=1$ and nonlinear term $V(x)|u|^p+W(x,t)\,(V(x)>0,\, W(x,t)\geq 0)$ is analyzed (see [24]). We refer readers to the works in [19,25,30,39] for more details and references therein.

Recently, the weakly coupled system of semilinear wave equation

$$\begin{cases} u_{tt}-\Delta u+b_1(t)(-\Delta)^{\theta_1}u_t=f_1(v,\, v_t), & x\in\mathbb{R}^n,\ t>0,\\ v_{tt}-\Delta v+b_2(t)(-\Delta)^{\theta_2}v_t=f_2(u,\, u_t), & x\in\mathbb{R}^n,\ t>0,\\ (u,\, u_t,\, v,\, v_t)(x,0)=\varepsilon(u_0,\, u_1,\, v_0,\, v_1)(x), & x\in\mathbb{R}^n \end{cases} \tag{1.6}$$

attracts extensive attention (see [1,3,7,12,16,35,40,41]). Chen et al. [1] establish upper bound lifespan estimates of solutions to problem (1.6) with $b_1(t)=b_2(t)=1$, $\theta_1=\theta_2=0$ and $f_1(v,\, v_t)=|v|^p$, $f_2(u,\, u_t)=|u|^q$ in the critical case by employing the test function method connected with nonlinear differential inequalities. Non-existence of global solutions to problem (1.6) with $b_1(t)=b_2(t)=1$, $\theta_1,\ \theta_2\in[0,\frac{1}{2}]$ and $f_1(v,\, v_t)=|v|^p$, $f_2(u,\, u_t)=|u|^q$ for general initial values are considered in [7], where the test function approach is performed. Blow-up dynamics and lifespan estimates of solutions to problem (1.6) with scattering damping terms ($b_1(t) = b_2(t) = \frac{\mu}{(1+t)^\beta}\,(\beta > 1)$, $\theta_1 = \theta_2 = 0$) and $f_1(v,\,v_t) = |v_t|^{p_1} + |v|^{q_1}$, $f_2(u,\,u_t) = |u_t|^{p_2}+|u|^{q_2}$ are obtained (see [35]). The proof is based on the test function technique and iteration method. Making use of an iteration method for unbounded multipliers with a slicing procedure, Chen et al. [3] verify blow-up of solutions to the weakly coupled system of the Nakao's problem with small initial values.

Enlightened by the works in [1, 2, 5, 7, 10, 22], our interest is to establish local existence and uniqueness of solutions to problem (1.1) with $f_1(v,\,v_t) = |v_t|^{p_1}+|v|^{q_1}$, $f_2(u,\,u_t) = |u_t|^{p_2} + |u|^{q_2}$, blow-up dynamics of solutions to problem (1.1) with $f_1(v,\,v_t) = |v|^p$, $f_2(u,\,u_t) = |u|^q$ and problem (1.2) with power nonlinearity $|u|^p$ on exterior domain in $n\,(n \geq 1)$ dimensions. It is worth to mention that existence of local solutions to semilinear wave equations with strong damping terms Δu_t and different types of nonlinear terms $|u|^p$, $|u_t|^p$, $|u_t|^p+|u|^q$ are investigated by applying the Banach fixed theorem (see [2, 10]). Blow-up results of solutions are derived by making use of the test function technique. We study local existence and uniqueness of solutions to problem (1.1) with combined nonlinearities $f_1(v,\,v_t) = |v_t|^{p_1}+|v|^{q_1}$, $f_2(u,\,u_t) = |u_t|^{p_2} + |u|^{q_2}$ by employing the Banach fixed theorem (see Theorem 1.1). Existence and non-existence of global solutions to the weakly coupled system of semilinear wave equations with structural dampings are obtained by utilizing the modified test function technique (see [7]). We present blow-up of solutions to problem (1.1) with $f_1(v,\,v_t) = |v|^p$, $f_2(u,\,u_t) = |u|^q$ on exterior domain by employing the test function method, which is different from the test functions in [2, 7, 10]. Moreover, Ikehata et al. [22] prove non-existence of global solution to problem (1.2) in the whole space $\mathbb{R}^n\,(n \geq 1)$ by making use of the test function method. Blow-up of solution to problem (1.2) on exterior domain ($n \geq 2$) is derived by exploiting the test function technique (see [5]). Nevertheless, there is no related result on exterior domain for $n = 1$. From this observation, taking advantage of the test function method ($\psi = \phi_0(x)\varphi_T^l(x)\eta_T^k(t)$) which is different from the test functions in [5, 22], we conclude non-existence of global solution to problem (1.2) on exterior domain in $n\,(n \geq 1)$ dimensions. To the best knowledge of authors, the results in Theorems 1.2-1.3 are new.

Definitions of mild solutions, weak solutions and the main results in this paper are presented as follows.

Definition 1.1. Let $((u_0, u_1), (v_0, v_1)) \in \left(H_0^1(\Omega^c) \times H^1(\Omega^c)\right)^2$ and

$$(u,\,v) \in \left(C([0,T), H_0^1(\Omega^c)) \cap C^1([0,T), H^1(\Omega^c))\right)^2$$

satisfy the integral equations

$$u(x,t) = R(t)(u_0,u_1)(x) + \int_0^t S(t-s)f_1(v,v_t)(x,s)ds, \tag{1.7}$$

$$v(x,t) = R(t)(v_0,v_1)(x) + \int_0^t S(t-s)f_2(u,u_t)(x,s)ds, \tag{1.8}$$

where $S(t)g(x) = R(t)(0,g)(x)$ for all $g \in L^2(\Omega^c)$. Then, $(u,\,v)$ are called mild solutions to problem (1.1) with $f_1(v,\,v_t) = |v_t|^{p_1} + |v|^{q_1}$, $f_2(u,\,u_t) = |u_t|^{p_2} + |u|^{q_2}$.

Definition 1.2. Let $(u,v) \in \left(C([0,T], H^1(\Omega^c)) \cap C^1([0,T], L^2(\Omega^c))\right)^2$. It holds that

$$\begin{aligned}&\int_0^T\int_{\Omega^c} f_1(v,v_t)\psi(x,s)dxds+\int_{\Omega^c}(u_0(x)+u_1(x)-\Delta u_0(x))\psi(x,0)dx\\&-\int_{\Omega^c}u_0(x)\psi_t(x,0)dx\\=&\int_0^T\int_{\Omega^c}u(x,s)(\psi_{tt}(x,s)+\Delta\psi_t(x,s)-\Delta\psi(x,s)-\psi_t(x,s))dxds\end{aligned}\tag{1.9}$$

and

$$\begin{aligned}&\int_0^T\int_{\Omega^c} f_2(u,u_t)\psi(x,s)dxds+\int_{\Omega^c}(v_0(x)+v_1(x)-\Delta v_0(x))\psi(x,0)dx\\&-\int_{\Omega^c}v_0(x)\psi_t(x,0)dx\\=&\int_0^T\int_{\Omega^c}v(x,s)(\psi_{tt}(x,s)+\Delta\psi_t(x,s)-\Delta\psi(x,s)-\psi_t(x,s))dxds,\end{aligned}\tag{1.10}$$

where $\psi \in C_0^\infty([0,T)\times\Omega^c)$. Then, $(u,\, v)$ are called global (in time) weak solutions to problem (1.1) with $f_1(v,\, v_t) = |v|^p$, $f_2(u,\, u_t) = |u|^q$.

