Journal of Applied Analysis and Computation
Volume 13, Number 3, June 2023, 1209–1224
Website:http://www.jaac-online.com
DOI:10.11948/20210497

DYNAMICAL BEHAVIORS OF A STOCHASTIC PREDATOR-PREY MODEL WITH ANTI-PREDATOR BEHAVIOR

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Abstract In this paper, a stochastic predator-prey model is proposed and studied, where the model has anti-predator behavior. By constructing a suitable Lyapunov function, combined with the Itô's formula and the stochastic comparison theorem, the existence and uniqueness of the global positive solution of the system are proved. Then the stochastic boundedness of the system is established, and we discussed the asymptotic behavior of the solution which fluctuates around the equilibrium point of the deterministic model. Moreover, we provide sufficient conditions for the persistence and extinction of the predator and prey. Finally, the results obtained in this paper are verified by numerical simulation, and the anti-predator behavior and stochastic perturbation are analyzed as well.

Keywords Stochastic predator-prey model, anti-predator behavior, boundedness, persistence, extinction.

MSC(2010) 34D20, 34F05, 92B05.

1. Introduction

The dynamic relationship between predator and prey has been one of the main themes of ecology and mathematical ecology for a long time. In 1910, Lotka and Volterra [13,20] first proposed and established the Lotka-Volterra model. After that, many researchers have conducted intensive research on the predator-prey system [8–12,14,15,22]. Although biologists often divide animals into predator and prey, some prey often cause harm to predators. In fact, the role exchange between predator and prey often occurs, that is, some prey have anti-predator behavior. In order to simulate the anti-predator behavior, the predator-prey models [7, 17–19, 21] with the influence of anti-predator behavior have been considered and studied. Ives and Dobson [19] proposed a predator-prey model with anti-predator behavior:

$$\begin{cases} \mathrm{d}x = [rx(1-\frac{x}{k}) - \dfrac{\beta xy}{a+x^2}]\mathrm{d}t, \\ \mathrm{d}y = [\dfrac{\mu\beta xy}{a+x^2} - by - \eta xy]\mathrm{d}t, \end{cases} \tag{1.1}$$

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where $x(t)$ and $y(t)$ are the densities of the prey and the predator at time t, all the parameters are positive constant, r is the inherent growth rate of the prey, k is the carrying capacity of the environment, β is the capture rate of the predator, μ is the conversion rate of the prey to the predator, b is the natural mortality rate of the predator population, and η is the ratio of prey to the anti-predation behavior of the predator population.

The authors [19] obtained these conclusions of system (1.1) as follows:

- System (1.1) always has a trivial equilibrium $E_0 = (0, 0)$ and a boundary equilibrium $E_k = (k, 0)$.
- If $\eta a - \mu\beta < 0, f(x_c) < 0$ and $x_1 < k \leq x_2$, then system (1.1) has only one positive equilibrium E_1.
- If $\eta a - \mu\beta < 0, f(x_c) < 0$ and $x_2 < k$, then there are two positive equilibria E_1 and E_2 of system (1.1).
- If $\eta a - \mu\beta < 0, f(x_c) = 0$, and $x_c < k$, then the positive equilibria E_1 and E_2 coincide into one positive equilibrium, which is denoted by E_c, with $E_c = (x_c, \frac{1}{\beta} r(1 - \frac{x_c}{k})(a + x_c^2))$.

Furthermore, environmental noise is an important part of the ecosystem [3, 4] and always affects the real world. When the environment fluctuates, many parameters of the system will show more or less random fluctuations. Therefore, it is interesting to study the impact of environmental noise on the model. Using the same method as Imhof and Walcher [5] to add white noise to system (1.1) to obtain a stochastic predator-prey model, we propose a stochastic predator-prey model with anti-predator behavior.

$$\begin{cases} dx = [rx(1 - \frac{x}{k}) - \dfrac{\beta xy}{a + x^2}]dt + \sigma_1 x dB_1(t), \\ dy = [\dfrac{\mu\beta xy}{a + x^2} - by - \eta xy]dt + \sigma_2 y dB_2(t), \end{cases} \tag{1.2}$$

where $B_i(t)$ are mutually independent standard Brownian motions with $B_i(0) = 0$, and $\sigma_i^2 > 0$ denoting the intensities of the white noise, $i = 1, 2$.

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\{\mathcal{F}_0\}$ contains all $\mathbb{P}$ -null sets). Let $B_i(t)$ be defined on the complete probability space, $i = 1, 2$. Define

$$\mathbb{R}_+^d = \{x = (x_1, ..., x_d) \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\},$$
$$\overline{\mathbb{R}}_+^d = \{x = (x_1, ..., x_d) \in \mathbb{R}^d : x_i \geq 0, 1 \leq i \leq d\}.$$

Next, we consider the d-dimensional stochastic differential equation

$$dX(t) = f(X(t))dt + g(X(t))dB(t), t \geq t_0, \tag{1.3}$$

with the initial value $X(0) = X_0 \in \mathbb{R}^d$, where $B(t)$ denotes a d-dimensional standard Brownian motion defined on the complete probability space. $C^2(\mathbb{R}^d; \overline{\mathbb{R}}_+)$ denotes the family of all real-valued nonnegative functions $V(X)$, and $V(X)$ defined on $\mathbb{R}^d$ such that they are continuously twice differentiable in X and once in t. The definition of the differential operator L of Eq.(1.3) is as follows [16]:

$$L = \sum_{i=1}^{d} f_i (X,t) \frac{\partial}{\partial X_i} + \frac{1}{2}\left[g^T (X,t) g (X,t)\right]_{ij} \frac{\partial^2}{\partial X_i \partial X_j}.$$

If L acts on a function $V \in C^2 \left(\mathbb{R}^d; \overline{\mathbb{R}}_+\right)$, then we have

$$LV (X) = V_X (X) f (X) + \frac{1}{2} trace \left[g^T (X) V_{XX} (X) g (X)\right],$$

where $V_X = \left(\frac{\partial V}{\partial X_1}, \cdots, \frac{\partial V}{\partial X_d}\right), V_{XX} = \left(\frac{\partial^2 V}{\partial X_i \partial X_j}\right)_{d\times d}$. By applying Itô's formula [16], then,

$$\mathrm{d}V (X(t)) = LV (X(t)) \,\mathrm{d}t + V_X (X(t)) g (X(t)) \,\mathrm{d}B(t), \ X(t) \in \mathbb{R}^d.$$

The paper is arranged as follows: In section 2, we prove that system (1.2) has a unique global positive solution for any initial value. In section 3, we show that the positive solutions of system are bounded mean and stochastic final bounded. Section 4 yields some results about the asymptotic behavior of the solution around the boundary equilibrium point $(k, 0)$ of system (1.1). We provide the conditions for the extinction and persistence of predator and prey in section 5. Finally, numerical simulations are provided to illustrate these conclusions in section 6.