Definition 1.3. Let $u \in C([0,T], H^1(\Omega^c)) \cap C^1([0,T], L^2(\Omega^c))$ and it satisfies

$$\begin{aligned}&\int_0^T\int_{\Omega^c} |u(x,s)|^p\psi(x,s)dxds+\int_{\Omega^c}(u_0(x)+u_1(x)-\Delta u_0(x))\psi(x,0)dx\\&-\int_{\Omega^c}u_0(x)\psi_t(x,0)dx\\=&\int_0^T\int_{\Omega^c}u(x,s)(\psi_{tt}(x,s)+\Delta\psi_t(x,s)-\Delta\psi(x,s)-\psi_t(x,s))dxds,\end{aligned}\tag{1.11}$$

where $\psi \in C_0^\infty([0,T)\times\Omega^c)$. Then, u is called a weak solution to problem (1.2).

Theorem 1.1. *Let*

$$\begin{cases}1 < p_1,\, q_1,\, p_2,\, q_2 < \infty, & n = 1,\, 2,\\ 1 < p_1,\, q_1,\, p_2,\, q_2 \le \dfrac{n}{n-2}, & n \ge 3.\end{cases}\tag{1.12}$$

The initial values satisfy $u_0,\, v_0 \in H_0^1(\Omega^c)$, $u_1,\, v_1 \in H^1(\Omega^c)$. Then, problem (1.1) with $f_1(v,\, v_t) = |v_t|^{p_1}+|v|^{q_1}$, $f_2(u,\, u_t) = |u_t|^{p_2}+|u|^{q_2}$ admits uniquely mild solutions

$$(u,\, v) \in \left(C([0,T_{max}), H_0^1(\Omega^c)) \cap C^1([0,T_{max}), H^1(\Omega^c))\right)^2,$$

where T_{max} is a positive constant $(0 < T_{max} \le \infty)$. It holds that

$$T_{max} = \infty \text{ or } \|u(t)\|_{H^1} + \|u_t(t)\|_{H^1} \to \infty \text{ as } t \to T_{max} < \infty \tag{1.13}$$

$$\text{or } \|v(t)\|_{H^1} + \|v_t(t)\|_{H^1} \to \infty \text{ as } t \to T_{max} < \infty. \tag{1.14}$$

Theorem 1.2. *Let p, q satisfy*

$$\begin{cases} \dfrac{n}{2} \le \dfrac{\max\{p,\, q\}+1}{pq-1}, & n \ge 3, \\ 1 < \dfrac{\max\{p,\, q\}+1}{pq-1}, \; 1 < p,\, q < 2, & n = 2, \\ \dfrac{2}{\alpha} \le \dfrac{\max\{p,\, q\}+1}{pq-1}, \; 1 < p,\, q \le 1 + \dfrac{\alpha}{2}, & n = 1, \end{cases} \tag{1.15}$$

where α is the positive root of $\alpha^2 - 2 = 0$. Assume that $((u_0,\, u_1),\, (v_0,\, v_1)) \in \big((W^{2,1} \cap W^{2,\infty}) \times (L^1 \cap L^\infty)\big)^2$. It holds that

$$\int_{\Omega^c} (u_0(x) + u_1(x) - \Delta u_0(x))\phi_0(x)dx > C, \tag{1.16}$$

$$\int_{\Omega^c} (v_0(x) + v_1(x) - \Delta v_0(x))\phi_0(x)dx > C, \tag{1.17}$$

where $C > 1$ is a constant. Then, the solutions $(u,\, v)$ to problem (1.1) with $f_1(v,\, v_t) = |v|^p$, $f_2(u,\, u_t) = |u|^q$ blow up in finite time.

Theorem 1.3. *Let p satisfies*

$$\begin{cases} 1 < p \le 1 + \dfrac{2}{n}, & n \ge 3, \\ 1 < p < 2, & n = 2, \\ 1 < p \le 1 + \dfrac{\alpha}{2}, & n = 1, \end{cases} \tag{1.18}$$

where α is the positive root of $\alpha^2 - 2 = 0$. Assume that $(u_0,\, u_1) \in (W^{2,1} \cap W^{2,\infty}) \times (L^1 \cap L^\infty)$. It holds that

$$\int_{\Omega^c} (u_0(x) + u_1(x))\phi_0(x)dx > 0. \tag{1.19}$$

Then, a solution u to problem (1.2) blows up in finite time.

Remark 1.1. It is worth pointing out that we obtain blow-up of solutions to problem (1.1) with $f_1(v,\, v_t) = |v|^p$, $f_2(u,\, u_t) = |u|^q$ on exterior domain in $n\,(n \ge 1)$ dimensions by utilizing the test function method, which is different from the test functions in [2, 7, 10].

Remark 1.2. We recognize that non-existence of global solution to problem (1.2) on exterior domain is obtained in [5]. But there is no related result for the case $n = 1$. We supplement blow-up of solution to problem (1.2) on exterior domain in $n\,(n \ge 1)$ dimensions by making use of the test function method ($\psi = \phi_0(x)\varphi_T^l(x)\eta_T^k(t)$), which is different from the test function in [5].

Remark 1.3. Concerning the Cauchy problem for weakly coupled system of semilinear wave equations (1.1) with $f_1(v,\, v_t) = |v|^p$, $f_2(u,\, u_t) = |u|^q$ in whole space $\mathbb{R}^n$, we suppose that p, q satisfy

$$\frac{n}{2} \le \frac{\max\{p,\, q\}+1}{pq-1}.$$

The initial values $u_i,\, v_i \in C_0^\infty(\mathbb{R}^n)\,(i=0,\,1)$ satisfy

$$\int_{\mathbb{R}^n} (u_0(x)+u_1(x)-\Delta u_0(x))dx > C, \tag{1.20}$$

$$\int_{\mathbb{R}^n} (v_0(x)+v_1(x)-\Delta v_0(x))dx > C, \tag{1.21}$$

where $C>1$ is a constant. We deduce non-existence of global solutions by applying test function function method.

2. Proof of Theorem 1.1

We state several lemmas related to the local existence and uniqueness of mild solutions to problem (1.1) with $f_1(v,\,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,\,u_t)=|u_t|^{p_2}+|u|^{q_2}$.

Lemma 2.1 ([2]). *Let* $((u_0,u_1),(v_0,v_1)) \in \left(H^2(\Omega^c)\times H_0^1(\Omega^c)\right)^2$ *and*

$$f_1(v,\,v_t),\; f_2(u,\,u_t) \in C([0,\infty), H^2(\Omega^c)\cap H_0^1(\Omega^c)) \cap C^1([0,\infty), L^2(\Omega^c)).$$

Then, there exist unique mild solutions (u,v) *to problem* (1.1) *with* $f_1(v,\,v_t) = |v_t|^{p_1}+|v|^{q_1}$, $f_2(u,\,u_t)=|u_t|^{p_2}+|u|^{q_2}$. *Moreover, the solutions* $(u,\,v)$ *satisfy*

$$\|(u_t,\nabla u)(t)\|_{L^2\times L^2} \le C\|(u_1,\nabla u_0)\|_{L^2\times L^2} + C\int_0^t \|f_1(s)\|_{L^2}ds, \tag{2.1}$$

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2} + C\int_0^t \big(\|(u_1,\nabla u_0)\|_{L^2\times L^2} + \int_0^s \|f_1(\tau)\|_{L^2}d\tau\big)ds, \tag{2.2}$$

$$\begin{aligned}\|\nabla u_t(t)\|_{L^2}^2 \le{}& C\|(\nabla u_0,u_1)\|_{L^2\times H^1}^2 + C\int_0^t \|f_1(s)\|_{L^2}^2 ds + C\|\nabla u(t)\|_{L^2}^2\\ &+ C\int_0^t \|f_1(s)\|_{L^2}\|u_s(s)\|_{L^2}ds\end{aligned} \tag{2.3}$$

and

$$\|(v_t,\nabla v)(t)\|_{L^2\times L^2} \le C\|(v_1,\nabla v_0)\|_{L^2\times L^2} + C\int_0^t \|f_2(s)\|_{L^2}ds, \tag{2.4}$$

$$\|v(t)\|_{L^2} \le \|v_0\|_{L^2} + C\int_0^t \big(\|(v_1,\nabla v_0)\|_{L^2\times L^2} + \int_0^s \|f_2(\tau)\|_{L^2}d\tau\big)ds, \tag{2.5}$$

$$\begin{aligned}\|\nabla v_t(t)\|_{L^2}^2 \le{}& C\|(\nabla v_0,v_1)\|_{L^2\times H^1}^2 + C\int_0^t \|f_2(s)\|_{L^2}^2 ds + C\|\nabla v(t)\|_{L^2}^2\\ &+ C\int_0^t \|f_2(s)\|_{L^2}\|v_s(s)\|_{L^2}ds.\end{aligned} \tag{2.6}$$

Proof. Making use of the Gagliardo-Nirenberg inequality and (1.12), we derive

$$\begin{aligned}\|f_1(v,v_t)\|_{L^2} &\le \|v_t\|_{L^{2p_1}}^{p_1} + \|v\|_{L^{2q_1}}^{q_1}\\ &\le C\|\nabla v_t\|_{L^2}^{\sigma_1 p_1}\|v_t\|_{L^2}^{(1-\sigma_1)p_1} + C\|\nabla v\|_{L^2}^{\sigma_2 q_1}\|v\|_{L^2}^{(1-\sigma_2)q_1}\\ &\le C\|v\|_{C^1([0,T],H^1(\Omega^c))}^{p_1} + C\|v\|_{C([0,T],H_0^1(\Omega^c))}^{q_1},\end{aligned} \tag{2.7}$$

$$\|f_2(u,u_t)\|_{L^2} \le \|u_t\|_{L^{2p_2}}^{p_2} + \|u\|_{L^{2q_2}}^{q_2}$$

$$\begin{aligned}&\le C\|\nabla u_t\|_{L^2}^{\sigma_3 p_2}\|u_t\|_{L^2}^{(1-\sigma_3)p_2}+C\|\nabla u\|_{L^2}^{\sigma_4 q_2}\|u\|_{L^2}^{(1-\sigma_4)q_2}\\&\le C\|u\|_{C^1([0,T],H^1(\Omega^c))}^{p_2}+C\|u\|_{C([0,T],H_0^1(\Omega^c))}^{q_2},\end{aligned}\tag{2.8}$$

where $\sigma_1=\frac{n(p_1-1)}{2p_1}\in(0,1]$, $\sigma_2=\frac{n(q_1-1)}{2q_1}\in(0,1]$, $\sigma_3=\frac{n(p_2-1)}{2p_2}\in(0,1]$, $\sigma_4=\frac{n(q_2-1)}{2q_2}\in(0,1]$. Therefore, we obtain $f_1(v,\,v_t),\ f_2(u,\,u_t)\in C([0,T],L^2(\Omega^c))$.

Let $X(T)=C([0,T],H_0^1(\Omega^c))\cap C^1([0,T],H^1(\Omega^c))$, $T>0$ and $R_1>0$. We define the functional space

$$Y_R(T)=\{U_\alpha\in X(T)\big|\ \|U_\alpha\|_{X(T)}\le 2R_1\},$$

where

$$\begin{aligned}\|U_\alpha\|_{X(T)}&=\|u_\alpha\|_{X(T)}+\|v_\alpha\|_{X(T)}\\&=\sup_{t\in[0,T]}(\|\partial_t u_\alpha(t)\|_{H^1}+\|u_\alpha(t)\|_{H^1})\\&\quad+\sup_{t\in[0,T]}(\|\partial_t v_\alpha(t)\|_{H^1}+\|v_\alpha(t)\|_{H^1}).\end{aligned}$$

Employing (2.7)-(2.8) and the Gagliardo-Nirenberg inequality yields

$$\begin{aligned}&v_\alpha\in Y_R(T)\to f_1(v_\alpha,\,\partial_t v_\alpha)=|\partial_t v_\alpha|^{p_1}+|v_\alpha|^{q_1}\in C([0,T],L^2(\Omega^c)),\\&u_\alpha\in Y_R(T)\to f_2(u_\alpha,\,\partial_t u_\alpha)=|\partial_t u_\alpha|^{p_2}+|u_\alpha|^{q_2}\in C([0,T],L^2(\Omega^c)).\end{aligned}$$

Thus, we define a mapping Φ: $Y_R(T)\times Y_R(T)\to X(T)$. Let $U=(u,\,v)=\Phi(U_\alpha)=\Phi(u_\alpha,\,v_\alpha)(x,t)$ be solutions to problem (1.1) with combined nonlinearities $f_1(v_\alpha,\,\partial_t v_\alpha)$, $f_2(u_\alpha,\,\partial_t u_\alpha)$.