2. Global positive solution

In this section, we establish the existence of global positive solution of system (1.2).

Theorem 2.1. *For any given initial value* $(x (0), y (0)) \in \mathbb{R}^2_+$ *, there is a unique solution* $(x(t), y(t)) \in \mathbb{R}^2_+$ *of system (1.2) on* $t \geq 0$ *, and the solution will remain in* $\mathbb{R}^2_+$ *with probability one.*

Proof. It is obviously, the coefficients of system (1.2) satisfy the local Lipschitz condition, so for any initial value $(x (0), y (0)) \in \mathbb{R}^2_+$, system (1.2) has a unique local solution $(x(t), y(t)) \in \mathbb{R}^2_+$ on $t \in [0, \tau_e)$ a.s., where τ_e is the explosion time. Next,we shall prove that this unique local solution is global, i.e. $\tau_e = \infty$ a.s.. Let nonnegative number n_0 be sufficiently large such that $x (0), y (0) \in \left[\frac{1}{n_0}, n_0\right]$. For any integer $n \geq n_0$, we define the stopping time by

$$\tau_n = \inf \left\{ t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{n}, n\right) \text{ or } y(t) \notin \left(\frac{1}{n}, n\right) \right\}.$$

Throughout the paper, we set $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty set). Obviously, τ_n is increasing as $n \to \infty$. Denote $\tau_\infty = \lim_{n\to\infty} \tau_n$, then $\tau_\infty \leq \tau_e$ a.s. Therefore, if $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(x(t), y(t)) \in \mathbb{R}^2_+$ a.s. for all $t \geq 0$. To this end, what we need to do is proving $\tau_\infty = \infty$ a.s. If the statement is not true, then there exist a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P} \{\tau_\infty \leq T\} \geq \varepsilon.$$

Therefore, there exists an integer $n_1 \geq n_0$ such that $\mathbb{P} \{\tau_n \leq T\} \geq \varepsilon$, for any $n \geq n_1$.

Next, define a C^2- function $V:\mathbb{R}_+^2\to\overline{\mathbb{R}}_+$ by

$$V\left(x,y\right)=\frac{1}{\beta}\left(x-1-\ln x\right)+\frac{1}{\mu\beta}\left(y-1-\ln y\right), \tag{2.1}$$

the nonnegativity of this function can be seen from $u-1-\ln u\geq 0,\forall u>0$.

Applying Itô's formula, one can derive:

$$\mathrm{d}V\left(x,y\right)=LV\left(x,y\right)\mathrm{d}t+\sigma_1\left(x-1\right)\mathrm{d}B_1(t)+\sigma_2\left(y-1\right)\mathrm{d}B_2(t), \tag{2.2}$$

where

$$\begin{aligned}LV=&\frac{1}{\beta}\left(1-\frac{1}{x}\right)\left[rx\left(1-\frac{x}{k}\right)-\frac{\beta xy}{a+x^2}\right]\\&+\frac{1}{\mu\beta}\left(1-\frac{1}{y}\right)\left(\frac{\mu\beta xy}{a+x^2}-by-\eta xy\right)+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2\\=&\frac{rx}{\beta}-\frac{rx^2}{\beta k}-\frac{r}{\beta}+\frac{rx}{\beta k}+\frac{y}{a+x^2}-\frac{by}{\mu\beta}-\frac{\eta xy}{\mu\beta}-\frac{x}{a+x^2}+\frac{b}{\mu\beta}+\frac{\eta x}{\mu\beta}+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2\\\leq&-\frac{rx^2}{\beta k}+\frac{\mu r+\mu kr+\eta k}{\mu\beta k}x+\frac{y}{a}+\frac{b}{\mu\beta}+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2\\\leq&\frac{\left(\mu r+\mu kr+\eta k\right)^2}{4r\beta k\mu^2}+\frac{y}{a}+\frac{b}{\mu\beta}+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2.\end{aligned}$$

Since $y<2(y-1-\ln y)+\ln 4$, thus

$$\begin{aligned}LV\leq&\frac{\left(\mu r+\mu kr+\eta k\right)^2}{4r\beta k\mu^2}+\frac{\ln 4}{a}+\frac{b}{\mu\beta}+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2+\frac{2}{a}V\left(x,y\right)\\\leq&c_1+c_2V(x,y),\end{aligned}$$

where $c_1=\frac{(\mu r+\mu kr+\eta k)^2}{4r\beta k\mu^2}+\frac{\ln 4}{a}+\frac{b}{\mu\beta}+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2, c_2=\frac{2}{a}$. Then

$$\mathrm{d}V\left(x,y\right)\leq\left(c_1+c_2V\left(x,y\right)\right)\mathrm{d}t+\sigma_1\left(x-1\right)\mathrm{d}B_1(t)+\sigma_2\left(y-1\right)\mathrm{d}B_2(t), \tag{2.3}$$

integrating both sides of the inequality (2.3) from 0 to $\tau_n\wedge T$, then taking the expectation, we can get