Firstly, we verify Φ: $Y_R(T)\times Y_R(T)\to Y_R(T)\times Y_R(T)$. From (2.7)-(2.8), we achieve

$$\begin{aligned}&\|f_1(v_\alpha,\,\partial_t v_\alpha)(s)\|_{L^2}+\|f_2(u_\alpha,\,\partial_t u_\alpha)(s)\|_{L^2}\\&\le C\|v_\alpha\|_{X(T)}^{p_1}+C\|v_\alpha\|_{X(T)}^{q_1}+C\|u_\alpha\|_{X(T)}^{p_2}+C\|u_\alpha\|_{X(T)}^{q_2}\\&\le C(2R_1)^{p_1}+C(2R_1)^{q_1}+C(2R_1)^{p_2}+C(2R_1)^{q_2}.\end{aligned}\tag{2.9}$$

Replacing $f_1(v,\,v_t)$, $f_2(u,\,u_t)$ in (2.1)-(2.6) by $f_1(v_\alpha,\,\partial_t v_\alpha)$, $f_2(u_\alpha,\,\partial_t u_\alpha)$ and applying (2.9) give rise to

$$\begin{aligned}&\|(u_t,\nabla u)(t)\|_{L^2\times L^2}+\|(v_t,\nabla v)(t)\|_{L^2\times L^2}\\&\le C(I_0+J_0)+C(2R_1)^{p_1}T+C(2R_1)^{q_1}T\\&\quad+C(2R_1)^{p_2}T+C(2R_1)^{q_2}T,\end{aligned}\tag{2.10}$$

$$\begin{aligned}&\|u(t)\|_{L^2}+\|v(t)\|_{L^2}\\&\le C(I_0+J_0)+C(2R_1)^{p_1}T+C(2R_1)^{q_1}T\\&\quad+C(2R_1)^{p_2}T+C(2R_1)^{q_2}T,\end{aligned}\tag{2.11}$$

$$\begin{aligned}&\|\nabla u_t(t)\|_{L^2}^2+\|\nabla v_t(t)\|_{L^2}^2\\&\le C(I_0^2+J_0^2)+C(2R_1)^{2p_1}T+C(2R_1)^{2q_1}T+C(2R_1)^{2p_2}T+C(2R_1)^{2q_2}T\\&\quad+C(I_0+(2R_1)^{p_1}T+(2R_1)^{q_1}T)((2R_1)^{p_1}+(2R_1)^{q_1})T\\&\quad+C(J_0+(2R_1)^{p_2}T+(2R_1)^{q_2}T)((2R_1)^{p_2}+(2R_1)^{q_2})T,\end{aligned}\tag{2.12}$$

where $I_0=\|(u_0,\,u_1)\|_{H^1\times H^1}$, $J_0=\|(v_0,\,v_1)\|_{H^1\times H^1}$ and $T\ll 1$. Thus, we choose sufficiently small positive constant T such that $\|U\|_{X(T)}\le 2R_1$ for some positive constant R_1. This indicates that Φ is a mapping from $Y_R(T) \times Y_R(T)$ to $Y_R(T) \times Y_R(T)$.

Secondly, let us prove that Φ is a contraction mapping.

Let U_α, $U_\beta \in Y_R(T)$, $U = \Phi(U_\alpha)$, $U_1 = \Phi(U_\beta)$ and $W = U - U_1$. Therefore, we obtain

$$\begin{aligned}
&\|f_1(v_\alpha, \partial_t v_\alpha) - f_1(v_\beta, \partial_t v_\beta)\|_{L^2} + \|f_2(u_\alpha, \partial_t u_\alpha) - f_2(u_\beta, \partial_t u_\beta)\|_{L^2}\\
\leq\ & \|\partial_t v_\alpha - \partial_t v_\beta\|_{L^{2p_1}} \left\| |\partial_t v_\alpha|^{p_1-1} + |\partial_t v_\beta|^{p_1-1} \right\|_{L^{\frac{2p_1}{p_1-1}}}\\
&+ \|v_\alpha - v_\beta\|_{L^{2q_1}} \left\| |v_\alpha|^{q_1-1} + |v_\beta|^{q_1-1} \right\|_{L^{\frac{2q_1}{q_1-1}}}\\
&+ \|\partial_t u_\alpha - \partial_t u_\beta\|_{L^{2p_2}} \left\| |\partial_t u_\alpha|^{p_2-1} + |\partial_t u_\beta|^{p_2-1} \right\|_{L^{\frac{2p_2}{p_2-1}}}\\
&+ \|u_\alpha - u_\beta\|_{L^{2q_2}} \left\| |u_\alpha|^{q_2-1} + |u_\beta|^{q_2-1} \right\|_{L^{\frac{2q_2}{q_2-1}}}\\
\leq\ & C\|v_\alpha - v_\beta\|_{X(T)} \left(\|v_\alpha\|^{p_1-1}_{X(T)} + \|v_\beta\|^{p_1-1}_{X(T)} + \|v_\alpha\|^{q_1-1}_{X(T)} + \|v_\beta\|^{q_1-1}_{X(T)} \right)\\
&+ C\|u_\alpha - u_\beta\|_{X(T)} \left(\|u_\alpha\|^{p_2-1}_{X(T)} + \|u_\beta\|^{p_2-1}_{X(T)} + \|u_\alpha\|^{q_2-1}_{X(T)} + \|u_\beta\|^{q_2-1}_{X(T)} \right)\\
\leq\ & C(2^{p_1} R_1^{p_1-1} + 2^{q_1} R_1^{q_1-1}) \|v_\alpha - v_\beta\|_{X(T)}\\
&+ C(2^{p_2} R_1^{p_2-1} + 2^{q_2} R_1^{q_2-1}) \|u_\alpha - u_\beta\|_{X(T)},
\end{aligned}$$

where we have applied the Holder inequality, the Sobolev embedding property $H^1(\Omega^c) \hookrightarrow L^{2r}(\Omega^c) (r > 1)$ and the inequality

$$||x|^r - |y|^r| \leq C(r)|x - y|(|x|^{r-1} + |y|^{r-1}) \quad for\ x,\, y \in \mathbb{R},\ r > 1.$$

Similar to the derivation in (2.10)-(2.12), choosing sufficiently small constant T, we acquire

$$\|W\|_{X(T)} \leq \frac{1}{2}(\|v_\alpha - v_\beta\|_{X(T)} + \|u_\alpha - u_\beta\|_{X(T)}), \tag{2.13}$$

which implies that Φ is a contraction mapping. Taking into account the Banach fixed point theorem, we conclude that there exists a unique mild solutions $U \in X(T)$ to problem (1.1) with $f_1(v,\, v_t) = |v_t|^{p_1} + |v|^{q_1}$, $f_2(u,\, u_t) = |u_t|^{p_2} + |u|^{q_2}$.

Moreover, we deduce that there exists a interval $[0, T_{max})$, where

$$T_{max} = \sup\left\{T > 0 \,\middle|\, U \in X(T)\right\} \leq \infty.$$

It follows that $\lim\limits_{t \to T_{max}} (\|v\|_{H_0^1} + \|v_t\|_{H^1} + \|u\|_{H_0^1} + \|u_t\|_{H^1}) = \infty$ when the lifespan T_{max} is finite. There exists a time sequence $\{t_m\}_{m \geq 0}$ tending to T_{max} as $m \to \infty$ such that

$$\sup_{m \in \mathbb{N}} (\|v(t_m)\|_{H_0^1} + \|v_t(t_m)\|_{H^1} + \|u(t_m)\|_{H_0^1} + \|u_t(t_m)\|_{H^1}) \leq M + 1$$

when $\lim\limits_{t \to T_{max}} (\|v(t)\|_{H_0^1} + \|v_t(t)\|_{H^1} + \|u(t)\|_{H_0^1} + \|u_t(t)\|_{H^1}) = M < \infty$. This contradicts to the definition of T_{max}. We finishes the proof of Theorem 1.1. $\square$

3. Proof of Theorem 1.2

In this section, we collect a related lemma which will be applied in the proof of blow-up results for solutions to problem (1.1) with $f_1(v,\, v_t) = |v|^p$, $f_2(u,\, u_t) = |u|^q$.

Lemma 3.1 ([2]). *Let $\phi_0(x) \in C^2(\Omega^c) \cap C(\overline{\Omega^c})$ satisfy*

$$\begin{cases} \Delta\phi_0(x) = 0, & x \in \Omega^c, \\ \phi_0(x) = 0, & x \in \partial\Omega^c, \end{cases}$$

$\phi_0(x) \to 1$ as $|x| \to \infty$ and $0 < \phi_0(x) < 1$ for all $x \in \Omega^c$ when $n \geq 3$. $\phi_0(x) \to \infty$ as $|x| \to \infty$ and $0 < \phi_0(x) \leq C\ln(|x|)$ for all $x \in \Omega^c$ when $n = 2$. $\phi_0(x) \to \infty$ as $|x| \to \infty$ and $C_1 x \leq \phi_0(x) \leq C_2 x$ for all $x > 0$ when $n = 1$, where C_1 and C_2 are positive constants. Moreover, it holds that $|\nabla\phi_0(x)| \leq C|x|^{1-n}$.

3.1. Proof of Theorem 1.2 for $n \geq 3$

Proof. We denote the functions $\eta_R(t) = \eta(\frac{t}{R^2})$ and $\varphi_R(x) = \Phi(\frac{|x|}{R})$ with

$$\eta(t) = \begin{cases} 1, \quad 0 \leq t \leq \dfrac{1}{2}, \\ \text{decreasing}, \quad \dfrac{1}{2} \leq t \leq 1, \\ 0, \quad t \geq 1, \end{cases} \qquad \Phi(r) = \begin{cases} 1, \quad 0 \leq r \leq 1, \\ 0, \quad r \geq 2. \end{cases}$$

It follows that $0 \leq \eta(t) \leq 1$, $|\eta'(t)| \leq C$. For $t \in [\frac{1}{2}, 1]$, we obtain

$$\eta^{-\frac{q'}{q}}(t)(|\eta'(t)|^{q'} + |\eta''(t)|^{q'}) \leq C, \tag{3.1}$$

where $q' = \frac{q}{q-1}$. Direct computation shows $0 \leq \Phi(r) \leq 1$, $|\Phi'(r)| \leq \frac{C}{r}$ and $|\Phi''(r)| \leq \frac{C}{r^2}$, where C is a positive constant.

We introduce the test function $\phi = \phi_0(x)\varphi_R^l(x)\eta_R^k(t)$.

We define

$$I_R = \int_0^T \int_{\Omega_1^c} |v(x,s)|^p \phi(x,s)dxds = \int_0^{R^2} \int_{\Omega_1^c} |v(x,s)|^p \phi(x,s)dxds,$$
$$J_R = \int_0^T \int_{\Omega_1^c} |u(x,s)|^q \phi(x,s)dxds = \int_0^{R^2} \int_{\Omega_1^c} |u(x,s)|^q \phi(x,s)dxds$$

and

$$I_{R,t} = \int_{\frac{R^2}{2}}^{R^2} \int_{\Omega_1^c} |v(x,s)|^p \phi(x,s)dxds, \quad J_{R,t} = \int_{\frac{R^2}{2}}^{R^2} \int_{\Omega_1^c} |u(x,s)|^q \phi(x,s)dxds,$$

where $\Omega_1^c = \{x \in \Omega^c \big| \, |x| \leq 2R\}$.

Replacing ψ in (1.9) by $\phi_0(x)\varphi_R^l(x)\eta_R^k(t)$, we deduce

$$\begin{aligned} &\int_0^{R^2} \int_{\Omega_1^c} |v(x,s)|^p \phi(x,s)dxds + \int_{\Omega_1^c} (u_0(x) + u_1(x) - \Delta u_0(x))\phi_0(x)\varphi_R^l(x)dx \\ &= \int_0^{R^2} \int_{\Omega_1^c} u(x,s)\phi_0(x)\varphi_R^l(x)\partial_t^2(\eta_R^k(s))dxds \\ &\quad + \int_0^{R^2} \int_{\Omega_1^c} u(x,s)\Delta(\phi_0(x)\varphi_R^l(x))\partial_t(\eta_R^k(s))dxds \end{aligned}$$

$$
\begin{aligned}
&-\int_0^{R^2}\int_{\Omega_1^c} u(x,s)\Delta(\phi_0(x)\varphi_R^l(x))\eta_R^k(s)dxds \\
&-\int_0^{R^2}\int_{\Omega_1^c} u(x,s)\phi_0(x)\varphi_R^l(x)\partial_t(\eta_R^k(s))dxds \\
=\ & J_1+J_2-J_3-J_4. \qquad (3.2)
\end{aligned}
$$

Applying the Holder inequality, Lemma 3.1, (3.1) and (3.2) gives rise to

$$|J_1| \lesssim J_{R,t}^{\frac{1}{q}} R^{-4+\frac{n+2}{q'}}, \quad |J_2| \lesssim J_{R,t}^{\frac{1}{q}} R^{-4+\frac{n+2}{q'}}. \tag{3.3}$$

Here, $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$.