$$\begin{aligned}&\mathbb{E}\left[V\left(x\left(\tau_n\wedge T\right),y\left(\tau_n\wedge T\right)\right)\right]\\\leq&V\left(x\left(0\right),y\left(0\right)\right)+c_1T+c_2\mathbb{E}\int_0^{\tau_n\wedge T}V\left(x(t),y(t)\right)\mathrm{d}t\\=&V\left(x\left(0\right),y\left(0\right)\right)+c_1T+c_2\mathbb{E}\int_0^{T}I_{[0,\tau_n]}(t)V\left(x(t),y(t)\right)\mathrm{d}t\\\leq&V\left(x\left(0\right),y\left(0\right)\right)+c_1T+c_2\mathbb{E}\int_0^{T}V\left(x\left(\tau_n\wedge T\right),y\left(\tau_n\wedge T\right)\right)\mathrm{d}t\\=&V\left(x\left(0\right),y\left(0\right)\right)+c_1T+c_2\int_0^{T}\mathbb{E}[V\left(x\left(\tau_n\wedge T\right),y\left(\tau_n\wedge T\right)\right)]\mathrm{d}t,\end{aligned}$$

where $I_A\left(\cdot\right)$ denotes the indicator function of A. Due to the Gronwall inequality, we deduce

$$\mathbb{E}\left[V\left(x\left(\tau_n\wedge T\right),y\left(\tau_n\wedge T\right)\right)\right]\leq\left(V\left(x\left(0\right),y\left(0\right)\right)+c_1T\right)e^{c_2T}. \tag{2.4}$$

Setting $\Omega_n = \{\tau_n \leq T\}$ for $n > n_1$, then for any $\omega \in \Omega_n$, there is $x(\tau_n, \omega)$ or $y(\tau_n, \omega)$ equals either $\frac{1}{n}$ or n. Therefore, we arrive at:

$$V\left(x\left(\tau_n, \omega\right), y\left(\tau_n, \omega\right)\right) \geq (n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right).$$

Letting $n \to \infty$, it then follows that

$$\infty > \left(V\left(x(0), y(0)\right) + c_1 T\right) e^{c_2 T} \geq \infty ,$$

which leads to the contradiction, thus we must have $\tau_e = \infty$ a.s.. This completes the proof. □

3. Stochastic boundedness

Since $x(t)$, $y(t)$ represent the population densities of prey and predator in system (1.2) at time t, respectively, the solution of the system is required to be positive. In this section, we have shown that the positive solutions of the system are bounded mean and the stochastic final bounded.

Now the following results is cited as a Lemma, considering the system as follows:

$$\begin{cases} \mathrm{d}\psi(t) = \psi(t)[\alpha - \beta\psi(t)]\mathrm{d}t + \sigma\psi(t)\mathrm{d}B(t), \\ \psi(0) = \psi_0, \end{cases} \tag{3.1}$$

where the parameters $\alpha, \beta > 0, B(t)$ are standard Brownian motion, with the following Lemma:

Lemma 3.1 ([6]). *Let $\psi(t)$ be the solution of system (3.1) with any initial value ψ_0, then $\limsup_{t\to\infty} E[\psi(t)] \leq \frac{\alpha}{\beta}$.*

Lemma 3.2 ([1]). *Let $\psi(t)$ be the solution of system (3.1) with any initial value ψ_0. Then for any $p > 1$, we have*

$$\limsup_{t\to\infty} E[\psi^p(t)] \leq \left[\frac{1}{x_0} e^{-\left(\alpha + \frac{p-1}{2}\sigma^2\right)t} + \frac{2\beta}{2\alpha + (p-1)\sigma^2}\left(1 - e^{-\left(\alpha + \frac{p-1}{2}\sigma^2\right)t}\right)\right]^{-p},$$

which implies $\limsup_{t\to\infty} E[\psi^p(t)] \leq \varsigma_p := \left(\frac{2\alpha + (p-1)\sigma^2}{2\beta}\right)^p$, for any $p > 1$.

Lemma 3.3 ([6]). *Let $\alpha > \frac{\sigma^2}{2}$, $\psi(t)$ be the solution of system (3.1) with any initial value ψ_0, then $\lim_{t\to\infty} \frac{\ln \psi(t)}{t} = 0$, a.s. and $\lim_{t\to\infty} \frac{1}{t}\int_0^t \psi(s)ds = \frac{\alpha - \frac{\sigma^2}{2}}{\beta}$, a.s..*

The following theorem is about boundedness of system (1.2).

Theorem 3.1. *Let $(x(t), y(t))$ be the solution of system (1.2) with any initial value $(x(0), y(0)) \in \mathbb{R}_+^2$, then the solution $(x(t), y(t))$ of system (1.2) satisfies*

$$\limsup_{t\to\infty} E[x(t)] \leq k, \ \limsup_{t\to\infty} (E[x(t)] + \frac{1}{\mu} E[y(t)]) \leq \frac{k(r+b)^2}{4rb}. \tag{3.2}$$

Proof. First, from the equation of $x(t)$ in system (1.2) we have

$$\mathrm{d}x(t) \leq x(t)(r - \frac{rx}{k})\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t).$$

Now, considering the following system:

$$\begin{cases} \mathrm{d}\phi(t) = \phi(t)(r - \frac{r\phi}{k})\mathrm{d}t + \sigma_1\phi(t)\mathrm{d}B_1(t), \\ \phi(0) = x(0). \end{cases} \tag{3.3}$$

According to Lemma 3.1, the solution $\phi(t)$ of system (3.3) satisfies $\limsup_{t\to\infty} E[\phi(t)] \leq k$, so that $\limsup_{t\to\infty} E[x(t)] \leq k$ is obtained from the stochastic comparison theorem.