In a similar way, we acquire

$$|J_3| \lesssim J_R^{\frac{1}{q}} R^{-2+\frac{n+2}{q'}}, \quad |J_4| \lesssim J_{R,t}^{\frac{1}{q}} R^{-2+\frac{n+2}{q'}}. \tag{3.4}$$

Combining (1.16) and (3.2)-(3.4), we derive

$$
\begin{aligned}
0 &< \int_{\Omega_1^c}(u_0(x)+u_1(x)-\Delta u_0(x))\phi_0(x)\varphi_R^l(x)dx \\
&\lesssim J_{R,t}^{\frac{1}{q}}(R^{-2+\frac{n+2}{q'}}+R^{-4+\frac{n+2}{q'}}) + J_R^{\frac{1}{q}} R^{-2+\frac{n+2}{q'}} - I_R \\
&\lesssim J_R^{\frac{1}{q}} R^{-2+\frac{n+2}{q'}} - I_R. \qquad (3.5)
\end{aligned}
$$

Similarly, we have

$$0 < \int_{\Omega_1^c}(v_0(x)+v_1(x)-\Delta v_0(x))\phi_0(x)\varphi_R^l(x)dx \lesssim I_R^{\frac{1}{p}} R^{-2+\frac{n+2}{p'}} - J_R. \tag{3.6}$$

It is deduced from (3.5) and (3.6) that

$$I_R \lesssim J_R^{\frac{1}{q}} R^{-2+\frac{n+2}{q'}}, \quad J_R \lesssim I_R^{\frac{1}{p}} R^{-2+\frac{n+2}{p'}}. \tag{3.7}$$

As a consequence, we conclude

$$I_R^{\frac{pq-1}{pq}} \lesssim R^{-2+\frac{n+2}{q'}+(-2+\frac{n+2}{p'})\frac{1}{q}} = R^{\gamma_1}, \tag{3.8}$$

$$J_R^{\frac{pq-1}{pq}} \lesssim R^{-2+\frac{n+2}{p'}+(-2+\frac{n+2}{q'})\frac{1}{p}} = R^{\gamma_2}. \tag{3.9}$$

It is worth noticing that $\frac{n}{2} \leq \frac{\max\{p,q\}+1}{pq-1}$ is equivalent to $\gamma_2 \leq 0$ when $p < q$. Thus, we shall divide our analysis into two cases.

In the sub-critical case $\gamma_2 < 0$, letting $R \to \infty$ in (3.9) yields

$$J_R = \int_0^{R^2}\int_{\Omega_1^c} |u(x,s)|^q \phi(x,s)dxds = 0, \tag{3.10}$$

which results in $u \equiv 0$. We arrive at a contradiction to (1.16).

In the critical case $\gamma_2 = 0$, taking into account (3.5) and (3.6), we achieve

$$J_R + C(v_0, v_1) \leq J_R^{\frac{1}{pq}}, \tag{3.11}$$

where $C(v_0,v_1)=\int_{\Omega_1^c}(v_0(x)+v_1(x)-\Delta v_0(x))\phi_0(x)\varphi_R^l(x)dx>0$.

That is

$$J_R\le J_R^{\frac{1}{pq}},\quad C(v_0,v_1)\le J_R^{\frac{1}{pq}}. \tag{3.12}$$

Plugging (3.12) into (3.11) and employing iteration argument for all $j\in\mathbb{N}^*$ give rise to

$$J_R\ge (C(v_0,v_1))^{(pq)^j}. \tag{3.13}$$

Sending $j\to\infty$ in (3.13) yields that $J_R\to\infty$, which contradicts to $J_R\le J_R^{\frac{1}{pq}}$ in (3.12). □

3.2. Proof of Theorem 1.2 for $n=2$

Proof. Similar to the derivation in (3.3)-(3.4), utilizing Lemma 3.1 and (3.2), we obtain

$$|J_1|\lesssim J_{R,t}^{\frac{1}{q}}(R^{-4+\frac{4}{q'}}(\ln R)^{\frac{1}{q'}}),\quad |J_2|\lesssim J_{R,t}^{\frac{1}{q}}(\frac{1}{3}R^{-4+\frac{4}{q'}}+\frac{2}{3}R^{-4+\frac{4}{q'}}(\ln R)^{\frac{1}{q'}}), \tag{3.14}$$

$$|J_3|\lesssim J_{R}^{\frac{1}{q}}(\frac{1}{3}R^{-2+\frac{4}{q'}}+\frac{2}{3}R^{-2+\frac{4}{q'}}(\ln R)^{\frac{1}{q'}}),\quad |J_4|\lesssim J_{R,t}^{\frac{1}{q}}(R^{-2+\frac{4}{q'}}(\ln R)^{\frac{1}{q'}}). \tag{3.15}$$

Taking advantage of (1.16), (3.2) and (3.14)-(3.15) leads to

$$0<\int_{\Omega_1^c}(u_0(x)+u_1(x)-\Delta u_0(x))\phi_0(x)\varphi_R^l(x)dx\lesssim J_R^{\frac{1}{q}}R^{-1+\frac{2}{q'}}-I_R, \tag{3.16}$$

where we have exploited the fact $\ln R\le CR^{q'-2}$ and $q<2$.

In a similar way, we derive

$$0<\int_{\Omega_1^c}(v_0(x)+v_1(x)-\Delta v_0(x))\phi_0(x)\varphi_R^l(x)dx\lesssim I_R^{\frac{1}{p}}R^{-1+\frac{2}{p'}}-J_R. \tag{3.17}$$

From (3.16) and (3.17), we acquire

$$I_R^{\frac{pq-1}{pq}}\lesssim R^{-1+\frac{2}{q'}+(-1+\frac{2}{p'})\frac{1}{q}}=R^{\gamma_1},\quad J_R^{\frac{pq-1}{pq}}\lesssim R^{-1+\frac{2}{p'}+(-1+\frac{2}{q'})\frac{1}{p}}=R^{\gamma_2}. \tag{3.18}$$

Similar to the derivation in the Subsection 3.1, we have blow-up results of solutions when $1<\frac{\max\{p,q\}+1}{pq-1}$ and $p,\,q<2$. □