Next, It is given below that $y(t)$ is also bounded mean. Letting

$$G(t) = x(t) + \frac{1}{\mu}y(t),$$

then

$$\begin{aligned} \mathrm{d}G(t) =& [rx(1 - \frac{x}{k}) - \frac{by}{\mu} - \frac{\eta xy}{\mu}]\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t) + \frac{\sigma_2}{\mu}y(t)\mathrm{d}B_2(t) \\ =& [(r+b)x - \frac{r}{k}x^2 - bG(t) - \frac{\eta xy}{\mu}]\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t) + \frac{\sigma_2}{\mu}y(t)\mathrm{d}B_2(t). \end{aligned}$$

Then integrating the equation from 0 to t, one obtains

$$\begin{aligned} G(t) =& G(0) + \int_0^t [(r+b)x(s) - \frac{r}{k}x^2(s) - bG(s) - \frac{\eta x(s)y(s)}{\mu}]\mathrm{d}s \\ &+ \sigma_1 \int_0^t x(s)\mathrm{d}B_1(s) + \sigma_2 \int_0^t \frac{1}{\mu}y(s)\mathrm{d}B_2(s). \end{aligned}$$

So there is

$$E[G(t)] = G(0) + \int_0^t E[(r+b)x(s) - \frac{r}{k}x^2(s) - bG(s) - \frac{\eta x(s)y(s)}{\mu}]\mathrm{d}s,$$

and we have

$$\begin{aligned} \frac{\mathrm{d}E[G(t)]}{\mathrm{d}t} =& (r+b)E[x(t)] - \frac{r}{k}E[x^2(t)] - bE[G(t)] - \frac{\eta}{\mu}E[x(t)y(t)] \\ \leq& (r+b)E[x(t)] - \frac{r}{k}(E[x(t)])^2 - bE[G(t)] \\ \leq& \frac{k(r+b)^2}{4r} - bE[G(t)]. \end{aligned}$$

By the stochastic comparison theorem, we can get $0 \leq \limsup_{t\to\infty} E[G(t)] \leq \frac{k(r+b)^2}{4rb}$, that is

$$\limsup_{t\to\infty} (E[x(t)] + \frac{1}{\mu}E[y(t)]) \leq \frac{k(r+b)^2}{4rb}.$$

Now the Theorem is proved. □

Remark 3.1. Combining the positivity of the solution of system (1.2) and inequality (3.3), it is obvious that

$$\limsup_{t\to\infty} E[y(t)] \le \frac{k\mu(r+b)^2}{4rb}.$$

Theorem 3.2. *For any $p>1$, we have*

$$\limsup_{t\to\infty} E[x^p(t)] \le (\frac{2rk+k(p-1)\sigma_1^2}{2r})^p;$$

for any $0<p\le 1$, we have

$$\limsup_{t\to\infty} E[x^p(t)] \le \frac{k(2r+\sigma_1^2)}{2r}^p.$$

Proof. First, for $x(t)$, we can get

$$\mathrm{d}x(t) \le x(t)(r-\frac{rx}{k})\mathrm{d}(t) + \sigma_1 x(t)\mathrm{d}B_1(t),$$

thus when $p>1$, combining Lemma 3.2 and the stochastic comparison theorem, the following inequality can be derived

$$\limsup_{t\to\infty} E[x^p(t)] \le (\frac{2rk+k(p-1)\sigma_1^2}{2r})^p.$$

Also, when $0<p\le 1$, according to Hölder inequality, we can get

$$\limsup_{t\to\infty} E[x^p(t)] \le \limsup_{t\to\infty} [E[x^2(t)]]^{\frac{p}{2}} \le \frac{k(2r+\sigma_1^2)}{2r}^p.$$

$\square$

Theorem 3.3. *If the condition $0<\sigma_2^2<2b-1$ holds, then for any initial value $(x(0),y(0))\in\mathbb{R}_+^2$, the solution of system (1.2) is stochastically finally bounded.*

Proof. Now defining a C^2 function $V:\mathbb{R}_+^2\to\overline{\mathbb{R}}_+$ by: $V(x,y)=(\mu x+y)^2$.

Applying Itô's formula to $V(x,y)$ implies,

$$\begin{aligned} LV(x,y) =& 2(\mu x+y)[r\mu x(1-\frac{x}{k}) - by - \eta xy] + \mu^2\sigma_1^2x^2 + \sigma_2^2y^2 \\ =& -\frac{2r\mu^2x^3}{k} + \mu^2\sigma_1^2x^2 + 2r\mu^2x^2 - 2b\mu xy - 2\mu\eta x^2y \\ &+ 2r\mu xy - \frac{2r\mu x^2y}{k} - 2by^2 - 2\eta xy^2 + \sigma_2^2y^2 \\ \le& -\frac{2r\mu^2x^3}{k} + \mu^2\sigma_1^2x^2 + 2r\mu^2x^2 + 2r\mu xy - 2\eta xy^2 - (2b-\sigma_2^2)y^2. \end{aligned}$$

Define a function $W=e^tV$, then

$$\begin{aligned} LW =& e^t(V+LV) \\ \le& e^t[\mu^2x^2 + 2\mu xy - \frac{2r\mu^2x^3}{k} + \mu^2\sigma_1^2x^2 + 2r\mu^2x^2 + 2r\mu xy \end{aligned}$$

$$-2\eta xy^2-(2b-\sigma_2^2-1)y^2],$$

therefore there is a positive constant H_1 such that $LW \leq H_1 e^t$, which yields

$$\mathrm{d}W \leq H_1 e^t + 2\mu e^t(\mu x+y)\sigma_1 x\mathrm{d}B_1(t) + 2e^t(\mu x+y)\sigma_2 y\mathrm{d}B_2(t), \tag{3.4}$$

integrating the equation (3.4) from 0 to t and taking expectation yields

$$E(e^t(\mu x+y)^2) \leq W(0) + H_1(e^t-1).$$

Noting that $\mu<1$, one observes that

$$\begin{aligned}E|X(t)|^2 =&E[x^2+y^2] \leq \frac{1}{\mu^2}E[(\mu x+y)^2]\\ \leq&\frac{1}{\mu^2}e^{-t}W(0)+\frac{1}{\mu^2}H_1(1-e^{-t})\\ \leq&\frac{1}{\mu^2}W(0) \triangleq H_2.\end{aligned}$$

An application of the Chebyshev inequality yields

$$P\{|X(t)|>M\} \leq \frac{E|X(t)|^2}{M^2}.$$

One can see that

$$\limsup_{t\to\infty} P\{|X(t)|>M\} \leq \frac{H_2}{M^2} = \frac{\varepsilon}{2} < \varepsilon \ \text{a.}s.,$$

where $\varepsilon \in (0,1)$, $M=\frac{\sqrt{2H_2}}{\sqrt{\varepsilon}}$. □

4. Asymptotic behavior of the solution

It is interesting for the stochastic system (1.2) to discuss the asymptotic behavior of the solution around the boundary equilibrium point $(k,0)$ of system (1.1). Then it yields the following results.

Theorem 4.1. *Suppose* $b>\max\{\frac{\mu\beta}{2\sqrt{a}}+\frac{\sigma_2^2}{2},\ \frac{k\mu\beta}{a}\}$, *and* $(x(t),y(t))$ *is the solution of system (1.2) with any initial value* $(x(0),y(0))\in\mathbb{R}_+^2$. *Then the following inequality holds:*

$$\limsup_{t\to\infty}\frac{1}{t}E[\int_0^t[(x(s)-k)^2+y^2(s)]\mathrm{d}s] \leq \frac{k\sigma_1^2}{2m_1},$$

where $m_1=\min\{\frac{r}{k}, b-\frac{\mu\beta}{2\sqrt{a}}-\frac{\sigma_2^2}{2}\}$.

Proof. Define a function $V:\mathbb{R}_+^2\to\overline{\mathbb{R}}_+$ by:

$$V(x,y)=x-k-\ln\frac{x}{k}+\frac{1}{2}y^2+\frac{1}{\mu}y. \tag{4.1}$$

Applying Itô's formula, we have:

$$LV(x,y)=(x-k)(r-\frac{rx}{k}-\frac{\beta y}{a+x^2})+\frac{k\sigma_1^2}{2}+y^2(\frac{\mu\beta x}{a+x^2}-b-\eta x)$$

$$
\begin{aligned}
&+\frac{\sigma_2^2 y^2}{2}+\frac{1}{\mu}(\frac{\mu\beta xy}{a+x^2}-by-\eta xy)\\
=&(x-k)(\frac{r}{k}k-\frac{rx}{k}-\frac{\beta y}{a+x^2})+\frac{k\sigma_1^2}{2}+y^2(\frac{\mu\beta x}{a+x^2}-b\\
&+\frac{\sigma_2^2}{2})+\frac{1}{\mu}(\frac{\mu\beta xy}{a+x^2}-by-\eta xy)-\eta xy^2\\
\leq&-\frac{r}{k}(x-k)^2+\frac{k\sigma_1^2}{2}-(b-\frac{\mu\beta}{2\sqrt{a}}-\frac{\sigma_2^2}{2})y^2+(\frac{k\beta}{a}-\frac{b}{\mu})y\\
\leq&-m_1[(x-k)^2+y^2]+\frac{k\sigma_1^2}{2},
\end{aligned}
$$

where $m_1=\min\{\frac{r}{k},b-\frac{\mu\beta}{2\sqrt{a}}-\frac{\sigma_2^2}{2}\}$. Then

$$
\mathrm{d}V(x,y)\leq -m_1[(x-k)^2+y^2]+\frac{k\sigma_1^2}{2}+\sigma_1(x-k)\mathrm{d}B_1(t)+\sigma_2(y^2+\frac{y}{\mu})\mathrm{d}B_2(t).
$$

Integrating the equation from 0 to t,

$$
\begin{aligned}
V(x,y)\leq& V(0,0)-m_1\int_0^t[(x(s)-k)^2+y^2(s)]\mathrm{d}s+\frac{k\sigma_1^2}{2}t+\sigma_1\int_0^t(x(s)\\
&-k)\mathrm{d}B_1(s)+\sigma_2\int_0^t(y(s)^2+\frac{y(s)}{\mu})\mathrm{d}B_2(s),
\end{aligned}
$$

next taking expectation on both sides,

$$
0\leq E[V(x,y)]\leq E[V(0,0)]-m_1E[\int_0^t[(x(s)-k)^2+y^2(s)]\mathrm{d}s]+\frac{k\sigma_1^2}{2}t,
$$

dividing both sides by t and taking the superior limit on both sides of the inequality, the following conclusion is got,

$$
\limsup_{t\to\infty}\frac{1}{t}E[\int_0^t[(x(s)-k)^2+y^2(s)]\mathrm{d}s]\leq\frac{k\sigma_1^2}{2m_1}. \tag{4.2}
$$

□

Remark 4.1. According to Theorem 4.2, although $(k,0)$ is no longer the equilibrium point of system (1.2), the solution of a stochastic system (1.2) oscillates up and down around the equilibrium point $(k,0)$ of its deterministic system. When the noise intensity σ_1 is small, the solution of the stochastic system oscillates within a small neighborhood at the equilibrium point $(k,0)$.

5. Extinction and persistence

Now, we will show the extinction and persistence of the stochastic system (1.2). These properties can be used to estimate and calculate the extinctions of stochastic systems of predator and prey populations. These conclusions will be useful in real-world .

Theorem 5.1. *Let* $(x(t),y(t))$ *be the solution of system (1.2) with any initial value* $(x(0),\ y(0))\in\mathbb{R}_+^2$*, The following conclusions are established:*

(i). *If* $r < \frac{\sigma_1^2}{2}$, *then the prey and predator populations will be extinct with probability one, that is to say,* $\lim\limits_{t\to\infty} x(t) = 0$ a.*s*. *and* $\lim\limits_{t\to\infty} y(t) = 0$ a.*s*.

(ii). *If* $r > \frac{\sigma_1^2}{2}$, $\mu\beta\frac{\sqrt{a}}{2a} < \frac{\sigma_2^2}{2} + b$, *then the predator populations will be extinct with probability one while the prey persists, which is* $\lim\limits_{t\to\infty} y(t) = 0$ a.*s*. *and* $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds = \frac{k\left(r-\frac{\sigma_1^2}{2}\right)}{r}$.

Proof.