3.3. Proof of Theorem 1.2 for $n=1$

Proof. Similar to the derivation in (3.3)-(3.4), we achieve

$$|J_1|\lesssim J_{R,t}^{\frac{1}{q}}(R^{-4+\frac{2\alpha+2}{q'}}),\quad |J_2|\lesssim J_{R,t}^{\frac{1}{q}}(\frac{1}{3}R^{-2-\alpha+\frac{\alpha+2}{q'}}+\frac{2}{3}R^{-2-2\alpha+\frac{2\alpha+2}{q'}}) \tag{3.19}$$

and

$$|J_3|\lesssim J_{R,t}^{\frac{1}{q}}(\frac{1}{3}R^{-\alpha+\frac{\alpha+2}{q'}}+\frac{2}{3}R^{-2\alpha+\frac{2\alpha+2}{q'}}),\quad |J_4|\lesssim J_{R,t}^{\frac{1}{q}}(R^{-2+\frac{2\alpha+2}{q'}}). \tag{3.20}$$

If $q < 1+\frac{\alpha}{2} < \frac{\alpha}{2(\alpha-1)}$, we have $-2q'+2\alpha+2 < -\alpha q'+\alpha+2$. Utilizing (1.16), (3.2) and (3.19)-(3.20) gives rise to

$$0 < \int_{\Omega_1^c} (u_0(x)+u_1(x)-\Delta u_0(x))\phi_0(x)\varphi_R^l(x)dx \lesssim J_R^{\frac{1}{q}} R^{-\alpha+\frac{\alpha+2}{q'}} - I_R. \quad (3.21)$$

In an analogous way, we obtain

$$0 < \int_{\Omega_1^c} (v_0(x)+v_1(x)-\Delta v_0(x))\phi_0(x)\varphi_R^l(x)dx \lesssim I_R^{\frac{1}{p}} R^{-\alpha+\frac{\alpha+2}{p'}} - J_R. \quad (3.22)$$

Similar to the derivation in the Subsection 3.1, we conclude that the solutions blow up in finite time when $\frac{2}{\alpha} \le \frac{\max\{p,q\}+1}{pq-1}$ and $p,\, q \le 1+\frac{\alpha}{2}$. This ends the proof of Theorem 1.2. □

4. Proof of Theorem 1.3

4.1. Proof of Theorem 1.3 for $n \ge 3$

Proof. Choosing the test function $\psi = \phi = \phi_0(x)\varphi_T^l(x)\eta_T^k(t)$ with $\eta_T(t) = \eta(\frac{t}{T^2})$, $\varphi_T(x) = \Phi(\frac{|x|}{T})$ in (1.11) leads to

$$\begin{aligned}
&\int_0^{T^2}\int_{\Omega_1^c} |u(x,s)|^p\phi(x,s)dxds + \int_{\Omega_1^c}(u_0(x)+u_1(x)-\Delta u_0(x))\phi_0(x)\varphi_T^l(x)dx\\
&= \int_0^{T^2}\int_{\Omega_1^c} u(x,s)\phi_0(x)\varphi_T^l(x)\partial_t^2(\eta_T^k(s))dxds\\
&\quad + \int_0^{T^2}\int_{\Omega_1^c} u(x,s)\Delta(\phi_0(x)\varphi_T^l(x))\partial_t(\eta_T^k(s))dxds\\
&\quad - \int_0^{T^2}\int_{\Omega_1^c} u(x,s)\Delta(\phi_0(x)\varphi_T^l(x))\eta_T^k(s)dxds\\
&\quad - \int_0^{T^2}\int_{\Omega_1^c} u(x,s)\phi_0(x)\varphi_T^l(x)\partial_t(\eta_T^k(s))dxds\\
&= I_1 + I_2 - I_3 - I_4,
\end{aligned} \quad (4.1)$$

where $\Omega_1^c = \{x \in \Omega^c \big|\, |x| \le 2T\}$.

It follows from Lemma 3.1 and (4.1) that

$$\begin{aligned}
I_1 &\le \frac{1}{8}\int_0^{T^2}\int_{\Omega_1^c} |u|^p\phi dxds\\
&\quad + C\int_0^{T^2}\int_{\Omega_1^c} \phi_0(x)\varphi_T^l(x)\eta_T(s)^{(k-2)p'}|\partial_t\eta_T(s)|^{2p'}dxds\\
&\quad + C\int_0^{T^2}\int_{\Omega_1^c} \phi_0(x)\varphi_T^l(x)\eta_T(s)^{(k-1)p'}|\partial_t^2\eta_T(s)|^{p'}dxds\\
&\le \frac{1}{8}\int_0^{T^2}\int_{\Omega_1^c} |u|^p\phi dxds + CT^{-4p'+n+2}.
\end{aligned} \quad (4.2)$$

Making use of the fact $|\nabla\phi_0(x)| \leq \frac{C}{|x|^{n-1}} \leq CT^{-1}$ and (4.1) gives rise to

$$\begin{aligned}
I_2 \leq& \frac{1}{8}\int_0^{T^2}\int_{\Omega_1^{c*}} |u|^p\phi dxds \\
&+C\int_0^{T^2}\int_{\Omega_1^{c*}} \varphi_T^{l-p'}(x)\eta_T^{k-p'}(s)|\nabla\phi_0(x)|^{p'}|\nabla\varphi_T(x)|^{p'}|\partial_t\eta_T(s)|^{p'}dxds \\
&+C\int_0^{T^2}\int_{\Omega_1^{c*}} \varphi_T^{l-2p'}(x)\eta_T^{k-p'}(s)\phi_0(x)|\nabla\varphi_T(x)|^{2p'}|\partial_t\eta_T(s)|^{p'}dxds \\
&+C\int_0^{T^2}\int_{\Omega_1^{c*}} \varphi_T^{l-p'}(x)\eta_T^{k-p'}(s)\phi_0(x)|\Delta\varphi_T(x)|^{p'}|\partial_t\eta_T(s)|^{p'}dxds \\
\leq& \frac{1}{8}\int_0^{T^2}\int_{\Omega_1^{c*}} |u|^p\phi dxds + CT^{-4p'+n+2}, \qquad (4.3)
\end{aligned}$$

where $\Omega_1^{c*} = \{x \in \Omega^c \big| T \leq |x| \leq 2T\}$.

In a similar way, we obtain

$$I_3 \leq \frac{1}{8}\int_0^{T^2}\int_{\Omega_1^{c*}} |u|^p\phi dxds + CT^{-2p'+n+2} \qquad (4.4)$$

and

$$I_4 \leq \frac{1}{8}\int_0^{T^2}\int_{\Omega_1^{c}} |u|^p\phi dxds + CT^{-2p'+n+2}. \qquad (4.5)$$

According to (4.1)-(4.5), we acquire

$$\begin{aligned}
&\int_{\Omega_1^c}(u_0(x)+u_1(x)-\Delta u_0(x))\phi_0(x)\varphi_T^l(x)dx \\
&\leq C(T^{-4p'+n+2}+T^{-2p'+n+2}) \\
&\leq CT^{-2p'+n+2}. \qquad (4.6)
\end{aligned}$$

It is worth to mention that $p \in (1, 1+\frac{2}{n}]$ is equivalent to $-2p'+n+2 \leq 0$. Thus, we divide our consideration into the following two cases.