(i). According to Itô's formula, we have

$$\mathrm{d}\ln x = \left[r\left(1-\frac{x}{k}\right) - \frac{\beta y}{a+x^2} - \frac{1}{2}\sigma_1^2\right]\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t).$$

Integrating the equation from 0 to t,

$$\begin{aligned}\ln x(t) - \ln x(0) &= \left(r - \frac{1}{2}\sigma_1^2\right)t - \frac{r}{k}\int_0^t x(s)\mathrm{d}s - \beta\int_0^t \frac{y(s)}{a+x^2(s)}\mathrm{d}s + \sigma_1 B_1(t)\\ &\le \left(r - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 B_1(t),\end{aligned}$$

then dividing both sides by t, and taking the superior limit on both sides, noting that $\lim\limits_{t\to\infty}\frac{B_1(t)}{t} = 0$ a.*s*., we have $\limsup\limits_{t\to\infty}\frac{\ln x(t)}{t} \le r - \frac{1}{2}\sigma_1^2 < 0$ a.*s*., which implies $\lim\limits_{t\to\infty} x(t) = 0$ a.*s*. Thus for any $\varepsilon \in \left(0, \frac{a\sigma_2^2}{2\mu\beta}\right)$, there exists a $T = T(\varepsilon) > 0$, so that when $t > T$, there is $x(t) < \varepsilon$. Then when $t > T$, for $y(t)$, we have

$$\begin{aligned}\mathrm{d}y &= \left[\frac{\mu\beta xy}{a+x^2} - by - \eta xy\right]\mathrm{d}t + \sigma_2 y\mathrm{d}B_2(t)\\ &\le \frac{\mu\beta\varepsilon y}{a}\mathrm{d}t + \sigma_2 y\mathrm{d}B_2(t).\end{aligned}$$

Applying Itô's formula to $\ln y$, it then follows that

$$\begin{aligned}\mathrm{d}\ln y &= \left(\frac{\mu\beta x}{a+x^2} - b - \eta x - \frac{1}{2}\sigma_2^2\right)\mathrm{d}t + \sigma_2\mathrm{d}B_2(t),\\ &\le \left(\frac{\mu\beta\varepsilon}{a} - \frac{\sigma_2^2}{2}\right)\mathrm{d}t + \sigma_2\mathrm{d}B_2(t),\end{aligned}$$

integrating the above inequality from 0 to t, and then dividing both sides by t, one may arrive at

$$\frac{\ln y}{t} \le \frac{\ln y(0)}{t} + \left(\frac{\mu\beta\varepsilon}{a} - \frac{\sigma_2^2}{2}\right) + \frac{\sigma_2 B_2(t)}{t}.$$

Now, taking the superior limit on both sides of the inequality and noting that is $\lim\limits_{t\to\infty}\frac{B_2(t)}{t} = 0$, thus $\limsup\limits_{t\to\infty}\frac{\ln y(t)}{t} \le \frac{\mu\beta\varepsilon}{a} - \frac{\sigma_2^2}{2} < 0$ which means $\lim\limits_{t\to\infty} y(t) = 0$ a.*s*..

(ii). Applying Itô's formula to $\ln y$, it then follows that

$$\mathrm{d}\ln y=\left(\frac{\mu\beta xy}{a+x^2}-b-\eta x-\frac{1}{2}\sigma_2^2\right)\mathrm{d}t+\sigma_2\mathrm{d}B_2(t),$$

integrating the above equation from 0 to t, and then dividing both sides by t, one may arrives at

$$\begin{aligned}\frac{\ln y(t)-\ln y\,(0)}{t}&=-\left(b+\frac{1}{2}\sigma_2^2\right)+\frac{\mu\beta\int_0^t\frac{x(s)}{a+x^2(s)}\mathrm{d}s}{t}-\frac{\eta\int_0^t x\,(s)\mathrm{d}s}{t}+\frac{\sigma_2B_2(t)}{t}\\&\leq-\left(b+\frac{1}{2}\sigma_2^2\right)+\frac{\mu\beta\int_0^t\frac{x(s)}{a+x^2(s)}\mathrm{d}s}{t}+\frac{\sigma_2B_2(t)}{t}\\&\leq-\left(b+\frac{1}{2}\sigma_2^2\right)+\frac{\sqrt{a}\mu\beta}{2a}+\frac{\sigma_2B_2(t)}{t}.\end{aligned}$$

Now, taking the superior limit on both sides of the inequality and noting that $\lim\limits_{t\to\infty}\frac{B_2(t)}{t}=0$, thus $\lim\limits_{t\to\infty}\sup\frac{\ln y(t)}{t}\leq-\left(b+\frac{1}{2}\sigma_2^2\right)+\mu\beta\frac{\sqrt{a}}{a}<0$ a.s. that is $\lim\limits_{t\to\infty}y(t)=0$ a.s. Therefore, for any $\varepsilon\in\left(0,\frac{a}{\beta}\left(r-\frac{\sigma_1^2}{2}\right)\right)$, there exists a $T=T\,(\varepsilon)>0$ such that when $t>T$, we have $y(t)<\varepsilon$. It then follows from the first equation of $x(t)$ in system (1.2) that when $t>T$,

$$\begin{aligned}\mathrm{d}x&=\left[rx\left(1-\frac{x}{k}\right)-\frac{\beta xy}{a+x^2}\right]\mathrm{d}t+\sigma_1x\mathrm{d}B_1(t)\\&\geq\left[rx\left(1-\frac{x}{k}\right)-\frac{\varepsilon\beta x}{a}\right]\mathrm{d}t+\sigma_1x\mathrm{d}B_1(t)\\&=\left[x\left(r-\frac{\varepsilon\beta}{a}-\frac{rx}{k}\right)\right]\mathrm{d}t+\sigma_1x\mathrm{d}B_1(t).\end{aligned}$$

It then follows from Lemma 3.3 and stochastic comparison theorem that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x\,(s)\mathrm{d}s\geq\frac{k\left(r-\frac{\beta\varepsilon}{a}-\frac{\sigma_1^2}{2}\right)}{r},$$

on the right-hand side of the above inequality, let $\varepsilon\to0$, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x\,(s)\mathrm{d}s\geq\frac{k\left(r-\frac{\sigma_1^2}{2}\right)}{r}.$$

On the other hand, note that

$$\begin{aligned}\mathrm{d}x&=\left[rx\left(1-\frac{x}{k}\right)-\frac{\beta xy}{a+x^2}\right]\mathrm{d}t+\sigma_1x\mathrm{d}B_1(t)\\&\leq\left[rx\left(1-\frac{x}{k}\right)\right]\mathrm{d}t+\sigma_1x\mathrm{d}B_1(t).\end{aligned}$$

By Lemma 3.3, it follows that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x\,(s)\mathrm{d}s\leq\frac{k\left(r-\frac{\sigma_1^2}{2}\right)}{r}.$$

Hence, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)\mathrm{d}s=\frac{k\left(r-\frac{\sigma_1^2}{2}\right)}{r}.$$

□

6. Numerical simulations

In this section, we mainly illustrate the effects of noise intensity on the model system (1.1). Applying the Milstein method [23] yields the discrete equation as follows:

$$\begin{cases} x_{k+1}=x_k+[rx_k(1-\frac{x_k}{k})-\frac{\beta x_k y_k}{a+x_k^2}]\triangle t+\sigma_1 x_k\sqrt{\triangle t}\xi_{1,k}+\frac{\sigma_1^2}{2}x_k(\xi_{1,k}^2-1)\triangle t,\\ y_{k+1}=y_k+[\frac{\mu\beta x_k y_k}{a+x_k^2}-by_k-\eta x_k y_k]\triangle t+\sigma_2 y_k\sqrt{\triangle t}\xi_{2,k}+\frac{\sigma_2^2}{2}y_k(\xi_{2,k}^2-1)\triangle t,\end{cases}$$

where time step $\triangle t>0,\xi_{i,k}$ $(i=1,2$ and $k=1,2,\cdots,n)$ are two independent Gaussian random variables, and obey the normal distribution with mean 0 and variance 1.

Choosing initial value $(x_0,y_0)=(0.45,0.45)$, $\triangle t=0.001$, $k=0.8$ [19], $r=0.05$, $\beta=0.6$, $a=0.8$, $\mu=0.8$, $b=0.24$, $\eta=0.01$.

Assumed to be disturbed by a small amount of white noise. The solution of a stochastic system (1.2) oscillates up and down around the equilibrium point of its deterministic system, and is persistent by Figure 1. In Figure 2, we change the

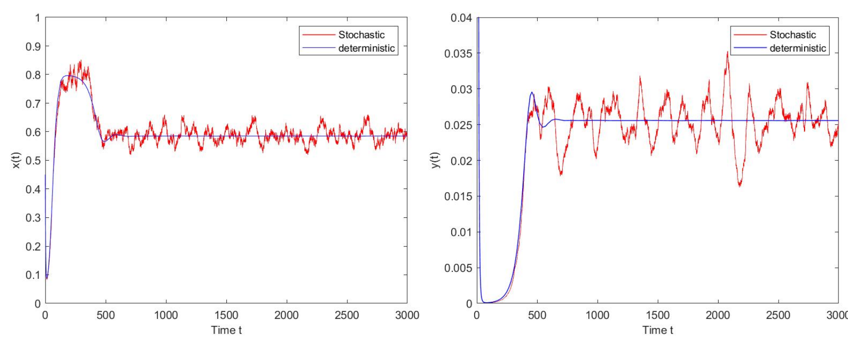

Figure 1. the paths of the populations $x(t)$ and $y(t)$ with different values of $\sigma_1=0.01,\sigma_2=0.01$.

parameters $\sigma_1=0.9$. This means the condition of $r<\frac{\sigma_1^2}{2}$ in Theorem 5.1 are satisfied, the curves of the population $x(t)$ of the prey and $y(t)$ of the predator tend to zero.

Choose $\sigma_1=0.1$, $\sigma_2=1.2$, then Theorem 5.1 holds because of $r>\frac{\sigma_1^2}{2}$, $\mu\beta\frac{\sqrt{a}}{2a}<\frac{\sigma_2^2}{2}+b$. According to Theorem 5.1, the predator dies out while the prey persists. This can be verified by Figure 3.

Now we choose $\sigma_1=0.8$, $\sigma_2=1.1$, the population of the prey and predator become extinct by Theorem 5.1. This conclusion can be verified by the curves of Figure 4. It is clear from these figures that changing the parameters σ_1, σ_2 leads to a range of dynamical behaviors.

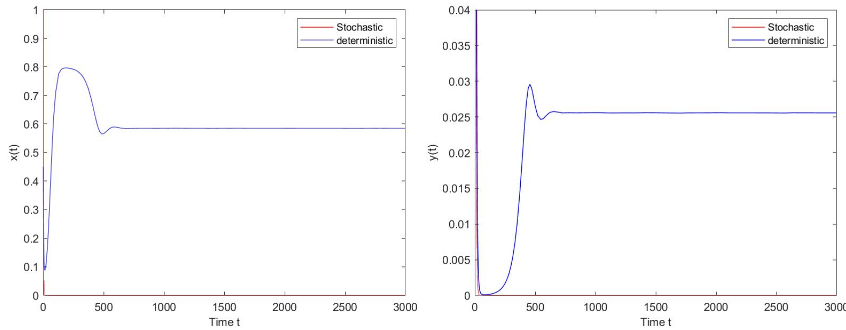

Figure 2. the paths of the populations $x(t)$ and $y(t)$ with $\sigma_1 = 0.9, \sigma_2 = 0.01$.