In the sub-critical case $-2p'+n+2 < 0$, letting $T \to \infty$ in (4.6), we arrive at a contradiction to (1.19).

In the critical case $-2p'+n+2 = 0$, from (4.1)-(4.5) and $p = 1+\frac{2}{n}$, we deduce that there exists a positive constant C such that $\int_0^{T^2}\int_{\Omega_1^c}|u|^p\phi dxds \leq C$, which results in

$$\int_{\frac{T^2}{2}}^{T^2}\int_{\Omega_1^c}|u|^p\phi dxds,\ \int_0^{T^2}\int_{\Omega_1^{c*}}|u|^p\phi dxds,\ \int_{\frac{T^2}{2}}^{T^2}\int_{\Omega_1^{c*}}|u|^p\phi dxds \to 0 \qquad (4.7)$$

as $T \to \infty$.

Utilizing the Holder inequality in $I_1 - I_4$ leads to

$$I_1 \leq CT^{-4+\frac{n+2}{p'}}(\int_{\frac{T^2}{2}}^{T^2}\int_{\Omega_1^c}|u|^p\phi dxds)^{\frac{1}{p}} \leq C(\int_{\frac{T^2}{2}}^{T^2}\int_{\Omega_1^c}|u|^p\phi dxds)^{\frac{1}{p}}. \qquad (4.8)$$

Similarly, we get

$$I_2 \leq C\Big(\int_{\frac{T^2}{2}}^{T^2} \int_{\Omega_1^{c*}} |u|^p \phi dxds\Big)^{\frac{1}{p}}, \quad I_3 \leq C\Big(\int_0^{T^2} \int_{\Omega_1^{c*}} |u|^p \phi dxds\Big)^{\frac{1}{p}} \tag{4.9}$$

and

$$I_4 \leq C\Big(\int_{\frac{T^2}{2}}^{T^2} \int_{\Omega_1^{c}} |u|^p \phi dxds\Big)^{\frac{1}{p}}. \tag{4.10}$$

Taking into account (4.1) and (4.8)-(4.10), we derive

$$\begin{aligned} &\int_{\Omega_1^c} (u_0(x) + u_1(x) - \Delta u_0(x))\phi_0(x)\varphi_T^l(x)dx \\ &\leq C\Big(\int_{\frac{T^2}{2}}^{T^2} \int_{\Omega_1^{c}} |u|^p \phi dxds\Big)^{\frac{1}{p}} + C\Big(\int_{\frac{T^2}{2}}^{T^2} \int_{\Omega_1^{c*}} |u|^p \phi dxds\Big)^{\frac{1}{p}} \\ &\quad + C\Big(\int_0^{T^2} \int_{\Omega_1^{c*}} |u|^p \phi dxds\Big)^{\frac{1}{p}}. \end{aligned} \tag{4.11}$$

Therefore, letting $T \to \infty$ in (4.11) and utilizing (4.7) yield a contradiction to (1.19). □

4.2. Proof of Theorem 1.3 for $n = 2$

Proof. An application of Lemma 3.1 and (4.1) shows

$$I_1 \leq \frac{1}{8}\int_0^{T^2} \int_{\Omega_1^c} |u|^p \phi dxds + CT^{-4p'+4}\ln T, \tag{4.12}$$

$$I_2 \leq \frac{1}{8}\int_0^{T^2} \int_{\Omega_1^{c*}} |u|^p \phi dxds + CT^{-4p'+4}\ln T + CT^{-4p'+4}, \tag{4.13}$$

$$I_3 \leq \frac{1}{8}\int_0^{T^2} \int_{\Omega_1^{c*}} |u|^p \phi dxds + CT^{-2p'+4}\ln T + CT^{-2p'+4} \tag{4.14}$$

and

$$I_4 \leq \frac{1}{8}\int_0^{T^2} \int_{\Omega_1^{c}} |u|^p \phi dxds + CT^{-2p'+4}\ln T. \tag{4.15}$$

Exploiting (4.1) and (4.12)-(4.15) gives rise to

$$\int_{\Omega_1^c} (u_0(x) + u_1(x) - \Delta u_0(x))\phi_0(x)\varphi_T^l(x)dx \leq CT^{-p'+2}, \tag{4.16}$$

where we have employed the fact $\ln T \leq CT^{p'-2}$ and $p < 2$. Letting $T \to \infty$ in (4.16) and applying $p < 2$, we conclude the desired contradiction to (1.19). □

4.3. Proof of Theorem 1.3 for $n = 1$

Proof. Making use of Lemma 3.1 and (4.1), we deduce

$$I_1 \leq \frac{1}{8}\int_0^{T^2}\int_{\Omega_1^c} |u|^p \phi dxds + CT^{-4p'+2\alpha+2}, \tag{4.17}$$

$$I_2 \leq \frac{1}{8}\int_0^{T^2}\int_{\Omega_1^{c*}} |u|^p \phi dxds + CT^{-\alpha p'-2p'+\alpha+2}, \tag{4.18}$$

$$I_3 \leq \frac{1}{8}\int_0^{T^2}\int_{\Omega_1^{c*}} |u|^p \phi dxds + CT^{-\alpha p'+\alpha+2} \tag{4.19}$$

and

$$I_4 \leq \frac{1}{8}\int_0^{T^2}\int_{\Omega_1^c} |u|^p \phi dxds + CT^{-2p'+2\alpha+2}, \tag{4.20}$$

where we have used the change of variables $y = T^{-\alpha}x$ and $t = T^{-2}s$. It is worth noticing that $p < 1 + \frac{\alpha}{2} < \frac{\alpha}{2(\alpha-1)}$ is equivalent to $-2p' + 2\alpha + 2 < -\alpha p' + \alpha + 2$.

From (4.1) and (4.17)-(4.20), we deduce

$$\int_{\Omega_1^c} (u_0(x) + u_1(x) - \Delta u_0(x))\phi_0(x)\varphi_T^l(x)dx \leq CT^{-\alpha p'+\alpha+2}, \tag{4.21}$$

which yields a contradiction to (1.19) by sending $T \to \infty$.

For the critical case $p = 1 + \frac{\alpha}{2}$, we conclude the contradiction by utilizing similar computation as in the Subsection 4.1, where we have employed the supports of $\nabla\varphi_T$, $\Delta\varphi_T$ and $\partial_t\eta_T$. This finishes the proof of Theorem 1.3. □

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