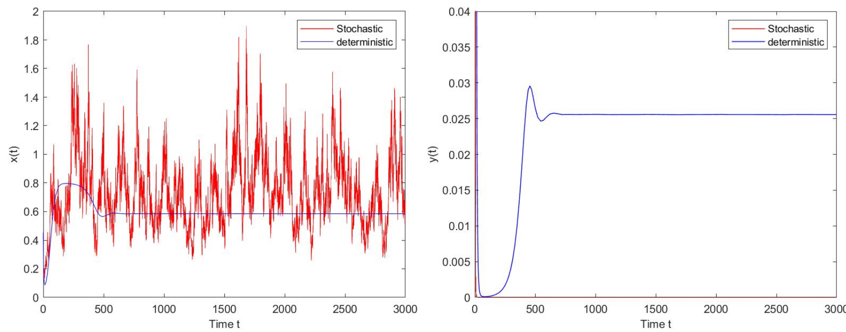

Figure 3. the paths of the populations $x(t)$ and $y(t)$ with $\sigma_1 = 0.1, \sigma_2 = 1.2$.

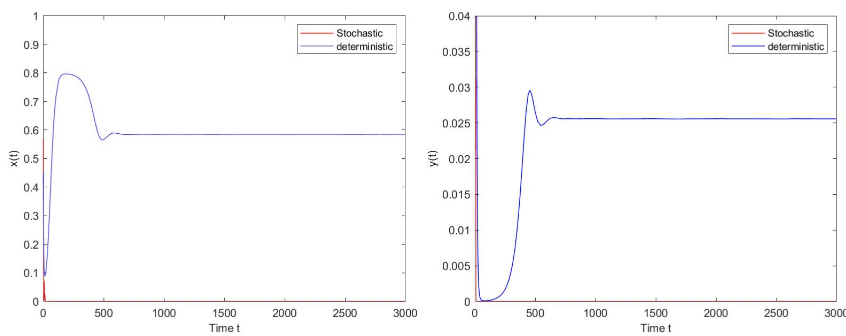

Figure 4. the paths of the populations $x(t)$ and $y(t)$ with $\sigma_1 = 0.8, \sigma_2 = 1.1$.

As we known, the anti-predation behavior of the prey does not directly affect the prey population, but only affects the growth of the predator population [2] which can illustrated by Figure 5.

Next, we choose a lager value of $\eta = 0.8$, this results in the equilibrium point E_1 of system (1.1) not being present. So, we discuss the point $(k, 0)$. As the Figure 6 show, stable population size, stable regions in stable states and amplitude of predator oscillations are affected to some extent.

Now, we choose lager value of $\sigma_1 = 0.1, \sigma_2 = 0.3$, the predator dies out while the prey persists from Theorem 5.1. Figure 7 indicates that the solution of the stochastic system vibrates up and down around the solution of its deterministic system.

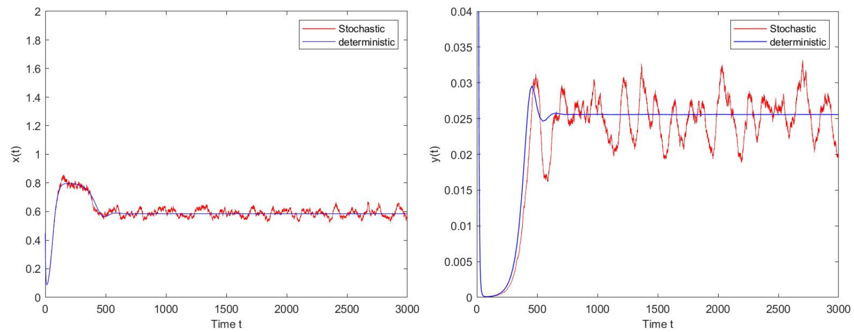

Figure 5. the paths of the populations $x(t)$ and $y(t)$ with $\sigma_1 = 0.01, \sigma_2 = 0.01, \eta = 0.01$.

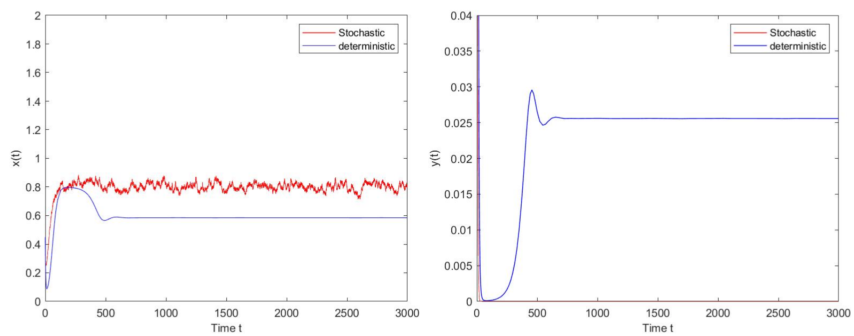

Figure 6. the paths of the populations $x(t)$ and $y(t)$ with $\eta = 0.8, \sigma_1 = 0.01, \sigma_2 = 0.01$.

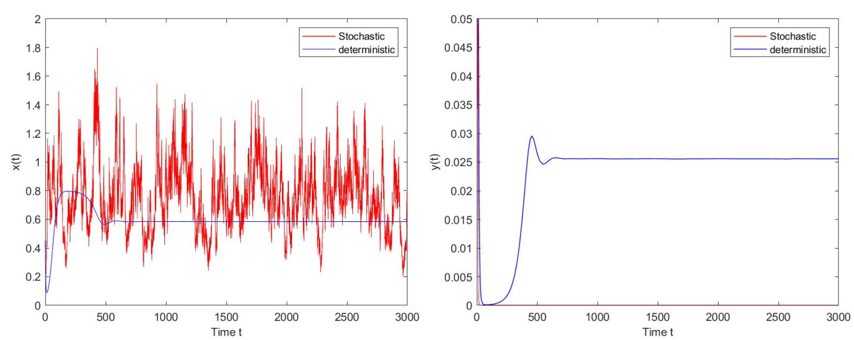

Figure 7. the paths of the populations $x(t)$ and $y(t)$ with $\eta = 0.8, \sigma_1 = 0.1, \sigma_2 = 0.3$.

Acknowledge

We would like to express our sincere appreciation to the reviewers for their valuable comments and suggestions to this manuscript.

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